Homework 3 (Due Sep 14, 2022)

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MATH 570 - Discrete Optimization - Fall 2022

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Justify all of your answers completely.

Exercise 1. Solve the following LP using the simplex algorithm with tableaux, then verify your answer graphically.

$$\begin{array}{ll} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 18 \\ & 2x_1 + 3x_2 \leq 42 \\ & 3x_1 + x_2 \leq 24 \\ & 0 \leq x_1, x_2 \end{array}$$

Answer: The LP as a tableau is

$$\left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 18 \\ 2 & 3 & 0 & 1 & 0 & 42 \\ 3 & 1 & 0 & 0 & 1 & 24 \\ 3 & 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivot element will be 1st column and 3rd row. Performing GE on this column results in

$$\left[\begin{array}{ccccc|c} 0 & \frac{1}{3} & 1 & 0 & \frac{-2}{3} & 2 \\ 0 & \frac{7}{3} & 0 & 1 & \frac{-2}{3} & 26 \\ 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 8 \\ 0 & 1 & 0 & 0 & -1 & -24 \end{array}\right]$$

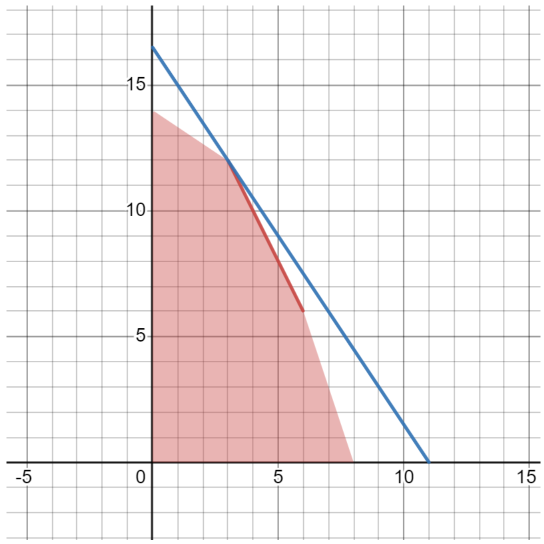
The next pivot element will be the 2nd column and 1st row. Performing GE on this column results in

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 0 & -2 & 6 \\ 0 & 0 & -7 & 1 & 4 & 12 \\ 1 & 0 & -1 & 0 & 1 & 6 \\ 0 & 0 & -3 & 0 & 1 & -30 \end{array}\right]$$

The next pivot element will be the 5th column and 2nd row. Performing GE on this column results in

$$\left[\begin{array}{ccccc|c} 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 12 \\ 0 & 0 & -\frac{7}{4} & \frac{1}{4} & 1 & 3 \\ 1 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 & 3 \\ 0 & 0 & -\frac{5}{4} & -\frac{1}{4} & 0 & -33 \end{array}\right]$$

This tell us that the optimal point is $x_1 = 3$ and $x_2 = 12$ with the optimal value being 33. Here is the LP graphed with the obj function set equal to 33, as verification of the answer.



Exercise 2. Dualize the LP from Exercise 1, and then solve it.
Since the primal in Exercise 1 was of the form

$$\begin{array}{ll} \text{maximize} & \vec{c}^T\vec{x} \\ \text{subject to} & A\vec{x} \leq \vec{b} \\ & 0 \leq \vec{x} \end{array}$$

Then the dual will be of the form

$$\begin{array}{ll} \text{minimize} & \vec{b}^T\vec{y} \\ \text{subject to} & A^T\vec{y} \geq \vec{c} \\ & 0 \leq \vec{y} \end{array}$$

So then the dual will be

$$\begin{array}{ll} \text{minimize} & 18y_1 + 42y_2 + 24y_3 \\ \text{subject to} & 2y_1 + 2y_2 + 3y_3 \geq 3 \\ & y_2 + 3y_2 + y_3 \geq 2 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

Since the dual is origin infeasible, my monkey brain doesn't know how to use the simplex method on this LP. Therefore, I must use the strong duality theorem, that uses the weak duality theorem, to say that since $\hat{x}$ solves the primal, $\hat{y}$ solves the dual. Looking at the last matrix in Exercise 1, the elements in the last row and in the 3rd through 5th columns are $-\hat{y}$. So $y_1 = \frac{5}{4}, y_2 = \frac{1}{4}$, and $y_3 = 0$ minimizes the dual, which does produce 33 like the primal did.

