Math 675 Spring 2026:
Homework 1 Unit ?
Due Tuesday, Janurary 3rd, 2026

Question 1:

For each system of differential equations in a-d below

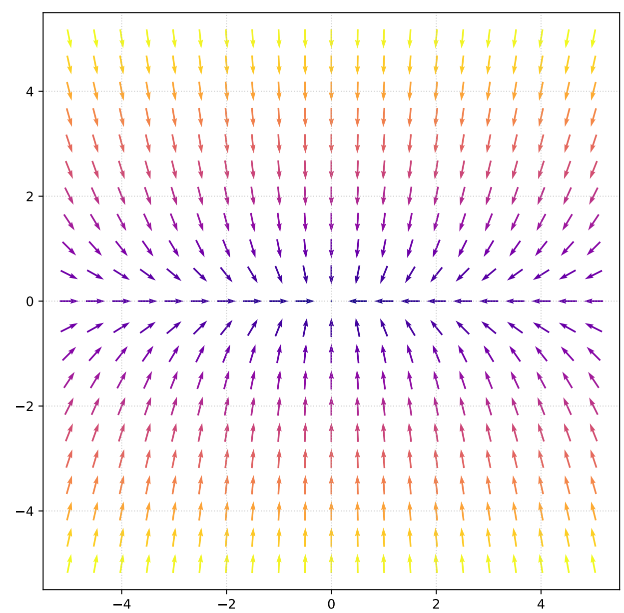
- Plot the change vector using the points $(1,1),(1,-1),(-1,-1),(-1,1)$
- Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

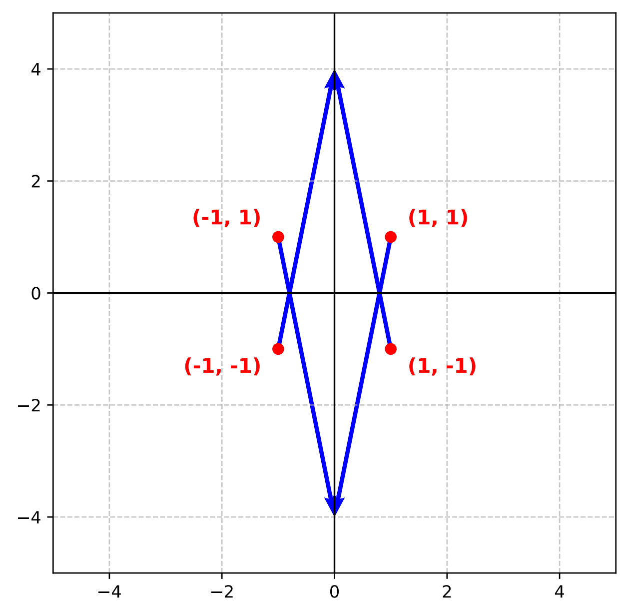
(a) $x' = -x, \quad y' = -5y$
Coupled: No; no dependance
EPs: Set $x' = 0, y' = 0$

$$\begin{cases} -x \quad\quad = 0 \\ \quad\quad -5y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the equilibrium point is at $(0,0)$ and is unstable

(x,y)	$(x',y') = (-x,-5y)$
$(1,1)$	$(-(1),-5(1)) = (-1,-5)$
$(1,-1)$	$(-(1),-5(-1)) = (-1,5)$
$(-1,-1)$	$(-(-1),-5(-1)) = (1,5)$
$(-1,1)$	$(-(-1),-5(1)) = (1,-5)$





(b) $x' = 4x - y, \quad y' = 2x + y$

Coupled: Yes;

