Homework 3 (Due Sept 11, 2023)

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MATH 554 - Analysis I - Fall 2023

October 15, 2023

Justify all of your answers completely.

2.29 Let $\delta > 0$. Assume that $|y - c| < \delta$. Show that

$$|(-5y + 42) - (-5c + 42)| < 5\delta$$

Proof. $|(-5y + 42) - (-5c + 42)| = |5y - 5c| = 5|y - c| < 5\delta$ ■

2.30 Show that if $|y - c| < \delta \le 2$, then

$$|y^4 - c^4| \le 4(2 + |c|)^3\delta$$

Proof. Note: $|y| = |y - c + c| \le |y - c| + |c| < \delta + |c|$.

$$|y^4 - c^4| = |(y - c)(y^3 + y^2c + yc^2 + c^3)| = |(y - c)||(y^3 + y^2c + yc^2 + c^3)|$$
$$\le \delta|(\delta + c)^3 + (\delta + c)^2c + (\delta + c)c^2 + c^3| = \delta|\delta^3 + 4\delta^2c + 6\delta c^2 + 4c^3|$$
$$\le \delta|2^3 + 4 \cdot 2^2c + 6 \cdot 2c^2 + 4c^3| = \delta|8 + 16c + 12c^2 + 4c^3|$$
$$\le 4\delta(2 + 4|c| + 3|c|^2 + |c|^3) \le 4\delta(8 + 12|c| + 6|c|^2 + |c|^3) = 4(2 + |c|)^3\delta$$

■

2.31 Assume $|a - x| < \delta, |y - b| < \delta, and \delta \le 1$. Show that

$$|xy - ab| < (1 + |a| + |b|)\delta$$

Proof. Note: $|x| = |x - a + a| \le |x - a| + |a| = |a - x| + |a| < \delta + |a|$
$|y| = |y - b + b| \le |y - b| + |b| < \delta + |b|$
$|xy - ab| \le |x||y| - |a||b| < |\delta + |a|| \cdot |\delta + |b|| - |a||b| \le (|\delta| + |a|)(|\delta| + |b|) - |a||b|$
$= |\delta|^2 + |\delta||b| + |\delta||a| + |a||b| - |a||b| = (\delta + |b| + |a|)\delta \le (1 + |b| + |a|)\delta$ ■

2.32 Assume $|x-a| < \delta \le |a|/2$. Show

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

and

$$|\frac{1}{x^2} - \frac{1}{a^2}| < \frac{10\delta}{|a|^3}$$

Proof. Let $|x-a| < \delta \le |a|/2$. First, let's show $\frac{|a|}{2} < |x|$.

$$|x| = |a + (x-a)| \ge |a| - |x-a| > |a| - |a|/2 = |a|/2$$

Now, let's show $|x| < 3|a|/2$.

$$|x| = |x - a + a| \le |x-a| + |a| < |a|/2 + |a| = 3|a|/2$$

Now to show $|\frac{1}{x^2} - \frac{1}{a^2}| < \frac{10\delta}{|a|^3}$

$$|\frac{1}{x^2} - \frac{1}{a^2}| = |\frac{a^2 - x^2}{x^2a^2}| = |\frac{1}{x^2a^2}| \cdot |a+x| \cdot |a-x| < \frac{\delta}{|x|^2|a|^2} \cdot |a+x|$$

$$\le \frac{\delta}{(\frac{|a|}{2})^2|a|^2} \cdot (|a| + \frac{3|a|}{2}) = \frac{10\delta}{|a|^3}$$

■

2.33 Prove for $a, b \in \mathbb{F}$

$$\max(a,b) = \frac{a+b+|a-b|}{2}$$

$$\min(a,b) = \frac{a+b-|a-b|}{2}$$

Proof. (max)
Case 1: $a \ge b$.

$$\frac{a+b+|a-b|}{2} = \frac{a+b+(a-b)}{2} = \frac{2a}{2} = a = \max(a,b)$$

Case 2: $a < b$.

$$\frac{a+b+|a-b|}{2} = \frac{a+b+|b-a|}{2} = \frac{a+b+(b-a)}{2} = \frac{2b}{2} = b = \max(a,b)$$

(min)
Case 1: $a \le b$.

$$\frac{a+b-|a-b|}{2} = \frac{a+b-|b-a|}{2} = \frac{a+b-(b-a)}{2} = \frac{2a}{2} = a = \min(a,b)$$

Case 2: $a > b$.

$$\frac{a+b-|a-b|}{2} = \frac{a+b-(a-b)}{2} = \frac{2b}{2} = b = \min(a,b)$$

■

2.34 Solve the following inequalities

(a) $5x - 9 < 7x + 21$

$5x - 9 < 7x + 21 \implies -15 < x$

(b) $x^2 - 10x + 9 < 16$

$x^2 - 10x + 9 < 16 \implies x^2 - 10x + 25 < 32 \implies (x-5)^2 < 32 \implies |x-5|^2 < 32$
$\implies -\sqrt{32} < x - 5 < \sqrt{32} \implies 5 - \sqrt{32} < x < 5 + \sqrt{32}$

(c) $\frac{x+2}{x-2} \leq 5$

Note: $x \neq 2$
$\frac{x+2}{x-2} \leq 5 \implies \frac{x+2-5(x-2)}{x-2} \leq 0 \implies \frac{4(-x+3)}{x-2} \leq 0 \implies \frac{x-3}{x-2} \geq 0 \implies x < 2$ or $x \geq 3$