Homework 8 (Due Nov 10, 2023)

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Justify all of your answers completely.

1. (a) List all the elements of A_4 that have order equal to 2.

   The decomposition types of S_4 are $\{(1),(2),(3),(4),(2,2)\}$. So then the only decomposition types of A_4 are 1, 3 and $2;2$ since the others are odd.
   The elements corresponding to decomposition types of 1 and 3 have order 1 and 3 respectively since they are just single cycles.
   So then the only possible elements of A_4 with order 2 can be ones corresponding to $2;2$, so two disjoint transpositions. This leaves us with:
   (1 2)(3 4), (1 3)(2 4), (1 4)(2 3).

   (b) Does A_4 have any cyclic subgroup of order 4?

   No, as the only elements in S_4 that has order 4 are the cycles of length 4, but those are odd permutations, so they aren't in A_4.

   (c) Does A_4 have any non-cyclic subgroup of order 4? Justify

   Yes, the group {(1), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)} is non cyclic and of order 4. It is clear that it is closed, and since they all have order 2, inverses also exist.

2. (a) List all the possible decomposition types of elements in A_8.

   We can refer back to hw 6 to find all the possible decomposition types of S_8. From that list, we want to take the decomps that have an even amount of even numbers in them to get decomps that will be in A_8. This gives us:
   {(1), (2,2), (2,2,2,2), (3), (3,2,2), (3,3), (4,2), (4,4), (5), (5,3), (6,2), (7)}

   (b) List all the possible orders of elements of A_8. For each possible order, give an example of an element that has that order.

   From (a), we find the lcm of the decomps to get the order:

1 – e	2 – (1 2)(3 4)	3 – (1 2 3)
4 – (1 2 3 4)(5 6 7 8)	5 – (1 2 3 4 5)	
6 – (1 2 3)(4 5)(6 7)	7 – (1 2 3 4 5 6 7)	
15 – (1 2 3 4 5)(6 7 8)		

3. (a) Let H be a subgroup of S_n. If $H \nsubseteq A_n$, prove that

$$|H \cap A_n| = \frac{H}{2}$$

Proof. Let O_n be the set of odd permutations of H. Denote $H \cap A_n$ as E_n for easy of use. Want to show that $|O_n| = |E_n|$.
Let $f : O_n \to E_n$ be defined as $f(\sigma) = \sigma(1\ 2)$. This takes any odd permutation and adds another transposition to it, which makes it even.
Showing that f is bijective proves $|O_n| = |E_n|$, which will be done by showing f is invertible by finding its inverse.
Since transpositions are their own inverses, f is its own inverse (just with different domains and codomains).
So let $f^{-1} : E_n \to O_n$, $f^{-1}(\sigma) = \sigma(1\ 2)$.
$f(f^{-1}(\sigma)) = \sigma(1\ 2)(1\ 2) = \sigma$.
$f^{-1}(f(\sigma)) = \sigma(1\ 2)(1\ 2) = \sigma$.
So f is invertible, which makes it bijective, which means $|O_n| = |H \cap A_n|$. Since $H = O_n \uplus (H \cap A_n)$, that means $|H| = |O_n| + |H \cap A_n| \implies |H \cap A_n| = \frac{H}{2}$ ■

(b) Using the result in part a., prove that if H is a subgroup of S_n and $|H|$ is an odd number, then $H \subseteq A_n$.

Proof. BWOC, assume $|H|$ is an odd numbers and $H \nsubseteq A_n$. Then by part (a), we know $|H \cap A_n| = \frac{H}{2}$.
But since $|H|$ is odd, then $\frac{H}{2}$ is not any integer, which is a contradiction.
So then $H \subseteq A_n$. ■

4. Prove that

$$A_4 = \{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\}$$

Proof. $A_4 = \{e$, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 2 3), (1 2 4), (1 3 2), (1 3 4), (1 4 2), (1 4 3), (2 3 4), (2 4 3)$\}$
For any cycle of length 3, (a b c), we can rewrite it as (a c b)(a c b) = (a c b)^2. So then the cycles of length 3 in A_4 are in $\{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\}$
For any pair of disjoint transpositions,(a b)(c d), we can rewrite it as (a c b d)^2. So then all the pairs of transpositions in A_4 is also in the set.
Obviously e is in the set too. So $A_4 \subseteq \{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\}$.
Let $\sigma \in \{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\}$. Then σ is even since no matter what parity τ has, τ^2 will be even. So $\sigma \in A_4$.

So $\{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\} \subseteq A_4$
So $A = \{\sigma \in S_4 : \exists \tau \in S_4 \text{ s.t. } \sigma = \tau^2\}$. ■

5. Let $a = (1\ 2)(3\ 4)$ and $b = (1\ 2\ 3)$. If H is a subgroup of A_4 with $a, b \in H$, prove that $H = A_4$.

   Proof. Just need to show that every element of A_4 can be made from compositions of a and b.
   $e = a^2$
   $(1\ 2)(3\ 4) = a$
   $(1\ 2\ 3) = b$
   $(1\ 2\ 4) = (1\ 2\ 3)(1\ 2)(3\ 4)(1\ 2\ 3) = bab$
   $(1\ 3\ 2) = (1\ 2\ 3)(1\ 2\ 3) = b^2$
   $(1\ 4\ 2) = (1\ 2)(3\ 4)(1\ 2\ 3)(1\ 2)(3\ 4) = aba$
   $(2\ 4\ 3) = (1\ 2)(3\ 4)(1\ 2\ 3) = ab$
   By Lagrange, we know the order of H must be 1, 2, 3, 4, 6, or 12. Since we found 7 elements, that means it can't be 6 or lower, so it must be 12. So then $H = A_4$. ■