Homework 7 (Due Oct 10, 2022)

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MATH 574 - Discrete Mathamatics - Fall 2022

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Justify all of your answers completely.

1. Recall that the Fibonacci numbers satisfy $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

   (a) Suppose that a sequence $\{b_n\}$ satisfies $b_n = b_{n-1} + b_{n-2}$ with initial conditions $b_0 = 1$ and $b_1 = 2$. Use induction to prove that for all $n \geq 0$, $b_n = f_{n+2}$.

   Base Cases:

   n=0
   $b_0 = 1 = f_2$
   n=1
   $b_1 = 2 = f_3$
   n=2
   $b_2 = 3 = f_4$

   Induction Step: Assume for some $n \in \mathbb{N}$ s.t. $n \geq 2$ we have $b_k = f_{k+2}$ for all $0 \leq k \leq n$.
   Looking at the $n + 1$ case:

   $$b_{n+1} = b_n + b_{n-1} = f_{n+2} + f_{n+1} = f_{n+3}$$

   ■

   (b) Suppose that a sequence $\{c_n\}$ satisfies $c_n = c_{n-1} + c_{n-2}$ with initial conditions $c_0 = 2$ and $c_1 = 1$. Use induction to prove that for all $n \geq 1$, $c_n = f_{n-1} + f_{n+1}$.

   Base Cases:

   n=1
   $c_1 = 1 = 0 + 1 = f_0 + f_2$
   n=2
   $c_2 = c_0 + c_1 = 2 + 1 = f_1 + f_3$

Induction Step: Assume for some $n \in \mathbb{N}$ s.t. $n \geq 2$ we have $c_k = f_{k-1} + f_{k+1}$ for all $1 \leq k \leq n$.
Looking at the $n+1$ case:

$$c_{n+1} = c_n + c_{n-1} = f_{n-1} + f_{n+1} + f_{n-2} + f_n = (f_{n-1} + f_{n-2}) + (f_{n+1} + f_n) = f_n + f_{n+2}$$

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2. Solve the recurrence relations together with the initial conditions given.

   (a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

   Finding the roots of the characteristic polynomial of the sequence:

   $$p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0$$

   $$\text{So } a_n = \alpha_1(3)^n + \alpha_2(2)^n$$

   Now to solve for the alpha's, we plug in the initial conditions.

   $$a_0 = \alpha_1(3)^0 + \alpha_2(2)^0 = \alpha_1 + \alpha_2 = 1$$

   $$a_1 = \alpha_1(3)^1 + \alpha_2(2)^1 = 3\alpha_1 + 2\alpha_2 = 0$$

   Solving this system of equations gives us $\alpha_1 = -2$ and $\alpha_2 = 3$.
   So $a_n = -2 \cdot 3^n + 3 \cdot 2^n$. ■

   (b) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

   Finding the roots of the characteristic polynomial of the sequence:

   $$p(\lambda) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0$$

   $$\text{So } a_n = \alpha_1(2)^n + \alpha_2(-2)^n$$

   Now to solve for the alpha's, we plug in the initial conditions.

   $$a_0 = \alpha_1(2)^0 + \alpha_2(-2)^0 = \alpha_1 + \alpha_2 = 0$$

   $$a_1 = \alpha_1(2)^1 + \alpha_2(-2)^1 = 2\alpha_1 - 2\alpha_2 = 4$$

   Solving this system of equations gives us $\alpha_1 = 1$ and $\alpha_2 = -1$.
   So $a_n = 2^n - (-2)^n$. ■

   (c) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6, a_1 = 8$

   Finding the roots of the characteristic polynomial of the sequence:

   $$p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

   $$\text{So } a_n = \alpha_1(2)^n + \alpha_2 n(2)^n$$

Now to solve for the alpha's, we plug in the initial conditions.

$$a_0 = \alpha_1(2)^0 + \alpha_2(0)(2)^0 = \alpha_1 = 6$$

$$a_1 = \alpha_1(2)^1 + \alpha_2(1)(2)^1 = 2\alpha_1 + 2\alpha_2 = 8$$

Solving this system of equations gives us $\alpha_1 = 6$ and $\alpha_2 = -2$.
So $a_n = 6(2)^n - n(2)^{n+1}$. ■

3. Let b_n be the number of bit strings of length n without 2 consecutive 0s. In class, we saw that $\{b_n\}$ satisfies the relation $b_n = b_{n-1} + b_{n-2}$ for $n \geq 2$. Find a solution of this recurrence relation using the initial conditions $b_0 = 1, b_1 = 2$.

$$p(\lambda) = \lambda^2 - \lambda - 1 = 0 \implies \lambda_{1,2} = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$b_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$$

$$b_0 = \alpha_1(\frac{1+\sqrt{5}}{2})^0 + \alpha_2(\frac{1-\sqrt{5}}{2})^0 = \alpha_1 + \alpha_2 = 1$$

$$b_1 = \alpha_1(\frac{1+\sqrt{5}}{2})^1 + \alpha_2(\frac{1-\sqrt{5}}{2})^1 = 2$$

Solving this system of equations gives us $\alpha_1 = \frac{5+3\sqrt{5}}{10}$ and $\alpha_2 = \frac{5-3\sqrt{5}}{10}$.
So

$$b_n = (\frac{5+3\sqrt{5}}{10})(\frac{1+\sqrt{5}}{2})^n + (\frac{5-3\sqrt{5}}{10})(\frac{1-\sqrt{5}}{2})^n$$

