Homework 9 (Due March 28, 2023)

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MATH 575 - Discrete Mathematics II - Spring 2023

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Justify all of your answers completely.

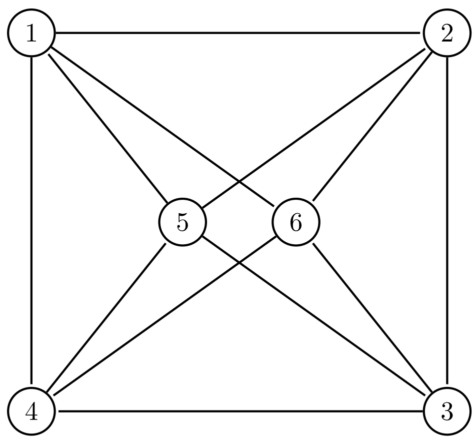
Collaborators: Chance, Sam, and Nathan

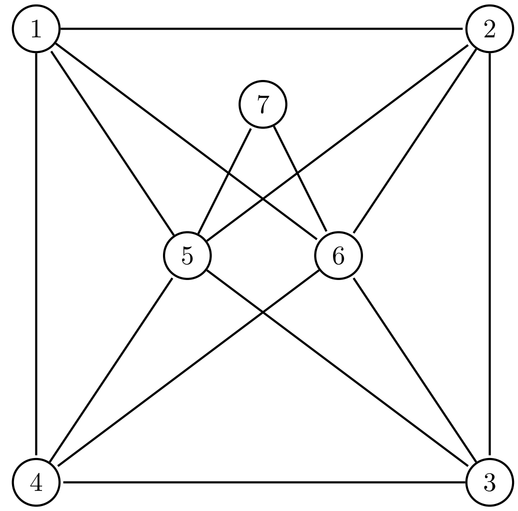
1. Use Kempe chains to prove that every planar graph with at most 11 vertices is 4-colorable.

   Proof. Assume that G is a 11-vertex planar graph, then there are at most $3(11) - 6 = 27$ edges. By handshaking lemma, we have 54 incidences to distribute between 11 vertices, guaranteeing a vertex with a degree of 4 or fewer. Now to consider a vertex with degree 4 or fewer. By the five color theorem, we know that every planar graph with at most 11 vertices is 5-colorable. Assume we have some 5 coloring. Let us consider every vertex of color 5, called v.
   Case 1: v has neighbors of only 3 or less other colors.
   Then we can color v a different color that is not color 5.
   Case 2: v has neighbors of all other 4 colors.
   For colors i, j, let $G_{i,j}$ be the subgraph induced by colors i and j.
   Case 2.1: v has 2 neighbors in different components in $G_{i,j}$ for some distinct $i, j \in [4]$
   WLOG, let those neighbors be color 1 and color 3. Then we can swap the vertices of color 1 and color 3 in the connected component containing the color 1 neighbor. Now we are at case 1.
   Case 2.2: v's neighbors are all in the same component of every $G_{i,j}$.
   So there must be Kempe chains between any two neighbors of v. This means the paths between 1 to 3 and 2 to 4 would cross, making it not planar.

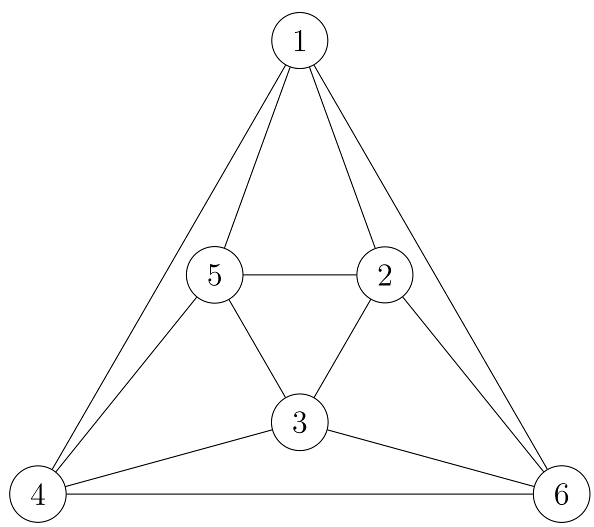
   ∎

2. Determine if the following graphs are planar or nonplanar. If it is planar, give a plane drawing. If it is nonplanar, demonstrate the existence of a K_5 or $K_{3,3}$ subdivision.

