Homework 4 (Due Feb 8, 2023)

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MATH 575 - Discrete Mathamatics II - Spring 2023

February 8, 2023

Justify all of your answers completely.

1. Let M be a maximum matching of a graph G, and let M' be a *maximal* matching of G. Prove that $|M'| \geq \frac{|M|}{2}$.

   Proof. Let M be a maximum matching of a graph G, and let M' be a *maximal* matching of G. Assume towards contradiction that $|M'| < \frac{|M|}{2}$. Let e be an edge in M'. Since M is maximum, then M must contain at least one, and at most 2 edges incident to e, because otherwise we could add e to M and then M wouldn't have been maximum. Let a be the number of edges in M that are incident to some saturated vertex in M'. So $a \leq 2|M'| < |M| \implies a < |M|$. So then there is some edge $e' \in M$ that is not incident to any vertex in M'. It transpires that we should be able to add e' to M', meaning M' wasn't maximal. 

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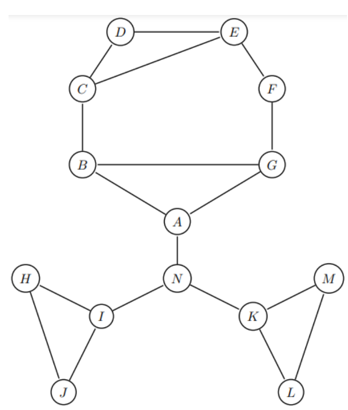
2. Consider the following graph and the matching given by the edges $M = \{CE, BG, HI, KM\}$.



   (a) Starting from the matching M, use a series of augmenting paths to find a matching in the graph of size 6. Write down the vertices for each augmenting path you use.

Starting with M, we see that (D, E, C, B, G, A) is an augmenting path. So now our new matching is $M = \{DE, CB, GA, HI, KM\}$. Now we have (F, E, D, C, B, G, A, N) is an augmenting path. Our new matching with 6 matches is now $M = \{FE, DC, BG, AN, HI, KM\}$.

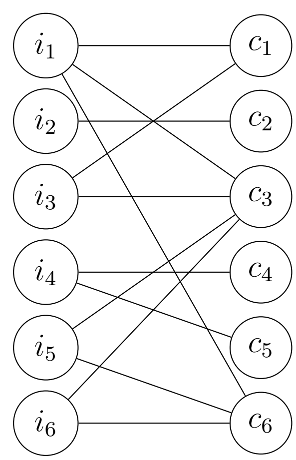
(b) Is the matching you found in part (a) maximum? If so, explain why. If not, find a larger matching.

Yes, because there are no augmenting paths left.

3. 6 instructors $i_1, i_2, \ldots, i_6$ must be assigned to teach 6 classes $c_1, c_2, \ldots, c_6$, 1 class per instructor. The table below shows an "x" if a teacher has taught a class in a previous semester. Suppose each teacher would prefer to teach a class they have taught in the past.

	c_1	c_2	c_3	c_4	c_5	c_6
i_1	x		x			x
i_2		x				
i_3	x		x			
i_4				x	x	
i_5			x			x
i_6			x			x

(a) Construct a bipartite graph G such that a matching in G corresponds to a (partial) assignment of instructors to classes.



(b) Find a maximum matching of G from part(a). Can every instructor be assigned a class of their preference? If not, find a subset of the instructors that violate Hall's condition.

The set $\{i_1, i_3, i_5, i_6\}$ has $\{c_1, c_3, c_6\}$ as neighbors, a subset that violates Hall's condition.

4. In a bipartite graph $G = X \cup Y$, the *deficiency* of a set $S \subseteq X$ is

$$\operatorname{def}(S) = \max\{0, |S| - |N(S)|\}.$$

Prove that a maximum matching in G has size $|X| - \max_{S \subseteq X} \operatorname{def}(S)$.

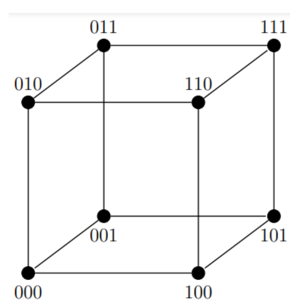
Proof. Let $G = X \cup Y$ be a bipartite graph. WLOG, let $|X| \geq |Y|$. Let us construct a graph G' with the following rules: G' is the same as G, except we add vertices to Y in a certain way. The set of vertices we will add shall be called W. $|W| = \max_{S \subseteq X} \operatorname{def}(S)$, and W and X form a complete bipartite subgraph.

This graph G' has a matching that saturates X iff G has a matching of size $|X| - \max_{S \subseteq X} \operatorname{def}(S)$. This is true because if G has a matching of the desired size, then there are $\max_{S \subseteq X} \operatorname{def}(S)$ vertices in X unmatched, which then can be matched to vertices in W. If G' has a matching saturating X, with how we constructed G', taking away W from G' leaves $\max_{S \subseteq X} \operatorname{def}(S)$ vertices in X unmatched, meaning G has a matching of the desired size.

We know G' always saturates X because the max deficiency finds the subset with the greatest difference in vertices with its neighbor vertices, which would leave $|S| - |N(S)|$ unmatched vertices in X in any maximum matching. So the addition of W to Y makes it so that every vertex in X can be match, fully saturating X. Hence, X is always fully saturated in G', meaning G has a matching of the desired size always. ■

5. For $d \in \mathbb{N}$, the d-dimensional hypercube Q_d is the 2^d-vertex graph in which every vertex is a binary string of length d, and two vertices are adjacent if their corresponding strings differ in exactly one coordinate.

Prove that for $d \geq 2$, Q_d has at least $2^{2^{d-2}}$ perfect matchings.



Proof. We shall induct on the number of dimensions, d, to show the above fact.
Base Case: d=2
It is clear that a $Q_2 = C_4$ which has $2 = 2^1 = 2^{2^{2-2}}$ perfect matchings.
Induction Step: Let $d \geq 2$ and Q_d have $2^{2^{d-2}}$ perfect matchings.
Consider the $d+1$ case. Q_{d+1} is created by making two Q_d's, and putting a 0 in front of the vertices of one, and a 1 in front of the vertices of the other. Before connecting these two components through the rule, we see that each component has $2^{2^{d-2}}$ perfect matchings. So a lower bound for the total perfect matchings for Q_{d+1} is to use the

multiplication rule of combinatorics and multiply $2^{2^{d-2}} \cdot 2^{2^{d-2}} = (2^{2^{d-2}})^2 = 2^{2 \cdot 2^{d-2}} = 2^{2^{d-1}}$. Therefore, by P.M.I., Q_d has at least $2^{2^{d-2}}$ perfect matchings. $\blacksquare$

I have a LaTeX package that has a Therefore command that will give a random funny phrase that is similar to the word therefore, fyi.