Homework 6 (Due Oct 25, 2023)

Jack Hyatt
MATH 546 - Algebraic Structures I - Fall 2023

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Justify all of your answers completely.

1. List all the possible orders of elements of the group $G = S_8$. For each possible order, give an example of an element that has that order. Explain why no other orders are possible.

   For every $\sigma \in S_8$, we can write it as $\sigma = \tau_1 \dots \tau_k$ for some $k \geq 1$, where τ_i is a cycle. The possible decomposition types of σ will tell us the lengths of each decomposed cycle, and calculating that is the same as calculating partitions. Once we get the decomposition types, we just take the lcm of the partitions.
   The possible partitions of 8 and their lcm is:

$$
\begin{aligned}
1+1+1+1+1+1+1+1 &: \operatorname{lcm}(1) = 1 & 2+1+1+1+1+1+1 &: \operatorname{lcm}(2,1) = 2 \\
2+2+1+1+1+1 &: \operatorname{lcm}(2,1) = 2 & 2+2+2+1+1 &: \operatorname{lcm}(2,1) = 2 \\
2+2+2+2 &: \operatorname{lcm}(2) = 2 & 3+1+1+1+1+1 &: \operatorname{lcm}(3,1) = 3 \\
3+2+1+1+1 &: \operatorname{lcm}(3,2,1) = 6 & 3+2+2+1 &: \operatorname{lcm}(3,2,1) = 6 \\
3+3+1+1 &: \operatorname{lcm}(3,2,1) = 6 & 3+3+2 &: \operatorname{lcm}(3,2) = 6 \\
4+1+1+1+1 &: \operatorname{lcm}(4,1) = 4 & 4+2+1+1 &: \operatorname{lcm}(4,2,1) = 4 \\
4+2+2 &: \operatorname{lcm}(4,2) = 4 & 4+3+1 &: \operatorname{lcm}(4,3,1) = 12 \\
4+4 &: \operatorname{lcm}(4) = 4 & 5+1+1+1 &: \operatorname{lcm}(5,1) = 5 \\
5+2+1 &: \operatorname{lcm}(5,2,1) = 10 & 5+3 &: \operatorname{lcm}(5,3) = 15 \\
6+1+1 &: \operatorname{lcm}(6,1) = 6 & 6+2 &: \operatorname{lcm}(6,2) = 6 \\
7+1 &: \operatorname{lcm}(7,1) = 7 & 8 &: \operatorname{lcm}(8) = 8
\end{aligned}
$$

   So then the possible orders for σ are: 1,2,3,4,5,6,7,8,10,12,15.
   An example of an element of order n is just $(1\ 2\ \dots\ n)$.

2. Let $\tau \in S_n$ be the cycle $(1\,2\,\dots\,k)$. Prove that for all $\sigma \in S_n$.

$$\sigma\tau\sigma^{-1} = (\sigma(1)\ \dots\ \sigma(k))$$

Proof. Let $\tau \in S_n$ be the cycle $(1\,2\,\ldots\,k)$, and $\sigma \in S_n$. Want to show an input of $\sigma(i)$ into $\sigma\tau\sigma^{-1}$ will be consistent with the above cycle.
Case 1: $1 \leq i < k$.
Then
$$\sigma\tau\sigma^{-1}\sigma(i) = \sigma\tau(i) = \sigma(i+1)$$
Case 2: $i = k$.
Then
$$\sigma\tau\sigma^{-1}\sigma(i) = \sigma\tau(i) = \sigma(1)$$
Case 3: $k < i \leq n$.
Then
$$\sigma\tau\sigma^{-1}\sigma(i) = \sigma\tau(i) = \sigma(i)$$
So then $\sigma\tau\sigma^{-1}$ is the $(\sigma(1)\,\ldots\,\sigma(k))$ cycle. ■

3. Let $G = S_4$, and let $H = \{\sigma \in S_4 : \sigma^2 = e\}$.

   (a) List all the elements of H.

   Obviously $e \in H$.
   Since we want the order of σ to be at most 2, then the decomposed cycles must be at most length 2, since the order of a permutation is the lcm of the orders of its cycles. This means the elements can be 2 disjoint cycles of length 2, or a cycle of length 2 with two cycles of length 1.
   So $H = \{e$, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 2), (1 3), (1 4), (2 3), (2 4), (3 4)$\}$.

   (b) Decide whether H is a subgroup of S_4 or not.

   Proof. It is not due to not being closed, since (1 2)$\circ$(1 3) produces (1 3 2), which is not in H. ■

4. Recall that for a group G, $Z(G)$ means the set
$$\{x \in G : ax = xa \; \forall a \in G\}.$$

   (a) Find $Z(G)$ for $G = S_3$.

   Proof. S_3 ={e, (1 2 3), (1 3 2), (1 2), (1 3), (2 3)}
   (1 2)$\circ$(2 3) = (1 2 3) and (2 3)$\circ$(1 2) = (1 3 2)
   So (1 2),(2 3) $\notin Z(S_3)$.
   (1 3)$\circ$(1 3 2) = (2 3) and (1 3 2)$\circ$(1 3) = (1 2)
   So (1 3),(1 3 2) $\notin Z(S_3)$.
   (1 2 3)$\circ$(1 2) = (1 3) and (1 2)$\circ$(1 2 3) = (2 3)
   So (1 2 3) $\notin Z(S_3)$.
   Obviously e commutes.
   So $Z(S_3) = \{e\}$. ■

   (b) If $G = S_4$ and x =(1 2 3 4), prove that $x \notin Z(G)$.

Proof. (1 2 3 4)∘(1 2) = (1 3 4) and (1 2)∘(1 2 3 4) = (2 3 4)
So (1 2 3 4) doesn't commute with (1 2), meaning (1 2 3 4) $\notin Z(G)$. ∎

5. Let $G = S_4$. Assume that H is a subgroup of S_4 such that every cycle of length 2 in S_4 belongs to H. Prove that H must be equal to the entire S_4.

   Proof. Since H contains all cycles of length 2 and H is a subgroup, then H is closed under the composition of those cycles. Since any permutation can be written as the product of cycles of length 2, every permutation in S_4 can be written using the length 2 cycles in H. Since H is closed, every permutation created using length 2 cycles must also be in H. So H contains all of S_4. ∎