Homework 14 (Due Nov 17, 2023)

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MATH 554 - Analysis I - Fall 2023

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Justify all of your answers completely.

1. Give an ϵ, δ proof that $f(x) = x^3 - x$ is continuous at all points a.

   Proof. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/(3|a|^2 + 3|a| + 2)\}$.
   Assume $|x - a| < \delta$. Note: $|x| = |a + x - a| \leq |a| + |x - a| < |a| + \delta \leq |a| + 1$

   $$|f(x) - f(a)| = |x^3 - x - a^3 + a| = |(x-a)(x^2 + xa + a^2) - (x - a)|$$

   $$= |x - a||x^2 + xa + a^2 - 1| \leq |x - a|(|x|^2 + |x||a| + |a|^2 + |1|)$$
   $$\leq |x - a|(3|a|^2 + 3|a| + 2) < \delta(3|a|^2 + 3|a| + 2) = \epsilon$$

   So $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. ■

2. Give an ϵ, δ proof that $f(x) = \sqrt{|x|}$ is continuous at $x = 0$.

   Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon^2$.
   Assume $|x - 0| < \delta$.

   $$|f(x) - f(0)| = |\sqrt{|x|}| = \sqrt{|x|} < \sqrt{\delta} = \epsilon$$

   So $|x - 0| < \delta \implies |f(x) - f(0)| < \epsilon$. ■

3. Give an ϵ, δ proof that $f(x) = \frac{x}{1+x}$ is continuous at all points $a \neq -1$.

Proof. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta = \min\{|1+a|/2, \epsilon(1+a)^2/2\}$.
Assume $|x-a| < \delta$.
Note: $|1+x| = |1+a+x-a| \geq |1+a| - |x-a| > |1+a| - \delta = |1+a|/2$
So $1/|1+x| < 2/|1+a|$.

$$|f(x)-f(a)| = \left|\frac{x}{1+x} - \frac{a}{1+a}\right| = \left|\frac{x-a}{(1+x)(1+a)}\right| = |x-a|\left|\frac{1}{(1+x)}\right|\left|\frac{1}{(1+a)}\right|$$

$$\leq |x-a|\frac{2}{|1+a|} \cdot \frac{1}{|1+a|} = |x-a|\frac{2}{(1+a)^2} < \delta\frac{2}{(1+a)^2} \leq \epsilon$$

So $|x-a| < \delta \implies |f(x)-f(a)| < \epsilon$. ■

4. Let $g : \mathbb{R} \to \mathbb{R}$ given be

$$g(x) = \begin{cases} 1, & \text{if x is rational;} \\ 0, & \text{if x is irrational} \end{cases} \tag{1}$$

Show that g is not continuous at any point.

Proof. Let a be rational. Then $g(a) = 1$.
Since there is an irrational number between any two real numbers, for each $n \in \mathbb{N}$, there exists an irrational number x_n s.t. $a < x_n < a + 1/n$.
So $\lim_{n\to\infty} x_n = a$. Since each x_n is irrational, $g(x_n) = 0$, which means, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 0 = 0 \neq 1 = g(a)$.
So g cannot be continuous at rational points by problem 7.
Let a be irrational. Then $g(a) = 0$.
Since there is a rational number between any two real numbers, for each $n \in \mathbb{N}$, there exists an rational number x_n s.t. $a < x_n < a + 1/n$.
So $\lim_{n\to\infty} x_n = a$. Since each x_n is rational, $g(x_n) = 1$, which means, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = g(a)$.
So g cannot be continuous at irrational points by problem 7.
So g cannot be continuous at any real point. ■

5. Define the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$ and $g(x,y) = y$. Show that f and g are continuous.

Proof. Since f and g are so gosh darn symmetric, showing f is enough as showing g would be a waste of everyone's time.

We have shown in class that $f(\vec{x}) = \vec{a}\cdot(\vec{x}) + b$ is continuous on all points in $\mathbb{R}^n$.
Let $\vec{a} = (1,0)$ and $b = 0$. Then we can write $f(x,y) = (x,y)\cdot\vec{a} + b$.
So then our function f is also continuous. ■

6. (a) Show that the function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x\cos(1/x), & x > 0 \end{cases} \tag{2}$$

is continuous at 0.

Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon$.
Assume $|x - 0| < \delta$.
Case 1: $x \le 0$.

$$|f(x) - f(0)| = |0 - 0| = 0 < \epsilon$$

Case 2: $x > 0$.

$$|f(x) - f(0)| = |x\cos(1/x)| = |x||\cos(1/x)| \le |x| < \delta = \epsilon$$

So $|x - 0| < \delta \implies |f(x) - f(0)| < \epsilon$. ■

(b) Show that the function

$$g(x) = \begin{cases} 0, & x \le 0 \\ \cos(1/x), & x > 0 \end{cases} \tag{3}$$

is not continuous at $x = 0$.

Proof. For $n \in \mathbb{N}$, let $x_n = 1/2\pi n$.
So $\lim_{n\to\infty} x_n = 0$.
$g(x_n) = \cos(2\pi n) = 1$.
So $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 1 \ne 0 = g(0)$.
So then g cannot be continuous by problem 7. ■

7. Let (E,d) and (E',d') be metric spaces and assume that f is continuous at the point $p_0 \in E$. Let $\langle p_n\rangle_{n=1}^{\infty}$ be a sequence in E with $\lim_{n\to\infty} p_n = p_0$. Prove that

$$\lim_{n\to\infty} f(p_n) = f(p_0)$$

Proof. Let $\epsilon > 0$. Since f is continuous at p_0, $\exists \delta > 0$ s.t.

$$d(p, p_0) < \delta \implies d'(f(p), f(p_0)) < \epsilon.$$

Since $\lim_{n\to\infty} p_n = p_0$, $\exists N$ s.t.

$$n \geq N \implies d(p_n, p_0) < \delta$$

So then $n \geq N$ also implies $d(f(p_n), f(p_0)) < \epsilon$, which means $\lim_{n\to\infty} f(p_n) = f(p_0)$. ■