Homework 2 (Due Jan 29, 2025)

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Justify all of your answers completely.

1. Let F be a field, $f(X) \in F[X]$ a polynomial, and $u \in F$ a root of f. Prove that u is a multiple root of $f(X)$ if and only if it is also a root of the derivative $f'(x)$.

Proof. For both directions, u is a root of f.
$(\implies)$
Assume u is a multiple root of $f(x)$.

Then $f(x) = g(x)(x-u)^k$ with $k \geq 2$.

Consider $f'(x)$.

$$f'(x) = g(x)k(x-u)^{k-1} + g'(x)(x-u)^k = (x-u)^{k-1}(kg(x) + g'(x)(x-u))$$

Since $k \geq 2$, we have $k-1 \geq 1$. So $x-u \mid f'$, making u a root of f'.

$(\impliedby)$
Assume u is a root of f' (still is also a root of f).

Then $x-u \mid f$, we have $f(x) = g(x)(x-u)$ for some polynomial g.

Consider $f'(x)$.

$$f'(x) = g(x) + g'(x)(x-u)$$
$$0 = f'(u) = g(u) + g'(u)(u-u) \implies g(u) = 0$$

Since u is a root of g, it also divides. So we have $g(u) = h(u)(x-u)$ for some function h. So then

$$f(x) = g(x)(x-u) = h(x)(x-u)(x-u)$$

making u a multiple root. ∎

2. Let $f(x) = x^5 + 4x^4 + 2x^3 + 3x^2 \in \mathbb{Z}_5[x]$ and $g(x) = x^2 + 3 \in \mathbb{Z}_5[x]$. Find the quotient and the remainder when $f(x)$ is divided by $g(x)$.

   First divide and subtract the first terms:

$$x^5/x^2 = x^3 \implies (x^5 + 4x^4 + 2x^3 + 3x^2) - (x^2 + 3)x^3 = 4x^4 + 4x^3 + 3x^2$$

   Next term:

$$4x^4/x^2 = 4x^2 \implies (4x^4 + 4x^3 + 3x^2) - (x^2 + 3)4x^2 = 4x^3 + x^2$$

   Next term:

$$4x^3/x^2 = 4x \implies (4x^3 + x^2) - (x^2 + 3)4x = x^2 + 3x$$

   Next term:

$$x^2/x^2 = 1 \implies (x^2 + 3x) - (x^2 + 3)1 = 3x + 2$$

   So then we finally have

$$x^5 + 4x^4 + 2x^3 + 3x^2 = (x^2 + 3)(x^3 + 4x^2 + 4x + 4) + (3x + 2)$$

3. Let $f(x) = x^4 - 5x^2 + 6$. Observe that $f(x)$ can be viewed as a polynomial in $\mathbb{Q}[x]$, or $\mathbb{R}[x]$, or $\mathbb{Z}_p[x]$. A different one of these fields is used in each of the parts below.

   (a) Find all the roots of f in $\mathbb{Q}$.

   Let us act like we are looking for roots in $\mathbb{R}$, and see if the values would land in $\mathbb{Q}$.

$$f(x) = (x^2)^2 - 5(x^2) + 6 \implies x^2 = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = 3, 2$$

$$x = \pm\sqrt{3}, \pm\sqrt{2}$$

   So there are no roots in $\mathbb{Q}$.

   (b) Find all the roots of f in $\mathbb{R}$.

   Found in part (a), $\{-\sqrt{3}, -\sqrt{2}, \sqrt{2}, \sqrt{3}\}$

   (c) Find all the roots of f in $\mathbb{Z}_3$.

   Since f is in $\mathbb{Z}_3[x]$, we can write it as

$$f(x) = x^4 + x^2 = x^2(x^2 + 1)$$

   which gives roots of just 0, since no element in Z_3 squares to 2 (i.e. -1).

   (d) Find all the roots of f in $\mathbb{Z}_5$.

   Since f is in $\mathbb{Z}_5[x]$, we can write it as

$$f(x) = x^4 + 1.$$

   We can just check all 5 elements to look for $x^4 = 4$.

$$((0)^4, (1)^4, (2)^4, (3)^4, (4)^4) = (0, 1, 1, 1, 1)$$

   So we have no roots.

4. Let $f(x) = x^4 + 5x^2 + 6 \in \mathbb{R}[x]$.

(a) Prove that $f(x)$ does not have any roots in $\mathbb{R}$.

$$x^2 = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)} = -3, -2$$

We can stop here, as we know that no elements in $\mathbb{R}$ can square to -3 or -2. So therefore, f does not have any roots in $\mathbb{R}$.

(b) Find all the roots of $f(x)$ in $\mathbb{C}$.

Continuing from part (a), we have

$$x = \pm\sqrt{-3}, \pm\sqrt{-2}$$

(c) Factor f into irreducible factors in $\mathbb{R}[x]$. Explain why the factors are irreducible.

Using our work from part (a), we can easily get

$$f(x) = (x^2 + 3)(x^2 + 2)$$

We know these terms are irreducible since they have degree less than or equal to 3, and no roots in $\mathbb{R}$.

(d) Factor f into irreducible factors in $\mathbb{C}[x]$. Explain why the factors are irreducible.

$$f(x) = (x - i\sqrt{3})(x + i\sqrt{3})(x + i\sqrt{2})(x - i\sqrt{2})$$

Since these are linear terms, they are obviously irreducible.