Homework 2 (Due Sept 8, 2023)

Jack Hyatt
MATH 554 - Analysis I - Fall 2023

December 28, 2023

Justify all of your answers completely.

2.10 Prove if $a, b \in \mathbb{F}$, then exactly one of the following holds:

$$a < b, \qquad a = b, \qquad a > b$$

Proof. Assume $a, b \in \mathbb{F}$. Since $\mathbb{F}$ is an ordered field, $b - a \in \mathbb{F}$.
Case 1: $(b-a) \in \mathbb{F}_+$. Then by definition of $<$, $a < b$.
Case 2: $b - a = 0$. Then by definition of $=$, $a = b$.
Case 3: $(b-a) \in \mathbb{F}_-$. Then by definition of $>$, $a > b$. ∎

2.11 Prove if $a < b$ and $b < c$, then $a < c$.

Proof. Assume $a < b$ and $b < c$. Then $(b-a) \in \mathbb{F}_+$ and $(c-b) \in \mathbb{F}_+$. $((c-b)+(b-a)) \in \mathbb{F}_+$ since $\mathbb{F}$ is an ordered field. $((c-b)+(b-a)) = (c-b+b-a) = (c-a) \in \mathbb{F}_+ \implies a < c$. ∎

2.12 Prove if $a < b$ and $c < d$, then $a + c < b + d$.

Proof. Assume $a < b$ and $c < d$. Then $(b-a) \in \mathbb{F}_+$ and $(d-c) \in \mathbb{F}_+$. $((d-c)+(b-a)) \in \mathbb{F}_+$ since $\mathbb{F}$ is an ordered field. $((d-c)+(b-a)) = (d-c+b-a) = (d+b)-(c+a) \in \mathbb{F}_+ \implies a + c < b + d$. ∎

2.13 Prove if $a < b$ and $c > 0$, then $ac < bc$.

Proof. Assume $a < b$ and $c > 0$. Then $(b-a) \in \mathbb{F}_+$ and $c \in \mathbb{F}_+$. $(c(b-a)) \in \mathbb{F}_+$ since $\mathbb{F}$ is an ordered field. $(c(b-a)) = (cb - ca) \in \mathbb{F}_+ \implies ac < bc$. ∎

2.14 Prove if $a < b$ and $c < 0$, then $ac > bc$.

Proof. Assume $a < b$ and $c < 0$. Then $(b-a) \in \mathbb{F}_+$ and $-c \in \mathbb{F}_+$. $(-c(b-a)) \in \mathbb{F}_+$ since $\mathbb{F}$ is an ordered field. $(-c(b-a)) = (-cb + ca) = (ac - bc) \in \mathbb{F}_+ \implies ac > bc$. ∎

2.15 Prove if $a < b$ and $c < d$, then $a + c > b + d$.

Proof. See 2.12 ∎

2.16 Prove if $0 < a < b$ and $0 < c \leq d$, then $ac < bd$.

Proof. Assume $0 < a < b$ and $0 < c \leq d$. Then $a, b, c, d \in \mathbb{F}_+$ and $(b - a), (d - c) \in \mathbb{F}_+$.
Case 1: $c = d$. See 2.13
Case 2: $c < d$. Since $\mathbb{F}$ is an ordered field and $b, c, (d-c), (b-a) \in \mathbb{F}_+$, $(b(d-c)+c(b-a))) \in \mathbb{F}_+$. $(b(d - c) + c(b - a))) = (bd - bc + bc - ac) = (bd - ac) \in \mathbb{F}_+ \implies ac < bd$. ∎

2.17 Prove if $a_1, a_2, \ldots, a_n > 0$, then $\prod_{i=1}^{n} a_i > 0$ and $\sum_{i=1}^{n} a_i > 0$.

Proof. Let us induct on n.
Base Case: $n = 1$ Obvious
$n = 2$ $a_1, a_2 \in \mathbb{F}_+ \implies (a_1 + a_2), (a_1 a_2) \in \mathbb{F}_+$ since $\mathbb{F}$ is an ordered field.
Induction Step: Assume $a_1, a_2, \ldots, a_n > 0 \implies \prod_{i=1}^{n} a_i > 0$ and $\sum_{i=1}^{n} a_i > 0$.
Consider $n + 1$. Assume $\forall i \in [n + 1], a_i > 0$. Consider $\prod_{i=1}^{n+1} a_i$ and $\sum_{i=1}^{n+1} a_i$. Since $\mathbb{F}$ is an ordered field, $\prod_{i=1}^{n+1} a_i = (\prod_{i=1}^{n} a_i) \cdot a_{n+1}$ and $(\sum_{i=1}^{n} a_i) + a_{n+1}$. By the Induction Hypothesis, $\prod_{i=1}^{n} a_i \in \mathbb{F}_+$ and $\sum_{i=1}^{n} a_i \in \mathbb{F}_+$. Since $a_{n+1} \in \mathbb{F}_+$ and $\mathbb{F}$ is an ordered field, $((\prod_{i=1}^{n} a_i) \cdot a_{n+1}) \in \mathbb{F}_+$ and $((\sum_{i=1}^{n} a_i) + a_{n+1}) \in \mathbb{F}_+$. ∎

2.18 Prove if $a \neq 0 \implies a^2 > 0$.

Proof. **Case 1**: $a > 0$. Since $\mathbb{F}$ is an ordered field, $a \in \mathbb{F}_+ \implies a \cdot a \in \mathbb{F}_+ \implies a^2 > 0$.
Case 2: $a < 0$. Since $\mathbb{F}$ is an ordered field, $a \in \mathbb{F}_- \implies -a \in \mathbb{F}_+ \implies$
$(-a)(-a) \in \mathbb{F}_+ \implies (-a)^2 \in \mathbb{F}_+ \implies a^2 > 0$. ∎

2.19 Prove if $a_1, \ldots, a_n \in \mathbb{F}$, then

$$\sum_{i=1}^{n} a_i^2 \geq 0$$

with equality iff $\forall i \in [n], a_i = 0$.

