

**Math 675 Spring 2026:**  
**Homework 1 Unit ?**  
**Due Tuesday, Janurary 3rd, 2026**

**Question 1:**

Consider the uncoupled system:

$$x' = 3x, y' = -y$$

- a) Solve the system explicitly for general initial conditions  $(x(0), y(0)) = (x_0, y_0)$ .

$$\begin{array}{ll} x'(t) = 3x & y'(t) = -y \\ \frac{dx}{dt} = 3x & \frac{dy}{dt} = -y \\ \int \frac{1}{x} dx = \int 3dt & \int \frac{1}{y} dy = \int -1dt \\ \ln|x| = 3t + c_1 & \ln|y| = -t + c_2 \\ e^{\ln|x|} = e^{3t+c_1} & e^{\ln|y|} = e^{-t+c_2} \\ x = e^{3t}e^{c_1} & y = e^{-t}e^{c_2} \\ \text{Solution: } x(t) = c_1 e^{3t} & \text{Solution: } y(t) = c_2 e^{-t} \\ \text{IC: } x(0) = x_0 = c_1 & \text{IC: } y(0) = y_0 = c_2 \\ \text{Final Solution: } x(t) = x_0 e^{3t} & \text{Final Solution: } y(t) = y_0 e^{-t} \end{array}$$

- b) Identify all equilibrium points of the system.

$$\begin{cases} 3x = 0 \\ -y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

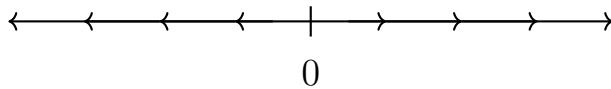
So like before, the only equilibrium point is at the origin  $(0, 0)$ .

- c) Using your explicit solution, determine the behavior of solutions as  $t \rightarrow +\infty$ :  
 Given out solutions from before:  $x(t) = x_0 e^{3t}$  and  $y(t) = y_0 e^{-t}$

- i) initial conditions on the x-axis

Using just  $x = x_0 e^{3t}$ , As  $t \rightarrow +\infty$ ,

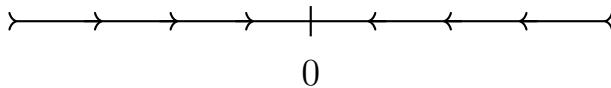
If  $x_0 > 0$  then  $x(t) \rightarrow +\infty$  and if  $x_0 < 0$   $x(t) \rightarrow -\infty$ .



- ii) initial conditions on the y-axis

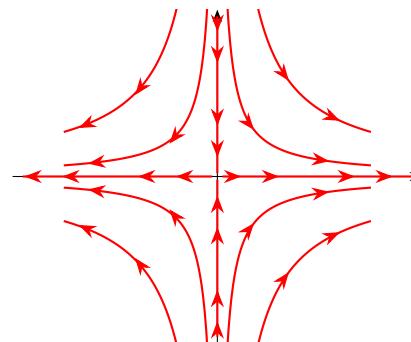
Using just  $y = y_0 e^{-t}$ , As  $t \rightarrow +\infty$ ,

If  $y_0 > 0$  then  $y(t) \rightarrow 0$  and if  $y_0 < 0$   $y(t) \rightarrow 0$ .



- iii) initial conditions not lying on either axis

- If  $t \rightarrow +\infty$  then solutions approach the positive x-axis



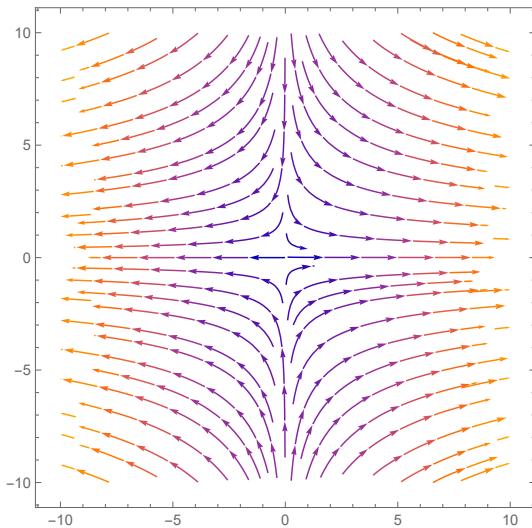
- If  $t \rightarrow -\infty$  then solutions approach the negative x-axis

- d) We can see from the behavior of the solutions as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$  that the equilibrium point at the origin is a saddle point. This is because along the x-axis, the solutions diverge away from the origin, while along the y-axis, the solutions converge towards the origin.

