Homework 15 (Due Nov 27, 2023)

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Justify all of your answers completely.

1. Let E and E' be metric spaces and $f : E \to E'$ a function. Prove the following are equivalent.

   (a) f is continuous.

   (b) for every $p_0 \in E$, the limit $\lim_{p \to p_0]} f(p) = f(p_0)$ holds.

   (c) if $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence in E with $\lim_{n\to\infty} p_n = p_0$, then $\lim_{n\to\infty} f(p_n) = f(p_0)$.

   (d) If V is an open subset of E', then the preimage $f^{-1}[V]$ is an open subset of E.

   Proof. It has been proven before in class that $(a) \iff (b) \iff (c)$, so showing $(a) \iff (d)$ is sufficient.
   $(a) \implies (d)$
   Assume f is continuous. Let $V \subseteq E'$ be an open set. Let $p_0 \in E$ with $f(p_0) \in V$.
   Since V is open, there is $\epsilon > 0$ s.t. $B_{E'}(f(p_0), \epsilon) \subseteq V$.
   Since f is continuous, $\exists \delta > 0$ s.t. $d(p_0, p) < \delta \implies d'(f(p_0), f(p)) < \epsilon$.
   So $p \in B_E(p_0, \delta) \implies f(p) \in B_{E'}(f(p_0), \epsilon) \subseteq V$.
   So $B_E(p_0, \delta) \subseteq f^{-1}[V]$. This means there is an open ball around every point in $f^{-1}[V]$, making it open.
   $(d) \implies (a)$
   Assume (d) and let $\epsilon > 0$. Let $p_0 \in E$ and let $V = B_{E'}(f(p_0), \epsilon)$.

By (d), $f^{-1}[V]$ contains an open ball around p_0, we'll denote $B_E(p_0, \delta) \subseteq f^{-1}[V]$.
So then

$$d(p, p_0) < \delta \implies p \in B_E(p_0, \delta) \implies p \in f^{-1}[V] \implies f(p) \in B_{E'}(f(p_0), \epsilon)$$
$$\implies d(f(p), f(p_0)) < \epsilon$$

So f is continuous. ■

2. Let $f : E \to E'$ be a continuous function between metric spaces and $\mathcal{U}$ an open cover of E'. Prove $\{f^{-1}[V] : V \in \mathcal{U}\}$ is an open cover of E.

   Proof. Assume f is continuous. Let $\mathcal{U}$ be an open cover of E' and towards contradiction, assume $S = \{f^{-1}[V] : V \in \mathcal{U}\}$ is not an open cover of E.
   By problem (1), we know that S a collection of open sets. So then since it isn't an open cover of E, $\exists p \in E$ s.t. $\forall s \in S$, $p \notin s$.
   So then $p \notin f^{-1}[V]$ for all $V \in \mathcal{U}$. This means $f(p) \notin V$ for all $V \in \mathcal{U}$.

   But $\mathcal{U}$ is an open cover of E' and $f(p) \in E'$. BOOM, A CONTRADICTION!!!

   ■

3. What is wrong with the following proof for $\lim_{x \to 1} \frac{1}{x-1} = 0$.

   Proof. Let $\epsilon > 0$ and set $\delta = |x-1|^2 \epsilon$. If $0 < |x-1| < \delta$, then

$$\begin{aligned} |\frac{1}{x-1} - 0| &= \frac{1}{|x-1|} \\ &= \frac{1}{|x-1|^2}|x-1| \\ &< \frac{1}{|x-1|^2}\delta \\ &= \frac{1}{|x-1|^2}|x-1|^2\epsilon \\ &= \epsilon \end{aligned}$$

   So if $f(x) = 1/(x-1)$ the inequality $0 < |x-1| < \delta$ implies $|f(x) - 0| < \epsilon$ which verifies that the definition of $\lim_{x \to 1} f(x) = 0$ holds. ■

The issue is that we have the definition of δ depending on x. That is a problem since in the definition of limits at points, we define δ before any x.

4. Show that

$$\lim_{x \to 1} \frac{1}{x-1}$$

does not exist.

Proof. BWOC, assume the limit exists and is equal to L.
Since the limit exists, it should hold that $\exists \delta > 0$ s.t.

$$|x-1| < \delta \implies \left| \frac{1}{x-1} - L \right| < 1$$

Let $k = \max(1+L, 1+1/\delta)$. Consider $x = 1 + 1/k$.
We have $|x-1| = |1+1/k-1| = 1/k < \delta$.
We also have $|\frac{1}{x-1} - L| = |\frac{1}{1+1/k-1} - L| = |k-L| \geq k - L \geq 1 + L - L = 1 = \epsilon$.
So $|x-1| < \delta$ and $|f(x) - L| \geq \epsilon$, but we assumed the limit exists, BOOM, A CONTRADCTION!!!. ∎