Derivatives depends on each other

EPs: Set $x' = 0, y' = 0$

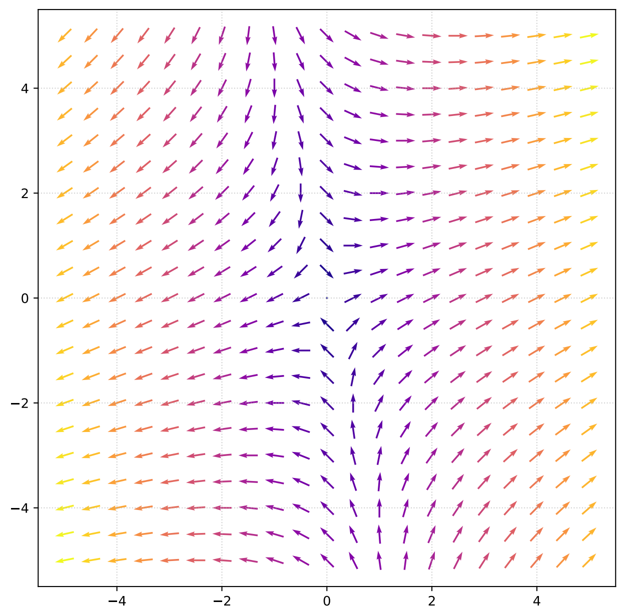
$$\begin{cases} 4x - y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \begin{cases} 4x \quad = y \\ 2x + y = 0 \end{cases}$$

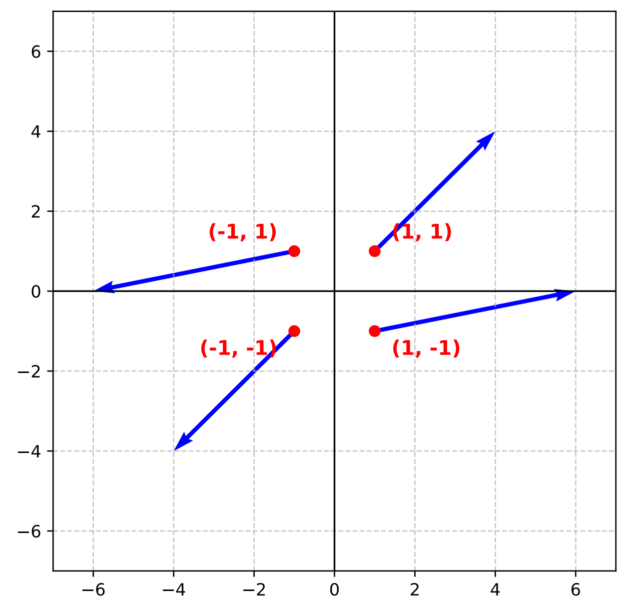
$$\begin{cases} 4x \quad = y \\ 2x + 4x = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 6x = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ x = 0 \end{cases} \rightarrow \begin{cases} 4(0) = 0 \\ x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x', y') = (4x - y, 2x + y)$
$(1, 1)$	$(4 - 1, 2 + 1) = (3, 3)$
$(1, -1)$	$(4 + 1, 2 - 1) = (5, 1)$
$(-1, -1)$	$(-4 + 1, -2 + 1) = (-3, -3)$
$(-1, 1)$	$(-4 - 1, -2 + 1) = (-5, -1)$





(c) $x_1' = x_2, \quad x_2' = -2x_1 - 3x_2$

Coupled: Yes;

