Homework 13 (Due Nov 10, 2023)

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Justify all of your answers completely.

1. (a) Show that the function

$$f(x) = \frac{x}{1+x^2}$$

is Lipschitz on the closed interval $[-b, b]$ for any $b > 0$.

Proof. Let $x, y \in [-b, b]$.
Then

$$|f(x) - f(y)| = \left|\frac{x}{1+x^2} - \frac{y}{1+y^2}\right| = \left|\frac{x + xy^2 - y - yx^2}{(1+x^2)(1+y^2)}\right|$$

$$= |x - y| \left|\frac{1 - xy}{x^2y^2 + x^2 + y^2 + 1}\right| \le |x - y||1 - xy| \le |x - y||1 + b^2|$$

Letting $1+b^2$ be the Lipschitz constant shows that $f(x)$ is Lipschitz on $[-b, b]$. ∎

(b) Use this to give a detailed N, ϵ proof that if $\langle x_n \rangle_{n=1}^{\infty}$ us a sequence of real numbers with $\lim_{n\to\infty} x_n = L$ that

$$\lim_{n\to\infty} \frac{x_n}{x_n^2 + 1} = \frac{L}{L^2 + 1}$$

Proof. Assume $\lim_{n\to\infty} x_n = L$.
Then $\exists N$ s.t. $n > N \implies |x_n - L| < \frac{\epsilon}{b^2+1}$.
Let $f(x)$ be on the interval $[-L, L]$, $\epsilon > 0$, and $n > N$.
Then

$$|f(x) - f(L)| = \left|\frac{x_n}{x_n^2+1} - \frac{L}{L^2+1}\right| \le (b^2+1)|x_n - L| < (b^2+1)\frac{\epsilon}{b^2+1}$$

So then

$$\left|\frac{x_n}{x_n^2+1} - \frac{L}{b^2+1}\right| < \epsilon$$

which proves the limit. ∎

2. Let (A, d_A) and (B, d_B) be metric spaces. Let $E := A \times B$ and d on E by

$$d((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + d_B(b_1, b_2)$$

(a) Prove this is a metric on $E = A \times B$

Proof. Clearly d is symmetric.
Since d_A and d_B are nonnegative, $d = d_A + d_B \ge 0$.
$d((a_1, b_1), (a_2, b_2)) = 0 \iff d_A(a_1, a_2) + d_B(b_1, b_2) = 0 \iff d_A(a_1, a_2) = 0$ and $d_B(b_1, b_2) = 0 \iff a_1 = a_2$ and $b_1 = b_2$.
Now to show triangle inequality:

$$d((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + d_B(b_1, b_2)$$

$$\le d_A(a_1, a') + d_A(a', a_2) + d_B(b_1, b') + d_B(b', b_2)$$
$$= d_A(a_1, a') + d_B(b_1, b') + d_A(a', a_2) + d_B(b', b_2)$$
$$= d((a_1, b_1), (a', b')) + d((a', b'), (a_2, b_2))$$

So all 4 requirements hold for d to be a metric on E. ∎

(b) Prove that if A and B are complete, then so is E.

Proof. Assume both A and B are complete. Then every Cauchy sequence in A and B will converge.
It has been shown previously that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty}(x_n, y_n) = (x, y)$.
Let $\langle (a_n, b_n) \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E.

So then $\forall \epsilon > 0$, $\exists N$ s.t. $m, \ell > N \implies d((a_m, b_m), (a_\ell, b_\ell)) < \epsilon$.
Since $d = d_A + d_B$, then $d_A(a_m, a_\ell), d_B(b_m, b_\ell) \leq d((a_m, b_m), (a_\ell, b_\ell))$.
So $d_A(a_m, a_\ell), d_B(b_m, b_\ell) < \epsilon$.
So then $\langle a_n \rangle_{n=1}^\infty$ and $\langle b_n \rangle_{n=1}^\infty$ are Cauchy sequences in A and B respectively.
So then $\langle a_n \rangle_{n=1}^\infty$ and $\langle b_n \rangle_{n=1}^\infty$ converge since A and B are complete.
So then $\langle (a_n, b_n) \rangle_{n=1}^\infty$ also converges, meaning E is also complete. ■

(c) Prove that if A and B are both sequentially compact, then so is E.

Proof. Assume both A and B are seq. compact. Then every sequence in A and B will have a convergent subsequence.
Let $\langle (a_n, b_n) \rangle_{n=1}^\infty$ be a sequence in E.
So then $\langle a_n \rangle_{n=1}^\infty$ has a convergent subsequence $\langle a_{n_j} \rangle_{j=1}^\infty$.
We can't say that $\langle b_{n_j} \rangle_{j=1}^\infty$ converges, but since B is seq. compact, can say that there exists $\langle b_{n_{j_k}} \rangle_{k=1}^\infty$ that does converge.
$\langle a_{n_{j_k}} \rangle_{k=1}^\infty$ is a subsequence of $\langle a_{n_j} \rangle_{j=1}^\infty$, which already converged. So $\langle a_{n_{j_k}} \rangle_{k=1}^\infty$ converges too.
So then $\langle (a_{n_{j_k}}, b_{n_{j_k}}) \rangle_{k=1}^\infty$ is a convergent subsequence of $\langle (a_n, b_n) \rangle_{n=1}^\infty$, proving E is complete. ■

3. Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be Lipschitz. Prove if $V \subseteq E'$ is an open set, then

$$U := f^{-1}[V] = \{p \in E : f(p) \in V\}$$

is also open.

