Optimal Harvest Constants

Jack Hanke

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1 Introduction

Let $X_{m,p} \sim B(tm, p)$ be a discrete random variable with parameters m, called the "speed", and p, called the "hit-rate". We are interested in the following function

$$\Lambda_{m,p}(n) = \max_t \frac{1}{t}\mathbb{P}(X_{m,p} \geq n) = \max_t \frac{1}{t}\left(1 - (1-p)^{mt-n+1}\sum_{k=0}^{n-1} p^k \binom{mt}{k}\right). \tag{1}$$

It turns out that $\Lambda_{3,2^{-12}}(n)$ is an important function in Minecraft farm optimization. For studying Minecraft, approximating $X_{m,p}$ with a Poisson random variable $Y \sim \text{Poisson}(t)$ is appropriate. We then define the corresponding function for Y, namely

$$\lambda(n) = \max_t \frac{1}{t}\mathbb{P}(Y \geq n) = \max_t \frac{1}{t}\left(1 - e^{-t}\sum_{k=0}^{n-1} \frac{t^k}{k!}\right) \tag{2}$$

We then have for large n and small p

$$\Lambda_{m,p}(n) \approx \frac{1}{mp}\lambda(n). \tag{3}$$

$$\begin{aligned} 0 &= (1-p)^{n-1}e^{-m\ln(1-p)t} + \frac{(p)^{n-1}}{(n-1)!}\ln(1-p)(mt)^n - 1 \\ &+ \sum_{k=1}^{n-1}\left(\frac{\ln(1-p)p^{k-1}}{(k-1)!} + \sum_{i=k}^{n-1}((k-1)s(i,k) + \ln(1-p)s(i,k-1))\frac{p^i}{i!}\right)(mt)^k, \end{aligned}$$

and similarly $\lambda(n)$ is the $t > 0$ that solves

$$0 = e^t - \frac{t^n}{(n-1)!} - \sum_{k=0}^{n-1}\frac{t^k}{k!}.$$

Clearly root finding algorithms provide solutions to the above equations. See below for small values of n

n	$\lambda(n)$	$(2^{12}/3)\lambda(n)$	$\Lambda_{3,2^{-12}}(n)$	Opimal Tick
1	0	0	0	1
2	1.7932821329	2448.4278721205	2449.1597630796	2449
3	3.3836342829	4619.7886741889	4620.6492653516	4621
4	4.8812774913	6664.5708681839	6665.5420888695	6666
5	6.3225055510	8632.3275790109	8633.3997579089	8633
6	7.7245836004	10546.6314757967	10547.7982629752	10548
7	9.0973444026	12420.9075576501	12422.1643509140	12422
8	10.4470306813	14263.6792235702	14265.0224812591	14265
9	11.7779066065	16080.7684866791	16082.1953698680	16082
10	13.0930424983	17876.3673576602	17877.8755221598	17878
11	14.3947389019	19653.6168474178	19655.2043132205	19655
12	15.6847741726	21414.9450036956	21416.6100682348	21417
13	16.9645579170	23162.2764093484	23164.0175872686	23164
14	18.2352307260	24897.1683511953	24898.9843310297	24899
15	19.4977315702	26620.9028372304	26622.7924493235	26623

We are interested in exact expressions for $\Lambda_{m,p}(n)$ and $\lambda(n)$. First note that both definitions are of the form

$$0 = b_1 e^{b_0 t} + \sum_{k=0}^{n} c_k t^k$$

for a given n. For $\Lambda_{m,p}(n)$ we have $b_0 = -m\ln(1-p)$, $b_1 = (1-p)^{n-1}$, $c_0 = -1$, $c_n = \frac{(mp)^{n-1}}{(n-1)!} m\ln(1-p)$, and $c_k = m^k(m\ln(1-p)+k-1)\sum_{j=k}^{n-1}\frac{p^j}{j!}s(j,k)$ for $2 \le k \le n-1$. For $\lambda(n)$ we have $b_0 = b_1 = 1$, $c_n = -((n-1)!)^{-1}$, and $c_k = -(k!)^{-1}$ for $0 \le k \le n-1$.

To solve Equation **??** exactly requires Lagrange Inversion.

