Homework 4 (Due Sept 18, 2023)

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Justify all of your answers completely.

2.36 Prove that in the real numbers, every nonempty set that is bounded below has a greatest lower bound.

Proof. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let b be a lower bound. Let $-S$ be

$$-S := \{-s : s \in S\}$$

So then $\forall s \in \S, b \leq s$. Then $-b \geq -s$. So $-b$ is an upper bound for $-S$, which means $c = \sup(-S)$ exists by least upper bound property. So $c \geq s \forall s \in S$.
$c \geq s \implies -c \leq -s$, and every $-s$ comprises up S. If $-c$ was not the infimum of S, then $-(-c)$ would be a better supremum of $-S$, which is a contradiction. ∎

2.37 Prove for any real number x, there is a natural n with $x < n$.

Proof. BWOC, assume $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is bounded above, so $b = \sup(\mathbb{N})$ exists.
So $n \leq b$. Since $n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$. So $\forall n \in \mathbb{N}, n + 1 \leq b \implies n \leq b - 1$.

So $b-1$ is an upper bound for $\mathbb{N}$. But $b-1 < b = \sup(\mathbb{N})$. BOOM, A CONTRADICTION!!!

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2.38 Let $a > 1$ be a real number. Prove that for any real number x, there is a natural number n such that $a^n > x$.

Proof. Let $N = \{a^n : n \in \mathbb{N}\}$. BWOC, assume $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a^n \leq x$. Then N is bounded above, so $b = \sup(N)$ exists.
So $a^n \leq b$. Since $a^n \in N \implies a^{n+1} \in N$. So $\forall a^n \in N, a^{n+1} \leq b \implies a^n \leq b/a$.

So b/a is an upper bound for N. But $b/a < b = \sup(N)$. BOOM, A CONTRADICTION!!!

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2.39 Prove $a > 0 \implies \exists n \in \mathbb{N} s.t. 1/n < a$ with the method specified in the notes.

Proof. Let $S = \{1/n : n \in \mathbb{N}\}$. This is bounded below by 0. Showing $\inf(S) = 0$ proves Archimedes axiom.
BWOC, assume $c := \inf(S) > 0$. Then $\forall n \in \mathbb{N}$ we have $c \leq 1/n$. But if $n \in \mathbb{N}$, then so is $2n$ and therefore $c \leq 1/(2n)$ which implies $2c \leq 1/n$. So $2c$ is also a lower bound for S and $c < 2c$. So c is not the infimum.
BOOM, A CONTRADICTION!!!

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2.40 Let a be a real number with $0 < a < 1$. Prove for any positive real number x, there is a natural number n such that $a^n < x$.

Proof. Let $N = \{a^n : n \in \mathbb{N}\}$. BWOC, assume $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a^n > x$. Then N is bounded below, so $b = \inf(N)$ exists.
So $a^n > b$. Since $a^n \in N \implies a^{n+1} \in N$. So $\forall a^n \in N, a^{n+1} > b \implies a^n > b/a$.

So b/a is a lower bound for N. But $b/a > b = \inf(N)$. BOOM, A CONTRADICTION!!!

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2.41 For any real number x, there is a unique integer n such that

$$n \leq x < n + 1$$

Proof. Want $m_0 \in \mathbb{Z}$ s.t. $m_0 < x$ **Case 1**: $x > 0$.
Let $m_0 = 0$. **Case 2**: $x \leq 0$.

So $-x \leq 0$, and by Archimedes Big Axiom, $\exists m_1 \in \mathbb{N} s.t. -x < m_1$. So let $m_0 = -m_1$. Now let

$$S := \{k \in \mathbb{Z} : m_0 \leq k \leq x\}$$

Clearly $m_0 \in S$. By Archimedes Big Axiom again, $\exists m_1 \in \mathbb{N} s.t. -x < m_1$. So $S \subseteq \{m_0, m_0+1, \ldots, m_1-1, m_1\}$. This set is finite, with $m_1 - m_0 + 1$ elements. So $n = \max(S) \in S$ exists.
Since $n \in S$, then $n \leq x$. Since $n = \max(S), n+1 \notin (S)$, which means $x < n+1$. So $n \leq x < n+1$.
Now showing uniqueness.
Let m, n be integers that both satisfy the desired inequality. Then $m, n \in (x-1, x]$. Since $x_1, x_2 \in (a,b) \implies |x_2 - x_1| < |b-a|$, we have $|m-n| < 1$. But since $m, n \in \mathbb{Z}$, m must equal n.

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2.42 Between any two real numbers, there is a rational number.

