Homework 5 (Due Oct 10, 2022)

Professor Cooper
MATH 570 - Discrete Optimization - Fall 2022

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Justify all of your answers completely.

Exercise 1. Use complementary slackness to verify that the vector $\mathbf{x} = [9/7, 0, 1/7]^\mathsf{T}$ is indeed the optimal solution to

$$\begin{array}{ll} \text{maximize} & x_1 - 2x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 - 2x_3 \leq 1 \\ & 2x_1 - x_2 - 3x_3 \leq 4 \\ & x_1 + x_2 + 5x_3 \leq 2 \\ & 0 \leq x_{1,2,3} \end{array}$$

The dual will be

$$\begin{array}{ll} \text{minimize} & y_1 + 4y_2 + 2y_3 \\ \text{subject to} & y_1 + 2y_2 + y_3 \geq 1 \\ & y_1 - y_2 + y_3 \geq -2 \\ & -2y_1 - 3y_2 + 5y_3 \geq 3 \\ & 0 \leq y_{1,2,3} \end{array}$$

If x is the solution, then the positive components of implies that the constrictions in the dual that match are equalities. We also see that the second constraint in the primal is not tight, so $y_2 = 0$. We are left with this system of equations:

$$\begin{array}{l} y_1 + y_3 = 1 \\ -2y_1 + 5y_3 = 3 \end{array}$$

which gives us $y_1 = \frac{2}{7}$ and $y_3 = \frac{5}{7}$. We see that $y = [\frac{2}{7}, 0, \frac{5}{7}]^T$ is $\mathbb{D}$-feasible. Therefore x does solve $\mathbb{P}$. We also observe that plugging in the vectors x and y into their respecting objective functions results the same value($\frac{12}{7}$), so now we are 100% confident.

Exercise 2. Directly dualize (but do not solve) the following system.

$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 - x_3 + x_4 + 2x_5 \\ \text{subject to} & x_1 \leq 10 \\ & x_1 + x_2 \leq 11 \\ & x_1 + x_2 + x_3 \leq 12 \\ & x_1 + x_2 + x_3 + x_4 + x_5 = 20 \\ & 0 \leq x_1, x_2, x_4, x_5 \end{array}$$

Rewriting the Primal to make it easier to see the dual:

$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 - x_3 + x_4 + 2x_5 \\ \text{subject to} & x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 \leq 10 \\ & x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \leq 11 \\ & x_1 + x_2 + x_3 + 0x_5 + 0x_5 \leq 12 \\ & x_1 + x_2 + x_3 + x_4 + x_5 = 20 \\ & 0 \leq x_1, x_2, x_4, x_5 \end{array}$$

Inequalities in the primal correspond to implicit constraints in the dual, and the opposite for equalities.

$$\begin{array}{ll} \text{minimize} & 10y_1 + 11y_2 + 12y_3 + 20y_4 \\ \text{subject to} & y_1 + y_2 + y_3 + y_4 \geq 1 \\ & y_2 + y_3 + y_4 \geq 3 \\ & y_3 + y_4 = -1 \\ & y_4 \geq 1 \\ & y_4 \geq 2 \\ & 0 \leq y_1, y_2, y_3, \end{array}$$

Simplification can be done, but this form shows where each part comes from, so it will be left like this.

Exercise 3. Prove that the intersection of any family of convex sets is convex.

Proof. Let C be the intersection of any family of convex sets. Assume ℓ_1 and ℓ_2 are two points that are in C. Then for each set, S, in the C, ℓ_1 and ℓ_2 are in S. Since S is convex, the line segment between the two, we'll call ℓ, is contained entirely in S. Since ℓ is entirely contained in each S, then ℓ is entirely contained in the intersection of all S's. So ℓ is entirely contained in C. No assumptions were made on ℓ_1 or ℓ_2 besides just being in C, so every possible line segment made from 2 points in C is entirely contained in C. Therefore, C is convex. ∎