Theorem 1. *The compositional inverse of f around a is given by*

$$f^{-1}(t) = a + \sum_{n\ge 1} \frac{g_n}{n!}(t - f(a))^n$$

where

$$g_n = \lim_{t\to a} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{t-a}{f(t)-f(a)}\right)^n .$$

Let $f_n(t)$ be

$$f_n(t) = b_1 e^{b_0 t} + \sum_{k=0}^{n} c_k t^k$$

and $h_n(t)$ be

$$h_n(t) = \frac{f_n(t) - f_n(a)}{t-a} = \frac{b_1 e^{b_0 t} - b_1 e^{b_0 a}}{t-a} - \sum_{k=1}^{n} c_k \sum_{j=0}^{k-1} x^i a^{k-1-i} = \frac{b_1 e^{b_0 t} - b_1 e^{b_0 a}}{t-a} - \sum_{k=0}^{n-1}\sum_{i=k+1}^{n} c_i a^{k-1-i} x^k.$$

We can rewrite g_n using Faa di Bruno's formula to give

$$g_n = \sum_{k=1}^{n-1} (-1)^k \frac{(n-1+k)!}{(n-1)! h_n(a)^{n+k}} B_{n-1,k}(h_n'(a), h_n''(a), \ldots, h_n^{(n-k)}(a))$$

where $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ are the partial Bell Polynomials. This gives a closed form following the computation of $\lim_{t \to a} h_n^{(j)}(t) = h_n^{(j)}(a)$, which can be written as

$$h_n^{(j)}(a) = \frac{b_1 b_0^j e^{b_0 a}}{j+1} - \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} \sum_{i=k}^{n-1} c_{i+1} a^{i-j}.$$

$$\begin{aligned}
\sum_{j \geq 1} h_n^{(j)}(a) \frac{x^j}{j!} &= \sum_{j \geq 1} \frac{b_1 b_0^j e^{b_0 a}}{j+1} \frac{x^j}{j!} - \sum_{j \geq 1} \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} \sum_{i=k}^{n-1} c_{i+1} a^{i-j} \frac{x^j}{j!} \\
&= \frac{b_1 e^{b_0 a}}{b_0 x} \sum_{j \geq 1} \frac{(b_0 x)^{j+1}}{(j+1)!} - \sum_{j=1}^{n-1} \sum_{k=j}^{n-1} \binom{k}{j} \sum_{i=k}^{n-1} c_{i+1} a^{i-j} x^j \\
&= \frac{b_1 e^{b_0 a}}{b_0 x} (\exp(b_0 x) - b_0 x - 1) - \sum_{j=1}^{n-1} \sum_{k=j}^{n-1} \binom{k}{j} \sum_{i=k}^{n-1} c_{i+1} a^{i-j} x^j
\end{aligned}$$

Example 2. For $\lambda(2)$ we have that

$$\sum_{j \geq 1} h_n^{(j)}(a) \frac{x^j}{j!} = \frac{1}{x} e^{x+2} + x - e^2 - \frac{e^2}{x}$$

Using partial Bell polynomial identities allows for exact expressions of $\lambda(2)$ and $\lambda(3)$, namely

$$\lambda(2) = \frac{e^2-3}{e^2-5} - \sum_{n \geq 1} \left(\sum_{i=1}^{n} \sum_{j=0}^{i} \sum_{k=0}^{j} \frac{(-1)^{n+k} (n+i)! S(n-i+j+k, k) e^{2j}}{(n+1)!(i-j)!(j-k)!(n-i+j+k)!(e^2-5)^i} \right) \left(\frac{e^2-7}{e^2-5} \right)^{n+1}$$

and

$$\lambda(3) = \frac{3e^4-71}{e^4-29} - \sum_{n \geq 1} g_n \left(\frac{e^4-45}{e^4-29} \right)^{n+1}$$

where

$$g_n = \sum_{i=1}^{n} \sum_{j=0}^{n} \sum_{k=0}^{i} \sum_{\ell=0}^{k-i} \frac{(-1)^{n-\ell} (n+i)! 13^{2k+j-n} S(j+\ell, \ell) e^{4(j-k)}}{(n+1)!(j+\ell)!(i-k-\ell)! k! 2^k (e^4-29)^i} \binom{k}{n-j-k}.$$

Here $\binom{k}{n-j-k} = 0$ if $k < n-j-k$ and $S(a,b)$ is the (a,b)-th Stirling Number of the second kind, which is defined as $S(a,b) = \frac{1}{b!}\sum_{k=0}^{b}(-1)^k\binom{b}{k}(b-k)^a$.

Simplifying Bell Polynomials for large n becomes increasingly unfeasible. To get an exact expression, generating functions and complex analysis can be used to get an exact expression. First the simpler case must be examined.