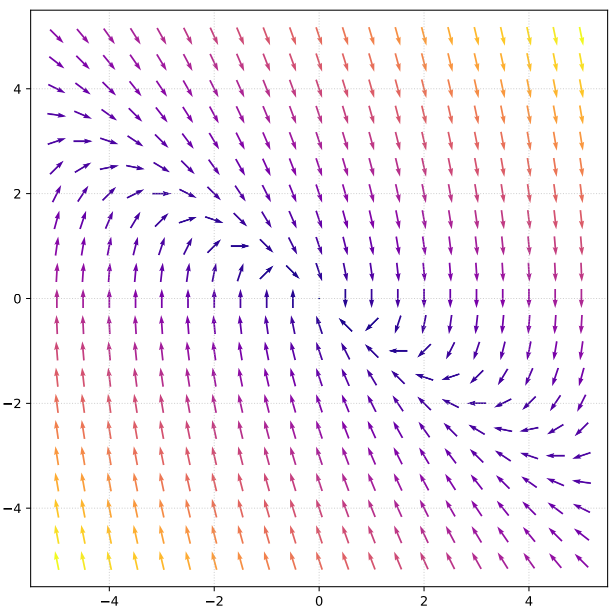
EPs: Set $x' = 0, y' = 0$

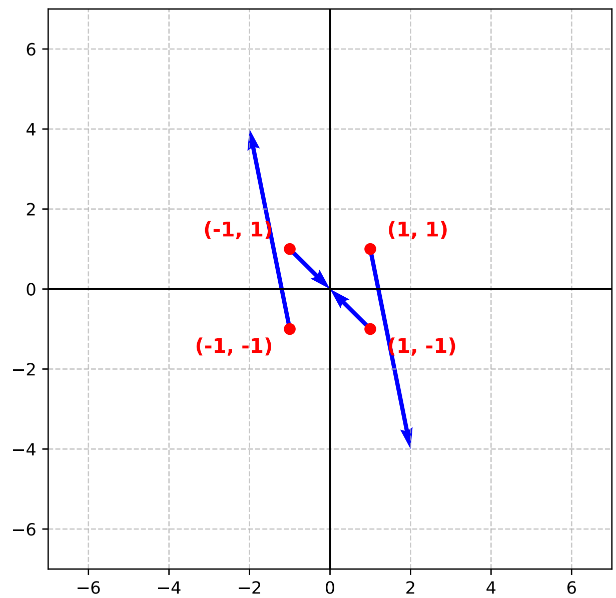
$$\begin{cases} -x_2 = 0 \\ -2x_1 - 3x_2 = 0 \end{cases}$$

$$\begin{cases} x_2 = 0 \\ -2x_1 + 0 = 0 \end{cases} \rightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases}$$

Thus, the EP is $(0,0)$

(x, y)	$(x_1', x_2') = (x_2, -2x_1 - 3x_2)$
$(1,1)$	$(-(1), -5(1)) = (-1, -5)$
$(1,-1)$	$(-(1), -5(-1)) = (-1, 5)$
$(-1,-1)$	$(-(-1), -5(-1)) = (1, 5)$
$(-1,1)$	$(-(-1), -5(1)) = (1, -5)$





(d) $x' = 5x + 2y, \quad y' = -17x - 5y$

Coupled: Yes;

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} 5x + 2y = 0 \mid \times 5 \\ -17x - 5y = 0 \mid \times 2 \end{cases}$$

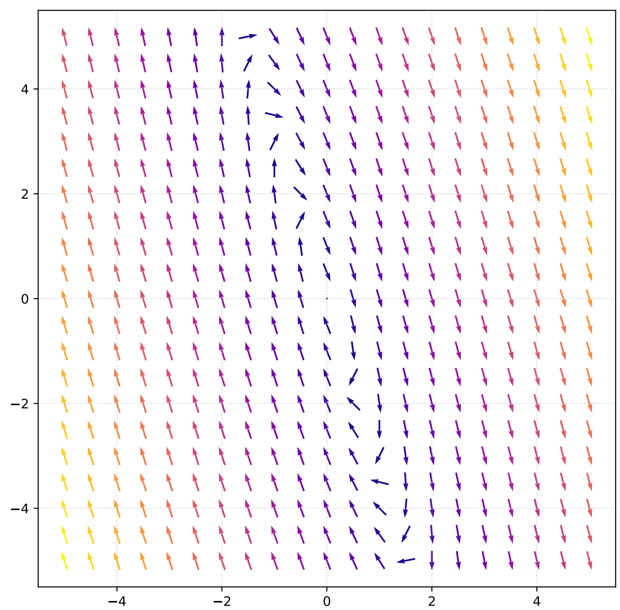
$$\begin{cases} 25x + 10y = 0 \\ -34x - 10y = 0 \end{cases}$$

