Math 675 Spring 2026:
Homework 1 Unit ?
Due Tuesday, Janurary 3rd, 2026

Question 1:

For each system of differential equations in a-d below

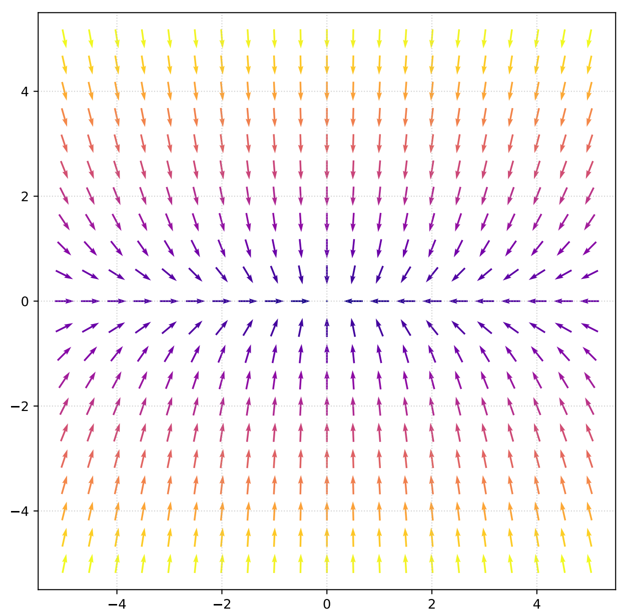
- Plot the change vector using the points $(1, 1), (1, -1), (-1, -1), (-1, 1)$
- Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

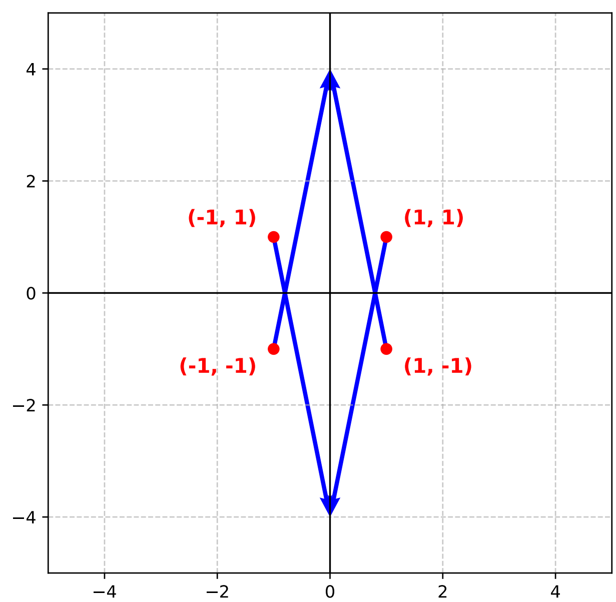
(a) $x' = -x, \quad y' = -5y$
Coupled: No; no dependance
EPs: Set $x' = 0, y' = 0$

$$\begin{cases} -x & = 0 \\ & -5y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Thus, the equilibrium point is at $(0, 0)$ and is unstable

(x, y)	$(x', y') = (-x, -5y)$
$(1, 1)$	$(-(1), -5(1)) = (-1, -5)$
$(1, -1)$	$(-(1), -5(-1)) = (-1, 5)$
$(-1, -1)$	$(-(-1), -5(-1)) = (1, 5)$
$(-1, 1)$	$(-(-1), -5(1)) = (1, -5)$





(b) $x' = 4x - y, \quad y' = 2x + y$

Coupled: Yes;

