# Homework 9 (Due Oct 18, 2023)

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Justify all of your answers completely.

1. Prove if $(E, d)$ is a metric space and $F$ is a closed subset of $E$, then $(F, d)$ is also a complete metric space.

   *Proof.* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $F$, and consequently in $E$. Since $E$ is complete, we know $p_n$ converges to some $p \in E$. We also know that a subset is closed iff the subset contains the limits of its sequences. So then $p \in F$. This shows $F$ is complete. ■

2. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^3$ with its usual metric. Let $p_n = (x_n, y_n, z_n)$.

   (a) Show that each of the sequence $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$, $\langle z_n \rangle_{n=1}^{\infty}$ are also Cauchy sequences and explain why this implies the limits $x := \lim_{n\to\infty} x_n$, $y := \lim_{n\to\infty} y_n$, $z := \lim_{n\to\infty} z_n$ exist.

   *Proof.*

$$|x_m - x_n| = \sqrt{(x_m - x_n)^2} \leq \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2 + (z_m - z_n)^2}$$
$$= d(p_m, p_n)$$

   Similar calculations can be done for $y$ and $z$.
   Since $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy, $\exists N > 0$ s.t. $m, n > N \implies d(p_m, p_n) < \epsilon$.
   The above inequalities give us $m, n > N \implies |x_m - x_n| < \epsilon$ and

similar ones for $y$ and $z$.
This means the $x$,$y$, and $z$ sequences are Cauchy. Since the individual sequences are in $\mathbb{R}$, which is complete, they will converge as they are Cauchy.

$\blacksquare$

(b) Let $p = (x, y, z)$ and show $\lim_{n\to\infty} p_n = p$.

*Proof.* Since the sequences in the part above converge, $\exists N_1 > 0$, $N_2 > 0$, and $N_3 > 0$ s.t.

$$n > N_1 \implies |x_n - x| < \frac{\epsilon}{\sqrt{3}}$$

$$n > N_2 \implies |y_n - y| < \frac{\epsilon}{\sqrt{3}}$$

$$n > N_3 \implies |z_n - z| < \frac{\epsilon}{\sqrt{3}}$$

Then if $N = \max\{N_1, N_2, N_3\}$ and $p = (x, y, z)$,

$$n > N \implies \|p_n - p\| = \sqrt{(x_n - x)^2 + (y_n - y)^2 + (z_n - z)^2}$$

$$< \sqrt{\left(\frac{\epsilon}{\sqrt{3}}\right)^2 + \left(\frac{\epsilon}{\sqrt{3}}\right)^2 + \left(\frac{\epsilon}{\sqrt{3}}\right)^2} = \epsilon$$

$\blacksquare$

(c) Conclude that $\mathbb{R}^3$ is a complete metric space.

*Proof.* The above parts show that an arbitrary Cauchy sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges, which means that $\mathbb{R}^3$ is complete. $\blacksquare$

3. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $E$. Prove that the sequence is bounded. That is show there is a ball $B(p, r)$ with $p_n \in B(p, r)$ for all $n$.

*Proof. For this problem, I'm too lazy to put a bar above the ball, but just know all the balls will be closed balls* Since $p_n$ is a Cauchy sequence, $\exists N$ s.t.

$$m, n > N \implies d(p_m, p_n) < \epsilon$$

So for any $n > m > N$, $p_n \in B(p_m, \epsilon)$.
Let $r = \max\{d(p_m, p_1), d(p_m, p_2), \ldots, d(p_m, p_N), \epsilon\}$.
So then $p_n \in B(p_m, r)$ for all $n$. ∎

4. Let $f : E \to E$ be a contraction and let $\lim_{n\to\infty} p_n = p$ in $E$. Show $\lim_{n\to\infty} f(p_n) = f(p)$.

*Proof.* Want to show $\forall \epsilon > 0$, $\exists N$ s.t. $n > N \implies d(f(p_n), f(p)) < \epsilon$.
Let $\epsilon > 0$. Since the limit exists in $E$, there is an $N_1$ s.t. $n > N_1 \implies d(p_n, p) < \epsilon$. Let $n > N_1$.
Since $f$ is a contraction, $d(f(p_n), f(p)) \le \rho d(p_n, p) \le \rho\epsilon < \epsilon$.
So $d(f(p_n), f(p)) < \epsilon$, which proves the limit. ∎

5. Prove the Banach Fixed Point Theorem following the outline given.

*Proof.* Let $p_0 \in E$ and define a sequence $\langle p_n \rangle_{n=1}^{\infty}$ where $p_n = f(p_{n-1})$.
Consider $d(p_k, p_{k+1})$ for $k \ge 1$.
$d(p_k, p_{k+1}) = d(f(p_{k-1}), f(p_k)) \le \rho d(p_{k-1}, p_k)$. Since we are big boys and girls, we can use our pattern recognition and see that induction will show $d(p_k, p_{k+1}) \le p^k d(p_0, p_1)$.
Let $m < n$. The triangle implies

$$d(p_m, p_n) \le \sum_{k=m}^{n-1} d(p_k, p_{k+1}) \le \sum_{k=m}^{n-1} \rho^k d(p_0, p_1) = d(p_0, p_1) \sum_{k=m}^{n-1} \rho^k$$

$$= d(p_0, p_1) \cdot \frac{\rho^m - \rho^n}{1 - \rho}$$

So $d(p_m, p_n) \le \frac{\rho^m - \rho^n}{1-\rho} d(p_0, p_1) \le \frac{\rho^m}{1-\rho} d(p_0, p_1)$
Let $m, n \ge N$. Since $0 \le \rho < 1$, $\rho^N \ge \rho^m$.
So $d(p_m, p_n) \le \frac{\rho^m}{1-\rho} d(p_0, p_1) \implies d(p_m, p_n) \le \frac{\rho^N}{1-\rho} d(p_0, p_1)$
Let $\epsilon > 0$. Since $\lim_{N\to\infty} \frac{\rho^N}{1-\rho} d(p_0, p_1) = 0$, $\exists N_1$ s.t.

