Homework 8 (Due March 22, 2023)

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MATH 575 - Discrete Mathematics II - Spring 2023

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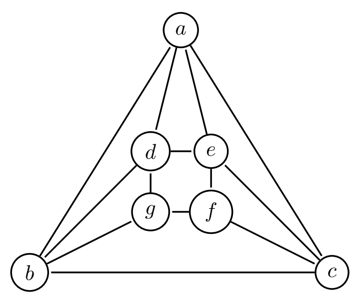
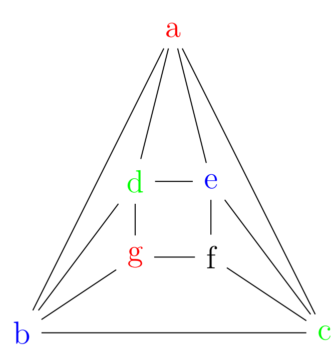
Justify all of your answers completely.

1. Let G be the graph below.



   (a) Determine $\chi(G)$.



I am not sure if the printer used in printing these sheets does color, but I colored the graph with 4 colors. To show there can't be less than 4, assume there is a coloring of 3. $\{a, b, d\}$ must all be different colors, and $\{b, d, g\}$ must all be different colors, so a and g must be the same color, we'll call color 1. Similar reasoning makes d and c the color 2, and b and e color 3. f is adjacent to g, e, and c, so f must be a color different from all of those. As Gauss proved, the chromatic number of this graph is 4. ■

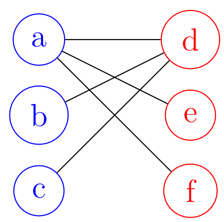
(b) Is G color-critical? If so, prove that $\chi(G - e) < \chi(G)$ for every edge $e \in E(G)$. If not, find a color-critical subgraph of G.

2. Prove that for any graph G, there exists an ordering of $V(G)$ for which the greedy algorithm uses exactly $\chi(G)$ colors.

3. Let G be a graph. Prove or disprove the following statements.

   (a) There exists a $\chi(G)$-coloring of G in which one color class contains $\alpha(G)$ vertices.

   I will disprove this conjecture by making a counter example.



   Each class has 3 vertices, and the largest independent set is $\{b, c, e, f\}$.

   (b) $\chi(G) \leq 1 + \overline{d}(G)$ where $\overline{d}(G)$ is the *average degree* in G.

   (c) $\chi(G) \leq n - \alpha(G) + 1$.

   Proof. Since the other two statements were disprovable, using the Theorem of human predictability and probabilistic method, we know this statement is provable.
   Let G be a n-vertex graph. Give every vertex it's own color. We have a coloring of n colors. Let us take the set of the largest independent set. Since the set has no edges between them, we can recolor those vertices as just 1 color. And now we have a valid coloring of $n - \alpha(G) + 1$. ■

4. Let G be an n-vertex graph. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n$, and use this to prove $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. For each n where $\sqrt{n}$ is an integer, construct a graph that achieves both equalities.

   Proof. ■

   Hint: using a $\chi(G)$-coloring of G and a $\chi(\overline{G})$-coloring of $\overline{G}$, construct a proper coloring of K_n. Also you may find it useful to use the AM-GM inequality which states that if x and y are non-negative real numbers, then $\sqrt{xy} \leq \frac{x+y}{2}$.

5. Let G be an n-vertex graph. Prove that $\chi(G) + \chi(\overline{G}) \leq n + 1$ and conclude that $\chi(G) \cdot \chi(\overline{G}) \leq [(n+1)/2]^2$. For each odd n, give an example of a graph G that achieves both equalities. [1]

   Hint: use induction to prove $\chi(G) + \chi(\overline{G}) \leq n + 1$.

[1] From questions 3 and 4, we obtain that for every graph G on n vertices, either G or its complement has chromatic number at least $\sqrt{n}$, and either G or its completement has chromatic number at most $(n+1)/2$.