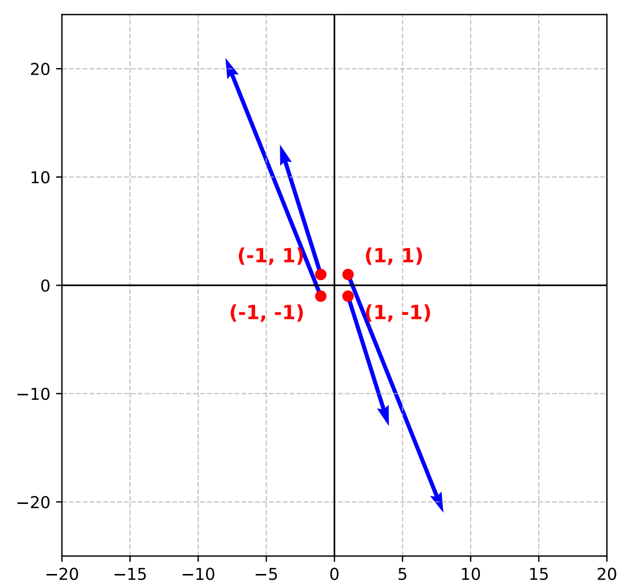
$$\begin{cases} -9x \quad = 0 \\ -34x - 10y = 0 \end{cases}$$

$$\begin{cases} x \quad = 0 \\ 0 - 10y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x_1', x_2') = (5x + 2y, -17x - 5y)$
$(1, 1)$	$(5 + 2, -17 - 5) = (7, -22)$
$(1, -1)$	$(5 - 2, -17 + 5) = (3, -12)$
$(-1, -1)$	$(-5 - 2, 17 + 5) = (-7, 22)$
$(-1, 1)$	$(-5 + 2, 17 + 5) = (-3, 12)$





Question 2:

Let $a \in \mathbb{R}$ be a parameter. Consider the system of differential equations

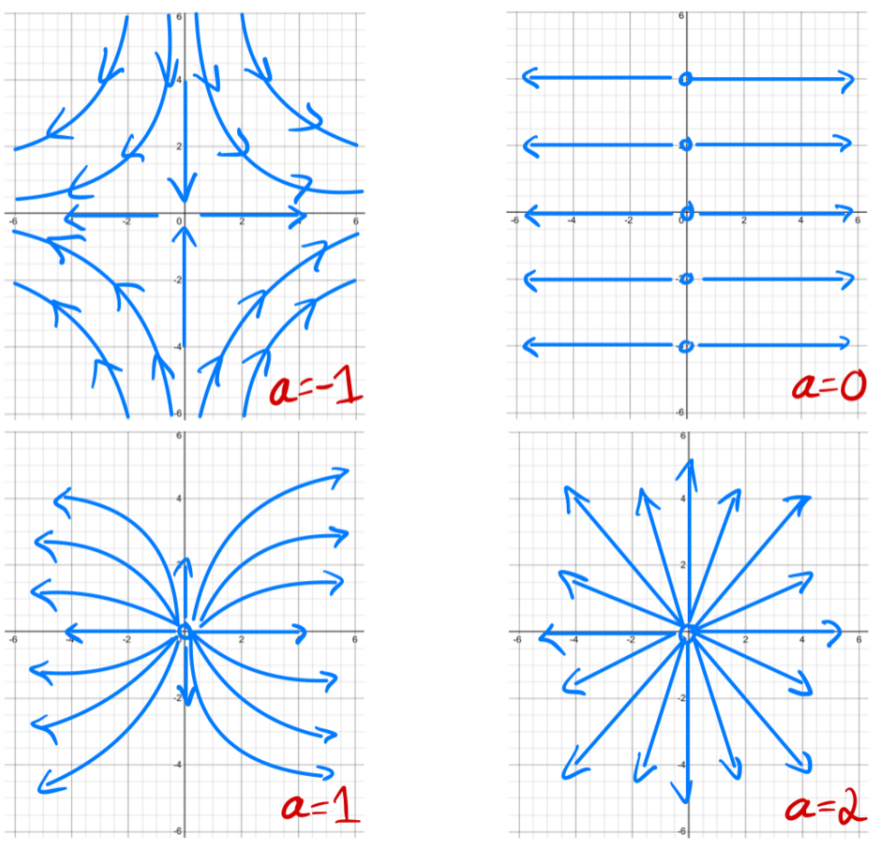
$$x'(t) = 2x, \qquad y'(t) = ay$$

(a) Use separation of variables to find the explicit solution for this system.

