Homework 1 (Due Jan 18, 2023)

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MATH 575 - Discrete Mathamatics II - Spring 2023

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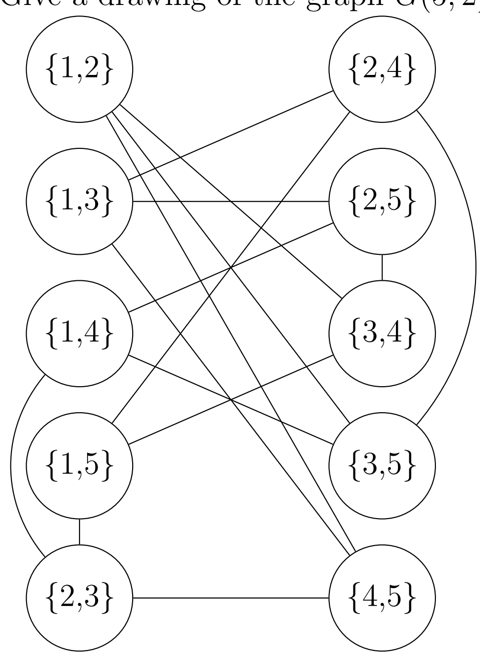
Justify all of your answers completely.

1. For positive integers n and k, consider the graph $G(n,k)$ which is defined as follows: the vertex set of $G(n,k)$ is the set of subsets of $[n]$ of size k, and two vertices are connected by an edge in $G(n,k)$ if and only if the corresponding subsets are disjoint. [1]

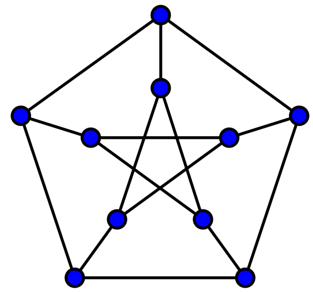
   (a) Give a drawing of the graph $G(5,2)$.



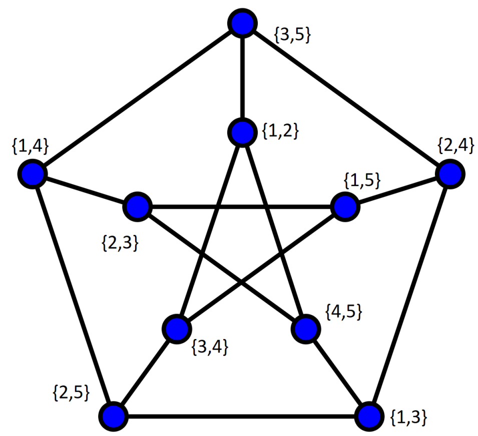
[1] E.g., $V(G(4,2)) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ and

$$E(G(4,2)) = \{\{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \{\{1,4\},\{2,3\}\}\}$$

(b) Let G be the graph drawn below. Show that $G(5,2)$ is isomorphic to G by relabelling the vertices of G in the drawing below.

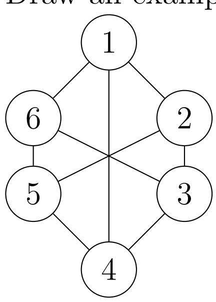


Here is the labeled graph

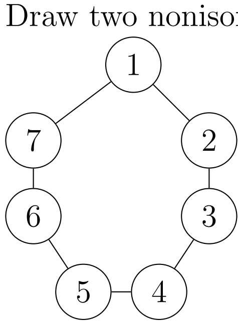


2. A graph is called k-**regular** if every vertex has degree k.

(a) Draw an example of a 3-regular graph on 6 vertices.



(b) Draw two nonisomorphic 2-regular graphs on 7 vertices.



(c) Prove that if k is odd, then there does not exist a k-regular graph with an odd number of vertices.

Proof. Assume we have a graph, G, that is k-regular, where k is odd. Assume towards contradiction that G has an odd number of vertices, namely n. By Handshaking lemma, summing up the degrees of the vertices will be an even number, but in G, it will be nk. This is an odd number multiplied by an odd number, which is not even. BOOM, A CONTRADICTION!!! ■

3. Prove that every graph G must contain a pair of vertices with the same degree.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of G, and d_i be the degree of v_i. There are n different possible numbers d_i can be, namely $0, ..., n-1$. If a vertex has a degree of $n-1$, then it is neighbors to every other vertex, making another vertex having a degree of 0 impossible. So a graph with every degree is impossible, meaning a graph can at most have $n-1$ different d_i's. So by PHP, matching n vertices to $n-1$ degrees must result in 2 vertices having the same degree. ■

4. Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Recall that the *adjacency matrix* of G is the $n \times n$ matrix A such that $A_{ij} = 1$ if $v_i v_j \in E(G)$ and $A_{ij} = 0$ otherwise. Use induction to prove that for all integers $k \geq 1$, the (i, j)-entry of A^k is equal to the number of v_i, v_j-walks of length k in G.