Exercise 4. Prove or disprove that (a) $\mathbf{x} \in \mathbb{R}^n$ is a vertex of a convex polytope C iff for every vector $\mathbf{y} \in \mathbb{R}^n$, if there is an $\epsilon > 0$ so that $\mathbf{x} + \epsilon\mathbf{y} \notin C$, then there exists a $\delta > 0$ so that $\mathbf{x} - \delta\mathbf{y} \in C$ and (b) $\mathbf{x} \in \mathbb{R}^n$ is a vertex of a convex polytope C iff for every vector $\mathbf{y} \in \mathbb{R}^n$, if there is an $\epsilon > 0$ so that $\mathbf{x}+\epsilon\mathbf{y} \in C$, then there exists a $\delta > 0$ so that $\mathbf{x}-\delta\mathbf{y} \notin C$.

(a) Disproving with counter example. Let C be the unit square centered at $(1, 1)$. Looking at the point$(1, 1)$, we see that it is clearly not a vertex. But for every vector y in 2D space, we just need to scale it such that the length is longer than $\frac{\sqrt{2}}{2}$ to get outside of the square. We can also scale y so that its length is less than $\frac{1}{2}$ to make sure we stay inside of the square. Therefore, we have shown that for this point $(1, 1)$, we can find a positive ϵ for every vector y in 2D space to not be in C, and finding a δ such that when subtracting y scaled by δ we stay in C.

(b) Disproving by triviality. In the *if* statement "if there is an $\epsilon > 0$ so that $\mathbf{x} + \epsilon\mathbf{y} \in C$, then there exists a $\delta > 0$ so that $\mathbf{x} - \delta\mathbf{y} \notin C$," the "there exists a $\delta > 0$ so that $\mathbf{x} - \delta\mathbf{y} \notin C$"

is always true. This is because you just have to make δ longer than the longest line segment possible in the polytope, which will always move the x vector outside of the polytope. This means the *if* statement is always true. Thus no matter when a point is a vertex or not, the *if* statement will be true. So the *iff* statement is not true.

Exercise 5. Let C be the feasible region for the LP below. Give the sequence of vertices of C that results from applying the simplex algorithm (with Bland's Rule) to the LP. For each such vertex/tableaux, what are the corresponding $\mathcal{I}$ and $\mathcal{J}$ from Corollary 1.1 of the "LP Geometry" chapter?

$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 + 2x_3 \\ \text{subject to} & 2x_1 + x_2 \leq 10 \\ & x_1 + 2x_2 - 2x_3 \leq 20 \\ & x_2 + 2x_3 \leq 5 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

The tableau looks like

$$\left[\begin{array}{cccccc|c} 2 & 1 & 0 & 1 & 0 & 0 & 10 \\ 1 & 2 & -2 & 0 & 1 & 0 & 20 \\ 0 & 1 & 2 & 0 & 0 & 1 & 5 \\ 2 & -1 & 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

We start at the (0,0,0) vertex, so $\mathcal{I} = \{\}$ and $\mathcal{J} = \{1, 2, 3\}$.

$$\left[\begin{array}{cccccc|c} 1 & 1/2 & 0 & 1/2 & 0 & 0 & 5 \\ 0 & 3/2 & -2 & -1/2 & 1 & 0 & 15 \\ 0 & 1 & 2 & 0 & 0 & 1 & 5 \\ 0 & -2 & 2 & -1 & 0 & 0 & -10 \end{array}\right]$$

We are now at the vertex (5,0,0), so $\mathcal{I} = \{1\}$ and $\mathcal{J} = \{2, 3\}$.

$$\left[\begin{array}{cccccc|c} 1 & 1/2 & 0 & 1/2 & 0 & 0 & 5 \\ 0 & 5/2 & 0 & -1/2 & 1 & 1 & 20 \\ 0 & 1/2 & 1 & 0 & 0 & 1/2 & 5/2 \\ 0 & -3 & 0 & -1 & 0 & -1 & -15 \end{array}\right]$$

We are now at the vertex $(5, 0, \frac{5}{2})$, so $\mathcal{I} = \{1, 3\}$ and $\mathcal{J} = \{3\}$. This vertex is the optimum for the LP.