Math 675 Spring 2026:
Homework 2
Due Tuesday, February 10th, 2026

Question 1:

Consider the uncoupled system:

$$x' = 3x, y' = -y$$

a) Solve the system explicitly for general initial conditions $(x(0), y(0)) = (x_0, y_0)$.

$$\begin{aligned} x'(t) &= 3x \\ \frac{dx}{dt} &= 3x \\ \int \frac{1}{x}dx &= \int 3dt \\ \ln|x| &= 3t + c_1 \\ e^{\ln|x|} &= e^{3t+c_1} \\ x &= e^{3t}e^{c_1} \\ \text{Solution: } x(t) &= c_1e^{3t} \\ \text{IC: } x(0) = x_0 &= c_1 \\ \text{Final Solution: } x(t) &= x_0e^{3t} \end{aligned}$$

$$\begin{aligned} y'(t) &= -y \\ \frac{dy}{dt} &= -y \\ \int \frac{1}{y}dy &= \int -1dt \\ \ln|y| &= -t + c_2 \\ e^{\ln|y|} &= e^{-t+c_2} \\ y &= e^{-t}e^{c_2} \\ \text{Solution: } y(t) &= c_2e^{-t} \\ \text{IC: } y(0) = y_0 &= c_2 \\ \text{Final Solution: } y(t) &= y_0e^{-t} \end{aligned}$$

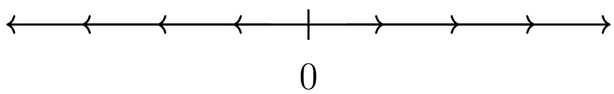
b) Identify all equilibrium points of the system.

$$\begin{cases} 3x = 0 \\ -y = 0 \end{cases} \to \begin{cases} x = 0 \\ y = 0 \end{cases}$$

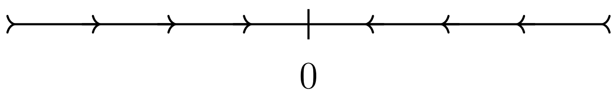
So like before, the only equilibrium point is at the origin $(0, 0)$.

c) Using your explicit solution, determine the behavior of solutions as $t \to +\infty$:
Given out solutions from before: $x(t) = x_0 e^{3t}$ and $y(t) = y_0 e^{-t}$

i) initial conditions on the x-axis
Using just $x = x_0 e^{3t}$, As $t \to +\infty$,
If $x_0 > 0$ then $x(t) \to +\infty$ and if $x_0 < 0$ $x(t) \to -\infty$.



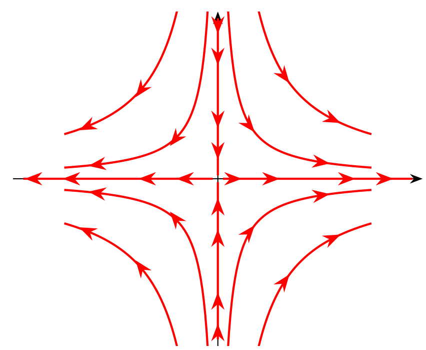
ii) initial conditions on the y-axis
Using just $y = y_0 e^{-t}$, As $t \to +\infty$,
If $y_0 > 0$ then $y(t) \to 0$ and if $y_0 < 0$ $y(t) \to 0$.



iii) initial conditions not lying on either axis

- If $t \to +\infty$ then solutions approach the positive x-axis

- If $t \to -\infty$ then solutions approach the negative x-axis



d) We can see from the behavior of the solutions as $t \to +\infty$ and $t \to -\infty$ that the equilibrium point at the origin is a saddle point. This is because along the x-axis, the solutions diverge away from the origin, while along the y-axis, the solutions converge towards the origin.

