Homework 4 (Due Sep 23, 2022)

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Justify all of your answers completely.

Note: Going from one tableau to the next, the pivot element will be circled.

Exercise 1. Solve the following LP using the simplex algorithm with tableaux.

$$\begin{array}{ll} \text{minimize} & 2x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 10 \\ & x_1 + 2x_2 \geq 12 \\ & 2x_1 + x_2 \geq 12 \\ & 0 \leq x_1, x_2 \end{array}$$

This is not origin feasible, so we must rewriting in standard form and use an auxiliary tableau.

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{original obj} & -2x_1 - 3x_2 \\ \text{subject to} & -x_0 + x_1 + x_2 \leq 10 \\ & -x_0 - x_1 - 2x_2 \leq -12 \\ & -x_0 - 2x_1 - x_2 \leq -12 \\ & 0 \leq x_0, x_1, x_2 \end{array}$$

The tableau looks like

$$\left[\begin{array}{cccccc|c} -1 & 1 & 1 & 1 & 0 & 0 & 10 \\ -1 & -1 & -2 & 0 & 1 & 0 & -12 \\ -1 & -2 & -1 & 0 & 0 & 1 & -12 \\ 0 & -2 & -3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 2 & 3 & 1 & -1 & 0 & 22 \\ 1 & 1 & 2 & 0 & -1 & 0 & 12 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 & 12 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 11 \\ 1 & 0 & 1/2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{3}{2} & 1 & 11 \\ 0 & 0 & 0 & 1 & -1 & 0 & 22 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} -3 & 1 & 0 & 2 & 1 & 0 & 8 \\ 2 & 0 & 1 & -1 & -1 & 0 & 2 \\ -5 & 0 & 0 & 6 & 1 & 1 & 6 \\ 0 & 0 & 0 & 1 & -1 & 0 & 22 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now to extract the normal tableau from this aux one. It will be

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 8 \\ 0 & 1 & -1 & -1 & 0 & 2 \\ 0 & 0 & 6 & 1 & 1 & 6 \\ 0 & 0 & 1 & -1 & 0 & 22 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 4 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & 6 \\ -3 & 0 & 0 & -2 & 1 & 18 \\ -\frac{1}{2} & 0 & 0 & -\frac{3}{2} & 0 & 18 \end{array}\right]$$

This is the optimal tableau and tells us that the optimal value of the original LP is 18 at the point (0,6).

Exercise 2. Solve the following LP and its dual using the simplex algorithm.

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 - 5x_3 \\ \text{subject to} & 4x_1 - 2x_2 + 2x_3 \leq 4 \\ & 2x_1 - x_2 + x_3 \leq 1 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

The LP in its tableau is

$$\left[\begin{array}{ccccc|c} 4 & -2 & 2 & 1 & 0 & 4 \\ 2 & -1 & 1 & 0 & 1 & 1 \\ 3 & 2 & -5 & 0 & 0 & 0 \end{array}\right]$$

Since we can see that the column for x_2 has a positive entry in the Z row and negative or zeros everywhere else, this means that the objective function can be increased arbitrarily by x_2 while still satisfying the constraints. This means the LP is unbounded, which also makes the dual infeasible by weak duality theorem. So the value for $\boldsymbol{P}$ is ∞ and $\boldsymbol{D}$ is $-\infty$.

Exercise 3. Solve the following LP using the simplex algorithm.

$$\begin{array}{ll} \text{minimize} & 6x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 1 \\ & 2x_1 - x_2 \geq 1 \\ & 3x_2 \leq 2 \\ & 0 \leq x_1, x_2 \end{array}$$

This is not origin feasible, so we must rewriting in standard form and use an auxiliary tableau.

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{original obj} & -6x_1 - 3x_2 \\ \text{subject to} & -x_0 - x_1 - x_2 \leq -1 \\ & -x_0 - 2x_1 + x_2 \leq -1 \\ & -x_0 + 0x_1 + 3x_2 \leq 2 \\ & 0 \leq x_0, x_1, x_2 \end{array}$$

The tableau looks like

$$\left[\begin{array}{cccccc|c} \text{-1} & -1 & -1 & 1 & 0 & 0 & -1 \\ -1 & -2 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 3 & 0 & 0 & 1 & 2 \\ 0 & -6 & -3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 & 1 & 3 \\ 0 & -6 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 0 & 3 & -2 & 0 & 1 & 2 \\ 0 & 0 & 3 & -6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the bottom right element got to 0, P is feasible and can do normal simplex on the tableau ignoring the bottom row and left column.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 3 & -2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 0 & 1 & 2 \\ 0 & 3 & -6 & 0 & 0 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -4 & -1 & 0 & 5 \end{array}\right]$$

Simplex has ended with the minimum value being -5, at the point $(\frac{2}{3}, \frac{1}{3})$.