We can also see this as we take limits to positive and negative infinity of  $x(t) = x_0 e^{3t}$  and  $y(t) = y_0 e^{-t}$  similar to what we have above,  
 As  $t \rightarrow +\infty$ ,  $x(t) \rightarrow +\infty$  and  $y(t) \rightarrow 0$ ,  
 As  $t \rightarrow -\infty$ ,  $x(t) \rightarrow -\infty$  and  $y(t) \rightarrow 0$ .

Because the limits for  $y(t)$  are both 0 yet the sign of infinity changes for  $x(t)$ , this tells us that this equilibrium point is a saddle point

- e) Now we want to use Mathematica to plot the full phase portrait of this system.



## Question 2:

Consider the uncoupled system:

$$x' = -5x, y' = -y$$

- a) Solve the system explicitly for general initial conditions  $(x(0), y(0)) = (x_0, y_0)$

$$\begin{array}{ll}
 x'(t) = -5x & y'(t) = -y \\
 \frac{dx}{dt} = -5x & \frac{dy}{dt} = -y \\
 \int \frac{1}{x} dx = \int -5 dt & \int \frac{1}{y} dy = \int -1 dt \\
 \ln|x| = -5t + c_1 & \ln|y| = -t + c_2 \\
 e^{\ln|x|} = e^{-5t+c_1} & e^{\ln|y|} = e^{-t+c_2} \\
 x = e^{-5t} e^{c_1} & y = e^{-t} e^{c_2} \\
 \text{Solution: } x(t) = c_1 e^{-5t} & \text{Solution: } y(t) = c_2 e^{-t} \\
 \text{IC: } x(0) = x_0 = c_1 & \text{IC: } y(0) = y_0 = c_2 \\
 \text{Final Solution: } x(t) = x_0 e^{-5t} & \text{Final Solution: } y(t) = y_0 e^{-t}
 \end{array}$$

- b) Determine which variable decays faster as  $t \rightarrow +\infty$ , and justify your answer using the explicit solutions.

As  $t \rightarrow +\infty$ , we can see both solutions  $x(t) = x_0 e^{-5t}$  and  $y(t) = y_0 e^{-t}$  approach zero. However, the solution for  $x(t)$  will decay faster than  $y(t)$  because  $x(t)$  has a larger negative exponent compared to  $y(t) = y_0 e^{-t}$ , which means that  $x(t)$  will approach zero faster than  $y(t)$  as time increases.

- c) Show that for any solution with  $x_0 \neq 0$

$$\lim_{t \rightarrow +\infty} \frac{y(t)}{x(t)} = \infty$$

Given our solutions  $x(t) = x_0 e^{-5t}$  and  $y(t) = y_0 e^{-t}$ , we can compute the limit:

$$\lim_{t \rightarrow +\infty} \frac{y(t)}{x(t)} = \lim_{t \rightarrow +\infty} \frac{y_0 e^{-t}}{x_0 e^{-5t}} = \lim_{t \rightarrow +\infty} \frac{y_0}{x_0} e^{4t}$$

Since  $e^{4t}$  increases to infinity as  $t \rightarrow +\infty$ , then the whole limit will also approach infinity. The only condition we need is that  $x_0 \neq 0$  to ensure that the denominator does not equal zero, which would make the limit undefined.

- d) Conclude that, except for the trajectory along the x-axis, all other solution curves asymptotically converge to the x-axis as  $t \rightarrow +\infty$ .

$$\lim_{t \rightarrow +\infty} x_0 e^{-5t} = 0 \quad \lim_{t \rightarrow +\infty} y_0 e^{-t} = 0$$

Since we see that both solutions converge to 0, this implies that the slopes also converge to horizontal lines, thus they become parallel to the x-axis.

