Homework 8 (Due Oct 13, 2023)

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Justify all of your answers completely.

3.35 Prove a set is closed iff it contains all its adherent points.

Proof. Let S be a set in the metric space E.
$(\implies)$
Assume S is closed and p is an adherent point of S. BWOC, assume that $p \notin S$. Then $p \in S^c$. Since S is closed, S^c is open. So then there is a ball for $r > 0$, that $B(p, r) \subseteq S^c$. So we have an open ball about p that doesn't contain any points in S, which makes p not an adherent point. BOOM, A CONTRADICTION!!!
$(\impliedby)$
Showing the contrapositive. Assume S is not closed. Then S^c is not open. So then $\exists p \in S^c$ s.t. $\forall r > 0$, $B(p, r) \nsubseteq S^c$. So then $\exists s \in B(p, r)$ s.t. $s \in S$. This makes $p \notin S$ an adherent point of S. ■

3.36 Let S be a set in a metric space and p and adherent point of S. Prove there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ from S that converges to p.

Proof. Let p be an adherent point of S. Then $\forall r > 0$, $B(p, r)$ contains a point of S. For every positive integer n let $p_n \in S$ be a point of S that is in the ball $B(p, 1/n)$.
Let $\epsilon > 0$, $N > 1/\epsilon$. Assume $n > N$.
Then since p_n is in the open ball of radius $1/n$ centered at p,

$d(p_n, p) < 1/n < 1/N = \epsilon$.
So $d(p_n, p) < \epsilon$, meaning p_n converges to p. ■

3.37 Let S be a set in a metric space and p a point that is a limit of a sequence of points from S. Prove p is an adherent point of S.

Proof. Let S be a set in the metric space E and let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S that converges to $p \in E$. Let $r > 0$. Since p_n converges to p, $\exists N$ s.t. $n > N \implies d(p_n, p) < r$.
So $p_n \in B(p, r)$. Since $p_n \in S$, $B(p, r)$ contains a point in S. So p is an adherent point. ■

3.38 Let S be a subset of the metric space E. Prove the following are equivalent:

(a) S is closed.

(b) S contains the limits of its sequences in the sense that if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence if points from S that converges, say $x = \lim_{n \to \infty}$, then $x \in S$.

Proof. $(a \implies b)$
Assume S is closed and that $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence of points from S that converge to the point p. Then from problem 3.37, we know that p is an adherent point of S. Since S is closed, we know from 3.35 that all its adherent points are in S. So $p \in S$.
$(b \implies a)$
Assume that S contains the limits of its sequences. Let p be an adherent point of S. So then by 3.36, there is a sequence of points $\langle p_n \rangle_{n=1}^{\infty}$ in S that converge to p. So then by the assumption, $p \in S$. So S contains all its adherent points. From 3.35, S is then closed. ■

3.39 Let F be a closed subset of $\mathbb{R}$ and f a polynomial. Show that

$$S := f^{-1}[F] = \{x : f(x) \in F\}$$

is a closed subset of $\mathbb{R}$.

Proof. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence of points from S that converge to p. So then $f(p_n) \in F$. We also have $\lim_{n\to\infty} f(p_n) = f(p)$. Since F is a closed, it contains the limit of its sequences. So from 3.38, $f(p) \in F$. Then $p \in f^{-1}[F] = S$. So S contains the limit of its sequences, meaning it's closed by 3.38. ∎

3.40 Prove every convergent sequence is a Cauchy sequence.

Proof. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space that converges to p. Let N be so that

$$n > N \implies d(p_n, p) < \frac{\epsilon}{2}$$

Similarly, that applies if we replace n with m.
So then we have $d(p_n, p) < \epsilon/2$ and $d(p_m, p) < \epsilon/2$. Adding together, we get $d(p_n, p) + d(p_m, p) < \epsilon$.
By triangle inequality, we get $d(p_n, p_m) \leq d(p_n, p) + d(p_m, p)$. So then $d(p_n, p_m) < \epsilon$.
By definition of Cauchy, our sequence is Cauchy. ∎

3.41 Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n \rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of E.

Proof. First let's show it's Cauchy. Let $N = 2/(\epsilon)$.
Let $m, n > N$. Then $0 < p_n < \epsilon/2$ and $0 < p_m < \epsilon/2$.
$|p_n - p_m| \leq |p_n| + |p_m| < \epsilon/2 + \epsilon/2 = \epsilon$.
So the sequence is Cauchy.
Now to show it is not convergent in E. BWOC, assume the sequence converges to a point in E, we'll call p.
Then for some N, $n > N \implies |1/n - p| < p/2$.
$p - 1/n \leq |p - 1/n| < p/2$, so then $p - 1/n < p/2 \implies -1/n < -p/2 \implies 1/n > p/2$.
Since p cannot be 0 due to the metric space, this means there is a real number smaller than $1/n$ for all n. This violates Archimedes Small Axiom, BOOM, A CONTRADICTION!!! . ∎

3.42 Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Prove the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.

Proof. Assume $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy. Assume $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges to p. Also assume $\epsilon > 0$
Then $\exists N_0$ s.t. $k > N_0 \implies d(p_{n_k}, p) < \epsilon/2$, since $n_k \geq k$ by Lemma 3.46 in the notes.
We also have $\exists N > N_0$ s.t. $n, m > N \implies d(p_m, p_n) < \epsilon/2$.
Choose $n_k \geq m$. Then we also have $d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \epsilon/2 + \epsilon/2 = \epsilon$.
So p_n converges to p. ■

3.43 Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space E. Let $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence of this sequence. Prove $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is also convergent and has the same limit as the original sequence.

Proof. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in the metric space, E, that converges to p, and $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence. Let $\epsilon > 0$
Then $\exists N$ s.t. $n > N \implies d(p_n, p) < \epsilon$.
So assume that $k > N$. Since p_{n_k} is in the original sequence and $n_k \geq k$, we have $d(p_{n_k}, p) < \epsilon$.
So $\forall \epsilon > 0$, $k > N \implies d(p_{n_k}, p) < \epsilon$. This means $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges to p. ■