Homework 5 (Due Sept 22, 2023)

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MATH 554 - Analysis I - Fall 2023

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Justify all of your answers completely.

1. Let $f : [a, b] \to \mathbb{R}$ be a function such that

$$f(a) \leq 0 \text{ and } f(b) \geq 0$$

and such that there is a $M > 0$ such that $\forall x_1, x_2 \in [a, b]$ the inequality

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_2|$$

holds. Prove that $\exists \xi [a, b]$ with

$$f(\xi) = 0$$

Proof. Let $S = \{x \in [a, b] : f(x) \leq 0\}$. Then we know $a \in S$ and S is bounded above by b, so $\xi = \sup(S)$ exists.
Let $\epsilon > 0$. Assume $f(a) < 0$ and $f(b) > 0$ since if either one was 0, then we are done.
Then $\exists x_1 \in S$ s.t. $\xi - \epsilon < x_1 \leq \xi$ since ξ wouldn't be a supremum of S otherwise. Also, $f(x_1) \leq 0$ since $x_1 \in S$.

$$f(\xi) = f(x_1) + (f(\xi) - f(x_1)) \leq 0 + |f(\xi) - f(x_1)|$$
$$\leq M|\xi - x_1| \leq M|\xi - (\xi - \epsilon)| = M|\epsilon| = M\epsilon$$

Note: $\exists x_2 \in [\xi, \xi + \epsilon]$ with $f(x_2) > 0$ since $x_2 > \xi \implies x_2 \notin S \implies f(x_2) > 0$ and an $x_2 > \xi$ exists since $\epsilon > 0$.
We have $f(x_2) > 0$ and $|\xi - x_2| \leq \epsilon$, then

$$f(\xi) = f(x_2) + (f(\xi) - f(x_2)) \geq 0 + |f(\xi) - f(x_2)|) \geq -M|\xi - x_2| \geq -M\epsilon$$

So then $-M\epsilon \leq f(\xi) \leq M\epsilon \implies |f(\xi)| = |f(\xi) - 0| \leq M\epsilon \forall \epsilon > 0$.
So then $f(\xi) = 0$.

■

2. Prove on a bounded interval $[a, b]$ the function $f(x) = x^n$ is Lipschitz for any positive integer n.

Proof. Let $x \in [a, b]$, then $|x| \leq \max(|a|, |b|)$. Let $C = \max(|a|, |b|)$.

$$|f(x_2) - f(x_1)| = |x_2^n - x_1^n| = |x_2 - x_1|(x_2^{n-1} + x_2^{n-2}x_1 + \ldots + x_2 x_1^{n-2} + x_1^{n-1})$$

$$\leq |x_2 - x_1|(|x_2|^{n-1} + |x_2|^{n-2}|x_1| + \ldots + |x_2||x_1|^{n-2} + |x_1|^{n-1}) \leq |x_2 - x_1|(nC^{n-1})$$
$$= M|x_2 - x_1|$$

where $M := nC^{n-1}$. Thus f is Lipschitz on $[a, b]$. ■

3. Show that $f(x) = x^2$ is not Lipschitz on the interval $[0, \infty)$.

Proof. BWOC, assume $\exists M$ s.t.

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2| \leq M|x_2 - x_1| \implies$$

$$|x_2 - x_1||x_2 + x_1| \leq M|x_2 - x_1|$$

Then $\forall x_1, x_2 \in [0, \infty), x_1 \neq x_2$,

$$|x_2 + x_1| \leq M$$

Let $x_2 = M, x_1 = M + 1$. Then $|x_2 + x_1| \leq M \implies 2M + 1 \leq M$ BOOM, A CONTRADICTION!!! . ■

4. Prove if n is a positive integer, then $x^n - c$ has a positive solution for all $c > 0$.

Proof. Let $p(x) = x^n - c$. Then $p(0) = -c < 0$. Consider $p(c+1) = (c+1)^n - c = (c^n + nc^{n-1} + \ldots + nc + 1) - c > 0$ since nc gets rid of the $-c$ since $n \geq 1$. ∎

5. Let $p(x) = x^3 + ax^2 + bx + c$. Prove that $p(x)$ has at least one real root.

   Proof.
$$p(x) = x^3(1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3})$$
   If $|x| \leq 1$, then
$$\frac{1}{|x|^3} \geq \frac{1}{|x|^2} \geq \frac{1}{|x|}$$
   If $|x| \geq 1$
$$|\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{|a|}{|x|} + \frac{|b|}{|x|^2} + \frac{|c|}{|x|^3} \leq \frac{|a| + |b| + |c|}{|x|}$$
   Let $S = |a| + |b| + |c|$.
   Looking at $|x| \geq \max(1, 2S)$, when $\max(1, 2S) = 1$ then $1 \geq 2S \implies S \leq 1/2$.
$$|\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{S}{|x|} \implies |\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{1}{2}$$
   When $\max(1, 2S) = 2S$
$$|\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{S}{|x|} \implies |\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{S}{2S} = \frac{1}{2}$$
   So
$$|\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}| \leq \frac{1}{2} \implies -\frac{1}{2} \leq \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \leq \frac{1}{2}$$
$$\implies \frac{1}{2} \leq 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \leq \frac{3}{2} \implies 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} > 0$$
   Finally, consider $\beta = 2S$ and $f(\beta)$.
   Then $f(\beta) = (2S)^3(\text{positive number}) > 0$
   Consider $\alpha = -2S$ and$f(\beta)$.
   Then $f(\beta) = (-2S)^3(\text{positive number}) = -(2S)^3(\text{positive number}) < 0$
   By Intermediate value theorem, $\exists \xi$ s.t. $f(\xi) = 0$. ∎