Homework 6 (Due Oct 3, 2022)

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MATH 574 - Discrete Mathamatics - Fall 2022

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Justify all of your answers completely.

1. In this problem, we will prove a one-sided Chebyshev-type bound in several steps. The goal is to fill in the details of the proof of the theorem.

   (a) Let X be a random variable and let $c \in \mathbb{R}$. Prove that $V(X) = V(X + c)$.

   Proof.

   $$V(X+c) = E((X+c)^2) - E(X+c)^2 = E(X^2 + 2Xc + c^2) - (E(X) + E(c))^2$$
   $$= E(X^2) + 2cE(X) + c^2 - (E(X)^2 + 2E(X)E(c) + E(c)^2)$$
   $$= E(X^2) + 2cE(X) + c^2 - E(X)^2 - 2cE(X) - c^2 = E(X^2) - E(X)^2 = V(X)$$

   ■

   (b) Prove that if Y is a random variable with $E(Y) = 0$, then for any constant $c \in \mathbb{R}$, $E((Y-c)^2) = V(Y) + c^2$.

   Proof.

   $$E((Y-c)^2) = E((Y-c)^2) - E(Y-c)^2 + E(Y-c)^2 = V(Y-c) + E(Y-c)^2$$
   $$= V(Y) + (E(Y) - c)^2 = V(Y) + (-c)^2 = V(Y) + c^2$$

   Since c can be any constant, this equality will also be true as $E((Y+c)^2) = V(Y) + c^2$. ■

   (c) Prove that for any random variable X, $E(X - E(X)) = 0$.

Proof.

$$E(X - E(X)) = E(X) - E(E(X))$$

Expected value results in a real number, and the expected value of a real number is that same real number, so taking the expected value twice is the same as taking it once. Therefore,

$$E(X) - E(E(X)) = E(X) - E(X) = 0$$

■

(d) Prove that for any $c \geq 0$ and random variable X, $p(X \geq c) \leq p(X^2 \geq c^2)$.

Proof. $P(X \geq c) \leq P(|X| \geq c)$ since the right side is an upper bound, since the non-negative values of X are unaffected and the negative values become positive and c is non-negative.
$P(|X| \geq c) = P(X^2 \geq c^2)$ since both sides of the inequality are positive, they are equivalent.
$\therefore P(X \geq c) \leq P(X^2 \geq c^2)$. ■

(e) Now prove the following theorem. A brief outline is given below, but you should write a full proof. You may cite the results proven above.

Theorem 1. *Let X be a random variable with variance σ^2. Then for any $k > 0$,*

$$p(X - E(X) \geq k) \leq \frac{\sigma^2}{\sigma^2 + k^2}.$$

Proof. Set $Y = X - E(X)$. Then

$$p(X - E(X) \geq k) = p(Y \geq k) = p(Y + x \geq k + x) \leq p((Y + x)^2 \geq (k + x)^2),$$

for any $x \geq 0$, using the inequality proved in (d).
Using Markov's inequality, we obtain

$$p((Y + x)^2 \geq (k + x)^2) \leq \frac{E((Y + x)^2)}{(k + x)^2}.$$

Using part (c), we know $E(Y) = 0$, and using (b) we conclude that

$$\frac{E((Y + x)^2)}{(k + x)^2} = \frac{V(Y) + x^2}{(k + x)^2} = \frac{\sigma^2 + x^2}{(k + x)^2}.$$

Therefore $p(X - E(X) \geq k) \leq \frac{\sigma^2 + x^2}{(k+x)^2}$, but this holds for *any* x. Hence this theorem is most useful when we minimize the function $\frac{\sigma^2 + x^2}{(k+x)^2}$.
Now we minimize the function $\frac{\sigma^2 + x^2}{(k+x)^2}$ with respect to x to get the best bound. The derivative is $\dfrac{2\,(kx - \sigma^2)}{(x + k)^3}$ with a zero at $x = \sigma^2/k$. Plugging in that value to the original gives $\frac{\sigma^2 + (\frac{\sigma^2}{k})^2}{(k + (\frac{\sigma^2}{k}))^2} = \frac{\sigma^2 k^2 + \sigma^4}{(k^2 + \sigma^2)^2} = \frac{\sigma^2}{\sigma^2 + k^2}$. ■

2. A biased coin has probability for heads $p = 0.75$. Suppose we flip the coin $1,000$ times. Give an upper bound for the probability that we flip at least 800 heads

   (a) using Markov's inequality.

   Letting X be the number of heads flipped, then $X \sim B(n,p)$, so $E(X) = 750$. So Markov's inequality gives us $p(X \geq 800) \leq \frac{750}{800}$.

   (b) using Chebyshev's inequality.

   $P(|X - E(X)| \geq k) \leq \frac{V(X)}{k^2} \Leftrightarrow P(|X - 750| \geq 50) \leq \frac{187.5}{2500}$

   (c) using Theorem 1 in the previous question.

   $P(X - 750 \geq 50) \leq \frac{187.5}{187.5+50^2} = \frac{3}{43}$.

3. Let a_n be the number of strings of length n with digits in $0-9$ that contain 2 or more consecutive 0s.

   (a) Find a recurrence relation for a_n.

   There are 2 cases for a string of length n, one where the last digit is 0 and one where it isn't 0. If the last digit is not 0, then there must be double 0's in the $n-1$ substring, which gives a term of $9a_{n-1}$. If the digit is 0, then the second to last digit can also be 0 or not 0. If that second to last digit is 0, this gives us a 10^{n-2} term since each digit up to the $n-2$ digit has 10 options. If that second to last digit isn't 0, then this gives us a term of a_{n-2}.
   So our recurrence relation is $a_n = 9a_{n-1} + 9a_{n-2} + 10^{n-2}$.

   (b) What are the initial conditions?

   a_1 is 0 since consecutive 0's aren't possible, and a_2 is 1 since only 00 works.

   (c) Determine a_7.

   We will have to calculate up to a_7 sequentially.
   $a_3 = 9(1) + 9(0) + 10 = 19$.
   $a_4 = 9(19) + 9(1) + 10^2 = 280$.
   $a_5 = 9(280) + 9(19) + 10^3 = 3691$.
   $a_6 = 9(3691) + 9(280) + 10^4 = 45739$.
   $a_7 = 9(45739) + 9(3691) + 10^5 = 544870$.

4. Let b_n be the number of strings of length n with digits in $0-9$ that contain no two repeated numbers in a row.

   (a) Determine b_1.

   This is trivially 10.

(b) Prove that b_n has the recurrence relation $b_n = 9b_{n-1}$ for all $n \geq 2$.

Let us assume we have a string of $n-1$ digits, we will call the last digit d. Adding on a last digit for a string of length n, we have 9 options to chose since d will be one of the 10 digits. ■

(c) Solve the recurrence relation with the initial condition in part (a).

$b_n = 9b_{n-1} = 9 \cdot 9b_{n-2} = \ldots = 9 \cdot \ldots \cdot 9 \cdot b_1$, and there will be $n-1$ 9's. $\therefore b_n = 9^{n-1}10.$■