Derivatives depends on each other

EPs: Set $x' = 0, y' = 0$

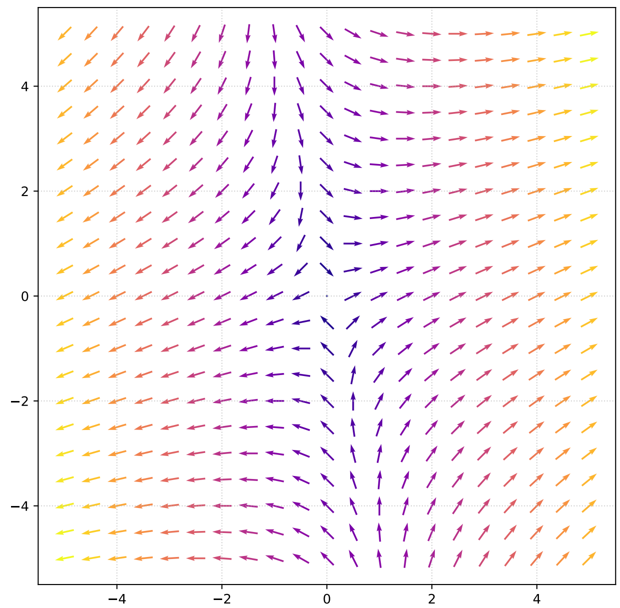
$$\begin{cases} 4x - y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \begin{cases} 4x \quad\quad = y \\ 2x + y = 0 \end{cases}$$

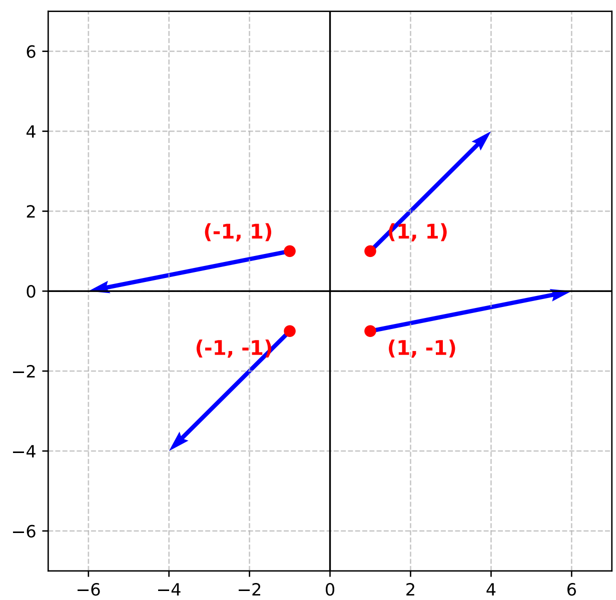
$$\begin{cases} 4x \quad\quad = y \\ 2x + 4x = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 6x = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ x = 0 \end{cases} \rightarrow \begin{cases} 4(0) = 0 \\ x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x', y') = (4x - y, 2x + y)$
$(1, 1)$	$(4 - 1, 2 + 1) = (3, 3)$
$(1, -1)$	$(4 + 1, 2 - 1) = (5, 1)$
$(-1, -1)$	$(-4 + 1, -2 + 1) = (-3, -3)$
$(-1, 1)$	$(-4 - 1, -2 + 1) = (-5, -1)$





(c) $x_1' = x_2, \quad x_2' = -2x_1 - 3x_2$

Coupled: Yes;

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} 4x - y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \begin{cases} 4x \quad = y \\ 2x + y = 0 \end{cases}$$