Theorem 3. *The compositional inverse of f around a can be written as*

$$f^{-1}(x) = a + G(x - f(a)) \tag{4}$$

where

$$G(x) \;=\; \frac{1}{\pi}\int_0^\infty e^{-\frac{u^2}{x}}\int_0^{2\pi}\exp\left(is + ue^{-is}\left(\sum_{j\geq 0} h^{(j)}(a)\frac{(ue^{is})^j}{j!}\right)\right)dsdu$$

and $h^{(j)}(a) = \lim_{x\to a}\frac{d^j}{dx^j}\frac{x-a}{f(x)-f(a)}$.

Proof. The Lagrange coefficient $g_n = \lim_{x\to a}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x-a}{f(x)-f(a)}\right)^n$, when written using Faa di Bruno's Formula gives

$$g_n = \sum_{k=0}^{n-1}\frac{n!}{(n-k)!}h(a)^{n-k}B_{n-1,k}(h'(a), h''(a), \ldots, h^{(n-k)}(a)).$$

We then let

$$\begin{aligned}
\hat{G}(x,y) &= \sum_{n,m\geq 0}\sum_{j=0}^{m-1}\binom{m}{j}j!B_{n-1,j}(x_1,\ldots)h(a)^{m-j}\frac{x^n}{n!}\frac{y^m}{m!} \\
&= \phi(x,y)(\exp(h(a)y)) \\
&= \exp\left(y\left(\sum_{j\geq 0}h^{(j)}(a)\frac{x^j}{j!}\right)\right)
\end{aligned}$$

To extract the off-diagonal we use compute the complex integral

$$\begin{aligned}
\bar{G}(x) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{is}}{\sqrt{x}}\hat{G}(\sqrt{x}e^{is}, \sqrt{x}e^{-is})ds \\
&= \frac{1}{2\pi\sqrt{x}}\int_0^{2\pi}\exp\left(is + \sqrt{x}e^{-is}\left(\sum_{j\geq 0}h^{(j)}(a)\frac{(\sqrt{x}e^{is})^j}{j!}\right)\right)ds,
\end{aligned}$$

which leaves the series with an extra $\frac{1}{n!}$ term. To remove this we convert the series using the following integral transform

$$\begin{aligned}
G(x) &= \sum_{n\geq 1} g_n \frac{x^n}{n!} = \int_0^\infty x e^{-u} \bar{G}(ux) du \\
&= 2\int_0^\infty u e^{-\frac{u^2}{x}} \bar{G}(u^2) du \\
&= \frac{1}{\pi}\int_0^\infty e^{-\frac{u^2}{x}} \int_0^{2\pi} \exp\left(is + ue^{-is}\left(\sum_{j\geq 0} h^{(j)}(a)\frac{(ue^{is})^j}{j!}\right)\right) dsdu
\end{aligned}$$

□

This result can be used to derive a double-integral expression for the Lambert W function.

Example 4. Let $f(x) = xe^x$ at $a = 0$, giving $h^{(j)}(a) = \lim_{x\to 0} \frac{d^j}{dx^j} e^{-x} = (-1)^j$. We then have that

$$G(x) = \frac{1}{\pi}\int_0^\infty e^{-\frac{u^2}{x}} \int_0^{2\pi} \exp(is + ue^{-is}\exp(-ue^{is}))dsdu$$

The following theorem is used to derive exact expressions for $\Lambda_{m,p}(n)$ and $\lambda(n)$, and is a modification of Theorem **??**.

Theorem 5. *The compositional inverse of f around a can be written as*

$$f^{-1}(x) = a + G(x - f(a)) \tag{5}$$

where

$$\begin{aligned}
G(x) &= \frac{h(a)}{\pi^2 x}\int_0^\infty \int_0^\infty uve^{-v^2-\frac{u^2}{x}} \int_0^{2\pi}\int_0^{2\pi} \exp\left(ive^{it}\left(\sum_{j\geq 1} h^{(j)}(a)\frac{(ue^{is})^j}{j!}\right)\right) \\
&* \left(h(a) - ue^{-is} - ive^{-it}\right)^{-1} dsdtdudv
\end{aligned}$$

and $h^{(j)}(a) = \lim_{x\to a} \frac{d^j}{dx^j}\frac{f(x)-f(a)}{x-a}$.

