Homework 11 (Due April 21, 2023)

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MATH 575 - Discrete Mathematics II - Spring 2023

April 24, 2023

Justify all of your answers completely.

Collaborators: Chance

1. Prove that if H is a graph with chromatic number $\chi(H) \geq 3$, then[1]

$$\text{ex}(n, H) \geq \left(1 - \frac{1}{\chi(H) - 1}\right)\frac{n^2}{2} + o(n^2).$$

Proof. Let k be $\chi(H) - 1$. If a graph is k colorable, that means H is not a subgraph, since the chromatic number of a graph is $\geq$ the chromatic number of every subgraph. Let G be a complete k-partite graph on n vertices where the partitions are as balanced as possible. Then the number of edges will be $\binom{k}{2}\frac{n^2}{k^2}$ since every pair of partitions will have roughly the n/k vertices map to the other n/k vertices. Since n^2 might not be divisible by k^2, we'll just tack on an $o(n^2)$.
$\binom{k}{2}\frac{n^2}{k^2}$ simplifies down to $\frac{k(k-1)}{2}\frac{n^2}{k^2}$ which simplifies to $(1 - \frac{1}{k})\frac{n^2}{2}$, our desired result. ∎

2. Prove that $\text{ex}(n, P_n) = \binom{n-1}{2}$.

Proof. To show that $\text{ex}(n, P_n) \geq \binom{n-1}{2}$, we can make a K_{n-1} graph and add a vertex with no edges. This n-vertex graph has $\binom{n-1}{2}$ edges and no P_n as a subgraph.
Showing $\text{ex}(n, P_n) \leq \binom{n-1}{2}$. BWOC, let G be a graph on n vertices with $|E(G)| \geq \binom{n-1}{2} + 1$. Let us then add an edge (ignoring the case when we can't since that would make the graph complete and trivial), we'll call e. So we now have $|E(G)| > \binom{n-1}{2} + 1$. Since we know $\text{ex}(n, C_n) = \binom{n-1}{2} + 1$, G must have a Hamiltonian cycle we'll call C. Let us now remove e from G. We now have $\geq \binom{n-1}{2} + 1$ edges.
Case 1: $e \in C$. Then we have turned C into a path that hits all the vertices, which is a P_n path.
Case 2: $e \notin C$. Then we still have a Hamiltonian path, which means we also have a

[1]We say a function $f(n)$ is $o(n^2)$ if $\lim_{n\to\infty} \frac{f(n)}{n^2} = 0$. I.e., we view this as a small (positive or negative) error that becomes negligible proportionally as n grows. Do not worry about calculating this error precisely.

P_n as a subgraph.
So for any graph with $\geq \binom{n-1}{2} + 1$ edges, we will have a P_n subgraph $\implies \text{ex}(n, P_n) \leq \binom{n-1}{2}$. ■

3. Prove that $\text{ex}(n, K_{1,m}) \leq (m-1)n/2$. Moreover, show that for any $m \in \mathbb{N}$, this bound is attainable when $n \geq m$ is even.

   Proof. To show $\text{ex}(n, K_{1,m}) \leq (m-1)n/2$, it suffices to show that a graph G with $> (m-1)n/2$ edges will have a vertex of deg $\geq m$.
   Let G be a n-vertex graph with $|E(G)| > \frac{(m-1)n}{2}$. We know that $|E(G)| = dn/2$, where d is the average degree. So we then have that the average degree, d, is greater than $m-1$. This means there must be at least one vertex that has degree $\geq m$. ■

   Hint: for the construction, recall that $\chi'(K_n) = n-1$ whenever n is even.

4. In this problem we will prove $\text{ex}(n, K_{2,m}) \leq \frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$.

   (a) Use pigeonhole principle to prove that if G is an n-vertex graph with $\sum_{v \in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$ as a subgraph.

   Proof. Let G be an n-vertex graph with $\sum_{v \in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$.
   What the left side of the inequality can be interpreted as doing is summing the number of pairs of neighbors a vertex has for all vertices. So the left side counts the number of distinct P_3 subgraphs.
   Let us define "boxes" for every pair of vertices. So there are $\binom{n}{2}$ boxes. Now let us put each distinct P_3 subgraph in a box based off of the end points of the P_3. We have more than $(m-1)\binom{n}{2}$ distinct P_3, so there must be a box with at least m paths. The union of all the paths in that box will be a $K_{2,\geq m}$. ■

   (b) Use Cauchy-Schwarz to prove that

$$\sum_{v \in V(G)} \binom{d(v)}{2} \geq e\left(\frac{2e}{n} - 1\right)$$

   where $e = |E(G)|$.

Proof.

$$\begin{aligned}
\sum_{v\in V(G)} \binom{d(v)}{2} &= \sum_{v\in V(G)} \frac{d(v)(d(v)-1)}{2} \\
&= \frac{1}{2}\sum_{v\in V(G)} d(v)^2 - \frac{1}{2}\sum_{v\in V(G)} d(v) \\
&= \frac{1}{2}\sum_{v\in V(G)} d(v)^2 - e \\
&= \frac{1}{n}(\frac{n}{2}\sum_{v\in V(G)} d(v)^2) - e \\
&= \frac{1}{n}(\frac{1}{2}\sum_{v\in V(G)} 1^2 \sum_{v\in V(G)} d(v)^2) - e \\
\text{By Cauchy-Schwarz} \qquad &\geq \frac{1}{2n}(\sum_{v\in V(G)} d(v))^2 - e \\
&= \frac{1}{2n}(2e)^2 - e \\
&= e(\frac{2e}{n} - 1)
\end{aligned}$$

∎

(c) Use parts (a) and (b) to conclude that if $e > \frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$, then G contains $K_{2,m}$.

Proof. Let G be a n-vertex graph with $e > \frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$. Then

$$\begin{aligned}
e(\frac{2e}{n} - 1) &> \left(\frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}\right)\left(\frac{(m-1)^{1/2}n^{3/2} + \frac{n}{2}}{n} - 1\right) \\
&= \left(\frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}\right)\left((m-1)^{1/2}n^{1/2} - \frac{1}{2}\right) \\
&= \frac{1}{2}(m-1)n^2 - \frac{n}{8} \\
&= (m-1)(\frac{n^2}{2} - \frac{n}{8(m-1)}) \\
&> (m-1)(\frac{n^2-n}{2}) \\
&= (m-1)\binom{n}{2}
\end{aligned}$$

So then using part (b), we get

$$\sum_{v\in V(G)} \binom{d(v)}{2} \geq e\left(\frac{2e}{n} - 1\right) > (m-1)\binom{n}{2}$$

So by part (a), we can say $\text{ex}(n, K_{2,m}) \leq \frac{1}{2}(m-1)^{1/2}n^{3/2} + \frac{n}{4}$. ∎

5. Let X be any set of n points in the plane $\mathbb{R}^2$. Use question 4 to prove that there are at most $\frac{\sqrt{2}}{2}n^{3/2} + \frac{n}{4}$ pairs of points in X that have (Euclidean) distance exactly 1.

   Proof. Let X be any set of n points in the plane $\mathbb{R}^2$ with edges between the points iff the pair has a Euclidean distance of exactly 1. Then to show the desired fact, it suffices to show there is no $K_{2,3}$ subgraph, since plugging in 3 into question 4's result will give that result.
   Let us choose 2 points in X that do not have an edge. We will now create a unit circle around each of those two points. If another point lies on the circle, that means it will have an edge between it and the center point. There would be a $K_{2,3}$ if the two circles intersected at 3 points, but this is not possible for 2 distinct circles. ∎