Homework 1 (Due March 22, 2023)

Jack Hyatt
MATH 554 - Analysis I - Fall 2023

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Justify all of your answers completely.

1.1 Let $a, r, n \in \mathbb{R}$, $r \neq 1$, and $n \geq 2$. Prove that

$$a + ar + ar^2 + \ldots + ar^n = \frac{a - ar^{n+1}}{1 - r}$$

Proof. Let $S = a + ar + ar^2 + \ldots + ar^n$. Then $rS = ar + ar^2 + \ldots + ar^{n+1}$. So $S - rS = a - ar^{n+1} \implies S = \frac{a - ar^{n+1}}{1-r}$. ■

1.2 What happens to geometric sum in 1.1 when $r = 1$?

If $r = 1$, then every term would become just a, and there are $n + 1$ terms. So we will be left with $a(n+1)$.

1.3 a) Find the sum of $S = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$.

$$S = \sum_{i=0}^{n-1} \frac{1}{2}\left(\frac{1}{2}\right)^i = \frac{\frac{1}{2} - \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n = \frac{2^n - 1}{2^n}$$

b) Find the sum of $S = P_0(1+r) + P_0(1+r)^2 + \ldots + P_0(1+r)^n$.

$$S = \sum_{i=0}^{n-1} P_0(1+r) \cdot (1+r)^i = \frac{P_0(1+r) - P_0(1+r)(1+r)^n}{1 - (1+r)} = P_0\left(1 + \frac{1}{r}\right)((1+r)^n - 1)$$

1.4 Multiply out $(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$ to see that you get $x^n - y^n$.

$$(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$$

$$= (x^n + x^{n-1}y + x^{n-2}y^2 + \ldots + x^2y^{n-2} + xy^{n-1}) - (x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3 + \ldots + xy^{n-1} + y^n) = x^n - y^n$$

1.5 Prove $x^n - y^n = (x-y)(\sum_{k=0}^{n-1} x^{n-1-k}y^k)$ with geometric sums.

Proof.

$$(x-y)(\sum_{k=0}^{n-1} x^{n-1-k}y^k) = (x-y)(\sum_{k=0}^{n-1} x^{n-1}(\frac{y}{x})^k) = (x-y)\frac{x^{n-1} - x^{n-1}(\frac{y}{x})^n}{1-\frac{y}{x}}$$

$$= (x-y)\frac{x^{n-1} - \frac{y^n}{x}}{1-\frac{y}{x}} \cdot \frac{x}{x} = (x-y)\frac{x^n - y^n}{x-y} = x^n - y^n$$

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1.6 Let $c_0, c_1, c_2, c_3 \in \mathbb{R}$, and $f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$. Simplify $\frac{f(x)-f(a)}{x-a}$.

$$\frac{c_3x^3 + c_2x^2 + c_1x + c_0 - (c_3a^3 + c_2a^2 + c_1a + c_0)}{x-a} = \frac{c_3x^3 + c_2x^2 + c_1x - c_3a^3 - c_2a^2 - c_1a}{x-a}$$

$$= c_3\frac{x^3-a^3}{x-a} + c_2\frac{x^2-a^2}{x-a} + c_1\frac{x-a}{x-a} = c_3\frac{(x-a)(x^2+xa+a^2)}{x-a} + c_2\frac{(x-a)(x+a)}{x-a} + c_1$$
$$= c_3(x^2+xa+a^2) + c_2(x+a) + c_1$$

and so the limit as x approaches a is $c_3(3a^2) + c_2(2a) + c_1$.

1.7 Use summation notation to derive a formula for the sum of the series

$$\sum_{k=0}^{n-1}(a+kd)$$

Proof. Let $S = \sum_{k=0}^{n-1}(a+kd) = \sum_{k=0}^{n-1}(a+(n-1-k)d)$. Then $2S = \sum_{k=0}^{n-1}(a+kd) + \sum_{k=0}^{n-1}(a+(n-1-k)d) = \sum_{k=0}^{n-1}(2a+(n-1)d) = (2a+(n-1)d)n \implies S = \frac{(2a+(n-1)d)n}{2}$ ■

