Homework 3 (Due Sept 22, 2023)

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Justify all of your answers completely.

1. Find the order of each of the following elements of $GL_2(\mathbb{R})$.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Note: $e = I_{2\times2}$.

$$A \neq I_{2\times2}$$

$$A^2 = \begin{bmatrix} 0\cdot1+1\cdot-1 & 0\cdot1+1\cdot0 \\ 0\cdot-1-1\cdot0 & 1\cdot-1+0\cdot0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_{2\times2} \neq I_{2\times2}$$

$$A^3 = A^2A = \begin{bmatrix} -1\cdot0+1\cdot0 & 0\cdot0+1\cdot-1 \\ -1\cdot-1+0\cdot0 & -1\cdot0+0\cdot-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A \neq I_{2\times2}$$

$$A^4 = A^2A^2 = (-I_{2\times2})(-I_{2\times2}) = I_{2\times2}$$

We will prove $o(B) = \infty$ through induction by showing $B^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$$B = B^1 = \begin{bmatrix} 1 & \mathbf{1} \\ 0 & 1 \end{bmatrix}$$

Assume $B^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for $k \geq 1$.

$$B^{k+1} = B^kB = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot1+k\cdot0 & 1\cdot1+k\cdot1 \\ 0\cdot1+1\cdot0 & 0\cdot(k+1)+1\cdot1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

By P.M.I., $B^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \neq I_{2\times2}$

2. Let n, k be integers, $n \leq 2$. Prove that $o([k]_n) = n/\gcd(n,k)$.

Proof. Need to check $[k]_n \cdot n/\gcd(n,k) = 0$, and $\forall 0 < \ell < \frac{n}{\gcd(n,k)}$, $[\ell]_n \cdot \frac{n}{\gcd(n,k)} \neq 0$.

$$[k]_n \cdot \frac{n}{\gcd(n,k)} = \left[n \cdot \frac{k}{\gcd(n,k)}\right]_n = [0]_n$$

since $k/\gcd(n,k)$ is an integer.
BWOC, assume $\exists 0 < \ell < \frac{n}{\gcd(n,k)}$ s.t. $o([k]_n) = \ell$. Then $\ell \cdot [k]_n = 0$. Then $\ell k = qn$ for some positive integer q.
Let $d = \gcd(n,k)$. Then $\exists k', n' \in \mathbb{Z}^+$ s.t. $k = dk'$ and $n = dn'$ and k' is coprime to n'.
$\ell k = qn \implies \ell dk' = qdn' \implies \ell k' = qn'$. So $n' | \ell k'$, and since $\gcd(k', n') = 1$, then $n' | \ell$.
So $\exists \alpha \in \mathbb{Z}^+$ s.t. $n'\alpha = \ell$. So $n' \cdot k$ is a multiple of n, and $n' < \ell$. But ℓ was the order of k, which is a contradiction. ■

3. Using the result from problem 2, list all the elements of $\mathbb{Z}_90$ that have order equal to 6.

   From problem 2, $\forall k,\ 0 < k < 90, o([k]_90) = 90/\gcd(90,k)$. So we want to find the values of k when $90/\gcd(90,k) = 6$, and these values will be the comprehensive list of what we want because of the equality.

$$\frac{90}{\gcd(90,k)} = 6 \implies \gcd(90,k) = 15$$

   Clearly $k \geq 15$ since a divisor can't be bigger than the number. Since $15|90$, we just need to find multiples of 15 that aren't multiples of 30, 45, or 90, since 30, 45, and 90 are bigger divisor of 90 than 15 is. So we are left with $k = 15, 75$.

4. Let $(G, *)$ be a group with the property that $\forall a, b \in G$, $(a * b)^2 = a^2 * b^2$. Prove that G is abelian.

   Proof. Assume the above. Then

$$(a * b)^2 = a^2 * b^2 \implies a * b * a * b = a * a * b * b$$
$$\implies a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1} \implies e * b * a * e = e * a * b * e$$
$$\implies b * a = a * b$$

   ■

5. Let $(G, *)$ be an abelian group, and let $a, b \in G$. Assume that $o(a) = 3$, $o(b) = 2$. Prove that $o(a * b) = 6$.

   Proof. Assume the above. Considering $a * b$, BWOC assume $a * b = e$. Then $b = a^{-1}$. But b and a have different orders, a contradiction. So $a * b \neq e$

$$\begin{aligned}
(a * b)^2 &= a^2 * b^2 = a^2 * e = a^2 \neq e && \text{since } o(a) = 3\\
(a * b)^3 &= a^3 * b^3 = e * b^2 * b = e * b = b \neq e && \text{since } o(b) = 2\\
(a * b)^4 &= a^4 * b^4 = a^3 * a * b^2 * b^2 = a \neq e && \text{since } o(a) = 3\\
(a * b)^5 &= a^5 * b^5 = a^3 * a^2 * b^2 * b^2 * b = a^2 * b \neq e
\end{aligned}$$

since $a^2 * b = e \implies b$ is the inverse of a^2, even though a is since $o(a) = 3$, and we know $a \neq b$ due to different orders.

$$(a * b)^6 = a^6 * b^6 = a^3 * a^3 * b^2 * b^2 * b^2 = e$$

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