Homework 9 (Due Oct 24, 2022)

Jack Hyatt
MATH 574 - Discrete Mathamatics - Fall 2022

October 24, 2022

Justify all of your answers completely.

1. Find a closed form[1] for the generating function for the following sequences.

   (a) $1, 2, 4, 8, 16, 32, \ldots$

$$\sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (2x)^k = \frac{1}{1-2k}$$

   (b) $\binom{7}{0}, 2^1\binom{7}{1}, 2^2\binom{7}{2}, 2^3\binom{7}{3}, 2^4\binom{7}{4}, 2^5\binom{7}{5}, \ldots$

$$\sum_{k=0}^{\infty} (2^k \binom{7}{k}) x^k = (1+2x)^7$$

   (c) $1, -1, 1, -1, 1, -1, \ldots$

$$\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$$

   (d) $1, 0, 1, 0, 1, 0, 1, 0, \ldots$

$$\sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$$

2. If $g(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for:

   (a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$

$$g^a(x) = \sum_{i=0}^{\infty} 2a_i x^i = 2\sum_{i=0}^{\infty} a_i x^i = 2g(x)$$

[1] By a **closed form** we mean an algebraic expression not involving a summation over a range of values or the use of ellipses. For instance, the closed form of the generating function $\sum_{k=0}^{n} x^k$ is $\frac{1}{1-x}$.

(b) $a_5, a_6, a_7, a_8, a_9, \ldots$

$$g^b(x) = \sum_{i=0}^{\infty} a_{i+5}x^i = \sum_{i=5}^{\infty} a_i x^{i-5} = \frac{\sum_{i=5}^{\infty} a_i x^i}{x^5} = \frac{\sum_{i=0}^{\infty} a_i x^i - \sum_{i=0}^{4} a_i x^i}{x^5}$$
$$= \frac{g(x) - \sum_{i=0}^{4} a_i x^i}{x^5}$$

(c) $a_1, 2a_2, 3a_3, 4a_4, \ldots$

$$g^c(x) = \sum_{i=1}^{\infty} i a_i x^{i-1} = \frac{d}{dx}\sum_{i=0}^{\infty} a_i x^i = \frac{d}{dx} g(x)$$

(d) $a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$

$$g^d(x) = \sum_{i=0}^{\infty}(a_i + a_{i+1})x^i = \sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} a_{i+1}x^i = \sum_{i=0}^{\infty} a_i x^i + \sum_{i=1}^{\infty} a_i x^{i-1} =$$
$$= g(x) + \frac{\sum_{i=0}^{\infty} a_i x^i - a_0}{x} = g(x) + \frac{g(x) - a_0}{x} = \frac{g(x)(x+1) - a_0}{x}$$

3. Let $\{a_k\}$ be the sequence with $a_k = (k+1)(k+2)$ for all $k \geq 0$. Find a closed form for the generating function $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

$$f(x) = \sum_{k=0}^{\infty}(k+1)(k+2)x^k = \sum_{k=2}^{\infty}(k-1)(k)x^{k-2} = \frac{d^2}{dx^2}\sum_{k=0}^{\infty} x^k = \frac{d^2}{dx^2}\frac{1}{1-x}$$
$$= \frac{2}{(1-x)^3}$$

4. For each of these generating functions, provide a closed formula for the sequence it determines. I.e., give a closed form for the coefficient of x^k for each k.

(a) $\frac{3x^2}{1+9x}$

$$\frac{3x^2}{1+9x} = 3x^2\frac{1}{1+9x} = 3x^2\sum_{k=0}^{\infty}(-9^k)x^k = \sum_{k=0}^{\infty} 3(-9)^k x^{k+2} = \sum_{k=2}^{\infty} 3(-9)^{k-2}x^k$$

So the kth term is $a_k = 3(-9)^{k-2}$ with initial conditions $a_0 = 0, a_1 = 0$.

(b) $(1+x^2)^4$

$$(1+(x^2))^4 = \sum_{k=0}^{4}\binom{4}{k}x^{2k}$$

By the table, the kth term is $\binom{4}{k/2}$ if 2 divides k, 0 if otherwise.

(c) $e^{4x} + e^{-4x}$

$$e^{4x} + e^{-4x} = \sum_{k=0}^{\infty} \frac{(4x)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-4x)^k}{k!} = \sum_{k=0}^{\infty} (\frac{4^k}{k!} + \frac{(-4)^k}{k!})x^k$$

So the kth term in the sequence is $\dfrac{4^k + (-4)^k}{k!}$

5. Find the coefficient of x^{12} in the power series of each of the following functions.

(a) $x/(1+3x)$

$$\frac{x}{1+3x} = x\sum_{k=0}^{\infty}(-3x)^k = \sum_{k=0}^{\infty}(-3x)^{k+1}$$

So the coeff of x^{12} is just $(-3)^{12}$.

