Homework 8 (Due Oct 17, 2022)

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Justify all of your answers completely.

Chapter 6 questions

1. How many solutions (x, y, z) of $x + y + z = 50$ are there with

   (a) x, y, z all non-negative integers?

   This is just distributing 50 indistinguishable objects to 3 distinguishable boxes. $\binom{52}{2}$

   (b) x, y, z all positive integers?

   Set x, y, z to 1 initially, leaving 47 numbers left to distribute. $\binom{49}{2}$

   (c) x, y, z all non-negative integers and $x \geq 10$?

   Give x 10 first, then distribute 40 numbers to 3 boxes. $\binom{42}{2}$

2. Find the coefficient of x^3y^4 in $(x + 5y^2)^5$.

$$(x + 5y^2)^5 = \sum_{k=0}^{5} \binom{5}{k} x^k (5y^2)^{5-k}$$

   So the coeff of x^3y^4 is $\binom{5}{3}5^{5-3} = \binom{5}{3}25$.

3. Give a double-counting proof of the following: for all positive integers n, a, b with $a + b \leq n$, $\binom{n}{a+b}\binom{a+b}{a} = \binom{n}{a}\binom{n-a}{b}$.

   LHS: The number of ways to chose a club of $a + b$ people from a group of n. Then choosing a committee of a people from that club.
   RHS: Choose a leading committee of a people from n, then choose the remaining b people to be in the club that is lead by that committee.
   Both sides are making a club of $a + b$ people with a people leading it out of a group of n people.

4. Prove that for all non-negative integers n, k with $k \leq n+1$, $\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$.

   LHS: Counts the ways to choose a subset of size k+1 from $\{1, \ldots, n+1\}$. RHS: Let $A \subset \{1, \ldots, n+1\}$ where $|A| = k+1$. Assume the largest element of A is $m+1$. Then there are k remaining elements and must be in the set $\{1, \ldots, m\}$. So $\binom{m}{k}$ counts the number of subsets of $\{1, \ldots, n+1\}$ of size k+1 where the largest element is m+1. Since every subset will satisfy that the largest element is m+1 for some $k \leq m \leq n$, the sum counts all subsets. ■

5. You select a set of 7 integers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Prove that there must be a pair of selected integers with sum equal to 13.

   There are 6 ways to add to 13 with two natural numbers. $\{1, 12\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}$. By pigeon hole theorem, choosing 7 distinct numbers means we must chose at least 2 from one of these groups. ■

Chapter 7 questions

1. What is the probability that a randomly selected function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ is one-to-one?

   The probability is injective functions over total functions, which is $\frac{n!}{n^n}$.

2. Suppose we flip n biased coins such that the probability of each heads is 1/4. A **run** is a maximal sequence of consecutive heads or tails (runs can have length 1). For instance, the outcome $TTTTHTHH$ has 4 runs: $TTTT, H, T, HH$. What is the expected number of runs achieved after n flips?

$$\text{Let } X_i = \begin{cases} 1 & \text{if flip i is the start of a run} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   This means that X_i is 1 in these 2 cases. 1) i is 1, 2) flip i-1 is the opposite of flip i. So for $i = 1, p(X_1 = 1) = 1$ and for $i > 1, p(X_i = 1) = (\frac{1}{4})(\frac{3}{4}) + (\frac{1}{4})(\frac{3}{4}) = \frac{3}{8}$.
   So $E(X_1) = 1$ and for $i > 1$ $E(X_i) = \frac{3}{8}$. So for n flips, we have $E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n) = 1 + (n-1)\frac{3}{8}$.

3. Suppose 10 fair dice are rolled. Compute the expected value and the variance of the sum of the results.

   Since the rolling of dice is independent, we can use $V(X_1 + \ldots + X_n) = V(X_1) + \ldots + V(X_n)$. It is known that expected value of a dice role is 3.5, so using Linearity of Expectation, we get our expected value is 35. We also know that the variance of a dice role is $\frac{35}{12}$, so our variance becomes $\frac{175}{6}$.