Proof. Let us induct on n.
Base Case: $n = 1$ Obvious
$n = 2$ Let $a_1, a_2 \in \mathbb{F}$.
Case 1: Let $a_1 = a_2 = 0$. Since $x \cdot 0 = 0 \quad \forall x \in \mathbb{F}$, $0^2 = 0$. So $a_1^2 + a_2^2 = 0^2 + 0^2 = 0 + 0 = 0$.
Case 2: WLOG, let $a_1 \neq 0$ and $a_2 = 0$. By the same logic in case 1, just remove the zero term, and we are left with a_1^2, which is greater than 0 by 2.18.
Case 3: Let both of a_1, a_2 be nonzero. By 2.18, $a_1^2 > 0$ and $a_2^2 > 0$. So $a_1^2, a_2^2 \in \mathbb{F}_+ \implies a_1^2 + a_2^2 \in \mathbb{F}_+$.
Induction Step: Assume if $a_1, \ldots, a_n \in \mathbb{F}$, then

$$\sum_{i=1}^{n} a_i^2 \geq 0$$

with equality iff $\forall i \in [n], a_i = 0$.
Let $a_{n+1} \in \mathbb{F}$. Consider $\sum_{i=1}^{n+1} a_i^2$.

$$\sum_{i=1}^{n+1} a_i^2 = (\sum_{i=1}^{n} a_i^2) + a_{n+1}^2$$

We know $\sum_{i=1}^{n} a_i^2 \in \mathbb{F}_+$ by the Induction Hypothesis, and $a_{n+1}^2 \in \mathbb{F}_+$ by 2.18. So by closure, $\sum_{i=1}^{n+1} a_i^2 \in \mathbb{F}_+$. ■

2.20 Prove if $a > 0$ then $1/a > 0$, and if $a < 0$ then $1/a < 0$.

Proof. BWOC, assume $a > 0$ and $1/a \le 0$. $1 = a(1/a) =$ (positive)(nonpositve) = (nonpositive). BOOM, A CONTRADICTION!!!
BWOC, assume $a < 0$ and $1/a \ge 0$. $1 = a(1/a) =$ (negative)(nonnegatvie) = (nonpositive). BOOM, A CONTRADICTION!!! ■

2.21 Prove if $0 < a < b$, then $1/b < 1/a$.

Proof. Assume $0 < a < b$. Then $a, b, (b - a) \in \mathbb{F}_+$. By 2.18, $ab \in \mathbb{F}_+$. By 2.20, $\frac{1}{ab} \in \mathbb{F}_+$. By 2.18, $\frac{b-a}{ba} \in \mathbb{F}_+$. So $\frac{b-a}{ba} = \frac{1}{a} - \frac{1}{b} \in \mathbb{F}_+ \implies 1/b < 1/a$ ■

2.22 Prove for $a \in \mathbb{F}$, $|a| \ge 0$ with equality iff $a = 0$.

Proof. **Case 1**: $a > 0$. Then $|a| = a > 0$ by definition of absolute value.
Case 2: $a < 0$. Then $|a| = -a > 0$ by definition of absolute value.
Case 3: $a = 0$. Then $|a| = 0$ by definition of absolute value. ■

2.23 Prove for $a \in \mathbb{F}$, $a \le |a|$.

Proof. **Case 1**: $a > 0$. Then $|a| = a \le a$ by definition of absolute value.
Case 2: $a < 0$. Then $|a| = -a > a$ by definition of absolute value.
Case 3: $a = 0$. Then $|a| = 0 \le a$ by definition of absolute value. ■

2.24 Prove for $a \in \mathbb{F}$, $a^2 = |a|^2$.

Proof. **Case 1**: $a > 0$. Then $|a| = a \implies |a| \cdot |a| = a \cdot |a| = a^2$ by definition of absolute value.
Case 2: $a < 0$. Then $|a| = -a \implies |a| \cdot |a| = -a \cdot |a| = a^2$ by definition of absolute value.
Case 3: $a = 0$. Then $|a| = 0 \implies |a|^2 = 0 = a^2$ by definition of absolute value. ■

2.25 Prove for $a, b \in \mathbb{F}$, the following are equivalent:

$$|a| = |b|, \qquad a = \pm b, \qquad a^2 = b^2$$

Proof. $(|a| = |b| \implies a = \pm b)$
Case 1: $a \geq 0$. Then $|a| = a = |b| = \pm b$.
Case 2: $a < 0$. Then $|a| = -a = |b| = \pm b \implies a = -(\pm b) \implies a = \pm b$.
$(a \pm b \implies a^2 = b^2)$
Case 1: $a = b$. Then $a \cdot a = b \cdot a = b \cdot b$.
Case 2: $a = -b$. Then $a \cdot a = -b \cdot a = (-b) \cdot (-b) = b^2$.
$(a^2 = b^2 \implies |a| = |b|)$
$a^2 = b^2 \implies |a|^2 = |b|^2$ by 2.24, $\implies |a| = |b|$. ∎