Homework 5 (Due Feb 15, 2023)

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MATH 575 - Discrete Mathamatics II - Spring 2023

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Justify all of your answers completely.

1. Let $n \geq 3$, and let G be an n-vertex graph. Prove that if $\kappa(G) = k$, then there exists $v \in V(G)$ such that $\kappa(G - v) = k - 1$. [1]

   Proof. It is already known that $\kappa(G - v) \geq \kappa(G) - 1$ for all $v \in V(G)$. So it suffices to show that $\kappa(G - v) \leq \kappa(G) - 1$ for some $v \in V(G)$.
   Let $S \subseteq G$ be a minimum separating set of G. Pick the vertex in the S that gets along swimmingly with all of the other vertices and makes their lives complete, name it v. Let us remove, no, KILL v from G, and then wipe v from the memories of the other vertices, leaving behind those vertices as empty shells with no emotion.
   Now we are left with $G - v$ and $S - v$. Since all we did was KILL a vertex, $S - v$ is a separating set of $G - v$. It most likely is a minimum separating set, but I don't need to provide any reasoning for that since I just need to show an inequality and not an equality. So the size of $S - v$ is $\kappa(G) - 1$, and it's a separating set of $G - v$. This means $\kappa(G - v) \leq \kappa(G) - 1$. ∎

2. Let G be a graph on $n \geq 3$ vertices. Prove that G is 2-connected if and only if for every three distinct vertices $x, y_1, y_2 \in V(G)$, there exists a y_1, y_2-path that passes through x.

   Proof. ($\Rightarrow$): Assume G is a 2-connected graph. Then we know by Whitney's theorem that there exists 2 internally disjoint paths for all pairs of vertices.
   Let x, y_1, y_2 be three distinct vertices. Let y' be a new vertex that is adjacent to y_1 and y_2, and we call this new graph G'. By Expansion Lemma, G' also is 2-connected. Then we know by Whitney's theorem that there exists 2 internally disjoint paths for x, y'. Since y' is only neighbors to y_1 and y_2, the paths must go through one of those each. So take those two paths and subtract y_1y' and y_2y' and we have a desired path.

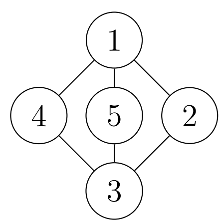
[1] We proved already (Midterm 1 Practice Problems) that $\kappa(G - v) \geq \kappa(G) - 1$ for all $v \in V(G)$. You may use this fact without repeating the proof.

($\Leftarrow$): Proof by contrapositive. Assume G is not 2-connected. Then it either is 1-connected or not connected at all. If it is not connected at all, then of course there are two vertices that don't contain an intermediate vertex, stoopid. Let us assume G is 1-connected.
Since G is not 2-connected, there exists a cut vertex. Let us call this vertex y_1. The removal of this vertex gives two connected components, A and B. Since y_1 connects A and B, it has at least one edge incident with a vertex in each component. Let x be a vertex in A adjacent to y_1 and y_2 similar but for B. Since the only path from B to A is through y_1, there is not a path that connects y_2 to y_1 that goes through x. ∎

3. Let G be an n-vertex graph. A *Hamiltonian cycle* in G is a cycle of length n, i.e., a cycle that covers all vertices of G. We say G is *Hamiltonian* if it contains a Hamiltonian cycle.

   (a) Prove or disprove: if G is 2-connected, then G is Hamiltonian.

   Disproof by counter example: BOOM!

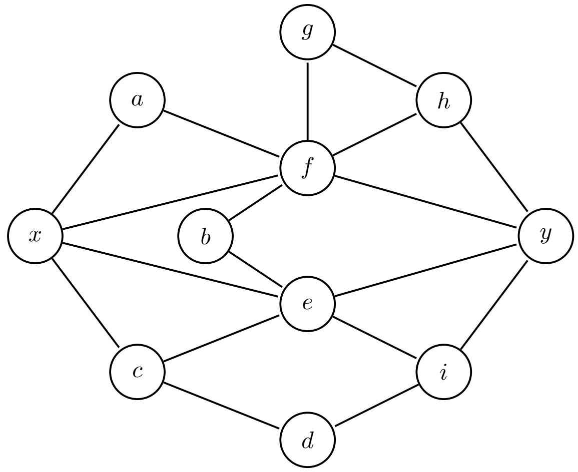
   (b) Prove or disprove: if G is Hamiltonian, then G is 2-connected.

   Proof. We know that a graph is 2 connected iff every pair of vertices is contained in a cycle. Every vertex is in the Hamiltonian cycle, so that's our cycle. BOOM! ∎

4. Let G be a k-connected graph and suppose A and B are disjoint subsets of $V(G)$ with $|A|, |B| \geq k$. Prove there exists k pairwise-disjoint A, B-paths.

   Proof. Let u be a G-connected graph and suppose w and v are disjoint subsets of $V(u)$ with $|w|, |v| \geq G$. Now let us create two new vertices, W and V. W is adjacent to all of w and similarly for V. Then by Expansion Lemma, we know that this new graph, we'll call u', is also G-connected.
   So then there exists G disjoint paths between W and V since u' is G-connected. Let π be one of those W, V-paths. Let Δ be the last vertex in π that is still in w, and Σ be the first vertex in π that is in v. The Δ, Σ-subpath of π is a w, v-path. Do this for each of the G different W, V-paths. We know these are pairwise-disjoint since the W, V-paths are internally disjoint. I hope we had fun doing math. ∎

5. Let G be the graph below.



(a) Determine $\kappa(x,y)$ and give an example of an x,y-cut of size $\kappa(x,y)$.

The set $\{f,e,d\}$ is a x,y-cut of size 3. There are also 3 disjoint paths from x to y, namely: x,f,y; x,e,y; x,c,d,i,y. Since there are 3 for both, we know by Mengar's theorem that $\kappa(x,y) = 3$.

(b) Determine $\kappa'(x,y)$ and give an example of an x,y-disconnecting set of size $\kappa'(x,y)$.

The set $\{xa,xf,xe,xc\}$ is a disconnecting set of size 4. There are 4 edge disjoint x,y-paths, namely: xa,af,fg,gh,hy; xf,fy; xe,ey; xc,cd,di,iy. So by a theorem that is similar to Mengar's theorem which is inappropriately named Mengar's II theorem (I think it should be named Mengar's$'$ Theorem because the functions are primed), we know that $\kappa'(x,y) = 4$.