Homework 1 (Due Jan 18, 2023)

Jack Hyatt

MATH 575 - Discrete Mathamatics II - Spring 2023

January 19, 2023

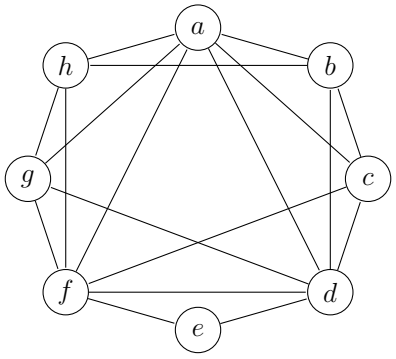
Justify all of your answers completely.

1. Recall that $\Delta(G)$ and $\delta(G)$ denote the maximum degree and minimum degree of a graph G respectively. Suppose G has n vertices.
   (a) Prove that if $\delta(G) \geq \lceil (n-1)/2 \rceil$, then G is connected.
   (b) For all $n \geq 3$, give an example of an n-vertex disconnected graph G with $\delta(G) = \lfloor (n-2)/2 \rfloor$.
   (c) Prove or disprove: if $\delta(G) = \lfloor (n-2)/2 \rfloor$ and $\Delta(G) \geq \lceil n/2 \rceil$, then G is connected.

2. Prove that if G is an n-vertex bipartite graph, then $|E(G)| \leq \frac{n^2}{4}$.

3. A graph is called k-partite if its vertex set can be partitioned into sets $V_1, V_2, \ldots, V_k$ such that for each $1 \leq i \leq k$, there are no edges between vertices in the set V_i. [1] Prove that for all integers $k \geq 2$ every graph G has a k-partite subgraph with at least $\frac{(k-1)|E(G)|}{k}$ edges.
   Hint: this can be done either by induction or by constructing an algorithm to find such a subgraph.

4. Consider the graph G below.



[1] For instance, 2-partite is the same as bipartite.

(a) Partition the edge set of G into a collection of edge-disjoint cycles. List the vertices of each cycle.

(b) Splice together the cycles in part (a) to find an Eulerian circuit of G.

5. Determine if each of the following sequences is graphic. If it is not, give a reason why. If it is, draw a graph that realizes the degree sequence.

   (a) $(4, 4, 3, 3, 2, 2, 1, 1, 1)$

   (b) $(4, 4, 3, 3, 2, 2, 1, 1)$

   (c) $(8, 7, 6, 5, 4, 3, 2, 1)$

   (d) $(7, 4, 4, 4, 4, 3, 3, 3)$