$$\begin{cases} 4x \quad = y \\ 2x + 4x = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 6x = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ x = 0 \end{cases} \rightarrow \begin{cases} 4(0) = 0 \\ x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x_1', x_2') = (x_2, -2x_1 - 3x_2 y)$
$(1, 1)$	$(-(1), -5(1)) = (-1, -5)$
$(1, -1)$	$(-(1), -5(-1)) = (-1, 5)$
$(-1, -1)$	$(-(-1), -5(-1)) = (1, 5)$
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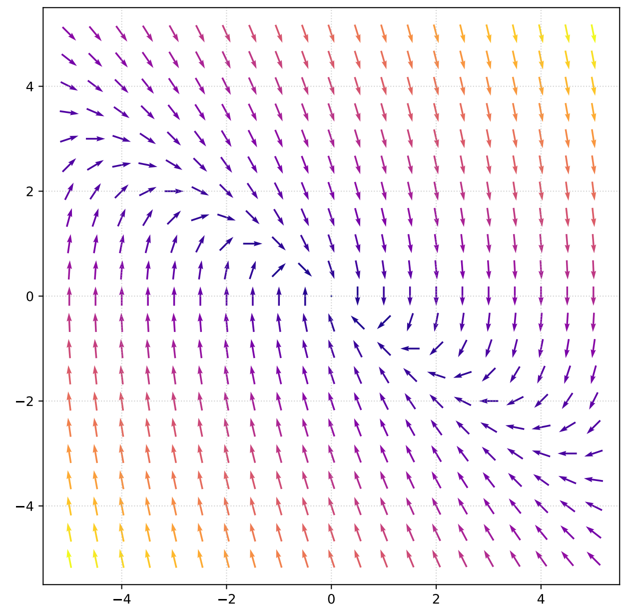

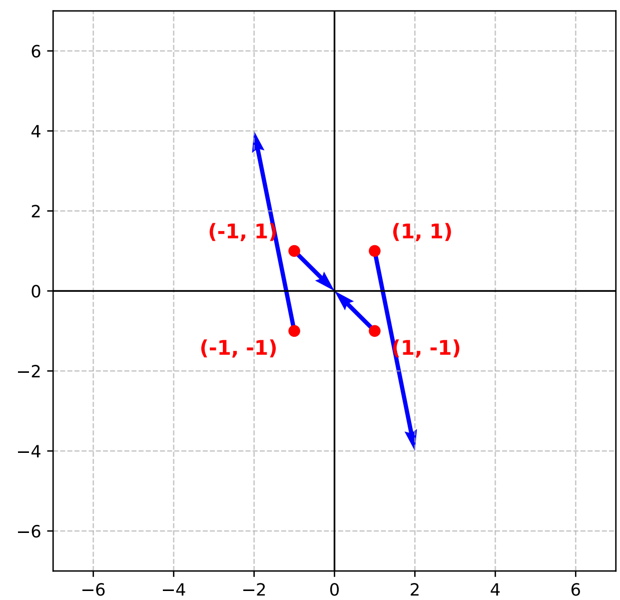

(d) $x' = 5x + 2y, \quad y' = -17x - 5y$

Coupled: Yes;

EPs: Set $x' = 0, y' = 0$

$$\begin{cases} 4x - y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 2x + y = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ 2x + 4x = 0 \end{cases} \rightarrow \begin{cases} 4x = y \\ 6x = 0 \end{cases}$$

$$\begin{cases} 4x = y \\ x = 0 \end{cases} \rightarrow \begin{cases} 4(0) = 0 \\ x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Thus, the EP is $(0, 0)$