(b) $1/(1-2x)^8$

$$\frac{1}{(1-2x)^8} = \sum_{k=0}^{\infty} \binom{7+k}{k} 2^k x^k$$

So the coeff of x^{12} is just $\binom{19}{12}(2)^{12}$.

6. Prove using generating functions that the number of ways to distribute n cookies among k children such that each child receives at least 2 cookies is $\binom{n-k-1}{k-1}$.

We can represent each child with the generating function $f_i(x) = a_2x^2 + a_3x^3 + \ldots$, where $f_i(x)$ is the ith child's generating function, each x^ℓ represents the number of cookies the child can get, and each a_ℓ represents the number of ways to give the child ℓ cookies. Since the cookies are indistinguishable, there is only ever 1 way to give ℓ cookies. So to find the total ways to give out the cookies to the kids, we just need to multiply all the kids' generating functions together. Note, each $f_i(x) = x^2 + x^3 + \ldots$

$$\prod_{i=0}^{k} f_i(x) = (x^2 + \ldots)^k = (\sum_{i=2}^{\infty} x^i)^k = (\sum_{i=0}^{\infty} x^{i+2})^k = (x^2\sum_{i=0}^{\infty} x^i)^k = \frac{x^{2k}}{(1-x)^k}$$

$$= x^{2k}\sum_{i=0}^{\infty}\binom{k+i-1}{i}x^i = \sum_{i=0}^{\infty}\binom{k+i-1}{i}x^{2k+i}$$

Since we must have at least 2k cookies, each term in the sum represents the number of ways to distribute the cookies for every possible n. So we then let $n = 2k+i$, which gives $i = n - 2k$.

$$\binom{k+i-1}{i}x^{2k+i} = \binom{k+n-2k-1}{n-2k}x^n = \binom{n-k-1}{n-2k}x^n = \binom{n-k-1}{(n-k-1)-(n-2k)}x^n$$

$$= \binom{n-k-1}{k-1}x^n \quad \blacksquare$$

7. Use generating functions to solve the recurrence relation $a_k = 2a_{k-1} - 7$ with $a_0 = 1$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}[2a_{n-1}x^n - 7x^n] = 2\sum_{n=1}^{\infty} a_{n-1}x^n - 7\sum_{n=1}^{\infty} x^n$$

$$= 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} - 7x\sum_{n=1}^{\infty} x^{n-1} = 2x\sum_{n=0}^{\infty} a_n x^n - 7x\sum_{n=0}^{\infty} x^n = 2xG(x) - \frac{7x}{1-x}$$

$$\text{So } G(x) - 1 = 2xG(x) - \frac{7x}{1-x} \implies G(x) = \frac{1-8x}{(1-x)(1-2x)} = \frac{7}{1-x} - \frac{6}{1-2x}$$

$$= \sum_{n=0}^{\infty} 7x^n - \sum_{n=0}^{\infty} 6 \cdot 2^n x^n = \sum_{n=0}^{\infty}(7 - 6 \cdot 2^n)x^n$$

So $a_n = 7 - 6(2^n)$

8. Let a_n denote the sum of the first n squares, i.e., $a_n = 0^2 + 1^2 + 2^2 + \ldots + n^2$.

   (a) Give a recurrence relation for $\{a_n\}$.

   $a_n = a_{n-1} + n^2$

   (b) Use part (a) to show that the generating function for $\{a_n\}$ is

   $$g(x) = (x^2 + x)/(1-x)^4.$$

   Note: $\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^n}{(1-x)^3}$

   $$g(x) - 0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}(a_{n-1} + n^2)x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} n^2 x^n = xg(x) + \frac{x+x^2}{(1-x)^3}$$

   $$\text{So } g(x) = xg(x) + \frac{x+x^2}{(1-x)^3} \implies g(x) = \frac{x^2+x}{(1-x)^4}$$

   (c) Use part (b) to find an explicit formula for the sum $1^2 + 2^2 + \ldots + n^2$.

   $$\frac{x^2+x}{(1-x)^4} = \frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4}$$

   $$= \sum_{n=0}^{\infty}\binom{n+1}{n}x^n - \sum_{n=0}^{\infty}3\binom{n+2}{n}x^n + \sum_{n=0}^{\infty}2\binom{n+3}{n}x^n$$

   $$= \sum_{n=0}^{\infty}\left(\binom{n+1}{n} - 3\binom{n+2}{n} + 2\binom{n+3}{n}\right)x^n$$

   So the explicit formula is $\left(\binom{n+1}{n} - 3\binom{n+2}{n} + 2\binom{n+3}{n}\right)$