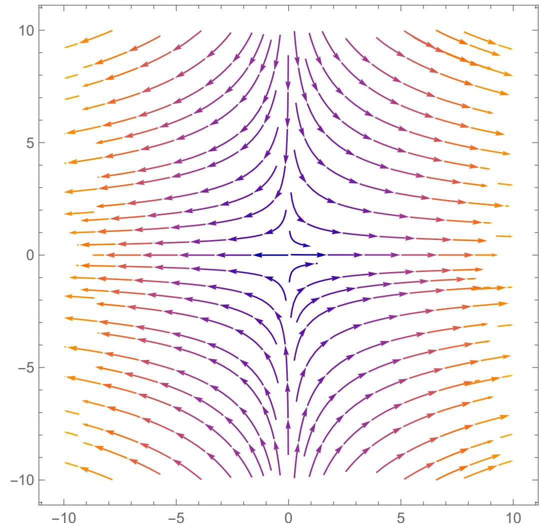
We can also see this as we take limits to positive and negitive infinity of $x(t) = x_0 e^{3t}$ and $y(t) = y_0 e^{-t}$ similar to what we have above,

As $t \to +\infty$, $x(t) \to +\infty$ and $y(t) \to 0$,

As $t \to -\infty$, $x(t) \to -\infty$ and $y(t) \to 0$.

Because the limits for $y(t)$ are both 0 yet the sign of infinity changes for $x(t)$, this tells us that this equilibrium point is a saddle point

e) Now we want to use Mathematica to plot the full phase portrait of this system.



Question 2:

Consider the uncoupled system:

$$x' = -5x, y' = -y$$

a) Solve the system explicitly for general initial conditions $(x(0), y(0)) = (x_0, y_0)$

$$\begin{aligned}
x'(t) &= -5x \\
\frac{dx}{dt} &= -5x \\
\int \frac{1}{x} dx &= \int -5dt \\
\ln|x| &= -5t + c_1 \\
e^{\ln|x|} &= e^{-5t+c_1} \\
x &= e^{-5t}e^{c_1} \\
\text{Solution: } x(t) &= c_1 e^{-5t} \\
\text{IC: } x(0) &= x_0 = c_1 \\
\text{Final Solution: } x(t) &= x_0 e^{-5t}
\end{aligned}$$

$$\begin{aligned}
y'(t) &= -y \\
\frac{dy}{dt} &= -y \\
\int \frac{1}{y} dy &= \int -1dt \\
\ln|y| &= -t + c_2 \\
e^{\ln|y|} &= e^{-t+c_2} \\
y &= e^{-t}e^{c_2} \\
\text{Solution: } y(t) &= c_2 e^{-t} \\
\text{IC: } y(0) &= y_0 = c_2 \\
\text{Final Solution: } y(t) &= y_0 e^{-t}
\end{aligned}$$

b) Determine which variable decays faster as $t \to +\infty$, and justify your answer using the explicit solutions.

As $t \to +\infty$, we can see both solutions $x(t) = x_0 e^{-5t}$ and $y(t) = y_0 e^{-t}$ approach zero. However, the solution for $x(t)$ will decay faster than $y(t)$ because $x(t)$ has a larger negative exponent compared to $y(t) = y_0 e^{-t}$, which means that $x(t)$ will approach zero faster than $y(t)$ as time increases.

c) Show that for any solution with $x_0 \neq 0$

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)} = \infty$$

Given our solutions $x(t) = x_0 e^{-5t}$ and $y(t) = y_0 e^{-t}$, we can compute the limit:

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)} = \lim_{t\to+\infty} \frac{y_0 e^{-t}}{x_0 e^{-5t}} = \lim_{t\to+\infty} \frac{y_0}{x_0} e^{4t}$$

Since e^{4t} increases to infinity as $t \to +\infty$, then the whole limit will also approach infinity. The only condition we need is that $x_0 \neq 0$ to ensure that the denominator does not equal zero, which would make the limit undefined.

d) Conclude that, except for the trajectory along the y-axis, all other solution curves asymtotically converge to the y-axis as $t \to +\infty$.

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)} = \lim_{t\to+\infty} \frac{y_0}{x_0} e^{4t} = \infty$$

We see that as long as $x_0 \neq 0$, which would technically make the slope already be parallel to the y-axis, this implies that the slopes also converge to vertical lines, thus they become parallel to the y-axis eventually.

e) Conclude that, except for the trajectory along the x-axis, all other solution curves asymtotically converge to become parallel to the x-axis as $t \to -\infty$.