$$N > N_1 \implies \frac{\rho^N}{1-\rho} d(p_0, p_1) < \epsilon.$$

So $d(p_m, p_n) < \epsilon$, meaning the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy.
Since $E$ is complete, this means a Cauchy sequence, like $\langle p_n \rangle_{n=1}^{\infty}$, converges.

Define $p_* := \lim_{n\to\infty} p_n$. By problem 4, $\lim_{n\to\infty} f(p_n) = f(p_*)$. We can say $f(p_n) = p_{n+1}$, and set $n' = n+1$. As $n \to \infty$, $n' \to \infty$.
So then $\lim_{n\to\infty} f(p_n) = f(p_*) \implies \lim_{n'\to\infty} p_{n'} = f(p_*) \implies p_* = f(p_*)$.
So $p_*$ is a fixed point of $f$.
To show that the fixed point is unique, assume that $p_{**}$ is a second fixed point of $f$. Then $d(p_*, p_{**}) = d(f(p_*), f(p_{**})) \le \rho d(p_*, p_{**})$. This can be repeated an arbitrary amount of times, and by pattern recognition again (since we are big kids) we see that it will approach 0.
So the distance between $p_*$ and $p_{**}$ is 0, meaning they are the same. ∎

6. Let $a \ge 1$ and define $f : [0,\infty) \to [0,\infty)$ by

$$f(x) = \sqrt{a+x}$$

   (a) Show for $x, y \in [0,\infty)$ that

$$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{a+x}+\sqrt{a+y}} \le \frac{|x-y|}{2\sqrt{a}} \le \frac{1}{2}|x-y|$$

   and therefore $f$ is a contraction. The space $[0,\infty)$ is a complete metric space as it is a closed subset of the complete space $\mathbb{R}$.

   *Proof.*

$$|f(x)-f(y)| = |\sqrt{a+x}-\sqrt{a+y}| = \left|\frac{(\sqrt{a+x}-\sqrt{a+y})((\sqrt{a+x}+\sqrt{a+y}))}{(\sqrt{a+x}+\sqrt{a+y})}\right|$$

$$= \frac{|x-y|}{\sqrt{a+x}+\sqrt{a+y}} \le \frac{|x-y|}{\sqrt{a}+\sqrt{a}} = \frac{|x-y|}{2\sqrt{a}} \le \frac{|x-y|}{2}$$

∎

   (b) Define a sequence $x_0 = a$ and $x_{n+1} = f(x_n)$. Find the fixed point.

   *Proof.* Want to find $\lim_{n\to\infty} x_n$, which can be thought of as finding $\sqrt{a+\sqrt{a+\sqrt{\dots}}}$.
   Since we know the limit exists by the Banach Fixed Point theorem, we can set the limit to $x$. Since $f(x) = x$ (shown in problem 5), we can do

$$x = \sqrt{a+x} \implies x^2 - x - a = 0 \implies x = \frac{1+\sqrt{1+4a}}{2}$$

$\blacksquare$

7. The Banach Fixed Point Theorem can be used to solve equations that at first glance are not fixed point problems. As an example let us compute numerically a root of the equation

$$x^3 - 5x - 1 = 0$$

We can rewrite this as

$$\frac{x^3 - 1}{5} = x$$

so we are looking for a fixed point of $f$ given by

$$f(x) = \frac{x^3 - 1}{5} = \frac{x^3}{5} - \frac{1}{5}.$$

Let $E = [-1, 1]$. This is a closed subspace of $\mathbb{R}$ and therefore is a complete metric space.

(a) If $|x| \le 1$ show

$$|f(x)| \le \frac{2}{5}$$

and therefore $f$ maps $E$ into $E$.

*Proof.* Let $|x| \le 1$. Then

$$|f(x)| = \left|\frac{x^3}{5} - \frac{1}{5}\right| \le \left|\frac{x^3}{5}\right| + \left|\frac{1}{5}\right| = \frac{|x|^3}{5} + \frac{1}{5} \le \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$\blacksquare$

(b) Show if $x, y \in E$ then

$$|f(x) - f(y)| \le \frac{3}{5}|x - y|$$

and therefore $f$ is a contraction on $E = [-1, 1]$.

*Proof.*

$$|f(x) - f(y)| = |\frac{x^3 - y^3}{5}| = |x - y|\frac{|x^2 + xy + y^2|}{5}$$

$$\le |x - y|\frac{|1^2 + 1 \cdot 1 + 1^2|}{5} = \frac{3}{5}|x - y|$$

$\blacksquare$

(c) This one does not seem like a problem to do, more like an example? So I shan't do it. $+$