

Math 675 Spring 2026:
Homework 1 Unit ?
Due Tuesday, January 3rd, 2026

Question 1:

For each system of differential equations in a-d below

- Plot the change vector using the points $(1, 1), (1, -1), (-1, -1), (-1, 1)$
- Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

(a) $x' = -x, \quad y' = -5y$

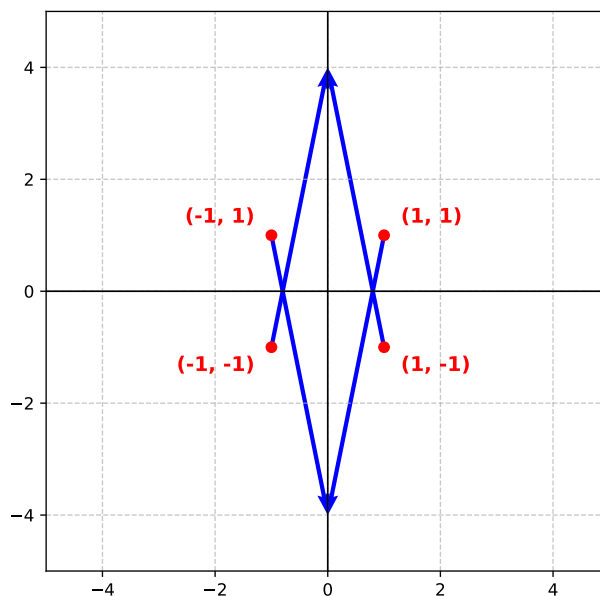
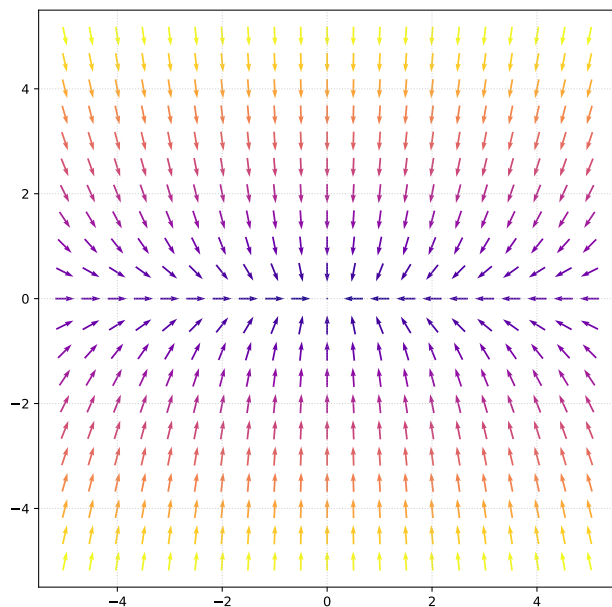
Coupled: No; no dependance

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} -x = 0 \\ -5y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the equilibrium point is at $(0, 0)$ and is unstable

(x, y)	$(x', y') = (-x, -5y)$
$(1, 1)$	$(-1, -5(1)) = (-1, -5)$
$(1, -1)$	$(-1, -5(-1)) = (-1, 5)$
$(-1, -1)$	$(-(-1), -5(-1)) = (1, 5)$
$(-1, 1)$	$(-(-1), -5(1)) = (1, -5)$



(b) $x' = 4x - y, \quad y' = 2x + y$

Coupled: Yes;