$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \lim_{t\to-\infty} \frac{y_0}{x_0} e^{4t} = 0$$

We see that as long as $y_0 \neq 0$, which would already make the slope be parallel to the x-axis, this implies that the slopes also converge to vertical lines, thus they become parallel to the x-axis eventually.

Question 3:

Show that for a linear, homogeneous system of ODE with $a, b, c, d \neq 0$, where x' and y' are not scalar multiples of each other, the only equilibrium point is $(0, 0)$.

$$x' = ax + by, \qquad y' = cx + dy$$

First, lets find the determinate of this system, since we are basically talking about linear independence. Let a matrix A, be the coeffient matrix of this system.
We want then force $\det(A) \neq 0$ so x' and y' are not scalar multiples of each other

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \tag{1}$$

Now, let's try to find the equilibrium points of this system. We set $x' = 0$ and $y' = 0$ and solve for x and y .

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} ax + \quad = -by \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} x + \quad = -\dfrac{by}{a} \\ cx + dy = 0 \end{cases}$$

Now lets substitute the first equation into the second equation to solve for y.

$$0 = cx + dy = c\left(-\frac{by}{a}\right) + dy = -\left(\frac{bc}{a}\right)y + dy = y\left(d - \frac{bc}{a}\right) \tag{2}$$

This is where things get interesting, since we're given $a, b, c, d \neq 0$ and $ad - bc \neq 0$, then we can see we have two solutions for this equation $y = 0$ or $d - \frac{bc}{a} = 0$.

$$0 = d - \frac{bc}{a} \implies ad - bc = 0$$

But we are given that $ad - bc \neq 0$ from equation (1), so the only valid solution for equation (2) is $y = 0$. Now we can substitute this back to solve for x.

$$x = -\frac{by}{a} = -\frac{b \cdot 0}{a} = 0$$

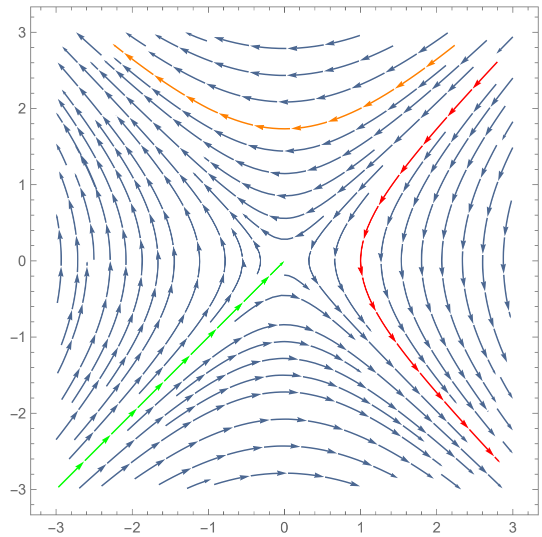
Therefore, the only equilibrium point is $(0, 0)$.

Question 4:

Consider the coupled ODE system

$$x' = -y, \qquad y' = -x$$

a) Use Mathematica to plot the phase portrait of this system.



b) Follow the outline below to prove that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$ where C is a constant.

i) Show that the system of ODEs implies $x\frac{dx}{dt} - y\frac{dy}{dt} = 0$.

$$x' = -y, \qquad y' = -x$$

$$\frac{x'}{y} = -1 \qquad \frac{y'}{x} = -1$$

$$\frac{x'}{y} = \frac{y'}{x}$$

$$xx' = yy'$$

$$x\frac{dx}{dt} - y\frac{dy}{dt} = 0$$

ii) Use the chain rule to show that $\frac{1}{2}\frac{d}{dt}(x^2) = x\frac{dx}{dt}$.

$$\frac{1}{2}\frac{d}{dt}(x(t))^2 = \frac{1}{2}(2)x(t)\frac{dx}{dt} = x(t)\frac{dx}{dt}$$

iii) Substitute the identity for both variables in part ii) into EQ(1)

$$x\frac{dx}{dt} - y\frac{dy}{dt} = 0$$

$$\frac{1}{2}\frac{d}{dt}(x^2) - \frac{1}{2}\frac{d}{dt}(y^2) = 0$$

iv) Integrate both sides of the equation and use the fundamental theorem of calculus to get the results

$$\frac{d}{dt}\left(\frac{x^2}{2} - \frac{y^2}{2}\right) = 0$$

$$\int \frac{d}{dt}\left(\frac{x^2}{2} - \frac{y^2}{2}\right) dt = \int 0 dt$$

$$\frac{x^2}{2} - \frac{y^2}{2} = C$$

$$x^2 - y^2 = C$$

Question 5:

