Homework 9 (Due March 28, 2023)

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MATH 575 - Discrete Mathematics II - Spring 2023

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Justify all of your answers completely.

Collaborators: Chance and Nathan. Anyone else that claimed I helped is just plain wrong.

1. Recall that Q_d is the d-dimensional hypercube. For every $d \geq 1$, give an explicit edge-coloring to prove $\chi'(Q_d) = \Delta(Q_d)$.

   Proof. Every vertex in a d-dimensional hypercube will have d neighbors, since there are d bits in the corresponding bitstring.
   Let us create a coloring by looking at a vertex. Since there will be an edge to another vertex iff a single bit in the bitstrings differ, the edge created is not adjacent to any other edge that exists from same bit differing. So let us color the edge color i, where i is the index of the differing bit. This gives us d colors for a vertex, and each color will never be adjacent to the same color. ∎

2. Recall that $\alpha'(G)$ denotes the size of a maximum matching in G.

   (a) Prove that $\chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}$.

   Proof. Considering a proper edge coloring, the color classes partition the graph into disjoint matchings. The minimum number of color classes would be in the case where every color class had the size of the biggest matching, since the classes are disjoint matchings themselves. So that case is $\frac{|E(G)|}{\alpha'(G)}$, which is the minimum case. Consequently, $\chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}$. ∎

   (b) Prove that if G is a k-regular graph with no perfect matching, then $\chi'(G) = \Delta(G) + 1$.

   Proof. BWOC, assume $\chi'(G) \neq \Delta(G) + 1$ and G is k-regular with no perfect matching. By Vizing's theorem, we can then assume $\chi'(G) = \Delta(G) = k$.
   We also know that $\chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}$. We'll then get $k \geq \frac{kn}{2\alpha'(G)} \implies \alpha'(G) \geq \frac{n}{2} \implies$ there exists a perfect matching, BOOM, A CONTRADICTION!!! ∎

3. Chromatic number versus edge-chromatic number

   (a) For every $n \geq 4$, construct a graph G on n-vertices with $\chi(G) < \chi'(G)$.

   $K_{1,n-1}$. It is bipartite, so $\chi(G) = 2$, but also $\chi'(G)$ will be the degree of the lone vertex in the smaller bipartition, which will be at least 3 since $n \geq 4$.

   (b) Give a characterization of all connected graphs for which $\chi(G) > \chi'(G)$. That is, prove a statement of the form:

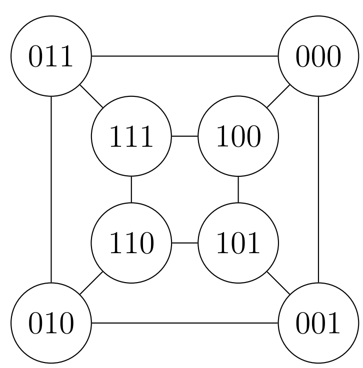
   "If G is a connected graph, then $\chi(G) > \chi'(G)$ if and only if __________."

   The claim I will make is $\chi(G) > \chi'(G)$ if and only if G is K_{2m}.

   Proof. ($\Longrightarrow$)
   Assume $\chi(G) > \chi'(G)$. Then by Vizing's theorem, we get $\chi(G) > \Delta(G)$. Then by Brooks' theorem, will get that G is an odd cycle or a complete graph. We know it isn't an odd cycle since $\chi'(C_{2n+1}) = 3 = \chi(C_{2n+1})$. So it must be a complete graph.
   We also know from class that $\chi'(K_{2m}) = \Delta(K_{2m})$ and $\chi'(K_{2m+1}) = \Delta(K_{2m+1}) + 1$. Then because of that and $\Delta(G) + 1 \geq \chi(G) > \chi'(G)$, G must be K_{2m}.
   ($\Longleftarrow$)
   Assume G is K_{2m}. Then $\chi(G) = 2m = \Delta(G) + 1$ since every node is adjacent to every other node. We also know from class that $\chi'(G) = \Delta(G) = 2m - 1$. So $\chi(G) > \chi'(G)$. ∎

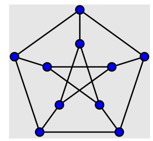
4. Determine for which values d the hypercube Q_d is planar and for which values it is non-planar.

   Q_1 and Q_2 are trivially planar. Below is Q_3.



   Q_4 is not planar since it has $\frac{dn}{2} = 32$ edges, and we showed in class any planar graph will satisfy the inequality $e \leq 2n - 4$, which Q_4 does not $(32 \geq 28)$. By extension, any $Q_{\geq 4}$ is not planar since it would contain a copy of Q_4.

5. Use Euler's Formula to prove that the Petersen graph (drawn below) is not planar. *Do not use Wagner's Theorem or Kuratowski's Theorem.*



Proof. BWOC, assume Euler's Formula holds true.
The graph has 10 vertices and 15 edges, so the number of faces must be 7. We also have that there are no cycles of length 3 or 4. So then with degree sum formula, we get that

$$2e = \sum_{f' \text{ in faces}} \ell(f') \geq 5f \implies f \leq 6$$

which is 

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