Homework 6 (Due March 5, 2025)

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Justify all of your answers completely.

1. Prove that $\frac{\mathbb{Z}[x]}{(x^2-2)}$ is isomorphic to $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

   Proof. Let $\phi : \mathbb{Z}[x] \to \mathbb{R}$ be the evaluation map $\phi(f(x)) = f(\sqrt{2})$. It is quite easy to see that $(x^2-2) \subseteq \ker\phi$. To show $(x^2-2) \supseteq \ker\phi$, we can see that any $f(x) \in \mathbb{Z}[x]$ with $f(\sqrt{2}) = 0$ will have $x^2 - 2$ as a factor, meaning it is a multiple, meaning it is in (x^2-2). So then $(x^2-2) = \ker\phi$.

   Since evaluation maps are homomorphisms, the F.H.T. gives us $\frac{\mathbb{Z}[x]}{(x^2-2)} \cong \operatorname{Im}\phi$. Now to find what $\operatorname{Im}\phi$ is.

   When a term in a polynomial has degree $0 \mod 2$, we get the term evaluated at $\sqrt{2}$ is an integer.

   When a term in a polynomial has degree $1 \mod 2$, we get the term evaluated at $\sqrt{2}$ is a multiple of $\sqrt{2}$.

   Putting this together, we can easily see that

   $$\operatorname{Im}\phi = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$$

   ∎

2. Prove that $I = (2, x)$ is a maximal ideal in $\mathbb{Z}[x]$.

   Proof. Let $\phi : \mathbb{Z}[x] \to \mathbb{Z}_2$ be the map $\phi(f(x)) = [f(0)]_2$. It is quite easy to see that $(2, x) \subseteq \ker\phi$, since the functions evaluated at 0 leave only the constant term, and that term is always even. To show $(2, x) \supseteq \ker\phi$, we can see that any $f(x) \in \mathbb{Z}[x]$ with $f(0) \equiv 0 \mod 2$ will have the constant be even, meaning it is in $(2, x)$. So then $(2, x) = \ker\phi$.

   Since evaluation maps are homomorphisms, and modulus is also a homomorphism, ϕ is a homomorphism. So the F.H.T. gives us $\frac{\mathbb{Z}[x]}{(2,x)} \cong \operatorname{Im}\phi$. It is trivial to check that $\operatorname{Im}\phi = \mathbb{Z}_2$ since there are only 2 elements.

   We know that any $\mathbb{Z}_q$ is a field as long as q is a prime power. So then $\mathbb{Z}_2$ is a field. That implies that $\frac{\mathbb{Z}[x]}{(2,x)}$ is also a field, meaning $(2, x)$ is a maximal ideal. ∎

3. Find a subring of $\mathbb{R}$ that is isomorphic to $\frac{\mathbb{Z}[x]}{(x^3-2)}$. Prove the isomorphism.

Proof. Let $\phi : \mathbb{Z}[x] \to \mathbb{R}$ be the evaluation map $\phi(f(x)) = f(\sqrt[3]{2})$. It is quite easy to see that $(x^3-2) \subseteq \ker\phi$. To show $(x^3-2) \supseteq \ker\phi$, we can see that any $f(x) \in \mathbb{Z}[x]$ with $f(\sqrt[3]{2}) = 0$ will have x^3-2 as a factor, meaning it is a multiple, meaning it is in (x^3-2). So then $(x^3-2) = \ker\phi$.

Since evaluation maps are homomorphisms, the F.H.T. gives us $\frac{\mathbb{Z}[x]}{(x^3-2)} \cong \operatorname{Im}\phi$. Now to find what $\operatorname{Im}\phi$ is.

When a term in a polynomial has degree $0 \mod 3$, we get the term evaluated at $\sqrt[3]{2}$ is an integer.

When a term in a polynomial has degree $1 \mod 3$, we get the term evaluated at $\sqrt[3]{2}$ is a multiple of $\sqrt[3]{2}$.

When a term in a polynomial has degree $2 \mod 3$, we get the term evaluated at $\sqrt[3]{2}$ is a multiple of $\sqrt[3]{4}$.

Putting this together, we can easily see that

$$\operatorname{Im}\phi = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Z}\}$$

■

4. Let R, S be commutative rings and let $F : R \to S$ be a ring isomorphism. Let I be an ideal of R. Consider
$$F(I) = \{F(x) : x \in I\}.$$

   a. Prove that $F(I)$ is an ideal of S and that R/I is isomorphic to $S/F(I)$.

   Proof. Let $F(x) \in F(I)$ and $s \in S$. Since F is a bijection, there is an $s' \in R$ s.t. $F(s') = s$. Since I is an ideal, then $s'x \in I$. So then $sF(x) = F(s')F(x) = F(s'x) \in F(I)$, making $F(I)$ an ideal of S.

   Define $\phi : R/I \to S/F(I)$ with $\phi(x+I) = F(x) + F(I)$. We need to check if ϕ is well defined.

   Let $r + I = r' + I$ be two representations of the same coset. Then $r - r' \in I$, which means $F(r-r') \in F(I)$. Since F is a isomorphism, we can get to $F(r) + F(I) = F(r') + F(I)$. So ϕ is well defined.

   It is also easy to see that ϕ is an isomorphism since addition and multiplication of cosets are preserved, and that F is an isomorphism.

   S0 then R/I is isomorphic to $S/F(I)$. ■

   b. Use the result from part a. to prove that

$$\frac{\mathbb{Q}[x]}{(x^2-2)} \cong \frac{\mathbb{Q}[x]}{(x^2+4x+2)}$$

Proof. Let $F : \mathbb{Q}[x] \to \mathbb{Q}[x]$ with $F(f(x)) = f(x+2)$. We can observe that $F(x^2 - 2) = (x+2)^2 - 2 = x^2 + 4x + 2$.

F has an inverse $F^{-1}(f(x)) = f(x-2)$. This is shown by $F(F^{-1}(f(x))) = F(f(x-2)) = f(x)$ and $F^{-1}(F(f(x))) = F^{-1}(f(x+2)) = f(x)$.

So then F is a bijection. We also know that function composition preserves structure through addition and multiplication, so that means F is an isomorphism.

So all together with part a., we have that

$$\frac{\mathbb{Q}[x]}{(x^2 - 2)} \cong \frac{\mathbb{Q}[x]}{(x^2 + 4x + 2)}$$

∎

5. Prove that the rings $R_1 = \frac{\mathbb{Z}[x]}{(x^2-2)}$ and $R_2 = \frac{\mathbb{Z}[x]}{(x^2-5)}$ are not isomorphic to each other.

Proof. BWOC, assume that $F : R_1 \to R_2$ is an isomorphism. Let $\bar{x} \in R_1$, meaning $\bar{x} = x + (x^2 - 2)$, making $\bar{x}^2 = 2$. Let $u := F(\bar{x}) \in R_2$. We then have

$$\begin{aligned} u^2 &= F(\bar{x})^2 \\ &= F(\bar{x}^2) \\ &= F(2) \\ &= 2 \end{aligned}$$

In R_2, any element is of the form $a + b\bar{y}$, where $\bar{y} = x + (x^2 - 5)$ satisfies $\bar{y}^2 = 5$.

Then we have that $u^2 = (a + b\bar{y})^2 = a^2 + 5b^2 + 2ab\bar{y}$. Since $u^2 = 2$, we have that $ab = 0$, giving either a or b is 0.

If $b = 0$, then $a^2 = 2$. This has no integer solutions however, a contradiction. If $a = 0$, then $5b^2 = 2$. This has no integer solutions however, a contradiction. ∎