Homework 2 (Due Sept 13, 2023)

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MATH 546 - Algebraic Structures I - Fall 2023

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Justify all of your answers completely.

Check if the four requirements for a Group holds for each of the following.

1. $G = \{x \in \mathbb{R} : x > 1\}$; operation defined by $a \star b = ab - a - b + 2$

   (Associativity)

$$a \star (b \star c) = a \star (bc - b - c + 2) = a(bc - b - c + 2) - a - (bc - b - c + 2) + 2$$
$$= abc - ab - ac + 2a - a - bc + b + c - 2 + 2 = abc - ac - bc + 2c - ab + a + b - 2 - c + 2$$
$$= (ab - a - b + 2)c - (ab - a - b + 2) - c + 2 = (ab - a - b + 2) \star c = (a \star b) \star c$$

   Set is associative over the operation.

   (Identity)
   $e = 2$ works.

$$a \star 2 = a \cdot 2 - a - 2 + 2 = a$$

   Operation is symmetric, so commutativity is obvious and let's us not check $e \star a$.
   Set contains an identity element.

   (Inverse)

$$e = 2 = a \star a^{-1} = aa^{-1} - a - a^{-1} + 2 \implies 2 = aa^{-1} - a - a^{-1} + 2$$
$$\implies a^{-1} = \frac{a}{a-1}$$

   Since $a > 1$, $a^{-1} \in G$.
   Operation is symmetric, so commutativity is obvious and let's us not check $a^{-1} \star a$.
   So there is an inverse for every element.

   (Closure)

$$a \star b = ab - a - b + 2 = a(b-1) - 1(b-1) + 1 = (b-1)(a-1) + 1 > ((1)-1)((1)-1) + 1 = 1$$
$$\text{So } a \star b > 1$$

   G is closed under the operation.

2. $G = \{x \in \mathbb{Z} : x \geq 8\}$; operation defined by $a \star b = \max(a, b)$

   (Associativity)
   $a \star (b \star c) = \max(a, \max(b, c))$
   $(a \star b) \star c = \max(\max(a, b), c)$
   Both return the max of all three numbers, so they are the same. Thus, set is associative over the operation.

   (Identity)
   $e = 8$ works.

   $$a \star 8 = \max(a, 8) = a \qquad \text{since } a \geq 8$$

   Operation is symmetric, so commutativity is obvious and let's us not check $e \star a$. Set contains an identity element.

   (Inverse)
   $e = 8 = a \star a^{-1} = \max(a, a^{-1})$
   There is no way to guarantee $\max(a, a^{-1}) = 8$, since if $a \geq 9$, then $\max(a, a^{-1}) \geq 9$. Set over the operation fails the inverse requirement.

   (Closure)
   $a \star b$ will either equal a or b, and both of them are in G. So $a \star b$ will also be in G. G is closed under the operation.

3. $G = \{x \in \mathbb{R} : x \geq 0\}$; operation defined by $a \star b = |a - b|$

   (Associativity)
   $1 \star (2 \star 3) = |1 - |2 - 3|| = 0$
   $(1 \star 2) \star 3 = ||1 - 2| - 3| = 2$
   G is not associative under the operation.

   (Identity)
   $e = 0$ works.
   $a \star 0 = |a - 0| = |a| = a$ since a is nonnegative.
   $0 \star a = |0 - a| = |a| = a$ since a is nonnegative.

   (Inverse)
   $e = 0 = a \star a^{-1} = |a - a^{-1}| \implies a^{-1} = a$.
   $e = 0 = a^{-1} \star a = |a^{-1} - a| \implies a^{-1} = a$. a^{-1} is in G since it equals a and a is in G.
   There is an inverse for every element.

   (Closure)
   $|a - b|$ will always be nonnegative by the definition of absolute value, so it is in the set. G is closed under the operation.

4. $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a \text{ and } b \text{ are both not } 0\}$; operation is normal multiplication.

   (Associativity)
   Multiplication is known to be associative.

(Identity)
$e = 1 + 0\sqrt{2}$ works.
$(a + b\sqrt{2})(1 + 0\sqrt{2}) = a + b\sqrt{2}$.
Multiplication is known to be communicative and let's us not check $e \star a$.
There is an identity element.

(Inverses)
$e = 1 = (a + b\sqrt{2})((a + b\sqrt{2}))^{-1} \implies (a + b\sqrt{2})^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$.
The denominator cannot equal 0 when a and b are rationals, so no values are excluded from having inverses.
Multiplication is known to be communicative and let's us not check $(a + b\sqrt{2})^{-1}(a + b\sqrt{2})$.
The set under the operation does have inverses for every element.

(Closure)

$$\begin{aligned}(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) &= a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + 2b_1b_2 \\ &= (a_1a_2 + 2b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2} \in G\end{aligned}$$

Set is closed under the operation.

5. G = set of all affine function $f_{m,b} : \mathbb{R} \to \mathbb{R}$, $f_{m,b} = mx + b$, where $m, b \in \mathbb{R}$ and $m \neq 0$; operation is normal composition of functions.

(Associativity)
Composition of functions is known to be associative.

(Identity)
$e = f_{1,0}$ works.
$f_{m,b} \circ f_{1,0} = m(x) + b = mx + b$
$f_{1,0} \circ f_{m,b} = (mx + b) = mx + b$

(Inverse)
$e = f_{1,0} = x = f_{m,b} \circ f^{-1}_{m,b} \implies x = m(f^{-1}_{m,b}) + b \implies f^{-1}_{m,b} = \frac{x}{m} - \frac{b}{m} \in G$
$f^{-1}_{m,b} \circ f_{m,b} = \frac{mx+b}{m} - \frac{b}{m} = x + \frac{b}{m} - \frac{b}{m} = x = e$
The set under the operation does have inverses for every element.

(Closure)
$f_{m_1,b_1} \circ f_{m_2,b_2} = m_1(m_2x + b_2) + b_1 = (m_1m_2)x + (m_1b_2 + b_1) \in G$, since $m_1m_2 \neq 0$.
Set is closed under the operation.