Derivatives depends on each other

EPs: Set $x' = 0, y' = 0$

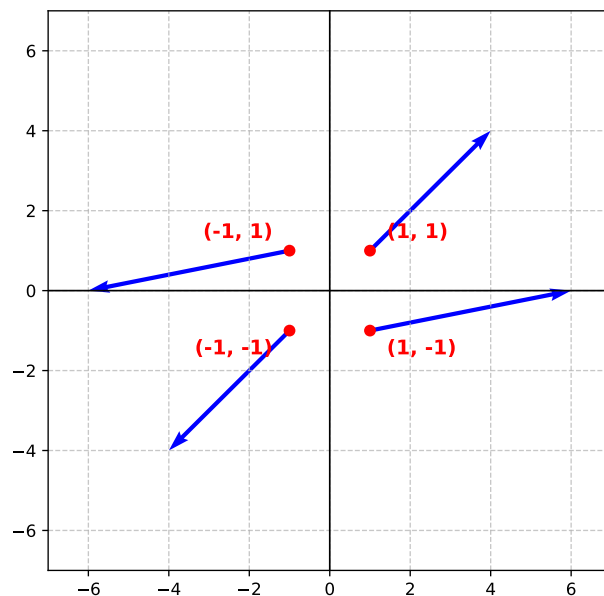
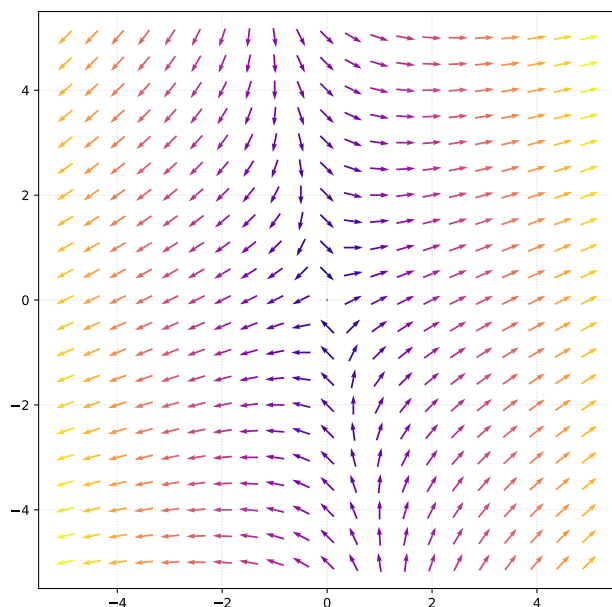
$$\begin{cases} 4x - y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 2x + y = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ 2x + 4x = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 6x = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ x = 0 \end{cases} \rightarrow \begin{cases} 4(0) = 0 \\ x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x', y') = (4x - y, 2x + y)$
$(1, 1)$	$(4 - 1, 2 + 1) = (3, 3)$
$(1, -1)$	$(4 + 1, 2 - 1) = (5, 1)$
$(-1, -1)$	$(-4 + 1, -2 + 1) = (-3, -3)$
$(-1, 1)$	$(-4 - 1, -2 + 1) = (-5, -1)$



(c) $x'_1 = x_2, \quad x'_2 = -2x_1 - 3x_2$

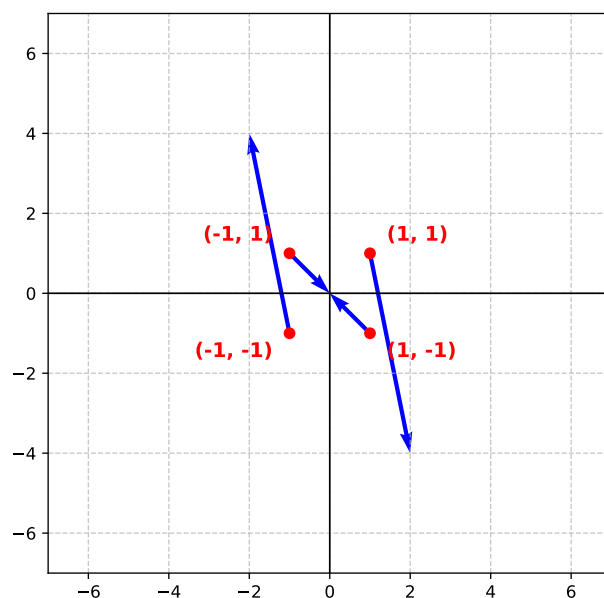
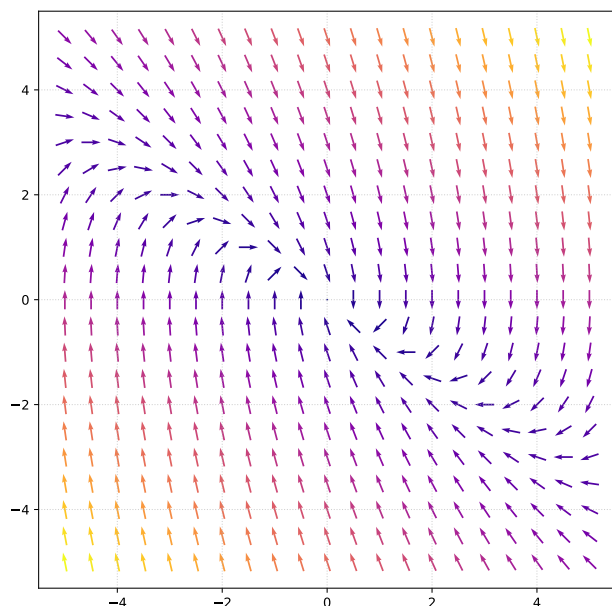
Coupled: Yes;