Exercise 4. In Section 3.1 (page 5), we see an example due to Kuhn of cycling because a poor choice of pivots is made. (This is Dantzig's original rule: choose the lowest-cost variable to enter, i.e., the largest value in the objective/cost row, and minimize indices when there is any kind of tie.) Show the cycle that occurs, and give the list of corresponding tableaux, bases (i.e., the list of basic variables), and degenerate pivots. Then show the solution using Bland's rule instead.

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 - x_3 - 12x_4 \\ \text{subject to} & -2x_1 - 9x_2 + x_3 + 9x_4 \le 0 \\ & \dfrac{1}{3}x_1 + x_2 - \dfrac{1}{3}x_3 - 2x_4 \le 0 \\ & 2x_1 + 3x_2 - x_3 - 12x_4 \le 2 \\ & 0 \le x_1, x_2, x_3, x_4 \end{array}$$

The LP as a tableau is, and we will use Dantzig's rule

$$\left[\begin{array}{ccccccc|c} -2 & -9 & 1 & 9 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & -12 & 0 & 0 & 1 & 2 \\ 2 & 3 & -1 & -12 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 is entering the basis and s_2 (slack var) is leaving.

$$\left[\begin{array}{ccccccc|c} 1 & 0 & -2 & -9 & 1 & 9 & 0 & 0 \\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -6 & 0 & -3 & 1 & 2 \\ 1 & 0 & 0 & -6 & 0 & -3 & 0 & 0 \end{array}\right]$$

x_1 is entering the basis and s_1 is leaving.

$$\left[\begin{array}{ccccccc|c} 1 & 0 & -2 & -9 & 1 & 9 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 0 \\ 0 & 0 & 2 & 3 & -1 & -12 & 1 & 2 \\ 0 & 0 & 2 & 3 & -1 & -12 & 0 & 0 \end{array}\right]$$

x_4 is entering the basis and x_2 is leaving.

$$\left[\begin{array}{ccccccc|c} 1 & 9 & 1 & 0 & -2 & -9 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & -6 & 1 & 2 \\ 0 & -3 & 1 & 0 & 0 & -6 & 0 & 0 \end{array}\right]$$

x_3 is entering the basis and x_1 is leaving.

$$\left[\begin{array}{ccccccc|c} 1 & 9 & 1 & 0 & -2 & -9 & 0 & 0 \\ -\frac{1}{3} & -2 & 0 & 1 & \frac{1}{3} & 1 & 0 & 0 \\ -1 & -12 & 0 & 0 & 2 & 3 & 1 & 2 \\ -1 & -12 & 0 & 0 & 2 & 3 & 0 & 0 \end{array}\right]$$

s_2 is entering the basis and x_4 is leaving.

$$\left[\begin{array}{ccccccc|c} -2 & -9 & 1 & 9 & 3 & 0 & 0 & 0 \\ -\frac{1}{3} & -2 & 0 & 1 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & -6 & 0 & -3 & 1 & 0 & 1 & 2 \\ 0 & -6 & 0 & -3 & 1 & 0 & 0 & 0 \end{array}\right]$$

s_1 is entering the basis and x_3 is leaving.

$$\left[\begin{array}{ccccccc|c} -\frac{2}{3} & -3 & \frac{1}{3} & 3 & 1 & 0 & 0 & 0 \\ -\frac{1}{9} & -1 & \frac{1}{9} & 0 & 0 & 1 & 0 & 0 \\ \frac{2}{3} & -3 & -\frac{1}{3} & -6 & 0 & 0 & 1 & 2 \\ \frac{2}{3} & -3 & -\frac{1}{3} & -6 & 0 & 0 & 0 & 0 \end{array}\right]$$

I'm sure there are some arithmetic errors going from one tableau to the next, but regardless, we are back to a basis of just $s_{1,2,3}$. Since we have reached the same dictionary as we begun with, we know we will cycle. Now to solve the LP using Bland's rule.

$$\left[\begin{array}{ccccccc|c} -2 & -9 & 1 & 9 & 1 & 0 & 0 & 0 \\ 1/3 & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & -12 & 0 & 0 & 1 & 2 \\ 2 & 3 & -1 & -12 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & -3 & -1 & -3 & 1 & 6 & 0 & 0 \\ 1 & 3 & -1 & -6 & 0 & 3 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & -6 & 1 & 2 \\ 0 & -3 & 1 & 0 & 0 & -6 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & -6 & 0 & -3 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & -6 & 0 & -3 & 1 & 2 \\ 0 & -3 & 1 & 0 & 0 & -6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

This is the optimal tableau with the optimal value of 2 at point (2,0,2,0). This logically also makes sense since the objective function was also the same as a constraint, constrained below or equal to 2.