(x, y)	$(x_1', x_2') = (x_2, -2x_1 - 3x_2 y)$
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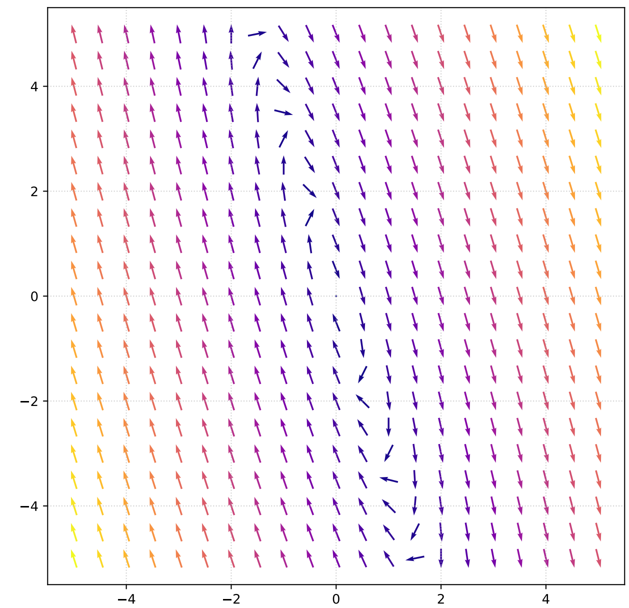
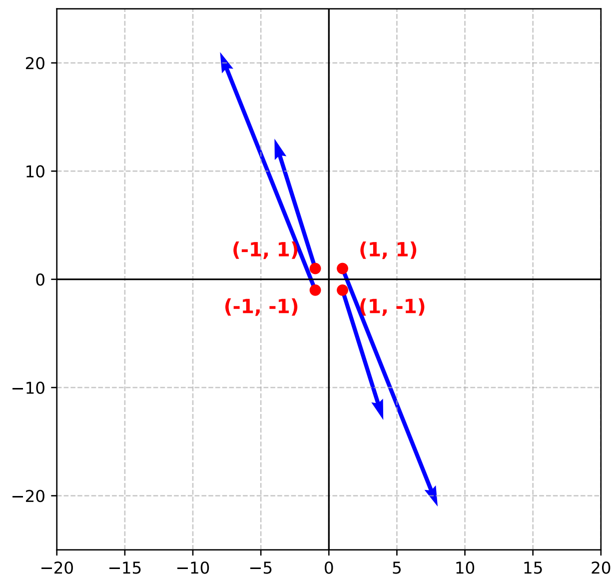




Question 2:

Let $a \in \mathbb{R}$ be a parameter. Consider the system of differential equations

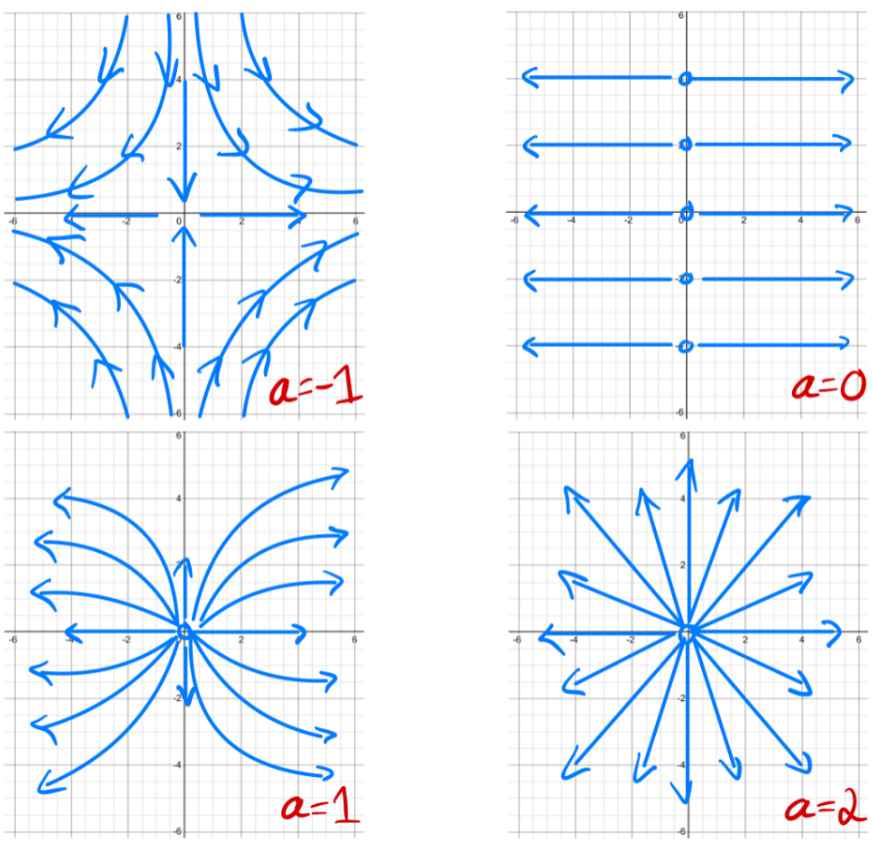
$$x'(t) = 2x, \qquad y'(t) = ay$$

(a) Use separation of variables to find the explicit solution for this system.