In this problem we consider the same coupled ODE system as in the previous problem

$$x' = -y, y' = -x$$

We want to decouple and solve this analytically. So please do the following:

a) Introduce new variables u and v defined by $u = x + y$ and $v = x - y$. Solve for the variables u, v in terms of x and y.

$$u + v = (x + y) + (x - y) \qquad u - v = (x + y) - (x - y)$$

$$u + v = 2x \qquad u - v = 2y$$

$$\frac{u+v}{2} = x \qquad \frac{u-v}{2} = y$$

b) Rewrite the ODE system in terms of u and v using the previous part We want to differentiate u and v with respect to t to get u' and v'. Then substitute the original ODEs for x' and y' to get the new ODEs in terms of u and v.

$$u' = x' + y' = -y - x = -(x + y) = -u$$

$$v' = x' - y' = -y + x = (x - y) = v$$

c) Solve for $u(t)$ and $v(t)$ starting from an arbirary initial condition (u_0, v_0).

$$\begin{aligned}
u'(t) &= -u \\
\frac{du}{dt} &= -u \\
\int \frac{1}{u} du &= \int -1 dt \\
\ln |u| &= -t + c_1 \\
e^{\ln |u|} &= e^{-t+c_1} \\
u &= e^{-t} e^{c_1} \\
\text{Solution: } u(t) &= c_1 e^{-t} \\
\text{IC: } u(0) = u_0 &= c_1 \\
\text{Final Solution: } u(t) &= u_0 e^{-t}
\end{aligned}$$

$$\begin{aligned}
v'(t) &= -v \\
\frac{dv}{dt} &= v \\
\int \frac{1}{v} dv &= \int 1 dt \\
\ln |v| &= t + c_2 \\
e^{\ln |v|} &= e^{t+c_2} \\
v &= e^{t} e^{c_2} \\
\text{Solution: } v(t) &= c_2 e^{t} \\
\text{IC: } v(0) = v_0 &= c_2 \\
\text{Final Solution: } v(t) &= v_0 e^{t}
\end{aligned}$$

d) Finally, using the answer above, write the general solution in terms of $x(t)$ and $y(t)$. Make sure to also write starting from the initial conditions (x_0, y_0). We can note that if $t = 0$ implies $u(0) = u_0$ and $v(0) = v_0$ then we can also conclude that we can obtain the initial conditions of u and v in terms of the initial conditions of x and y.

$$u(0) = x(0) + y(0) = x_0 + y_0 \qquad v(0) = x(0) - y(0) = x_0 - y_0$$

Now, substitute these modified initial conditions into our solutions to get:

$$\begin{aligned}
x(t) &= \frac{1}{2}\left[u(t) + v(t)\right] \\
&= \frac{1}{2}\left[u_0 e^{-t} + v_0 e^{t}\right] \\
&= \frac{1}{2}\left[(x_0 + y_0)e^{-t} + (x_0 - y_0)e^{t}\right] \\
&= \frac{1}{2}\left[x_0(e^{-t} + e^{t}) + y_0(e^{-t} - e^{t})\right] \\
&= x_0 \cosh(t) - y_0 \sinh(t)
\end{aligned}$$

$$\begin{aligned}
y(t) &= \frac{1}{2}\left[u(t) - v(t)\right] \\
&= \frac{1}{2}\left[u_0 e^{-t} - v_0 e^{t}\right] \\
&= \frac{1}{2}\left[(x_0 + y_0)e^{-t} - (x_0 - y_0)e^{t}\right] \\
&= \frac{1}{2}\left[x_0(e^{-t} - e^{t}) + y_0(e^{-t} + e^{t})\right] \\
&= -x_0 \sinh(t) + y_0 \cosh(t)
\end{aligned}$$