- e) Conclude that, except for the trajectory along the x-axis, all other solution curves asymptotically converge to the y-axis as  $t \rightarrow -\infty$ .

$$\lim_{t \rightarrow -\infty} x_0 e^{-5t} = \infty \quad \lim_{t \rightarrow -\infty} y_0 e^{-t} = \infty$$

Since we see that both solutions converge to  $\infty$ , this implies that the slopes also converge to vertical lines, thus they become parallel to the y-axis.

### Question 3:

Show that for a linear, homogeneous system of ODE with  $a, b, c, d \neq 0$ , where  $x'$  and  $y'$  are not scalar multiples of each other, the only equilibrium point is  $(0, 0)$ .

$$x' = ax + by, \quad y' = cx + dy$$

First, lets find the determinate of this system, since we are basically talking about linear independence. Let a matrix  $A$ , be the coefficient matrix of this system.

We want then force  $\det(A) \neq 0$  so  $x'$  and  $y'$  are not scalar multiples of each other

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \quad (1)$$

Now, let's try to find the equilibrium points of this system. We set  $x' = 0$  and  $y' = 0$  and solve for  $x$  and  $y$ .

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} ax + -by = 0 \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} x + -\frac{by}{a} = 0 \\ cx + dy = 0 \end{cases}$$

Now lets substitute the first equation into the second equation to solve for  $y$ .

$$0 = cx + dy = c\left(-\frac{by}{a}\right) + dy = -\left(\frac{bc}{a}\right)y + dy = y\left(d - \frac{bc}{a}\right) \quad (2)$$

This is where things get interesting, since we're given  $a, b, c, d \neq 0$  and  $ad - bc \neq 0$ , then we can see we have two solutions for this equation  $y = 0$  or  $d - \frac{bc}{a} = 0$ .

$$0 = d - \frac{bc}{a} \implies ad - bc = 0$$

But we are given that  $ad - bc \neq 0$  from equation (1), so the only valid solution for equation (2) is  $y = 0$ . Now we can substitute this back to solve for  $x$ .

$$x = -\frac{by}{a} = -\frac{b \cdot 0}{a} = 0$$

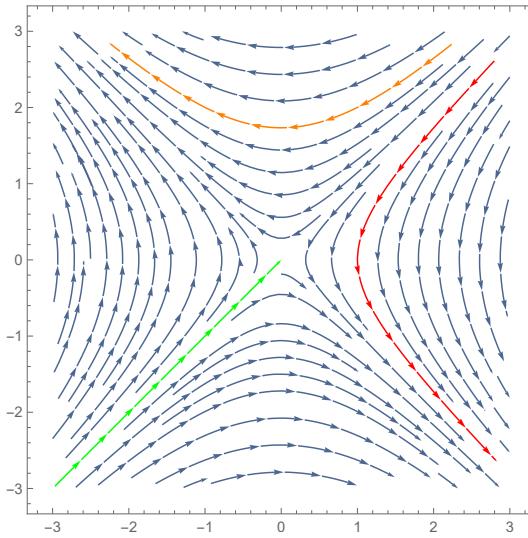
Therefore, the only equilibrium point is  $(0, 0)$ .

## Question 4:

Consider the coupled ODE system

$$x' = -y, \quad y' = -x$$

- a) Use Mathematica to plot the phase portrait of this system.



- b) Follow the outline below to prove that the trajectories of the system are hyperbolas of the form  $x^2 - y^2 = C$  where  $C$  is a constant.

- i) Show that the system of ODEs implies  $x \frac{dx}{dt} - y \frac{dy}{dt} = 0$ .

$$x' = -y, \quad y' = -x$$

$$\frac{x'}{y} = -1 \quad \frac{y'}{x} = -1$$

$$\frac{x'}{y} = \frac{y'}{x}$$

$$xx' = yy'$$

$$x \frac{dx}{dt} - y \frac{dy}{dt} = 0$$

- ii) Use the chain rule to show that  $\frac{1}{2} \frac{d}{dt}(x^2) = x \frac{dx}{dt}$ .