Proof. The Lagrange coefficient $g_n = \lim_{x\to a}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{f(x)-f(a)}{x-a}\right)^{-n}$, when written using Faa di Bruno's Formula gives

$$g_n = \sum_{k=0}^{n-1}(-1)^k \frac{(n-1+k)!}{(n-1)!h(a)^{n+k}} B_{n-1,k}(h'(a), h''(a), h'''(a), \ldots, h^{(n-k)}(a))$$

We then let

$$
\begin{aligned}
\hat{G}(x,y) &= \sum_{n,m\geq 0}\binom{n+m}{n}\frac{1}{n!}B_{n,m}(x_1,\dots)h(a)^{-n-m}x^n y^m \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(\sqrt{x}e^{is},\sqrt{y}e^{it})\left(1-\frac{\sqrt{x}e^{-is}}{h(a)}-\frac{\sqrt{y}e^{-it}}{h(a)}\right)^{-1}dsdt \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\exp\left(\sqrt{y}e^{it}\left(\sum_{j\geq 1}h^{(j)}(a)\frac{(\sqrt{x}e^{is})^j}{j!}\right)\right)\left(1-\frac{\sqrt{x}e^{-is}}{h(a)}-\frac{\sqrt{y}e^{-it}}{h(a)}\right)^{-1}dsdt
\end{aligned}
$$

We then have that

$$
\begin{aligned}
G(x) &= \sum_{n\geq 0}g_n\frac{x^n}{n!} = \int_0^\infty\int_0^\infty \hat{G}(ux,-v)e^{-u-v}dudv \\
&= \frac{1}{4\pi^2}\int_0^\infty\int_0^\infty e^{-u-v}\int_0^{2\pi}\int_0^{2\pi}\exp\left(i\sqrt{v}e^{it}\left(\sum_{j\geq 1}h^{(j)}(a)\frac{(\sqrt{ux}e^{is})^j}{j!}\right)\right) \\
&* \left(1-\frac{\sqrt{ux}e^{-is}}{h(a)}-\frac{i\sqrt{v}e^{-it}}{h(a)}\right)^{-1}dsdtdudv
\end{aligned}
$$

after the substitutions $u\to\frac{u^2}{x}$ and $v\to v^2$ we have

$$
\begin{aligned}
G(x) &= \frac{h(a)}{\pi^2 x}\int_0^\infty\int_0^\infty uve^{-v^2-\frac{u^2}{x}}\int_0^{2\pi}\int_0^{2\pi}\exp\left(ive^{it}\left(\sum_{j\geq 1}h^{(j)}(a)\frac{(ue^{is})^j}{j!}\right)\right) \\
&* \left(h(a)-ue^{-is}-ive^{-it}\right)^{-1}dsdtdudv.
\end{aligned}
$$

$\square$

We can use Theorem **??** to get an exact expression for

Example 6. For $\lambda(2)$ we have that

$$
\sum_{j\geq 1}h_n^{(j)}(a)\frac{x^j}{j!} = \frac{1}{x}e^{x+2}+x-e^2-\frac{e^2}{x}
$$

and using $a=2$, $h(2)=e^2-5$ and $x=7-e^2$ we can write

$$
\begin{aligned}
\lambda(2) &= \frac{h(2)}{\pi^2 x}\int_0^\infty\int_0^\infty uve^{-v^2-\frac{u^2}{x}}\int_0^{2\pi}\int_0^{2\pi}\exp\left(ive^{it}(ue^{-is}e^{ue^{is}+2}+ue^{is}-e^2-e^2ue^{-is})\right) \\
&* \left(h(2)-ue^{-is}-ive^{-it}\right)^{-1}dsdtdudv.
\end{aligned}
$$

2 Asymptotics for $\lambda(\mathbf{n})$

The computation of large values of $\Lambda_{m,p}(n)$ and $\lambda(n)$ can be difficult for the above exact expressions and for modern root finding algorithms. This motivates the following theorem.

Theorem 1.

$$\lambda(n) = n + \sqrt{n \ln\left(\frac{n}{2\pi}\right)} + \dots$$

Proof. Using the continuous analog of the defining equation of $\lambda(n)$, namely

$$1 = \int_0^1 e^{(n\ln(1-u)+\lambda(n)u)} du, \tag{6}$$

we can use Laplace's method of approximation to derive leading terms of $\lambda(n)$.

$$1 = \int_0^1 e^{(n\ln(t)+\lambda(n)(1-t))} dt = \int_0^1 e^{nf(n,t))} dt \tag{7}$$

there $f(n,t) = \ln(t) + \frac{\lambda(n)}{n}(1-t)$. We start by writing a taylor explansion for $f(n,t)$.

$$f(n,t) = \ln\left(\frac{n}{\lambda(n)}\right) + \frac{\lambda(n)}{n} - 1 + \sum_{k\geq 2}\left(\frac{\lambda(n)}{n}\right)^k \frac{(-1)^{k-1}}{k}\left(t - \frac{n}{\lambda(n)}\right)^k \tag{8}$$

Equation **??** is valid for $t \in (\frac{n}{\lambda(n)} - 1, \frac{n}{\lambda(n)} + 1)$. As $\frac{n}{\lambda(n)} > 0$, we have $(0,1) \subset (\frac{n}{\lambda(n)} - 1, \frac{n}{\lambda(n)} + 1)$ and can integrate over the region $(0,1)$.