4. A standard deck of 52 cards is shuffled. We draw cards *with replacement* until we encounter an ace. That is, every time we draw a card that is not an ace, we put the card back in the deck and reshuffle it. Let X be the number of cards we drew.

(a) Determine $E(X)$.

X clearly has geometric distribution, with the probability, p, being $\frac{1}{13}$. Thus, $E(X) = \frac{1}{p} = 13$.

(b) Determine $V(X)$.

Variance for geometric distribution is $\frac{1-p}{p^2}$, so $V(X) = 156$.

(c) Use Chebyshev's inequality to give an upperbound on the probability that we draw at least 49 cards.

$$p(|X - E(X)| \geq 49 - E(X)) \leq \frac{V(X)}{(49-E(X))^2} \implies p(|X - 13| \geq 36) \leq \frac{156}{36^2} \approx 0.12$$

Chapter 8 questions

1. Let a_n be the number of ways to tile a $1 \times n$ row using red, blue, and green tiles such that a blue tile never comes directly after a green tile. Give a recurrence relation for a_n and determine the initial conditions.

   This can be told in a ternary string scenario. Let's have it be that we cant have a 2 directly follow a 1. Looking at a length n case, we could have the last digit be a 2, so then second to last digit must be a 0. This gives us an a_{n-2} term. We could also have the last digit be a 0 or 1. If this is the case, then there is no restriction on the previous number. That'll give us a term of $2a_{n-1}$. So we have $a_n = 2a_{n-1} + a_{n-2}$.
   $a_0 = 1$ and $a_1 = 3$.

2. Solve the recurrence relation $a_n = 25a_{n-2}$ with initial conditions $a_0 = 2, a_1 = 4$.

$$p(\lambda) = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5) \implies \lambda_{1,2} = -5, 5$$
$$\text{So } a_n = \alpha_1(-5)^n + \alpha_2 5^n$$
$$a_0 = \alpha_1(-5)^0 + \alpha_2 5^0 = \alpha_1 + \alpha_2 = 2$$
$$a_1 = \alpha_1(-5)^1 + \alpha_2 5^1 = -5\alpha_1 + 5\alpha_2 = 4$$

   This gives us that $\alpha_{1,2} = \frac{3}{5}, \frac{7}{5}$. So $a_n = \frac{3}{5}(-5)^n + \frac{7}{5}5^n$.

3. Solve the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 0, a_1 = 0, a_2 = 1$. You may use the fact that the roots of the polynomial $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ are $1, 2$, and 3.

   We have that $\lambda 1, 2, 3 = 1, 2, 3$

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$$

$$a_0 = \alpha_1 1^0 + \alpha_2 2^0 + \alpha_3 3^0 = \alpha_1 + \alpha_2 + \alpha_3 = 0$$
$$a_1 = \alpha_1 1^1 + \alpha_2 2^1 + \alpha_3 3^1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$$

$$a_2 = \alpha_1 1^2 + \alpha_2 2^2 + \alpha_3 3^2 = \alpha_1 + 4\alpha_2 + 9\alpha_3 = 1$$

Solving this gives us $\alpha_{1,2,3} = \frac{1}{2}, -1, \frac{1}{2}$. So

$$a_n = \frac{1}{2} - 2^n + \frac{1}{2}3^n$$

4. Find the solution of the reccurence relation $a_n = 2a_{n-1} + 3^n$ with the initial condition $a_0 = 5$.

   The homogeneous part of the relation is $a_n^{(h)} = \alpha 2^n$.
   The particular part will be of the form $a_n^{(p)} = c3^n$.
   To solve for c, we plug $a_n^{(p)}$ in the recurrence relation.

$$c3^n = 2(c3^{n-1}) + 3^n \implies 3c = 2c + 3 \implies c = 3$$

   So $a_n = \alpha 2^n + 3^{n+1}$.

$$a_0 = \alpha 2^0 + 3^1 = \alpha + 3 = 5 \implies \alpha = 2$$

$$a_n = 2^{n+1} + 3^{n+1}$$