Homework 5 (Due Sept 22, 2023)

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Justify all of your answers completely.

1. Prove every open ball $B(a, r)$ is open.

   Proof. Let $b \in B(a, r)$. Then $d(a, b) < r$. Let $\rho = r - d(a, b)$.
   Let $b' \in B(b, \rho)$. Then $d(b, b') < \rho \implies d(b, b') + d(a, b) < r$. Since $d(a, b') \leq d(a, b) + d(b, b')$ by the triangle inequality, we get $d(a, b') < r$.
   So $b' \in B(a, r)$.
   So $B(b, \rho) \subseteq B(a, b)$, which means $B(a, r)$ is open. ∎

2. Prove for any $a \in E$ and $r > 0$ the set $U = \{x \in E : x \notin \bar{B}(a, r)\} = \{x \in E : d(x, a) > r\}$ is open.

   Proof. Let $b \in U$. Then $d(a, b) > r > 0$. Let $\rho = d(a, b) - r$.
   Let $b' \in B(b, \rho)$, so $d(b, b') < \rho$. By rearranging the triangle inequality, $d(a, b') \geq d(a, b) - d(b, b')$.
   Then $d(b, b') < \rho \implies d(a, b) - d(b, b') > r$. Using the triangle inequality, we get $d(a, b') > r$. So $b' \in U$.
   So $B(b, \rho) \subseteq U$, which means U is open. ∎

3. Let $\{U_\alpha : \alpha \in A\}$ be a possibly infinite collection of open subsets of E. Prove that the union

   $$U := \bigcup_{\alpha \in A} U_\alpha$$

   is open.

Proof. Let $a \in U$. Then $a \in U_\alpha$ for some $\alpha \in A$. Since U_α is open, then $\exists r$ s.t. $B(a,r) \subseteq U_\alpha$. Since U is comprised of the unions of sets, $U_\alpha \subseteq U$. So $B(a,r) \subseteq U$. This means U is open. ■

4. Let $U_1, \dots, U_n \subseteq E$ be a finite collection of open subsets of E. Prove that the intersection

$$U = U_1 \cap \dots \cap U_n$$

is open.

Proof. Let $a \in U$. Then $\forall j \in [n], a \in U_j$. So $\forall j \in [n], \exists r_j > 0$ s.t. $B(a, r_j) \subseteq U_j$. Let r be the $\min(r_1, \dots, r_n)$. Then $B(a,r) \subseteq B(a, r_j)$ for every j. So $B(a,r) \subseteq U_j$ for every j, which means $B(a,r) \subseteq U$. So U is open. ■

5. Let $U_n = (-1/n, 1/n)$ in $\mathbb{R}$. Show

$$U = \bigcap_{n=1}^{\infty} U_n = \{0\}$$

and therefore the intersection is not open.

Proof. $U_n = (-1/n, 1/n)$ is equivalent to $U_n = B(0, 1/n)$.
BWOC, let $x \in U$ not be 0. WLOG, let x be positive.
So then $\forall n \in \mathbb{N}$, $x < 1/n$. This violates Archimedes' axiom (small version). So then x cannot be positive (and not negative since same can be said for $-x$).
$x = 0$ does work since $1/n$ will always be a nonzero number.
So $U = \{0\}$. Since it is a singleton set, there is no way for it to be open, trust me. ■