Homework 9 (Due Nov 17, 2023)

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Justify all of your answers completely.

1. Let $G_1 = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \ : \ f_{m,b}(x) = mx + b, \ m \neq 0\}$ be the group of affine functions, with composition of functions as operation, and let

$$G_2 = \{\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} : m, b \in \mathbb{R}, \ m \neq 0\}$$

with multiplication of matrices as operation. Prove that $G_1 \cong G_2$.

Proof. Let $\phi : G_1 \to G_2$ defined by $\phi(f_{m,b}) = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$.
Clearly ϕ is well defined.
Showing that ϕ preserves linearity.

$$\phi(f_{m_1,b_1} \circ f_{m_2,b_2}) = \phi(m_1(m_2x + b_2) + (b_1)) = \phi(m_1m_2x + (m_1b_2 + b_1))$$
$$= \begin{bmatrix} m_1m_2 & m_1b_2 + b_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_2 & b_2 \\ 0 & 1 \end{bmatrix} = \phi(f_{m_1,b_1}) \cdot \phi(f_{m_2,b_2})$$

Now to show that ϕ is a bijection, by showing it has an inverse.
Let $\phi^{-1}(\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}) = mx + b$.
Clearly ϕ^{-1} is well defined.
Clearly composing ϕ into ϕ^{-1} or visa versa will result in the original input, meaning that ϕ and ϕ^{-1} are inverses, proving ϕ is a bijection.
So then ϕ is an isomorphism between G_1 and G_2, meaning $G_1 \cong G_2$. ■

2. Let $C = \{-1, 1\}$ with multiplication as operation. Let $G_1 = \mathbb{R}^*$, and let $G_2 = C \times \mathbb{R}^+$. Prove that $G_1 \cong G_2$.

Proof. Let $\phi : \mathbb{R}^* \to C \times \mathbb{R}^+$ be defined by $\phi(x) = (\frac{x}{|x|}, |x|)$.
Since $\frac{x}{|x|}$ results in the sign of x and $|x| \in \mathbb{R}^+$, ϕ is well defined.
Showing ϕ preserves linearity.

$$\phi(x \cdot y) = \left(\frac{xy}{|xy|}, |xy| \right) = \left(\frac{x}{|x|} \cdot \frac{y}{|y|}, |x| \cdot |y| \right) = \left(\frac{x}{|x|}, |x| \right) \cdot \left(\frac{y}{|y|}, |y| \right) = \phi(x) \cdot \phi(y)$$

Now to show ϕ is bijective by finding an inverse.
Let $\phi^{-1} : C \times \mathbb{R}^+ \to \mathbb{R}^*$ defined by $\phi^{-1}((c, x)) = cx$.
Clearly ϕ^{-1} is well defined.
Now to show ϕ^{-1} is indeed the inverse of ϕ.

$$\phi(\phi^{-1}((c, x))) = \phi(cx) = (\frac{cx}{|cx|}, |cx|) = (\frac{cx}{x}, |x|) = (c, x)$$

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$$\phi^{-1}(\phi(x)) = \phi^{-1}((\frac{x}{|x|}, |x|)) = (\frac{x}{|x|} \cdot |x|) = x$$

. So ϕ is an isomorphism between G_1 and G_2, meaning $G_1 \cong G_2$. ∎

3. Let G_1 be $\mathbb{R}$ with operation $*$ defined by $a * b = a + b - 1$. Prove that G_1 is isomorphic to $\mathbb{R}$.

Proof. Let $\phi : G_1 \to \mathbb{R}$ be defined by $\phi(x) = x - 1$.
Clearly ϕ is well defined.
Showing ϕ preserves linearity.

$$\phi(x * y) = \phi(x + y - 1) = x + y - 2 = (x - 1) + (y - 1) = \phi(x) + \phi(y)$$

Now to show that ϕ is bijective by finding an inverse.
Let $\phi^{-1} : \mathbb{R} \to G_1$ be defined by $\phi^{-1}(x) = x + 1$.
Clearly ϕ^{-1} is well defined.
It is trivial to show composing ϕ and ϕ^{-1} is x and visa versa.
So ϕ is an isomorphism between G_1 and $\mathbb{R}$, meaning $G_1 \cong \mathbb{R}$. ∎

4. Let $G = \mathbb{R} \backslash \{-1\}$, with operation defined by $ab = a + b + ab$. Prove that G is isomorphic to $\mathbb{R}^*$.

Proof. Let $\phi : G \to \mathbb{R}^*$ be defined by $\phi(x) = x + 1$.
Clearly ϕ is well defined since $x + 1$ cannot be 0 since x cannot be -1.
Showing ϕ preserves linearity.

$$\phi(xy) = \phi(x + y + xy) = x + y + xy + 1 = x(y + 1) + y + 1 = (x + 1)(y + 1) = \phi(x) \cdot \phi(y)$$

Now to show that ϕ is bijective by finding an inverse.
Let $\phi^{-1} : \mathbb{R}^* \to G$ be defined by $\phi^{-1}(x) = x - 1$.
Since $x \neq 0$, then $x - 1 \neq -1$, meaning $(x - 1) \in G$. So ϕ^{-1} is well defined.
It is trivial to show composing ϕ and ϕ^{-1} is x and visa versa.
So ϕ is an isomorphism between G and $\mathbb{R}^*$, meaning $G \cong \mathbb{R}^*$. ∎

5. Let $G = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod 7\}$, with component-wise addition as operation. Prove that $G \cong \mathbb{Z} \times \mathbb{Z}$.

Proof. Let $\phi : G \to \mathbb{Z} \times \mathbb{Z}$ be defined by $\phi(a,b) = (a, \frac{a-b}{7})$.
ϕ is well defined since $(a,b) \in G \implies 7 \mid (b-a)$.
Showing ϕ preserves linearity.

$$\phi((a_1,b_1) + (a_2,b_2)) = \phi(a_1 + a_2, b_1 + b_2) = \left((a_1 + a_2), \frac{(a_1 + a_2) - (b_1 + b_2)}{7}\right)$$

$$= \left(a_1 + a_2, \frac{a_1 - b_1}{7} + \frac{a_2 - b_2}{7}\right) = \left(a_1, \frac{a_1 - b_1}{7}\right) + \left(a_2, \frac{a_2 - b_2}{7}\right) = \phi(a_1, b_1) + \phi(a_2, b_2)$$

Now to show that ϕ is bijective by finding an inverse.
Let $\phi^{-1} : \mathbb{Z} \times \mathbb{Z} \to G$ be defined by $\phi^{-1}(a,b) = (a, a - 7b)$.
First, we need to show that $(a, a-7b) \in G$.
$(a - 7b) - a = -7b$, so $7 \mid ((a - 7b) - a)$, meaning $(a, a - 7b) \in G$.
Now to show ϕ^{-1} is indeed the inverse of ϕ.

$$\phi^{-1}(\phi(a,b)) = \phi^{-1}\left(a, \frac{a-b}{7}\right) = \left(a, a - 7\left(\frac{a-b}{7}\right)\right) = (a, a - (a-b)) = (a,b)$$

$$\phi(\phi^{-1}(a,b)) = \phi(a, a - 7b) = \left(a, \frac{a - (a - 7b)}{7}\right) = \left(a, \frac{7b}{7}\right) = (a,b)$$

So ϕ is an isomorphism between G and $\mathbb{Z} \times \mathbb{Z}$, meaning $G \cong \mathbb{Z} \times \mathbb{Z}$. ∎