$$\frac{1}{2} \frac{d}{dt}(x(t))^2 = \frac{1}{2}(2)x(t) \frac{dx}{dt} = x(t) \frac{dx}{dt}$$

iii) Substitute the identity for both variables in part ii) into EQ(1)

$$x \frac{dx}{dt} - y \frac{dy}{dt} = 0$$

$$\frac{1}{2} \frac{d}{dt}(x^2) - \frac{1}{2} \frac{d}{dt}(y^2) = 0$$

iv) Integrate both sides of the equation and use the fundamental theorem of calculus to get the results

$$\frac{d}{dt} \left( \frac{x^2}{2} - \frac{y^2}{2} \right) = 0$$

$$\int \frac{d}{dt} \left( \frac{x^2}{2} - \frac{y^2}{2} \right) dt = \int 0 dt$$

$$\frac{x^2}{2} - \frac{y^2}{2} = C$$

$$x^2 - y^2 = C$$

## Question 5:

In this problem we consider the same coupled ODE system as in the previous problem

$$x' = -y, y' = -x$$

We want to decouple and solve this analytically. So please do the following:

- a) Introduce new variables  $u$  and  $v$  defined by  $u = x + y$  and  $v = x - y$ .  
Solve for the variables  $u, v$  in terms of  $x$  and  $y$ .

$$u + v = (x + y) + (x - y) \quad u - v = (x + y) - (x - y)$$

$$u + v = 2x \quad u - v = 2y$$

$$\frac{u + v}{2} = x \quad \frac{u - v}{2} = y$$

- b) Rewrite the ODE system in terms of  $u$  and  $v$  using the previous part. We want to differentiate  $u$  and  $v$  with respect to  $t$  to get  $u'$  and  $v'$ . Then substitute the original ODEs for  $x'$  and  $y'$  to get the new ODEs in terms of  $u$  and  $v$ .

$$u' = x' + y' = -y - x = -(x + y) = -u$$

$$v' = x' - y' = -y + x = (x - y) = v$$

c) Solve for  $u(t)$  and  $v(t)$  starting from an arbirary initial condition  $(u_0, v_0)$ .

$$\begin{aligned}
 u'(t) &= -u & v'(t) &= -v \\
 \frac{du}{dt} &= -u & \frac{dv}{dt} &= v \\
 \int \frac{1}{u} du &= \int -1 dt & \int \frac{1}{v} dv &= \int 1 dt \\
 \ln |u| &= -t + c_1 & \ln |v| &= t + c_2 \\
 e^{\ln |u|} &= e^{-t+c_1} & e^{\ln |v|} &= e^{t+c_2} \\
 u &= e^{-t} e^{c_1} & v &= e^t e^{c_2} \\
 \text{Solution: } u(t) &= c_1 e^{-t} & \text{Solution: } v(t) &= c_2 e^t \\
 \text{IC: } u(0) &= u_0 = c_1 & \text{IC: } v(0) &= v_0 = c_2 \\
 \text{Final Solution: } u(t) &= u_0 e^{-t} & \text{Final Solution: } v(t) &= v_0 e^t
 \end{aligned}$$

d) Finally, using the answer above, write the general solution in terms of  $x(t)$  and  $y(t)$ . Make sure to also write starting from the initial conditions  $(x_0, y_0)$ . We can note that if  $t = 0$  implies  $u(0) = u_0$  and  $v(0) = v_0$  then we can also conclude that we can obtain the initial conditions of  $u$  and  $v$  in terms of the initial conditions of  $x$  and  $y$ .

$$u(0) = x(0) + y(0) = x_0 + y_0 \quad v(0) = x(0) - y(0) = x_0 - y_0$$

Now, substitute these modified initial conditions into our solutions to get:

$$\begin{aligned}
 x(t) &= \frac{1}{2} [u(t) + v(t)] & y(t) &= \frac{1}{2} [u(t) - v(t)] \\
 &= \frac{1}{2} [u_0 e^{-t} + v_0 e^t] & &= \frac{1}{2} [u_0 e^{-t} - v_0 e^t] \\
 &= \frac{1}{2} [(x_0 + y_0)e^{-t} + (x_0 - y_0)e^t] & &= \frac{1}{2} [(x_0 + y_0)e^{-t} - (x_0 - y_0)e^t] \\
 &= \frac{1}{2} [x_0(e^{-t} + e^t) + y_0(e^{-t} - e^t)] & &= \frac{1}{2} [x_0(e^{-t} - e^t) + y_0(e^{-t} + e^t)] \\
 &= x_0 \cosh(t) - y_0 \sinh(t) & &= -x_0 \sinh(t) + y_0 \cosh(t)
 \end{aligned}$$