Exercise 3. Solve the following LP.

$$\begin{array}{ll} \text{minimize} & 2x+3y+4z \\ \text{subject to} & x+y+z \le 10 \\ & x \ge -3 \\ & y \ge -5 \\ & 3 \ge z \ge 0 \\ & 4x-y-2z=2 \end{array}$$

Solving the equality for z and substituting it everywhere else yields us with

$$\begin{array}{ll} \text{minimize} & 10x+y \\ \text{subject to} & 3x+\frac{y}{2} \le 11 \\ & 4 \ge 2x-\frac{y}{2} \ge 1 \\ & x \ge -3 \\ & y \ge -5 \end{array}$$

Getting to (almost) standard forms looks like

$$\begin{array}{ll} \text{Let} & x = x'-3 \\ & y = y'-5 \\ \text{minimize} & 10x'+y' \\ \text{subject to} & 3x'+\frac{y'}{2} \le \frac{45}{2} \\ & 2x'-\frac{y'}{2} \le \frac{15}{2} \\ & -2x'+\frac{y'}{2} \le -\frac{9}{2} \\ & 0 \le x', y' \end{array}$$

This looks to not work with the simplex method on the primal (if I changed it to max, then the coeff would be negative and that messes it up), so therefore I will find the dual and use the simplex method on that.

The dual in standard form will be

$$\begin{array}{ll} \text{maximize} & -\frac{9}{2}u+\frac{15}{2}v+\frac{45}{2}w \\ \text{subject to} & -\frac{1}{2}u+\frac{1}{2}v-\frac{1}{2}w \le -1 \\ & 2u+2v-3w \le -10 \\ & 0 \le u,v,w \end{array}$$

The LP as a tableau is

$$\left[\begin{array}{ccccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & -1 \\ 2 & 2 & -3 & 0 & 1 & -10 \\ -\frac{9}{2} & \frac{15}{2} & \frac{45}{2} & 0 & 0 & 0 \end{array}\right]$$

The first pivot element will be 2nd col and 2nd row.

$$\left[\begin{array}{ccccc|c} -1 & 0 & 1/4 & 1 & -1/4 & 3/2 \\ 1 & 1 & -3/2 & 0 & 1/2 & -5 \\ -12 & 0 & 45/4 & 0 & -15/4 & 75/2 \end{array}\right]$$

The next pivot element will be 3rd col and 1st row.

$$\left[\begin{array}{ccccc|c} -4 & 0 & 1 & 4 & -1 & 6 \\ -5 & 1 & 0 & 6 & -1 & 4 \\ 33 & 0 & 0 & -45 & 15/2 & -30 \end{array}\right]$$

The simplex method has now ended. Looking at the last row, the two right most elements before the augment line gives us our answer for the primal.

Exercise 4. Solve the following LP, and provide the Gaussian-elimination matrix G which, via left-multiplication, performs the full simplex algorithm on a tableau. Then, say what the record matrix R is.

$$\begin{array}{ll} \text{maximize} & x + y + z \\ \text{subject to} & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ & 0 \leq z \leq 1 \end{array}$$

The LP in standard form is

$$\begin{array}{ll} \text{maximize} & x + y + z \\ \text{subject to} & x \leq 1 \\ & y \leq 1 \\ & z \leq 1 \\ & 0 \leq x, y, z \end{array}$$

The LP as a tableau is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

By inspection, the optimal tableau (called T_k) will look like this

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -3 \end{array}\right]$$

And T_k has this form

$$\left[\begin{array}{cc|c} RA & R & R\vec{b} \\ \vec{c}^T - \hat{y}^T A & -\hat{y}^T & -\hat{y}^T b \end{array}\right]$$

and by inspecting T_k, we see that R is the same as the 3x3 identity matrix.
G has the form

$$\begin{bmatrix} R & \vec{0} \\ -\hat{y}^T & 1 \end{bmatrix}$$

and equals

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$