Proof. Let $a, b \in \mathbb{R}$ and WLOG assume $a < b$. So $(b-a) > 0$. So by Archimedes axiom, $\exists N \in \mathbb{N}, \frac{1}{N} < (b-a) \implies Na + 1 < Nb$.
Let $n = \lfloor Na \rfloor$. Then

$$n \leq Na < n+1 \leq Na+1 < Nb$$

So

$$Na < n+1 < Nb \implies a < \frac{n+1}{N} < b$$

And $(n+1)/N$ is a rational number. ■

2.43 Prove between any two rational numbers, there is a irrational number.

Proof. Let a, b be distinct rational numbers. WLOG, let $a < b$. Consider the irrational number $a + (b-a)/\sqrt{2}$.
Since $b > a$, then

$$(b-a)/\sqrt{2} > 0 \implies a + (b-a)/\sqrt{2} > a$$

$$b = a + (b-a) > a + (b-a)/\sqrt{2} \implies a + (b-a)/\sqrt{2} < b$$

So $a < a + (b-a)/\sqrt{2} < b$. ■

2.44 Let $y_0, y_1 \in \mathbb{R}$ and assume that there is a number $M > 0$ such that $\forall \epsilon > 0, |y_1 - y_0| \leq M\epsilon$. Prove $y_0 = y_1$.

Proof. BWOC, assume $y_0 \neq y_1$. Let $\epsilon = |y_1 - y_0|/2M$. Then

$$|y_1 - y_0| \leq M\epsilon \implies |y_1 - y_0| \leq M\frac{|y_1 - y_0|}{2M} \implies |y_1 - y_0| \leq \frac{|y_1 - y_0|}{2}$$

which only is true iff $|y_1 - y_0| = 0$, but we assumed $y_0 \neq y_1$. BOOM, A CONTRADICTION!!!

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2.45 Prove if $f : [a, b] \to \mathbb{R}$ is *Lipschitz*, with *Lipschitz constant* M, then for any $x, x_0 \in [a, b]$, the inequalities

$$-M[x - x_0] \leq f(x) - f(x_0) \leq M|x - x_0|$$

and

$$f(x_0) - M[x - x_0] \leq f(x) \leq f(x_0) + M|x - x_0|$$

hold.

Proof. Assume $f : [a, b] \to \mathbb{R}$ is *Lipschitz*, with *Lipschitz constant* M. Then

$$\forall x_1, x_2 \in [a, b] \qquad |f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

Since $|x| \leq a$ iff $-a \leq x \leq a$,

$$|f(x_2) - f(x_1) \leq M|x_2 - x_1| \implies -M|x_2 - x_1| \leq f(x_2) - f(x_1) \leq M|x_2 - x_1|$$

Adding $f(x_1)$ to every side, we get

$$f(x_1) - M|x_2 - x_1| \leq f(x_2) \leq f(x_1) + M|x_2 - x_1|$$

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Problem 1 Let A and B be nonempty subsets of $\mathbb{R}$ that are each bounded above. Let

$$S = A + B = \{a + b : a \in A \text{ and } b \in B\}$$

(a) show that S is bounded above.

(b) Prove

$$\sup(S) = \sup(A) + \sup(B)$$

Proof. Since A and B are both bounded above, they both have a supremum s_a, s_b respectively. So $\forall a \in A, a \leq s_a$ and $\forall b \in B, b \leq s_b$. Then $a + b \leq s_a + s_b$. So then S is bounded above.

Let $\epsilon > 0$. Let $a \in A$ s.t. $a > s_a - \epsilon/2$ and $b \in B$ s.t. $b > s_b - \epsilon/2$. We know that a and b exist since A and B are nonempty, and s_a, s_b are the best supremums for A and B.
So then $a+b > s_a+s_b-\epsilon$, which makes anything smaller than s_a+s_b not an upper bound for S. ∎

Problem 2 Let $S \subseteq \mathbb{R}$ be a subset that satisfies the two conditions

(a) S is bounded above.

(b) If $s_1, s_2 \in S$ with $s_1 \neq s_2$, then

$$|s_1 - s_2| \geq 1$$

Show $\sup(S) \in S$ and therefore S has a maximum.

Proof. Since S is bounded above, $s = \sup(S)$ exists. BWOC, let $s \notin S$. Then for $0 < \epsilon < 1, s - \epsilon \in S$ as that would be a better $\sup(S)$ otherwise. However, there can only be one ϵ that satisfies that, since a second ϵ would mean the (b) condition is violated. So $s - \epsilon \geq s_1, \forall s_1 \in S$, but then $s - \epsilon$ is a better supremum than s.

BOOM, A CONTRADICTION!!!

So $s \in S$, which is a maximum element. ∎