Homework 17 (Due Dec 6, 2023)

Jack Hyatt
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Justify all of your answers completely.

1. Let $f : A \to B$ be a bijection between sets. Prove there is a function $g : B \to A$ s.t. $\forall a \in A$ and $\forall b \in B$, $f(g(b)) = b$ and $g(f(a)) = a$, and g is unique.

   Proof. Since f is surjective and injective, $\forall b \in B,\ \exists! a \in A$ s.t. $f(a) = b$.
   Then we can define $g : B \to A$ with $g(b) = a$ without any ambiguity.
   So then $f(g(b)) = f(a) = b$ and $g(f(a)) = g(b) = a$.
   Now to show g is unique. Assume g and h are both inverses of f. Let $I_A : A \to A$ and $I_B : B \to B$ be the identity functions for their respective sets.
   Then $f \circ g = I_B = f \circ h$ and $g \circ f = I_A = h \circ f$, meaning

   $$f \circ g = f \circ h \implies g \circ (f \circ g) = g \circ (f \circ h)$$
   $$\implies (g \circ f) \circ g = (g \circ f) \circ h \implies I_A \circ g = I_A \circ h$$
   $$\implies g = h$$

   So the inverse is unique. ■

2. Prove that if $f : A \to B$ is a bijection, then the inverse $f^{-1} : B \to A$ is also a bijection and $(f^{-1})^{-1} = f$.

Proof. First to prove is $(f^{-1})^{-1} = f$.
Since f is bijective, then f^{-1} exists and is unique.
So $\forall a \in A,\ f^{-1}(f(a)) = a$ and $\forall b \in B,\ f(f^{-1}(b)) = b$.
So then f satisfies the definition of inverse for f^{-1}. So $(f^{-1})^{-1} = f$.
To prove f^{-1} is injective, let $x, y \in B$ and assume $f^{-1}(x) = f^{-1}(y)$.

$$f^{-1}(x) = f^{-1}(y) \implies f(f^{-1}(x)) = f(f^{-1}(y)) \implies x = y$$

To prove f^{-1} is surjective, let $a \in A$. We can choose some $b \in B$ s.t. $f(a) = b$.
So $f^{-1}(b) = f^{-1}(f(a)) = a$. So every element in the codomain has a matching input.
So f^{-1} is bijective. ■

3. Let $f : E \to E'$ be a continuous bijection between metric spaces with E compact. Prove the inverse $f^{-1} : E' \to E$ is continuous.

   Proof. Assume f is a continuous bijection and E is compact.
   The continuous image of a compact set is compact, and since f is surjective, that means E' is compact as well.
   Showing f^{-1} is continuous is equivalent to showing the preimage of closed sets of f^{1} is closed.
   Let $K \subseteq E$. The preimage of K is $(f^{-1})^{-1}(K)$.
   That is just $f(K)$, and we know that a continuous image of a closed set is closed. So then $f(K)$ is closed, meaning f^{-1} is continuous. ■

4. Let $f : [a, b] \to [\alpha, \beta]$ be an increasing continuous function with $f(a) = \alpha$ and $f(b) = \beta$. Prove that f is bijective and that the inverse, f^{-1}, is continuous.

   Proof. Since f is continuous, f is surjective by Intermediate Value Theorem. Now to show f is injective.
   Let $x, y \in [a, b]$ and assume $x \neq y$. WLOG, assume $x < y$.
   Since f is an increasing function, $f(x) < f(y)$. So then $f(x) \neq f(y)$.
   So f is injective.
   So f is bijective. Since f is a bijective continuous function, problem 3 tells us that the inverse is continuous. ■

5. Let K be a closed bounded subset of $\mathbb{R}^2$. Show that there exists $x_*, y_*, z_* \in K$ so that the triangle $\triangle x_* y_* z_*$ has maximum area of triangles with vertices in K. That is $A(x,y,z) \leq A(x_*, y_*, z_*)$ for all $x, y, z \in K$.

   Proof. The set of possible triangles in K is a subset of $K \times K \times K$, which is a compact subset of $\mathbb{R}^6$. So a triangle in K can be represented as a point in $\mathbb{R}^6$.
   The area function of triangles, $A(x,y,z)$, is a continuous function from $\mathbb{R}^6$ (really $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ but they are isomorphic) to $\mathbb{R}$.
   Those facts together means that the image of $A[K \times K \times K]$ is compact, meaning it is closed and bounded.
   Since it is closed and bounded, that means it contains a maximum element.
   So there exists a triangle in K such that it has a maximum area. ∎