Proof. Let $p \in U$. Then $f(p) \in V$.
Since V is open, that means $\exists r > 0$ s.t. $B(f(p), r) \subseteq V$ by definition of open.
Let M be the Lipschitz constant for f. Let $\delta = r/M$.
If $q \in B(p, \delta)$, then $d(p, q) < r/M \implies Md(p, q) < r$.
Since $d'(f(p), f(q)) \leq Md(p, q)$, we know $d'(f(p), f(q)) < r$ which means $f(q) \in B(f(p), r) \subseteq V$.
So then $q \in B(p, \delta) \implies f(q) \in V \implies q \in U$.
So then $B(p, \delta) \subseteq U$, which means U is open by definition of open. ■

4. Let $\mathbb{R}^2$ have its usual metric.

   (a) Let $\vec{a}, \vec{b} \in \mathbb{R}^2$. Prove the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (1 - t)\vec{a} + t\vec{b}$ is Lipschitz.

   Proof. Let $x, y \in \mathbb{R}$. Then

$$||f(x) - f(y)|| = ||(1-x)\vec{a} + x\vec{b} - (1-y)\vec{a} - y\vec{b}||$$

$$= ||(y-x)\vec{a} + (x-y)\vec{b}|| = |x-y| \cdot ||\vec{b} - \vec{a}||$$

   Since $||\vec{b} - \vec{a}|| \geq 0$, f is Lipschitz. ■

   (b) For $\vec{a}, \vec{b} \in \mathbb{R}^2$ define the segment with endpoints $\vec{a}$ and $\vec{b}$ as

$$[\vec{a}, \vec{b}] = \{(1-t)\vec{a} + t\vec{b} : 0 \leq t \leq 1\}$$

   Prove $[\vec{a}, \vec{b}]$ is connected.

   Proof. BWOC, let $[\vec{a}, \vec{b}] = A \uplus B$ be a disconnection.
   Let $A_0 = \{t \in [0,1] : (1-t)\vec{a} + t\vec{b} \in A\}$ and similarly for B_0.
   We can rewrite A_0 as $\{t \in [0,1] : f(t) \in A\}$ and similarly for B_0.
   We know that f is Lipschitz from (a) and since both A and B are open (know from them being a disconnection), Problem 2 tells us that the preimage of A and B through f is also open. So A_0 and B_0 are both open.
   Since A and B are disjoint, so will A_0 and B_0 since taking the preimage preserves that property.
   Obviously $A_0 \neq \varnothing \neq B_0$ and $A_0 \cup B_0 = [0,1]$.
   So then $A_0 \uplus B_0$ is a disconnection of $[0,1]$, which is a contradiction since closed intervals of the reals are connected.
   So then the disconnection $A \uplus B$ cannot exist, making $\vec{a}, \vec{b}]$ connected. ■

5. Let E be a metric space.

   (a) Show that if E is compact, then any any collection, $\mathcal{F}$, of closed subsets of E with the finite intersection property has nonempty intersection. That is if $\mathcal{F}$ has the finite intersection property, then

$$\bigcap \mathcal{F} \neq \varnothing$$

Proof. BWOC, assume $\mathcal{F}$ has the finite intersection property and $\bigcap \mathcal{F} = \varnothing$.
Let $\mathcal{U} = \{F^c : F \in \mathcal{F}\}$.
Since F is closed $\forall F \in \mathcal{F}$, then F^c is open.
Since $\bigcap \mathcal{F} = \varnothing$, then there does not exist any element that is in all $F \in \mathcal{F}$. So then $\forall x \in E,\ \exists F \in \mathcal{F}$ s.t. $x \in F^c$.
So then $\mathcal{U}$ is an open cover of E.
Since E is compact, then $\exists\ \{F_1^c, \dots, F_k^c\}$ that is a finite subcover of E.
Since $E = \bigcup_{i=1}^k F_i^c$, then $E^c = (\bigcup_{i=1}^k F_i^c)^c \implies \varnothing = \bigcap_{i=1}^k F_i$.
But that contradicts that $\mathcal{F}$ has the finite intersection property, so $\bigcap \mathcal{F} \neq \varnothing$. ■

(b) Conversely, show if every collection of closed sets, $\mathcal{F}$, of E with the finite intersection property has nonempty intersection, then E is compact.

Proof. Let $\mathcal{U}$ be an open cover of E. BWOC, assume $\mathcal{U}$ does not have a finite subcover.
Let $\mathcal{F} := \{U^c : U \in \mathcal{U}\}$.
Since U are open sets, then U^c are closed, meaning $\mathcal{F}$ is a collection of closed sets.
Since no finite subset of $\mathcal{U}$ can cover E, that means for every finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ there is an element $x \in E$ s.t. $\forall U \in \mathcal{U}_0,\ x \notin U$. So then $\forall U \in \mathcal{U}_0,\ x \in U^c$.
So then for any $\mathcal{F}_0 = \mathcal{U}_0$, there exists an element $x \in E$ s.t. $x \in \bigcap \mathcal{F}_0$.
This means $\mathcal{F}$ has the finite intersection property.
Then by our assumption in the problem, $\bigcap \mathcal{F} \neq \varnothing$. So there is an element in E that is in every element of $\mathcal{F}$, meaning there is an element of E that is not in every element of U, which contradicts that U is an open cover of E.
So then U must have a finite sub cover, meaning E is compact. ■