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} -x_2 = 0 \\ -2x_1 - 3x_2 = 0 \end{cases} \rightarrow \begin{cases} x_2 = 0 \\ -2x_1 + 0 = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x'_1, x'_2) = (x_2, -2x_1 - 3x_2y)$
$(1, 1)$	$(-1, -5(1)) = (-1, -5)$
$(1, -1)$	$(-1, -5(-1)) = (-1, 5)$
$(-1, -1)$	$(-(-1), -5(-1)) = (1, 5)$
$(-1, 1)$	$(-(-1), -5(1)) = (1, -5)$



(d) $x' = 5x + 2y, \quad y' = -17x - 5y$

Coupled: Yes;

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} 5x + 2y = 0 & | \times 5 \\ -17x - 5y = 0 & | \times 2 \end{cases}$$

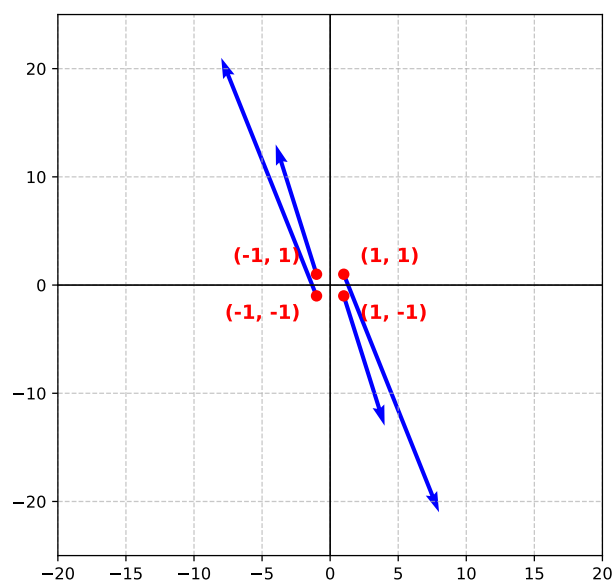
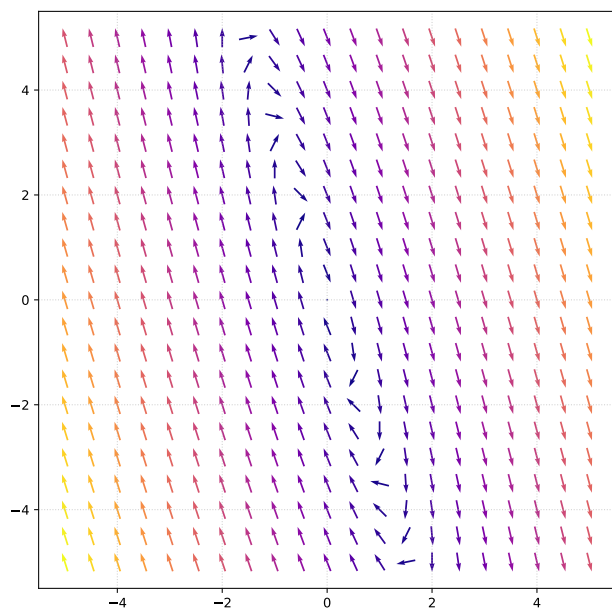
$$\begin{cases} 25x + 10y = 0 \\ -34x - 10y = 0 \end{cases}$$

$$\begin{cases} -9x = 0 \\ -34x - 10y = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ 0 - 10y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x'_1, x'_2) = (x_2, -2x_1 - 3x_2y)$
$(1, 1)$	$(-1, -5(1)) = (-1, -5)$
$(1, -1)$	$(-1, -5(-1)) = (-1, 5)$
$(-1, -1)$	$(-(-1), -5(-1)) = (1, 5)$
$(-1, 1)$	$(-(-1), -5(1)) = (1, -5)$



Question 2:

Let $a \in \mathbb{R}$ be a parameter. Consider the system of differential equations

$$x'(t) = 2x, \quad y'(t) = ay$$

- (a) Use separation of variables to find the explicit solution for this system.