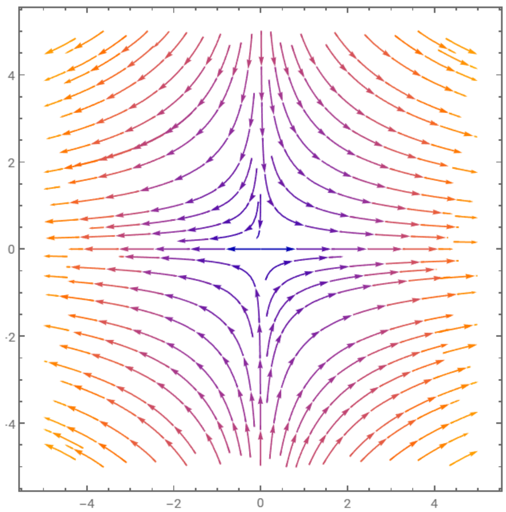
$$\begin{aligned} x'(t) &= 2x \\ \frac{dx}{dt} &= 2x \\ \int \frac{1}{x}dx &= \int 2dt \\ \ln|x| &= 2t + c_1 \\ e^{\ln|x|} &= e^{2t+c_1} \\ x &= e^{2t}e^{c_1} \\ \text{Solution: } x(t) &= c_1 e^{2t} \end{aligned}$$

$$\begin{aligned} y'(t) &= -ay \\ \frac{dy}{dt} &= ay \\ \int \frac{1}{y}dy &= \int adt \\ \ln|y| &= at + c_1 \\ e^{\ln|y|} &= e^{at+c_1} \\ y &= e^{at}e^{c_1} \\ \text{Solution: } y(t) &= c_2 e^{at} \end{aligned}$$

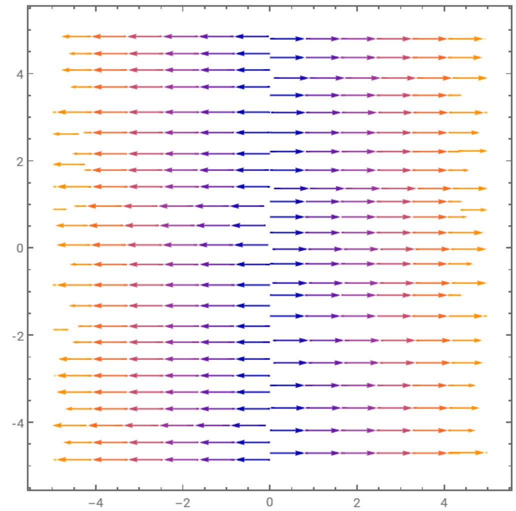
(b) Sketch the phase portrait for this system for the following values of a: -1, 0, 1, 2. Do not use technology for this part.



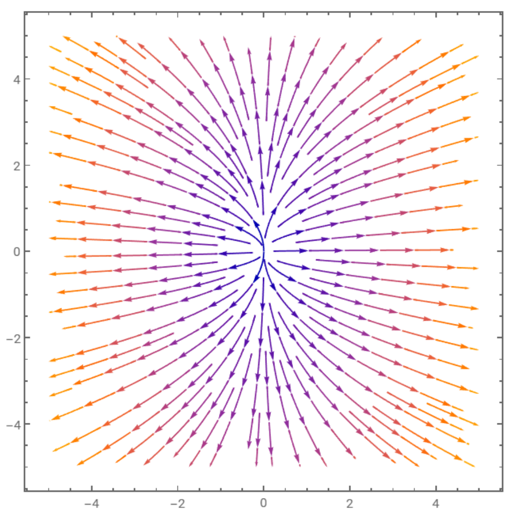
(c) Now use tech (Mathematica, Python, etc) to plot a complete vector field and some trajectories.