1.9 Show $\binom{n}{k} := \frac{n!}{k!(n-k)!} \implies \binom{n}{k} = \binom{n}{n-k}$.

Proof. Let $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. Set $k = n-k$. We now have $\frac{n!}{k!(n-k)!} \implies \frac{n!}{(n-k)!(n-(n-k))!} \implies \frac{n!}{(n-k)!(k)! = \binom{n}{n-k}}$ ■

1.10 Prove

$$\binom{n}{0} = 1$$
$$\binom{n}{1} = n$$
$$\binom{n}{2} = \frac{n(n-1)}{2}$$
$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Proof.

$$\binom{n}{0} = \frac{n!}{0!n!} = \frac{n!}{n!} = 1$$
$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = n$$
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{\frac{n(n-1)...(2)(1)}{(n-2)(n-3)...(2)(1)}}{2} = \frac{n(n-1)}{2}$$
$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{\frac{n(n-1)...(2)(1)}{(n-3)(n-4)...(2)(1)}}{3!} = \frac{n(n-1)(n-2)}{6}$$

■

1.11 Prove

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-k+1)!}{k!} = \frac{n^{\underline{k}}}{k!}$. Like I don't know what else you want from me, it's that straight forward. ■

1.12 Prove that for $k, n \in \mathbb{Z}, 1 \le k \le n$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!k + n!(n-k+1)}{(k)!(n-k+1)!}$$
$$= \frac{n!(k+n-k+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

■

1.13 Let k, n be nonnegative integers with $0 \le k \le n$. Prove $\binom{n}{k} \in \mathbb{Z}$.

Proof. Let us induct on n.
Base case: $n = 1$, $\binom{1}{0} = \binom{1}{1} = 0$.
Induction Step: Assume $\forall k, n \in \mathbb{Z}, 0 \le k \le n, n \ge 1, \binom{n}{k} \in \mathbb{Z}$.
Consider $n+1$. $\binom{n+1}{k} \in \mathbb{Z}$ when $k = 0$ or $k = n+1$ as those values will make the expression equal to 1. Now assume $1 \le k \le n$.
By Pascal's Identity, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. By the induction hypothesis, we know both $\binom{n}{k-1}$ and $\binom{n}{k}$ are integers, which means $\binom{n+1}{k}$ is also an integer. So $\binom{n+1}{k} \in \mathbb{Z}$ for all values $0 \le k \le n+1$. ■

1.17 Use induction and the Pascal Identity to prove the Binomial Theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof. Let us induct on n.
Base case: $n = 1$, $(x+y)^1 = \binom{1}{0}x + \binom{1}{1}y = \sum_{k=0}^{1} \binom{1}{k} x^k y^{n-k}$.
Induction Step: Assume $n \geq 1, (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
Consider $n+1$.

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = x\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right)+y\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right)$$

$$= \left(\sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k}\right) + \left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1}\right) = \left(\sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n-k+1}\right) + \left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1}\right)$$

$$= \left(\sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n-k+1}\right) + \binom{n}{-1}x^0 y^{n+1} + \left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1}\right) + \binom{n}{n+1}x^{n+1}y^0$$

$$= \sum_{k=0}^{n+1}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=0}^{n+1}\binom{n}{k}x^k y^{n+1-k} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}$$

$\blacksquare$

1. Show that $x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$ and use it to find a formula for $\sum_{k=1}^{n} k^3$.

$x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} = (x)(x-1)(x-2) + 3(x)(x-1) + x = (x^3 - 3x^2 + 2x) + (3x^2 - 3x) + x = x^3$

This makes computing $\sum_{k=1}^{n} k^3$ easy with $\frac{1}{p+1}x^{\underline{p+1}}$ being the anti difference of $x^{\underline{p}}$, along with the fundamental theorem of summations.

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}(k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) = \frac{(n+1)^{\underline{4}}}{4} + (n+1)^{\underline{3}} + \frac{(n+1)^{\underline{2}}}{2} - \frac{(1)^{\underline{4}}}{4} - (1)^{\underline{3}} - \frac{(1)^{\underline{2}}}{2}$$

$$= \frac{(n+1)(n)(n-1)(n-2)}{4} + (n+1)(n)(n-1) + \frac{(n+1)(n)}{2}$$

$$= (n+1)(n)\left(\frac{(n-1)(n-2)}{4} + (n-1) + \frac{1}{2}\right) = (n+1)(n)\left(\frac{n^2 - 3n + 2 + (4n-4) + 2}{4}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$