Then for equation **??** we have

$$\int_0^1 \exp(nf(n,t))dt = \left(\frac{n}{\lambda(n)}\right)^n e^{\lambda(n)} e^{-n} \int_0^1 \exp\left(n\sum_{k\geq 2}\frac{(-1)^{k-1}}{k}\left(\frac{\lambda(n)}{n}t - 1\right)^k\right) dt \tag{9}$$

Let us call the last integral

$$I(n,\lambda(n)) = \int_0^1 \exp\left(n\sum_{k\geq 2}\frac{(-1)^{k-1}}{k}\left(\frac{\lambda(n)}{n}t - 1\right)^k\right) dt \tag{10}$$

and the truncated series

$$I_m(n,\lambda(n)) = \int_0^1 \exp\left(n\sum_{k=2}^{m}\frac{(-1)^{k-1}}{k}\left(\frac{\lambda(n)}{n}t - 1\right)^k\right) dt \tag{11}$$

Following Laplace's method, we approximate I with I_2. We can then say that, for $n \to \infty$, that

$$\left(\frac{n}{\lambda(n)}\right)^n e^{\lambda(n)} e^{-n} I_2(n,\lambda(n)) \to 1$$

or

$$n\ln(n) - n\ln(\lambda(n)) + \lambda(n) - n + \ln(I_2(n,\lambda(n))) \to 0. \tag{12}$$

We then suppose that $\lambda(n) = n + n\delta_1(n)$. We can then say that $\delta_1(n) \to 0$ as $n \to \infty$.

$$-n\ln(1+\delta_1(n)) + n\delta_1(n) + \ln(I_2(n,\lambda(n)) \to 0 \tag{13}$$

What can be said about I_2?

$$I_2(n,\lambda(n)) = \int_0^1 \exp\left(-\left(\frac{\lambda(n)}{\sqrt{2n}}t - \sqrt{\frac{n}{2}}\right)^2\right)dt \tag{14}$$

and written in terms of common functions

$$I_2(n,\lambda(n)) = \sqrt{\frac{\pi n}{2}}\frac{1}{\lambda(n)}\left(\operatorname{erf}\left(\sqrt{\frac{n}{2}}\delta_1(n)\right) + \operatorname{erf}\left(\sqrt{\frac{n}{2}}\right)\right). \tag{15}$$

Therefore, we have

$$-n\ln(1+\delta_1(n)) + n\delta_1(n) + \frac{1}{2}\ln\left(\frac{\pi}{2}\right) + \frac{1}{2}\ln(n) - \ln(\lambda(n)) + \ln(2) \to 0 \tag{16}$$

as

$$\operatorname{erf}\left(\sqrt{\frac{n}{2}}\delta_1(n)\right) + \operatorname{erf}\left(\sqrt{\frac{n}{2}}\right) \to 2$$

quickly. We can then write

$$-n\ln(1+\delta_1(n)) + n\delta_1(n) - \frac{1}{2}\ln(n) - \ln(1+\delta_1(n)) + \frac{1}{2}\ln(2\pi) \to 0. \tag{17}$$

We can again use the Taylor expansion of $\ln(1+x)$ to get the further approximate

$$\frac{n\delta_1^2(n)}{2} + \frac{1}{2}\ln\left(\frac{2\pi}{n}\right) - \delta_1(n) + \frac{\delta_1^2(n)}{2} \to 0. \tag{18}$$

Finally, completing the square gives the following expression for $\delta_1(n)$

$$\delta_1(n) \approx \frac{1}{n+1}\left(1 + \sqrt{1 + (n+1)\ln\left(\frac{n}{2\pi}\right)}\right) \tag{19}$$

and ignoring the $-\ln(1+\delta_1(n))$ in Equation **??** gives the cleaner but less accurate expression

$$\delta_1(n) \approx \sqrt{\frac{1}{n}\ln\left(\frac{n}{2\pi}\right)} \tag{20}$$

□