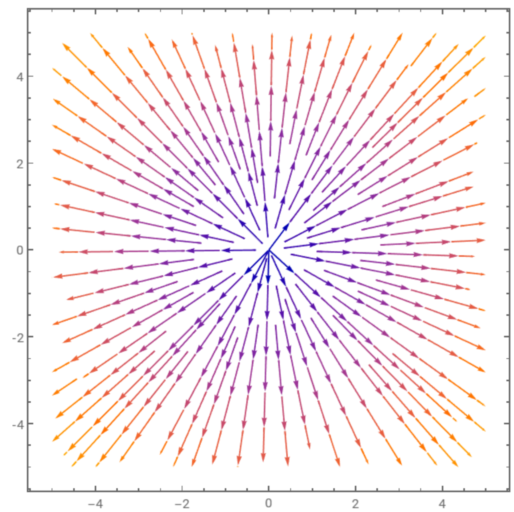
Vector Field: $a = -1$



Vector Field: $a = 0$



Vector Field: $a = 1$



Vector Field: $a = 2$

(d) Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

- ($a = -1$): EP: $(0, 0)$ and this is an unstable saddle point
- ($a = 0$): EP: $(0, 0)$ and this is unstable
- ($a = 1$): EP: $(0, 0)$ and this is unstable
- ($a = 2$): EP: $(0, 0)$ and this is unstable

Question 3:

Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the first order ODE

$$x' = a(t)x$$

(a) Find a formula involving integrals for the solution of this system.

$$\begin{aligned} \frac{dx}{dt} &= a(t)x(t) \\ \int \frac{1}{x(t)} dx &= \int a(t) dt \\ \ln |x(t)| &= \int a(t) dt + C \\ x(t) &= e^{\int a(t) dt + C} \\ x(t) &= Ce^{\int a(t) dt} \end{aligned}$$

Future knowledge: Hey look integrating factor!!

(b) Prove that your formula gives the general solution of this system. Hint: follow the sketch of proof done on the first day of class

First, let $u(t)$ be any solution and let's differentiate $f(t) = u(t)e^{\int -a(t)dt}$
Given that $x(t) = Ce^{\int a(t)dt}$, want to show that $f(t) = 0$
First, we show existence:

$$\begin{aligned} x'(t) &= a(t)x(t) \\ \frac{dx}{dt}\left(Ce^{\int a(t)dt}\right) &= a(t)\left(Ce^{\int a(t)dt}\right) \\ \frac{dx}{dt}\left(\int a(t)dt\right)\left(Ce^{\int a(t)dt}\right) &= a(t)\left(Ce^{\int a(t)dt}\right) \\ a(t)\left(Ce^{\int a(t)dt}\right) &= a(t)\left(Ce^{\int a(t)dt}\right) \\ RHS &= LHS \end{aligned}$$

Thus, since both sides are equal, then $x(t) = Ce^{\int a(t)dt}$ is in fact the general solution. Although, I do sense some explotation of the Fundamental Theorem of Calculus, since we differentiated an integral a bit too easily.

Second, we show uniqueness:

$$\begin{aligned} \frac{d}{dt}f(t) &= \frac{d}{dt}u(t)e^{\int a(t)dt} \\ &= \frac{d}{dt}(u(t))\,e^{\int -a(t)dt} + u(t)\left(\frac{d}{dt}e^{\int -a(t)dt}\right) \\ &= u'(t)e^{\int -a(t)dt} + u(t)\left(\frac{d}{dt}\int -a(t)dt\right)e^{\int -a(t)dt} \\ &= u'(t)e^{\int -a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\ &= \left(a(t)Ce^{\int a(t)dt}\right)e^{\int -a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\ &= a(t)u(t)e^{\int a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\ \frac{d}{dt}f(t) &= a(t)f(t) - a(t)f(t) = 0 \end{aligned}$$

This, our solution does in fact give the general solution to this system above! I'm about 95% confident in the reasoning, but I feel like I am missing the ultimate punchline to solidify why the derivative being zero means something impactful or why the solution multiplied by the integrating factor is a constant