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4. Find the solution to the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with initial conditions $a_0 = 3, a_1 = 6, a_2 = 0$.

$$p(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-1)(\lambda^2-\lambda+2) = (\lambda-1)(\lambda+1)(\lambda-2) = 0 \implies \lambda_{1,2,3} = -1, 1, 2$$

$$a_n = \alpha_1(-1)^n + \alpha_2(1)^n + \alpha_3(2)^n$$

$$a_0 = \alpha_1(-1)^0 + \alpha_2(1)^0 + \alpha_3(2)^0 = \alpha_1 + \alpha_2 + \alpha_3 = 3$$

$$a_1 = \alpha_1(-1)^1 + \alpha_2(1)^1 + \alpha_3(2)^1 = -\alpha_1 + \alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 = \alpha_1 + \alpha_2 + 4\alpha_3 = 0$$

Solving this system of equations gives us $\alpha_1 = -2$, $\alpha_2 = 6$, and $\alpha_3 = -1$.
So $a_n = 2(-1)^{n+1} - 2^n + 6$. ■

5. Find the solution to the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Use your solution to calculate the value of a_{20}.

$p(\lambda) = \lambda^2 - 2\lambda + 2 = 0$. Using the quadratic formula gives us $\lambda_{1,2} = 1 + i, 1 - i$.

$$a_n = \alpha_1(1+i)^n + \alpha_2(1-i)^n$$

$$a_0 = \alpha_1(1+i)^0 + \alpha_2(1-i)^0 = \alpha_1 + \alpha_2 = 1$$
$$a_1 = \alpha_1(1+i)^1 + \alpha_2(1-i)^1 = \alpha_1 + \alpha_2 + i\alpha_1 - i\alpha_2 = 2$$

Solving this system of equations gives us $\alpha_1 = \frac{1}{2} - \frac{1}{2}i$ and $\alpha_2 = \frac{1}{2} + \frac{1}{2}i$.
So $a_n = (\frac{1}{2} - \frac{1}{2}i)(1+i)^n + (\frac{1}{2} + \frac{1}{2}i)(1-i)^n$.
$a_{20} = (\frac{1}{2} - \frac{1}{2}i)(1+i)^{20} + (\frac{1}{2} + \frac{1}{2}i)(1-i)^{20} = -1024$ ■

6. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

   (a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

   $L_n = (L_{n-1} + L_{n-2})/2 = L_{n-1}/2 + L_{n-2}/2$.

   (b) Find L_n if $4,000$ lobsters were caught in year 1 and $10,000$ were caught in year 2.

   $p(\lambda) = \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = (\lambda - 1)(\lambda + \frac{1}{2}) \implies \lambda_{1,2} = 1, -\frac{1}{2}$
   So $L_n = \alpha_1(1)^n + \alpha_2(-\frac{1}{2})^n$. Solving for $\alpha_{1,2}$:

   $$L_1 = \alpha_1(1)^1 + \alpha_2(-\frac{1}{2})^1 = \alpha_1 - \alpha_2/2 = 4000$$
   $$L_2 = \alpha_1(1)^2 + \alpha_2(-\frac{1}{2})^2 = \alpha_1 + \alpha_2/4 = 10000$$

   Solving this system of equations gives us $\alpha_1 = 8,000$ and $\alpha_2 = 8,000$.
   So $L_n = 8000(1)^n + 8000(-\frac{1}{2})^n = 8000(1 + (-\frac{1}{2})^n)$. ■

   (c) What is the long-term behavior of L_n? That is, what is $\lim_{n\to\infty} L_n$?

   By inspection of L_n, we see that as n approaches infinity, L_n approaches 8000.

7. Let a_n be the number of ways a $2 \times n$ rectangular chessboard can be tiled using 1×2 and 2×2 pieces.

   (a) Determine a_1 and a_2.

   $a_1 = 1$ and $a_2 = 3$.

(b) Find a recurrence relation for $\{a_n\}$.

Imagining the situation for a_n, we consider the possible ways to fill the last column with tiles. The first way is just with one 1×2 piece. This leaves a $2 \times n-1$ board left, so we have a term of a_{n-1}. The second way is to cover the last two columns with a 2×2 piece, leaving the rest of the $2 \times n-2$ board untouched. This gives us a a_{n-2} term. Then finally we could cover each squared with its own 2×1 piece, leaving again the board $2 \times n-2$ untouched, giving a a_{n-2} term. $\therefore a_n = a_{n-1} + 2a_{n-2}$.

(c) Find a solution of the recurrence relation in part (b) using the initial conditions in part (a).

$$p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \implies \lambda_{1,2} = 2, -1$$
$$a_n = \alpha_1(2)^n + \alpha_2(-1)^n$$
$$a_1 = \alpha_1(2)^1 + \alpha_2(-1)^1 = 2\alpha_1 - \alpha_2 = 1$$
$$a_2 = \alpha_1(2)^2 + \alpha_2(-1)^2 = 4\alpha_1 + \alpha_2 = 3$$

Solving this system of equations gives us $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$.

$$a_n = \frac{2}{3}(2)^n + \frac{1}{3}(-1)^n$$

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