$$x'(t) = 2x$$

$$\frac{dx}{dt} = 2x$$

$$\int \frac{1}{x} dx = \int 2 dt$$

$$\ln |x| = 2t + c_1$$

$$e^{\ln |x|} = e^{2t+c_1}$$

$$x = e^{2t} e^{c_1}$$

$$\text{Solution: } x(t) = c_1 e^{2t}$$

$$y'(t) = -ay$$

$$\frac{dy}{dt} = ay$$

$$\int \frac{1}{y} dy = \int a dt$$

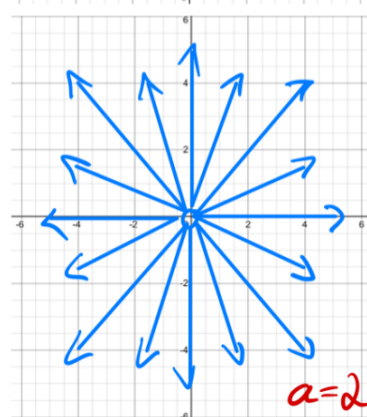
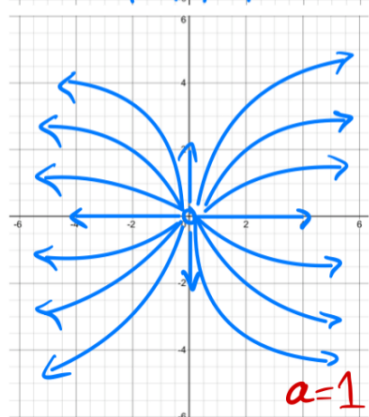
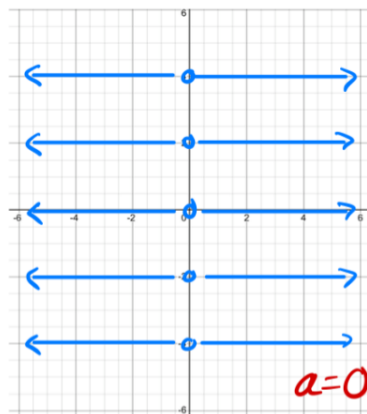
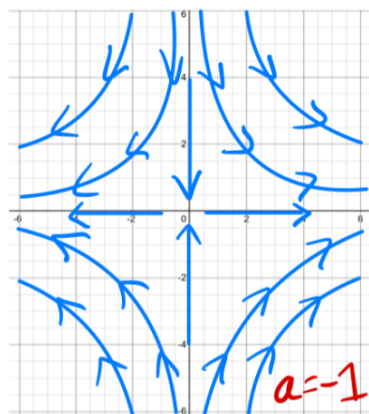
$$\ln |y| = at + c_1$$

$$e^{\ln |y|} = e^{at+c_1}$$

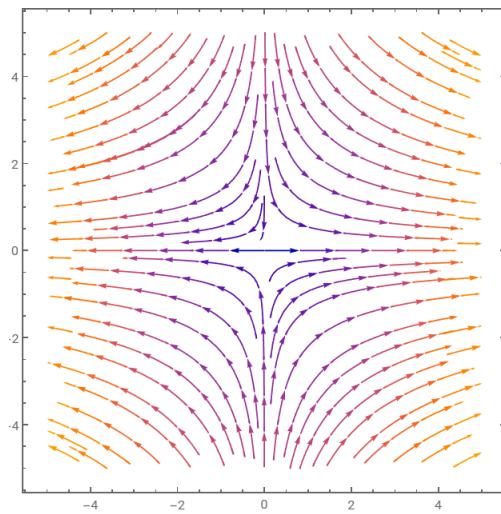
$$y = e^{at} e^{c_1}$$

$$\text{Solution: } y(t) = c_2 e^{at}$$

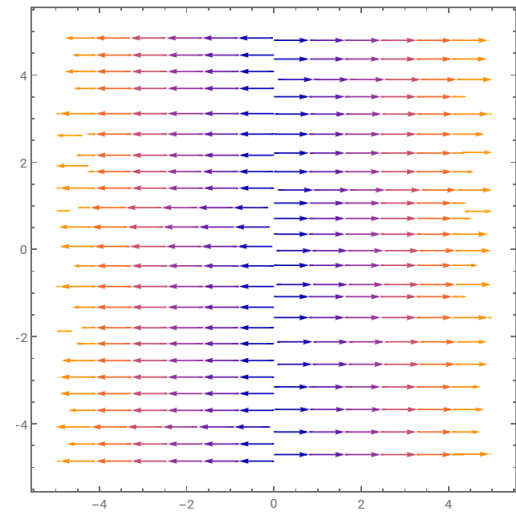
- (b) Sketch the phase portrait for this system for the following values of a : -1, 0, 1, 2. Do not use technology for this part.