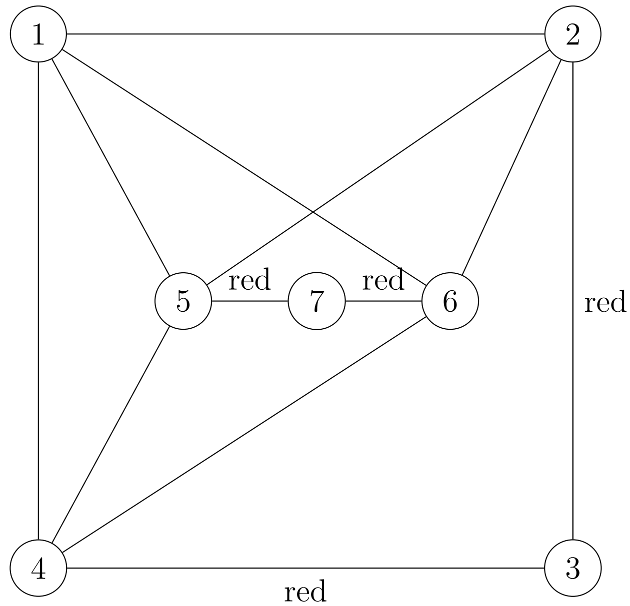




Here is the first one drawn planar:



Here is the second one drawn with the K_5 subdivision, with subdivided edges highlighted red:



3. Let X be a set of n points in $\mathbb{R}^2$ such that the (Euclidean) distance between any pair of distinct points in X is at least 1. Prove that there are at most $3n-6$ pairs of points with distance exactly 1.

   Proof. Let X be a set of n points in $\mathbb{R}^2$ such that the (Euclidean) distance between any pair of distinct points in X is at least 1, and let G be the graph of those points with the edges between points when there are distance 1 apart. BWOC, assume there are more than $3n-6$ edges. This means G is nonplanar, which means there is a K_5 or $K_{3,3}$ minor.
   There cannot be a K_5 minor since if we had a K_3 (3 points each dist 1 away), we can center a unit circle at each vertex and the three circles only intersect at 3 points. That means no other vertex could have an edge between all 3.
   There cannot be a $K_{3,3}$ for similar reasoning. If we have a vertex adjacent to 3 other vertices, then we can draw a unit circle centered at that vertex. Those 3 vertices are on the circle, and we know that a circle is uniquely defined by 3 points on it, so no other vertex can be adjacent to all three of those vertices. ■

4. Recall that a graph is outerplanar if it has a plane drawing with all of its vertices touching the outer face.

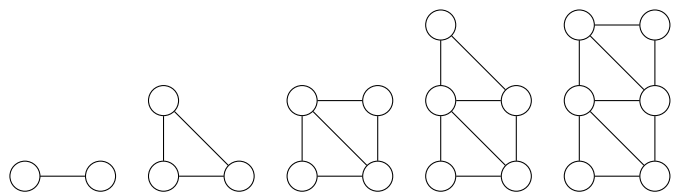
   (a) Let $n \geq 2$. Prove that every n-vertex outerplanar graph has at most $2n-3$ edges.

   Proof. Let us induct on n
   Base case: n=2
   This has at most 1 edge $(2n-3)$ and is outerplanar.✓
   Induction step:
   Let $n > 2$. Assume the proposition for any outplanar graph with $n' < n$ vertices.

Let G be a n-vertex outerplanar graph. Let us remove a vertex, v, with degree ≤ 2 (we know one exists from class). By induction hypothesis, $G - v$ has at most $2n - 5$ edges. If we add back v, we'd be at most adding back 2 edges, so we'd have at most $2n - 5 + 2 = 2n - 3$ edges. ■

(b) Show that part (a) is best possible for all $n \geq 2$ by iteratively constructing graphs $G_2, G_3, G_4, \ldots$ such that G_n is an n-vertex, outerplanar graph with $2n - 3$ edges.

An iterative definition is to take the drawing, and then add a vertex outside of the drawing, connecting it to two vertices that are adjacent by an edge on the outerface.



5. Let G be an n-vertex graph. Suppose for some $t \in \mathbb{N}$ that $d(u) + d(v) \geq n - t$ for every pair of distinct non-adjacent vertices $u, v \in V(G)$. Prove that the vertices of G can be partitioned into at most t pairwise-disjoint paths.

Proof. Let G be an n vertex graph and suppose $\exists t \in \mathbb{N}$ that $d(u) + d(v) \geq n - t$ for every pair of distinct non-adjacent vertices $u, v \in V(G)$. Let us add t vertices that are adjacent to every vertex. Then every vertex's degree goes up by t, which will satisfy Ore's condition. By Ore's theorem, we have that this graph is Hamiltonian. Removing the t vertices add will now make turn the cycle into at most t paths. ■