Homework 7 (Due Nov 3, 2023)

Jack Hyatt
MATH 546 - Algebraic Structures I - Fall 2023

November 3, 2023

Justify all of your answers completely.

1. In D_4 with t representing a reflection along a diagonal, which one of the elements $e, r, r^2, r^3, t, rt, r^2t, r^3t$ is equal to the reflection about the other diagonal? Justify.

   WLOG, we can assign t to be $(2\ 4)$. Then we want to get $(1\ 3)$.
   Since $r = (1\ 2\ 3\ 4)$, then $r^2 = (1\ 3)(2\ 4)$. So we want to undo that $(2\ 4)$ in r^2. Luckily, cycles of length 2 are their own inverses, and $t = (2\ 4)$. So then $r^2t = (1\ 3)$. We found our other reflection.

2. Prove that $Z(D_4) = \{e, r^2\}$.

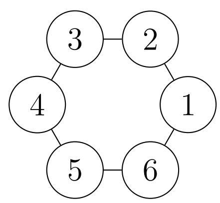
   Proof. Note: $r^2 = (1\ 3)(2\ 4)$ Obviously $e \in Z(D_4)$.
   Consider r and t. $rt = (1\ 2\ 3\ 4)(2\ 4) = (1\ 2)(3\ 4)$. $tr = (2\ 4)(1\ 2\ 3\ 4) = (1\ 4)(2\ 3)$. So $rt \neq tr$, meaning $r, t \notin Z(D_4)$.
   Consider r^3 and rt. $r^3rt = t = (2\ 4)$. $rtr^3 = tr^2 = r^2t = (1\ 3)$. So $r^3rt \neq rtr^3$, meaning $r^3t, rt \notin Z(D_4)$.
   Consider r^2t and r^3t. $r^2tr^3t = rtr^2t = trt = r^3$. $r^3tr^2t = r^2trt = r$. Since $r \neq r^3$, $r^2tr^3t \neq r^3tr^2t$, meaning $r^2t, r^3t \notin Z(D_4)$.
   Now to show $r^2 \in Z(D_4)$.
   Obviously r^2 commutes with other powers of r.
   $r^2(t) = r^6t = r^3tr = (t)r^2$
   $r^2(rt) = r^3t = tr = r^4tr = (rt)r^2$
   $r^2(r^2t) = r^4t = t = tr^4 = r^3tr^3 = r^6tr^2 = (r^2t)r^2$
   $r^2(r^3t) = r^2tr = r^6tr = (r^3t)r^2$.
   So r^2 commutes with every element, meaning $r^2 \in Z(D_4)$. ∎

3. Find all the subgroups of D_4. For each subgroup, list the elements of the subgroup and explain why it is a subgroup. Also explain why there are no other subgroups. You may use some of the results of the calculations you have done for problem 1. without reproving them.

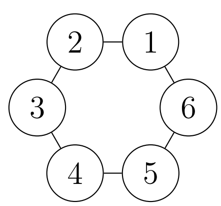
   We know that $|D_4| = 8$, so any subgroup must be of size $1, 2, 4, 8$.

Obviously, $\{e\}$ and D_4 are subgroups.
For subgroups of size 2, the element other than e must have order 2. The only elements with that property are r^2, t, rt, r^2t, r^3t. So the subgroups with e and any one of those elements are valid.
For subgroups of size 4, since e must be in it, then the other 3 must be closed and have order 2 or 4.
If r is in the group, then r^2 and r^3 must also be in the group. So $\{e, r, r^2, r^3\}$ works, and no other subgroup of order 4 can contain r.
$\{r^2, t, r^2t\}$ works, along with $\{r^2, rt, r^3t\}$, which is all the work with r^2.
So then t can't be with rt since it would have r with closure, and t can't be with r^3t since that would put r^3 in the subgroup along with r^2, which is more than 4.
So the last trio to check is rt, r^2t, r^3t, which is not closed since rtr^2t creates r^3.
So the only subgroups are:
e, D_4, $\{e, r^2\}$, $\{e, t\}$, $\{e, rt\}$, $\{e, r^2t\}$, $\{e, r^3t\}$, $\{e, r, r^2, r^3\}$, $\{e, r^2, t, r^2t\}$, $\{e, r^2, rt, r^3t\}$.