Proof. Let us induct on the degree of adjacency matrix A.
Base case: $\boldsymbol{k = 1}$
Obvious by the definition of adjacency matrix.

Induction step: Assume that A is an adjacency matrix of G and that the (i, j)-entry of A^k is equal to the number of v_i, v_j-walks of length k in G. Let us right multiply A to get $A^k \cdot A = A^{k+1}$. Looking at the (i, j)-entry of A^{k+1}, we get it from the dot product of the ith row of A^k and the jth column of A. Each element in the ith row of A^k is the number of walks from vertex i to each intermediate vertex q, and those entries will be multiplied by 0 if that intermediate vertex q is not adjacent to j. This means those walks of length k starting at vertex i that cannot reach j by adding one more edge to the end are not added. The walks of length k that do reach j after adding an edge to the end of the walk are added, since the number they are being multiplied by is 1 in the jth column of A. So we are left with the number of walks from vertex i to an intermediate vertex q (walk of length k) to vertex j, which gives a walk of length $k+1$.

■

5. Let G be an n-vertex graph with degree sequence $(d_1, d_2, \ldots, d_n)$.

(a) What is the degree sequence of $\overline{G}$?
$(n-1-d_1, n-1-d_2, \ldots, n-1-d_n)$ is the sequence.

(b) A graph G is called *self-complementary* if it is isormophic to its complement. Prove that if G is self-complementary, then either n or $n-1$ is divisible by 4.

Proof. Assume G is self-complementary. This means that $|E(G)|$ is half of the total number of edges for the complete graph with the same number of vertices, since $G \cup \overline{G} = K_n$. So $|E(K_n)| = \frac{n(n-1)}{2} \implies |E(G)| = \frac{n(n-1)}{4}$. And since $|E(G)|$ is a whole number, 4 must divide n or $n-1$. ■

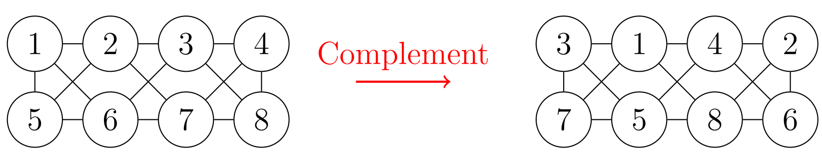
(c) Show that for all n divisible by 4, there exists a self-complementary graph on n vertices. [2]

Proof. The rule set for enumerated vertices: v_iv_j is an edge in the graph

i. if $v_i \equiv 1 \mod 4 \;\wedge\; (v_j \equiv 1 \mod 4 \vee v_j \equiv 2 \mod 4)$
ii. if $v_i \equiv 2 \mod 4 \;\wedge\; (v_j \equiv 1 \mod 4 \vee v_j \equiv 3 \mod 4)$
iii. if $v_i \equiv 3 \mod 4 \;\wedge\; (v_j \equiv 2 \mod 4 \vee v_j \equiv 0 \mod 4)$
iv. if $v_i \equiv 0 \mod 4 \;\wedge\; (v_j \equiv 3 \mod 4 \vee v_j \equiv 0 \mod 4)$

gives a self complementary graph on $4k$ vertices.
We will show the above statement by inducting on $n = 4k$.
Base case: $\boldsymbol{k = 1}$
The P_4 graph is self-complementary.

$\boldsymbol{k = 2}$
Layering P_4 graphs with the rule set results in self-complementary graphs.



Induction Step: Assume we have a simple graph, G, such that there are $4k$ vertices $\{v_1, \ldots, v_{4k}\}$, with the rules stated in the beginning, where $k \geq 2$. (This is the pattern shown in base case $k = 2$)
Let us add 4 vertices to the graph and apply the rule to those vertices. We know from the base case $k = 2$ that the subgraph of this new "layer" and any other layer will be self-complementary. Since the vertices in the new layer are self complementary with every other layer, and the other layers were already self-complementary in total, this new graph with $4(k+1)$ vertices is also self complementary. ■

[2] Hint: generalize the structure of the path P_4.

(d) Show that for all n such that $n-1$ is divisible by 4, there exists a self-complementary graph on n vertices. [3]

In part (c), we constructed a graph with 4k vertices that was self-complementary. Let us add a single vertex to our graph. It is now not self-complementary. To make it self complementary, we must connect the vertex to the "outer" vertices in each "row". This makes the graph self-complementary, since the vertices our new vertex is adjacent to in the complement graph are now the "outer" vertices (i.e. 2 and 3 become the outer vertices), and we still have half the total possible edges. So now we have a self-complementary construction for every graph with n vertices where $4|(n-1)$.

[3]Hint: add a vertex to a construction in part (c).