Homework 10 (Due Dec 8, 2023)

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Justify all of your answers completely.

1. Find a subgroup of S_4 that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by carrying through the procedure we used to prove Cayley's theorem.

   Denote elements of $Z_2 \times Z_2$ as $g_1 = ([0],[0]), g_2 = ([1],[0]), g_3 = ([0],[1]), g_4 = ([1],[1])$.
   For each g_i, associate a σ_i by constructing $\sigma_i(\ell) = j$ if $g_i + g_\ell = g_j$.
   For g_1: $g_1 + g_i = g_i$ for all i. So $\sigma_1 = e$
   For g_2: $g_2+g_1 = g_2$, $g_2+g_2 = g_1$, $g_2+g_3 = g_4$, $g_2+g_4 = g_3$. So $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 4)$.
   For g_3: $g_3+g_1 = g_3$, $g_3+g_2 = g_4$, $g_3+g_3 = g_1$, $g_3+g_4 = g_2$. So $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\ 3)(2\ 4)$.
   For g_4: $g_4+g_1 = g_4$, $g_4+g_2 = g_3$, $g_4+g_3 = g_2$, $g_4+g_4 = g_1$. So $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 4)(2\ 3)$.
   So the subgroup, H, of S_4 that is isomorphic to $Z_2 \times Z_2$ is
   $H = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 4)\}$.

2. Cayley's theorem tells us that there exists a subgroup of S_6 that is isomorphic to Z_6.

   (a) Give an example of such a subgroup and justify the isomorphism.

   We know that cyclic groups of the same order are isomorphic. Z_6 is a cyclic group of order 6, so we want a cyclic subgroup of S_6 also with order 6. The subgroup $\langle(1\ 2\ 3\ 4\ 5\ 6)\rangle$ is a simple example of that.

   (b) Does there exist any $n < 6$ such that $\mathbb{Z}_6$ is isomorphic to a subgroup of S_n? Find the smallest such value of n.

   Since Z_6 is cyclic of order 6, the subgroup of S_n it would be isomorphic to would also have to be cyclic of order 6.
   So we are looking for values of n that when partitioned, the lcm of the partitions can be 6.
   $n = 5$ works since you can partition 5 into 2+3, so a cyclic subgroup generated from permutation decomposed into cycles of length 2 and 3 would be isomorphic.
   $n < 5$ cannot be partitioned for the lcm to equal 6, so 5 is the lowest.

3. For the group G and the subgroup H, list all the cosets with respect to H. For each coset, list the elements of the coset. How many distinct cosets are there?

   (a) $G = S_3,\ H = \{e, (1\ 2)\}$

   $G = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$
   $eH = \{e,\ (1\ 2)\}, \quad (1\ 2)H = \{(1\ 2),\ e\}, \quad (1\ 3)H = \{(1\ 3), (1\ 2\ 3)\}$
   $(2\ 3)H = \{(2\ 3), (1\ 3\ 2)\}, \quad (1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$
   $(1\ 3\ 2)H = \{(1\ 3\ 2), (2\ 3)\}$
   So there are 3 distinct cosets.

   (b) $G = \mathbb{Z}_4 \times \mathbb{Z}_4,\ H = \langle([1]_4, [1]_4)\rangle$.

   Since distinct cosets are disjoint, this can cut our search space down a lot.
   $e + H = \{([0]_4, [0]_4), ([1]_4, [1]_4), ([2]_4, [2]_4), ([3]_4, [3]_4)\}$
   $([1]_4, [0]_4) + H = \{([1]_4, [0]_4), ([2]_4, [1]_4), ([3]_4, [2]_4), ([0]_4, [3]_4)\}$
   $([2]_4, [0]_4) + H = \{([2]_4, [0]_4), ([3]_4, [1]_4), ([0]_4, [2]_4), ([1]_4, [3]_4)\}$
   $([3]_4, [0]_4) + H = \{([3]_4, [0]_4), ([0]_4, [1]_4), ([1]_4, [2]_4), ([2]_4, [3]_4)\}$
   That gives us 4 cosets, each with 4 elements in them, giving 16 total elements. That covers all the elements in $\mathbb{Z}_4 \times \mathbb{Z}_4$, so we know no more distinct cosets exists.

4. For the group G and the subgroup H, decide whether H is a normal subgroup of G or not.

   (a) $G = S_3,\ H = \{e, (1\ 2)\}$

   It is not since $(2\ 3) \in G$ and $(1\ 2) \in G$ is a counter example.
   The inverse of $(2\ 3)$ is itself. So $(2\ 3)(1\ 2)(2\ 3) = (1\ 3) \notin H$.
   So H is not a normal subgroup of G.

   (b) $G = S_4,\ H = A_4$

   Proof. Let $g \in G$ and $h \in H$.
   Let us represent g and h as transpositions.
   $g = \tau_1 \dots \tau_n$ and $h = \tau_1' \dots \tau_{2k}'$. h has $2k$ transpositions since $h \in A_4$.
   Since the inverse of transpositions is just the order reversed, we get $g^{-1} = \tau_n \dots \tau_1$.
   So then $ghg^{-1} = (\tau_1 \dots \tau_n)(\tau_1' \dots \tau_{2k}')(\tau_n \dots \tau_1)$.
   ghg^{-1} has $n + 2k + n = 2(n + k)$ transpositions, which is an even amount.
   So $ghg^{-1} \in H$, making H a normal subgroup. ∎

5. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$, and let $H = \langle([1]_4, [0]_6)\rangle$. Consider the factor group G/H.

   (a) What is the order of the element $([1]_4, [2]_6)$ as an element of G?

   $[1]_4$ has order 4 and $[2]_6$ has order 3, so the order of $([1]_4, [2]_6)$ will be the $\text{lcm}(3, 4)$, which is 12.

   (b) What is the order of the element $([1]_4, [2]_6) + H$ as an element of G/H?

   To find the order of the element, want to find the smallest n s.t. $n \cdot (([1]_4, [2]_6) + H)$ is the identity of G/H, which is $e + H = H$.

$(([1]_4, [2]_6) + H) \neq H$.
$(([1]_4, [2]_6) + H) + (([1]_4, [2]_6) + H) = (([2]_4, [4]_6) + H) \neq H$.
$(([1]_4, [2]_6) + H) + (([1]_4, [2]_6) + H) + (([1]_4, [2]_6) + H) = (([2]_4, [4]_6) + H) + (([1]_4, [2]_6) + H) = (([3]_4, [0]_6) + H) = H$ since $([3]_4, [0]_6) \in H$.
So the order is 3.