RISK

Jack Hanke

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Let $t(a,b)$ be the probability that a attackers defeats b defenders in the board game RISK.

Theorem 1. *$t(a,b)$ can be exactly calculated.*

Proof. The success chance $t(a,b)$ has the following complicated recurrence relation.

Let

$$T(x,y) = \sum_{a,b\geq 0} t(a,b)x^a y^b.$$

Relation	Domain
$t(0,b) = 0$	$a = 0, b \geq 0$
$t(1,b) = 0$	$a = 1, b \geq 0$
$t(a,0) = 1$	$a \geq 2, b = 0$
$t(2,b) = \left(\frac{55}{216}\right)^{b-1}\left(\frac{15}{36}\right)$	$a = 2, b \geq 1$
$t(a,1) = 1 - \left(\frac{441}{1296}\right)^{a-3}\left(\frac{91}{216}\right)\left(\frac{21}{36}\right)$	$a \geq 3, b = 1$
$t(3,b) = \frac{295}{1296}t(3,b-2) + \left(\frac{55}{216}\right)^{b-2}\left(\frac{420}{1296}\right)\left(\frac{15}{36}\right)$	$a = 3, b \geq 2$
$t(a,b) = \frac{2890}{7776}t(a,b-2) + \frac{2611}{7776}t(a-1,b-1) + \frac{2275}{7776}t(a-2,b)$	$a \geq 4, b \geq 2$

It can be shown using the above relation that that

$$T(x,y) = \frac{x^2 P(x,y)}{Q(x,y)}$$

where

$P(x,y) = 9043409375x^4y^4 - 35515935000x^4y^3 + 1666848925x^3y^4 - 52636437000xy^5 - 39729690000x^4y^2 + 64068612300x^3y^3 - 117017327700x^2y^4 + 207323109000xy^5 + 156029328000x^4y + 662907777840x^3y^2 - 564006330720x^2y^3 + 900580161600xy^4 - 154686672000y^5 + 932460984000x^3y - 1870090286400x^2y^2 - 235779828480xy^3 - 954637747200y^4 - 1123411161600x^3 - 1287684324864x^2y - 3612316538880xy^2 + 1095781478400y^3 + 3301453209600x^2 + 601880315904xy + 6762537123840y^2 + 3839844040704x - 1828497162240y - 11284439629824$

and

$Q(x,y) = 36(x-1)(49x-144)(55y-216)(295y^2-1296)(2275x^2+2611xy+2890y^2-7776).$

□

Let $T(x,y) = T_1(x,y) + T_2(x,y) + T_3(x,y) + T_4(x,y) + T_5(x,y)$, where T_i is defined by the following.

$$T_1(x,y) = -x^2\left(\frac{7776 + 7776x + 3240y + 3254xy}{(7776 - 2275x^2 - 2611xy - 2890y^2)}\right) \tag{1}$$

$$T_2(x,y) = x^2\left(\frac{7776 - 2611xy - 2275x^2}{(1-x)(7776 - 2275x^2 - 2611xy - 2890y^2)}\right) \tag{2}$$

$$T_3(x,y) = x^2\left(\frac{7776 + 1260y - 2611xy - 2890y^2 - \frac{91385}{216}xy^2 - \frac{50575}{108}y^3}{(1 - \frac{55}{216}y)(7776 - 2275x^2 - 2611xy - 2890y^2)}\right) \tag{3}$$

$$T_4(x,y) = x^2\left(\frac{3240y + \frac{3045}{2}xy - \frac{6965}{12}x^2y - \frac{2309125}{5184}x^3y - \frac{557375}{5184}x^4y}{(1 - \frac{49}{144}x)(1-x)(7776 - 2275x^2 - 2611xy - 2890y^2)}\right) \tag{4}$$

$$T_5(x,y) = x^2\left(\frac{7776x + 3885xy - \frac{240005}{72}xy^2 - \frac{1871275}{1296}xy^3 + \frac{46131625}{279936}xy^4}{(1 - \frac{55}{216}y)(1 - \frac{295}{1296}y^2)(7776 - 2275x^2 - 2611xy - 2890y^2)}\right) \tag{5}$$

Now we define the following notation to solve the generating function.

$$F(x,y) = \frac{1}{7776 - 2275x^2 - 2611xy - 2890y^2} \tag{6}$$

This function has coefficients

$$t_1(a,b) = F(x,y)[x^a][y^b] = \frac{1}{7776}\left(\frac{2275}{7776}\right)^{\frac{a}{2}}\left(\frac{2890}{7776}\right)^{\frac{b}{2}}\sum_{j=0}^{\frac{a+b}{2}}\binom{\frac{a+b}{2}}{\frac{a-b}{2}+2j}\binom{\frac{a-b}{2}+2j}{j}\left(\frac{2611}{85\sqrt{910}}\right)^{b-2j} \tag{7}$$

for even $a+b$ and 0 otherwise. We similarly define

$$t_2(a,b) = \frac{1}{(1-x)}F(x,y)[x^a][y^b] = \sum_{i=0}^{a} t_1(i,b) \tag{8}$$

$$t_3(a,b) = \frac{1}{(1-\frac{55}{216}y)}F(x,y)[x^a][y^b] = \sum_{j=0}^{b} t_1(a,j)\left(\frac{55}{216}\right)^{b-j} \tag{9}$$

$$t_4(a,b) = \frac{1}{(1-x)(1-\frac{49}{144}x)}F(x,y)[x^a][y^b] = \sum_{i_2=0}^{a}\sum_{i_1=0}^{i_2} t_1(i_1,b)\left(\frac{49}{144}\right)^{i_2-i_1} \tag{10}$$

$$t_5(a,b) = \frac{1}{(1-\frac{55}{216}y)(1-\frac{295}{1296}y^2)}F(x,y)[x^a][y^b] = \frac{1}{2}\sum_{i_2=0}^{b}\sum_{i_1=0}^{i_2} t_1(a,i_1)\left(\frac{55}{216}\right)^{b-i_2}\left(\left(\sqrt{\frac{295}{1296}}\right)^{i_2-i_1}+\left(-\sqrt{\frac{295}{1296}}\right)^{i_2-i_1}\right)$$

Then

$$
\begin{aligned}
t(a+2,b) \quad = \quad & -7776t_1(a,b) - 7776t_1(a-1,b) - 3240t_1(a,b-1) - 3254t_1(a-1,b-1) \\
& + 7776t_2(a,b) - 2611t_2(a-1,b-1) - 2275t_2(a-2,b) \\
& + 7776t_3(a,b) + 1260t_3(a,b-1) - 2611t_3(a-1,b-1) - 2890t_3(a,b-2) \\
& - \frac{91385}{216}t_3(a-1,b-2) - \frac{50575}{108}t_3(a,b-3) \\
& + 3240t_4(a,b-1) + \frac{3045}{2}t_4(a-1,b-1) - \frac{6965}{12}t_4(a-2,b-1) \\
& - \frac{2309125}{5184}t_4(a-3,b-1) - \frac{557375}{5184}t_4(a-4,b-1) \\
& + 7776t_5(a-1,b) + 3885t_5(a-1,b-1) - \frac{240005}{72}t_5(a-1,b-2) \\
& - \frac{1871275}{1296}t_5(a-1,b-3) + \frac{46131625}{279936}t_5(a-1,b-4)
\end{aligned}
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