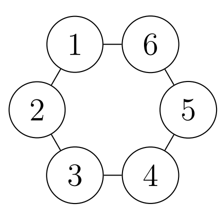
4. Let D_6 be the group of symmetries (rotations and reflections) of a regular hexagon. Using the numbers $1, \ldots, 6$ to label the vertices of the hexagon, write each element of D_6 as a permutation, explaining which elements are rotations and by what angle, and which elements are reflections and about what axis of symmetry.



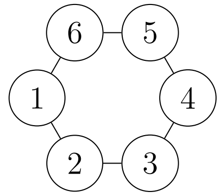
Let the base rotation of $\pi/3$ counterclockwise be $r = (1\ 6\ 5\ 4\ 3\ 2)$.



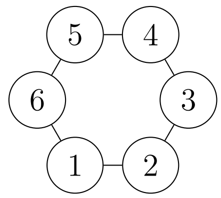
Then r^2 is a ccw rotation of $2\pi/3$ as $(1\ 5\ 3)(2\ 6\ 4)$.



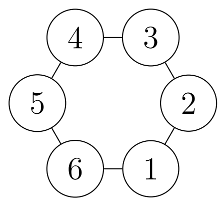
Then r^3 is a ccw rotation of π as $(1\ 4)(2\ 5)(3\ 6)$.



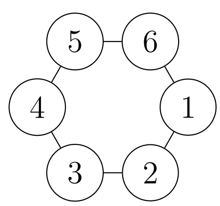
Then r^4 is a ccw rotation of $4\pi/3$ as $(1\ 3\ 5)(2\ 4\ 6)$.



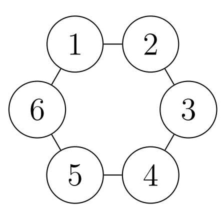
Then r^5 is a ccw rotation of $5\pi/3$ as $(1\ 2\ 3\ 4\ 5\ 6)$.



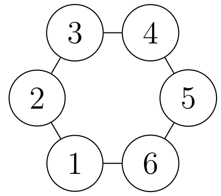
Let the base reflection along the diagonal 1-4 be $t = (2\ 6)(3\ 5)$.



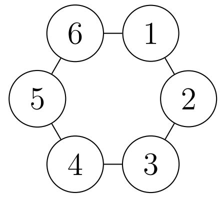
Reflection along 2-5 $= (1\ 3)(4\ 6)$:



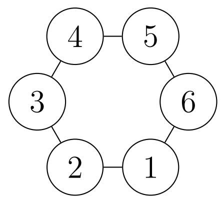
Reflection along 3-6 $= (1\ 4)(2\ 5)$:



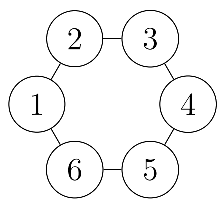
Reflection along slope of $1 = (1\ 2)(3\ 6)(4\ 5)$:



Reflection along slope of -1 $= (1\ 6)(2\ 5)(3\ 4)$:



Reflection along vertical slope $= (1\ 4)(2\ 3)(5\ 6)$:



5. Let r denote the rotation of the regular hexagon by an angle of $\pi/3$, and let t denote the reflection about one of the diagonals. Prove that each of the elements of D_6 that you found in problem 4 can be obtained as $r^i t^j$ with $i \in \{0, 1, 2, 3, 4, 5\}$ and $j \in \{0, 1\}$.

   Let $r = (1\ 6\ 5\ 4\ 3\ 2)$ and $t = (2\ 6)(3\ 5)$.
   The elements of only rotations found in problem 4 can obviously be made out of r^i, as well as e and t being made out of desired powers.
   $rt = (1\ 6\ 5\ 4\ 3\ 2)(2\ 6)(3\ 5) = (1\ 6)(2\ 5)(3\ 4) =$ reflection along slope of -1
   $r^2t = (1\ 5\ 3)(2\ 6\ 4)(2\ 6)(3\ 5) = (1\ 4)(2\ 5) =$ reflection along 3-6
   $r^3t = (1\ 4)(2\ 5)(3\ 6)(2\ 6)(3\ 5) = (1\ 4)(2\ 3)(5\ 6) =$ reflection along vertical slope.
   $r^4t = (1\ 3\ 5)(2\ 4\ 6)(2\ 6)(3\ 5) = (1\ 3)(4\ 6) =$ reflection along slope of 2-5
   $r^5t = (1\ 2\ 3\ 4\ 5\ 6)(2\ 6)(3\ 5) = (1\ 2)(3\ 6)(4\ 5) =$ reflection along slope of 1.