Homework 3 (Due Feb 1, 2023)

Jack Hyatt
MATH 575 - Discrete Mathamatics II - Spring 2023

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Justify all of your answers completely.

1. Prove that a forest with n vertices and c components contains exactly $n - c$ edges.

   Proof. We shall induct on the number of components.
   Base case: $c = 1$
   Since $c = 1$, that means we have just one tree. A tree has $|E| = n - 1 = n - c$.

   Induction Step: Suppose we have a forest, F, with n vertices and $c > 1$ components, and let every forest with n' vertices and $c' < c$ components have exactly $n' - c'$ edges.
   Then F has c trees. Let us remove 1 tree, we'll call this removed tree T. Assume T has n_1 vertices, which means it has $n_1 - 1$ edges. So $F \setminus T$ has $n - n_1$ vertices and $c - 1$ components. By the induction hypothesis, we know $F \setminus T$ has $n - n_1 - (c - 1) = n - n_1 - c + 1$ edges. Let us now add back T. So we added back n_1 vertices and $n_1 - 1$ edges.

   $$E(F) = E(F \setminus T) + E(T) = (n - n_1 - c + 1) + (n - 1) = n - c$$

   QED by P.M.I. ∎

2. Let G be an n-vertex tree with $\Delta(G) \leq 2$. Prove that G must be isomorphic to the path P_n.

   Proof. We shall induct on the number of vertices.
   Base case: $n = 1$
   Trivially obvious.
   Induction Step: Suppose we have a tree, G, with $n > 1$ vertices and $\Delta(G) \leq 2$, and let every graph, G' with $n' < n$ vertices and $\Delta(G') \leq 2$ be isomorphic to $P_{n'}$.
   We know that every tree has at least 1 leaf vertex. Let ℓ be a leaf vertex of G, and we shall remove it from G. G will still be a tree and has less than n vertices. Since we only removed a vertex, $\Delta(G - \ell)$ could not have gone up. So we can apply our

induction hypothesis to $G - \ell$, showing $G - \ell$ is isomorphic to P_{n-1}.
Since $G - \ell$ is isomorphic to P_{n-1}, adding ℓ to $G - \ell$ is equivalent to adding ℓ to P_{n-1}. We know that adding ℓ would not create a vertex of degree 3 or more due to $\Delta(G) \leq 2$, so ℓ would be added to a vertex in P_{n-1} with degree less than 2. This means ℓ is added to an end point vertex in P_{n-1}, which lengthens the path to P_n.
QED by P.M.I. ■

3. Suppose G is a graph with the property that deleting any single vertex (and its incident edges) results in a tree.

   (a) What can we say about the number of edges in G and the degrees of its vertices?

   Let n be the number of vertices in G. Since deleting any single vertex results in a tree, and those trees will *always* have $(n-1)-1$ edges, we know that every vertex has the same degree. So G is k-regular. So G has $nk/2$ edges. Since removing a single vertex results in $n-2$ edges and that vertex has degree k, G also has $n-2+k$ edges.

   (b) Use part (a) to determine G itself.

$$\frac{nk}{2} = n - 2 + k \implies k(n-2) = 2(n-2) \implies k = 2$$

   So G is 2-regular, and connected since the sub-trees are connected. This and the above algebra means that G is a cycle graph when $n \neq 2$. However, the case where $n = 1$ or $n = 2$ still works, even if it is 2 disconnected vertices. Since they're crappy edge cases that work, lets just throw them in with the "G must be a C_n" conclusion.

4. Let G be a connected, weighted graph with n vertices. Let xy be an edge of largest weight in G.

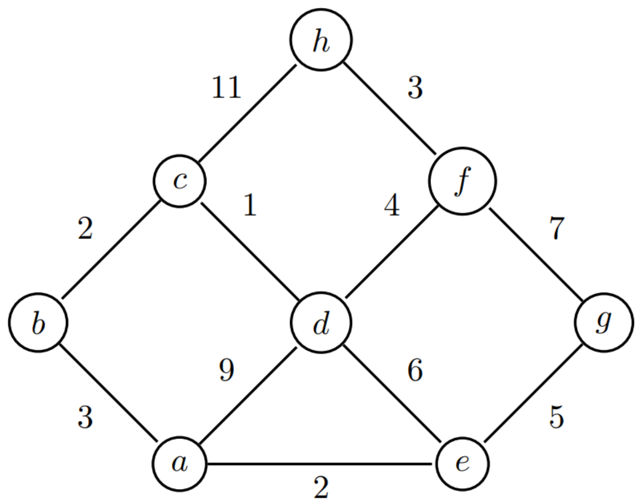
   (a) Prove or disprove: there must exist a MST of G that avoids the edge xy.

   Disproving by counter example: Let G be just the vertices x and y. Then the *only* MST of G must contain xy.

   (b) Prove that if xy is contained in a cycle in G, then there exists a MST of G that avoids the edge xy.

   Proof. Let T be a spanning tree containing xy. If we delete xy, we will get 2 subtrees, with one containing x and the other containing y. Let's denote these subtrees as T_x and T_y respectively. In T_x, there must be one vertex with an edge in G that connects to T_y that is not xy, since xy was part of a cycle. Let us add that edge to connect T_x and T_y. This edge must have weight less than or equal to xy since xy had the highest weight. So the weight could not have increased. This means that if we found a MST s.t. xy was contained, we can swap it out with the above technique. ■

5. Use Dijkstra's Algorithm to find a shortest path tree starting with the vertex a. At each step of the algorithm, write the edge that was added to the tree.



	a	b	c	d	e	f	g	h	Chosen Vertex
Step 1, Prev		(3,a)	∞	(9,a)	(2,a)	∞	∞	∞	e
Step 2, Prev		(3,a)	∞	(8,e)		∞	(7,e)	∞	b
Step 3, Prev			(5,b)	(8,e)		∞	(7,e)	∞	c
Step 4, Prev				(6,c)		∞	(7,e)	(16,c)	d
Step 5, Prev						(10,d)	(7,e)	(16,c)	g
Step 6, Prev						(10,d)		(16,c)	f
Step 7, Prev								(13,f)	h