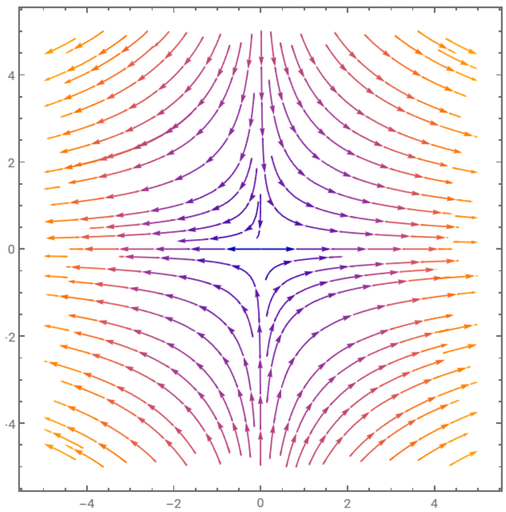
$$\begin{aligned} x'(t) &= 2x \\ \frac{dx}{dt} &= 2x \\ \int \frac{1}{x}dx &= \int 2dt \\ \ln|x| &= 2t + c_1 \\ e^{\ln|x|} &= e^{2t+c_1} \\ x &= e^{2t}e^{c_1} \\ \text{Solution: } x(t) &= c_1e^{2t} \end{aligned}$$

$$\begin{aligned} y'(t) &= -ay \\ \frac{dy}{dt} &= ay \\ \int \frac{1}{y}dy &= \int adt \\ \ln|y| &= at + c_1 \\ e^{\ln|y|} &= e^{at+c_1} \\ y &= e^{at}e^{c_1} \\ \text{Solution: } y(t) &= c_2e^{at} \end{aligned}$$

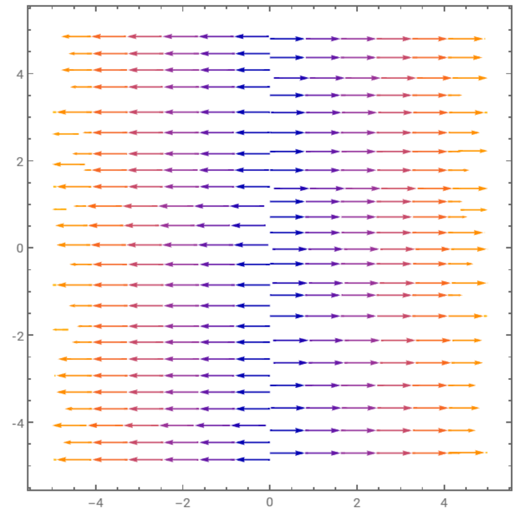
(b) Sketch the phase portrait for this system for the following values of a: -1, 0, 1, 2. Do not use technology for this part.



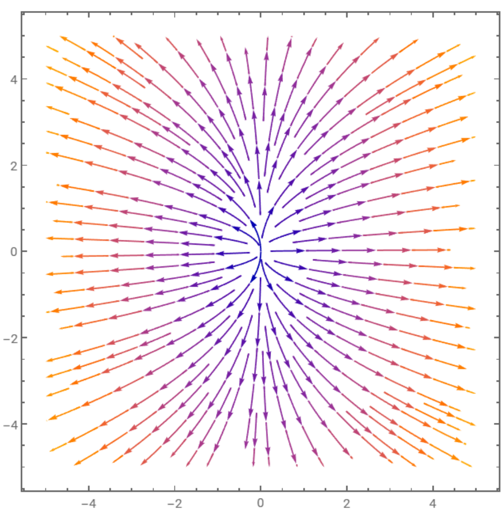
(c) Now use tech (Mathematica, Python, etc) to plot a complete vector field and some trajectories.



