Homework 10 (Due Oct 25, 2023)

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Justify all of your answers completely.

1. Let E be a metric space, $p \in E$ and $r_1, \dots, r_n$ positive real numbers. Let $\mathcal{U}$ be the collection of open balls

$$\mathcal{U} = \{B(p, r_1), \dots, B(p, r_n)\}.$$

Let $r_{\max} = \max(r_1, \dots, r_n)$ and $r_{\min} = \min(r_1, \dots, r_n)$. Prove

$$\bigcup \mathcal{U} = B(p, r_{\max}), \qquad \bigcap \mathcal{U} = B(p, r_{\min})$$

Proof. Let $\beta \in \bigcup \mathcal{U}$. Then $\beta \in B(p, r_j)$ for some $j \in [n]$. So then $\beta \in B(p, r_{\max})$ since $d(p, \beta) < r_j \leq r_{\max}$. So $\bigcup \mathcal{U} \subseteq B(p, r_{\max})$.
Let $\beta \in B(p, r_{\max})$. Since $r_{\max} = r_j$ for some $j \in [n]$, $\beta \in \bigcup \mathcal{U}$. So $B(p, r_{\max}) \subseteq \bigcup \mathcal{U}$.
So $\bigcup \mathcal{U} = B(p, r_{\max})$.

Let $\beta \in \bigcap \mathcal{U}$. Then $\beta \in B(p, r_j)$ for all $j \in [n]$. So then $\beta \in B(p, r_{\min})$ since the min ball has the radius of at least one of the r_j's. So $\bigcap \mathcal{U} \subseteq B(p, r_{\min})$.
Let $\beta \in B(p, r_{\min})$. So then $d(p, \beta) < r_{\min} \leq r_j$ for all $j \in [n]$. So $\beta \in \bigcap \mathcal{U}$. So $B(p, r_j) \subseteq \bigcap \mathcal{U}$.
So $\bigcap \mathcal{U} = B(p, r_{\min})$.

■

2. Let E be a metric space and $p \in E$. Let R be a set of positive real numbers which is bounded above and set

$$\mathcal{U} = \{B(p, r) : r \in R\}.$$

Let

$$r_0 = \inf R, \qquad r_1 = \sup R.$$

Show

$$\bigcup \mathcal{U} = B(p, r_1).$$

Give an example where

$$\bigcap \mathcal{U} \neq B(p, r_0).$$

Proof. Let $\beta \in \bigcup \mathcal{U}$. Then $\beta \in B(p, r)$ for some $r \in R$. Since $d(p, \beta) < r < r_1$, $\beta \in B(p, r_1)$. So $\bigcup \mathcal{U} \subseteq B(p, r_1)$.
Let $\beta \in B(p, r_1)$. So $d(p, \beta) < r_1$. Since r_1 is the supremum, then $\beta \in B(p, r)$ for some $r \in R$ since $d(p, \beta)$ would be a better supremum otherwise. So $B(p, r_1) \subseteq \bigcup \mathcal{U}$.
So $\bigcup \mathcal{U} = B(p, r_1)$. ■

An example where $\bigcap \mathcal{U} \neq B(p, r_0)$ holds is $R = (0, \infty)$ and $p = 0$. r_0 will be 0, making the ball $B(0, 0) = \varnothing$ and $\bigcap \mathcal{U} = \{0\}$.

3. Let $\mathcal{U} = \{(-n, n) : n \in N\}$. Show the following are equivalent:

   (a) $\mathcal{U}$ is an open cover of $\mathbb{R}$.

   (b) Archimedes' Axiom.

Proof. ($a \implies b$)
Since $\mathcal{U}$ is an open cover of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $\exists n \in N$ s.t. $x \in (-n, n)$. So then $x < n$.
So then for every real number, x, there exists a natural number, n, such that $x < n$. This is Archimedes' Axiom
($b \implies a$)
Let $\mathcal{U} = \{(-n, n) : n \in N\}$. Let $x \in \mathbb{R}$. By Archimedes' Axiom, $\exists n_1 \in \mathbb{N}$ s.t. $x < n_1$, and similar for $-x < n_2$. Then $x \in (-n', n')$ for $n' = \max(n_1, n_2)$. Since $(-n', n') \in \mathcal{U}$ and $(-n', n')$ is an open subset of $\mathbb{R}$, $\mathcal{U}$ is an open cover of $\mathbb{R}$. ■

4. Let S be a subset of the metric space E. For each $p \in S$, let r_p be a positive number. Prove $\mathcal{U} = \{B(p, r_p) : p \in S\}$ is an open cover of S.

   Proof. Let $p \in S$. Then $r_p > 0$. Also, $d(p,p) = 0 < r_p$. So then $p \in B(p, r_p)$. Since $B(p, r_p) \in \mathcal{U}$, $S \subseteq \bigcup \mathcal{U}$.
   Obviously, each element of $\mathcal{U}$ is an open ball, which is an open subset of E.
   So $\mathcal{U}$ is an open cover of S. ∎

5. Let E be a metric space and $S \subseteq E$. Let $p \in E$. Prove $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S.

   Proof. Let $s \in S$. Then $d(p,s) + 1 > 0$. So then $s \in B(p, d(p,s) + 1)$ and $B(p, d(p,s) + 1) \in \mathcal{U}$. So $S \subseteq \bigcup \mathcal{U}$.
   Obviously, every element in $\mathcal{U}$ is an open subset of E.
   So $\mathcal{U}$ is an open cover of S. ∎

6. Let S be a set that has a finite open cover $\mathcal{U}$. Assume that for each $U \in \mathcal{U}$ that $U \cap S$ is finite. Prove S is finite.

   Proof. Since for each $U \in \mathcal{U}$, $U \cap S$ is finite and $\mathcal{U}$ covers S, we can create an upper bound for $|S|$ with $\sum_{U \in \mathcal{U}} |U \cap S|$. Since $\mathcal{U}$ is finite, there are finite terms in that sum. So then $|S|$ is finite. ∎

7. Let S be a subset of the metric space E that has has the property that if $\mathcal{U}$ is an open cover of S, then $\mathcal{U}$ has a finite subset $\mathcal{U}_0$ which is also a cover of S. Prove that S is bounded.

   Proof. Let $p \in E$. Then from problem 5, $\mathcal{U} = \{B(p, r) : r > 0\}$ is an open cover of S. So then by the property of S, $\mathcal{U}_0 = \{B(p, r_1), \ldots, B(p, r_n)\}$ is a cover of S. So $S \subseteq \bigcup \mathcal{U}_0$.
   Let $r_{\max} = \max\{r_1, \ldots, r_n\}$. Then by problem 1, $\bigcup \mathcal{U}_0 = B(p, r_{\max})$.
   So $S \subseteq B(p, r_{\max})$, meaning S is bounded. ∎