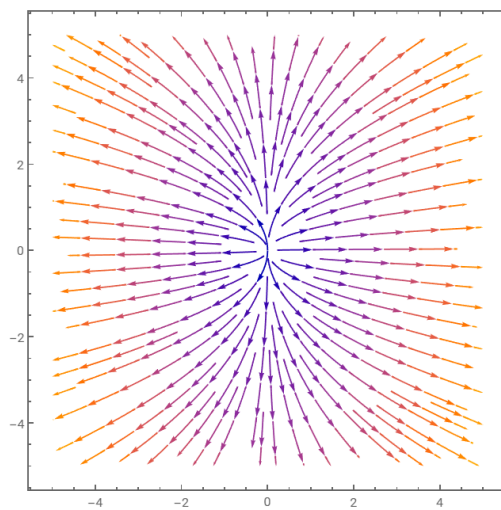
(c) Now use tech (Mathematica, Python, etc) to plot a complete vector field and some trajectories.



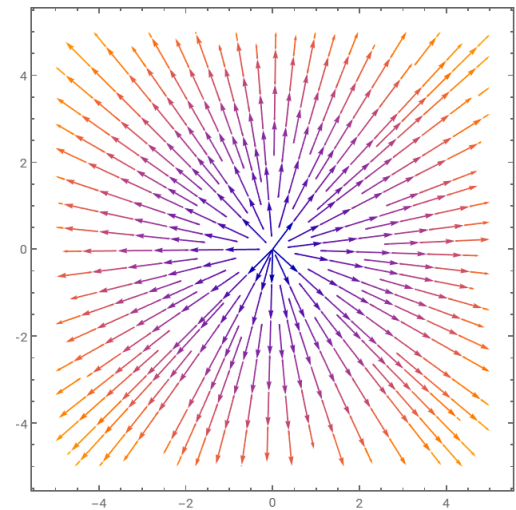
Vector Field: $a = -1$



Vector Field: $a = 0$



Vector Field: $a = 1$



Vector Field: $a = 2$

(d) Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

- ($a = -1$): EP: $(0, 0)$ and this is an unstable saddle point
- ($a = 0$): EP: $(0, 0)$ and this is unstable
- ($a = 1$): EP: $(0, 0)$ and this is unstable
- ($a = 2$): EP: $(0, 0)$ and this is unstable

Question 3:

Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Consider the first order ODE

$$x' = a(t)x$$

- (a) Find a formula involving integrals for the solution of this system.

$$\begin{aligned}\frac{dx}{dt} &= a(t)x(t) \\ \int \frac{1}{x(t)} dx &= \int a(t) dt \\ \ln |x(t)| &= \int a(t) dt + C \\ x(t) &= e^{\int a(t) dt + C} \\ x(t) &= Ce^{\int a(t) dt}\end{aligned}$$

Future knowledge: Hey look integrating factor!!

- (b) Prove that your formula gives the general solution of this system. Hint: follow the sketch of proof done on the first day of class

First, let $u(t)$ be any solution and let's differentiate $f(t) = u(t)e^{\int -a(t) dt}$
Given that $x(t) = Ce^{\int a(t) dt}$, want to show that $f(t) = 0$

Second, we show uniqueness:

$$\begin{aligned}\frac{d}{dt}f(t) &= \frac{d}{dt}u(t)e^{\int a(t) dt} \\ &= \frac{d}{dt}(u(t))e^{\int -a(t) dt} + u(t)\left(\frac{d}{dt}e^{\int -a(t) dt}\right) \\ &= u'(t)e^{\int -a(t) dt} + u(t)\left(\frac{d}{dt}\int -a(t) dt\right)e^{\int -a(t) dt} \\ &= u'(t)e^{\int -a(t) dt} - u(t)a(t)e^{\int -a(t) dt} \\ &= \left(a(t)Ce^{\int a(t) dt}\right)e^{\int -a(t) dt} - u(t)a(t)e^{\int -a(t) dt} \\ &= a(t)u(t)e^{\int a(t) dt} - u(t)a(t)e^{\int -a(t) dt} \\ \frac{d}{dt}f(t) &= a(t)f(t) - a(t)f(t) = 0\end{aligned}$$

Definition of The Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$ and $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$

First, we show existence:

$$\begin{aligned}
 x'(t) &= a(t)x(t) \\
 \frac{dx}{dt} \left(C e^{\int a(t)dt} \right) &= a(t) \left(C e^{\int a(t)dt} \right) \\
 \frac{dx}{dt} \left(\int a(t)dt \right) \left(C e^{\int a(t)dt} \right) &= a(t) \left(C e^{\int a(t)dt} \right) \\
 a(t) \left(C e^{\int a(t)dt} \right) &= a(t) \left(C e^{\int a(t)dt} \right) \\
 RHS &= LHS
 \end{aligned}$$

Thus, since both sides are equal, then $x(t) = C e^{\int a(t)dt}$ is in fact the general solution. Although, I do sense some exploitation of the Fundamental Theorem of Calculus, since we differentiated an integral a bit too easily.