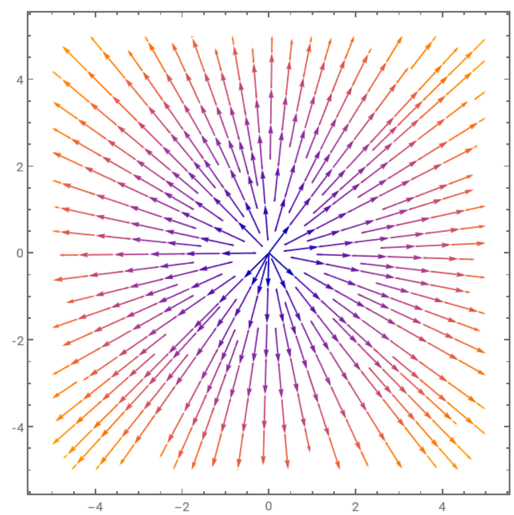
Vector Field: $a = -1$



Vector Field: $a = 0$



Vector Field: $a = 1$



Vector Field: $a = 2$

(d) Use the vector field to classify the equilibrium point(s) as stable, unstable, a saddle point, an unstable spiral, a stable spiral, or a center.

- $(a = -1)$: EP: $(0, 0)$ and this is an unstable saddle point
- $(a = 0)$: EP: $(0, 0)$ and this is unstable
- $(a = 1)$: EP: $(0, 0)$ and this is unstable
- $(a = 2)$: EP: $(0, 0)$ and this is unstable

Question 3:

Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the first order ODE

$$x' = a(t)x$$

(a) Find a formula involving integrals for the solution of this system.

$$\begin{aligned}
\frac{dx}{dt} &= a(t)x(t) \\
\int \frac{1}{x(t)}dx &= \int a(t)dt \\
\ln|x(t)| &= \int a(t)dt + C \\
x(t) &= e^{\int a(t)dt+C} \\
x(t) &= Ce^{\int a(t)dt}
\end{aligned}$$

Future knowledge: Hey look integrating factor!!

(b) Prove that your formula gives the general solution of this system. Hint: follow the sketch of proof done on the first day of class

First, let $u(t)$ be any solution and let's differentiate $f(t) = u(t)e^{\int -a(t)dt}$
Given that $x(t) = Ce^{\int a(t)dt}$, want to show that $f(t) = 0$
Second, we show uniqueness:

$$\begin{aligned}
\frac{d}{dt}f(t) &= \frac{d}{dt}u(t)e^{\int a(t)dt} \\
&= \frac{d}{dt}\left(u(t)\right)e^{\int -a(t)dt} + u(t)\left(\frac{d}{dt}e^{\int -a(t)dt}\right) \\
&= u'(t)e^{\int -a(t)dt} + u(t)\left(\frac{d}{dt}\int -a(t)dt\right)e^{\int -a(t)dt} \\
&= u'(t)e^{\int -a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\
&= \left(a(t)Ce^{\int a(t)dt}\right)e^{\int -a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\
&= a(t)u(t)e^{\int a(t)dt} - u(t)a(t)e^{\int -a(t)dt} \\
\frac{d}{dt}f(t) &= a(t)f(t) - a(t)f(t) = 0
\end{aligned}$$

Definition of The Fundamental Theorem of Calculus:
If f is continuous on $[a,b]$ and $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$
First, we show existence:

$$x'(t) = a(t)x(t)$$
$$\frac{dx}{dt}\left(Ce^{\int a(t)dt}\right) = a(t)\left(Ce^{\int a(t)dt}\right)$$
$$\frac{dx}{dt}\left(\int a(t)dt\right)\left(Ce^{\int a(t)dt}\right) = a(t)\left(Ce^{\int a(t)dt}\right)$$
$$a(t)\left(Ce^{\int a(t)dt}\right) = a(t)\left(Ce^{\int a(t)dt}\right)$$
$$RHS = LHS$$

Thus, since both sides are equal, then $x(t) = Ce^{\int a(t)dt}$ is in fact the general solution. Although, I do sense some explotation of the Fundamental Theorem of Calculus, since we differentiated an integral a bit too easily.