

Math 675 Spring 2026:
Homework 1 Unit ?
Due Tuesday, January 3rd, 2026

Question 1:

Consider the uncoupled system:

$$x' = 3x, y' = -y$$

a) Solve the system explicitly for general initial conditions $(x(0), y(0)) = (x_0, y_0)$.

$x'(t) = 3x$	$y'(t) = -y$
$\frac{dx}{dt} = 3x$	$\frac{dy}{dt} = -y$
$\int \frac{1}{x} dx = \int 3 dt$	$\int \frac{1}{y} dy = \int -1 dt$
$\ln x = 3t + c_1$	$\ln y = -t + c_2$
$e^{\ln x } = e^{3t+c_1}$	$e^{\ln y } = e^{-t+c_2}$
$x = e^{3t} e^{c_1}$	$y = e^{-t} e^{c_2}$
Solution: $x(t) = c_1 e^{3t}$	Solution: $y(t) = c_2 e^{-t}$
IC: $x(0) = x_0 = c_1$	IC: $y(0) = y_0 = c_2$
Final Solution: $x(t) = x_0 e^{3t}$	Final Solution: $y(t) = y_0 e^{-t}$

b) Identify all equilibrium points of the system.

$$\begin{cases} 3x = 0 \\ -y = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

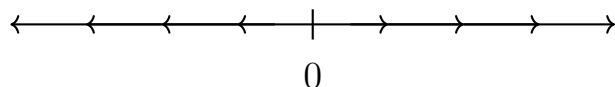
So like before, the only equilibrium point is at the origin $(0, 0)$.

- c) Using your explicit solution, determine the behavior of solutions as $t \rightarrow +\infty$:
 Given out solutions from before: $x(t) = x_0 e^{3t}$ and $y(t) = y_0 e^{-t}$

- i) initial conditions on the x-axis

Using just $x = x_0 e^{3t}$, As $t \rightarrow +\infty$,

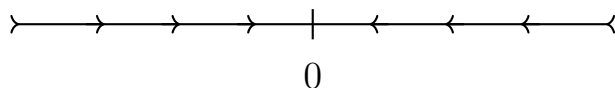
If $x_0 > 0$ then $x(t) \rightarrow +\infty$ and if $x_0 < 0$ $x(t) \rightarrow -\infty$.



- ii) initial conditions on the y-axis

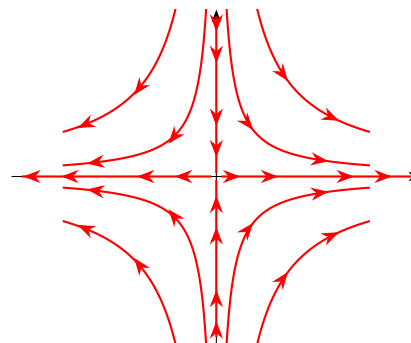
Using just $y = y_0 e^{-t}$, As $t \rightarrow +\infty$,

If $y_0 > 0$ then $y(t) \rightarrow 0$ and if $y_0 < 0$ $y(t) \rightarrow 0$.



- iii) initial conditions not lying on either axis

- If $t \rightarrow +\infty$ then solutions approach the positive x-axis



- If $t \rightarrow -\infty$ then solutions approach the negative x-axis

- d) We can see from the behavior of the solutions as $t \rightarrow +\infty$ and $t \rightarrow -\infty$ that the equilibrium point at the origin is a saddle point. This is because along the x-axis, the solutions diverge away from the origin, while along the y-axis, the solutions converge towards the origin.

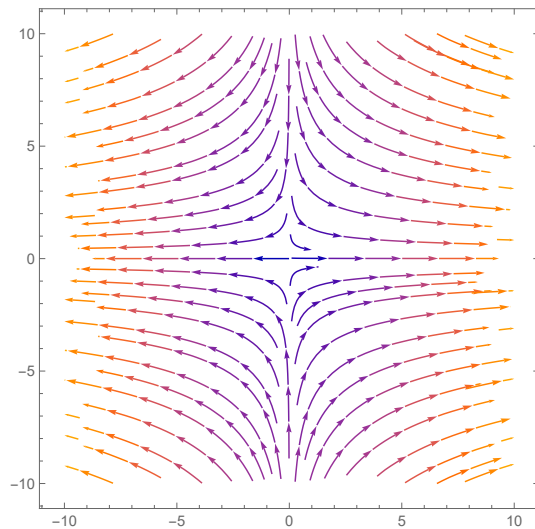
We can also see this as we take limits to positive and negative infinity of $x(t) = x_0 e^{3t}$ and $y(t) = y_0 e^{-t}$ similar to what we have above,

As $t \rightarrow +\infty$, $x(t) \rightarrow +\infty$ and $y(t) \rightarrow 0$,

As $t \rightarrow -\infty$, $x(t) \rightarrow -\infty$ and $y(t) \rightarrow 0$.

Because the limits for $y(t)$ are both 0 yet the sign of infinity changes for $x(t)$, this tells us that this equilibrium point is a saddle point

- e) Now we want to use Mathematica to plot the full phase portrait of this system.



Question 2:

Consider the uncoupled system:

$$x' = -5x, y' = -y$$

- a) Solve the system explicitly for general initial conditions $(x(0), y(0)) = (x_0, y_0)$

$$\begin{aligned} x'(t) &= -5x \\ \frac{dx}{dt} &= -5x \\ \int \frac{1}{x} dx &= \int -5 dt \\ \ln |x| &= -5t + c_1 \\ e^{\ln |x|} &= e^{-5t+c_1} \\ x &= e^{-5t} e^{c_1} \end{aligned}$$

$$\text{Solution: } x(t) = c_1 e^{-5t}$$

$$\text{IC: } x(0) = x_0 = c_1$$

$$\text{Final Solution: } x(t) = x_0 e^{-5t}$$

$$\begin{aligned} y'(t) &= -y \\ \frac{dy}{dt} &= -y \\ \int \frac{1}{y} dy &= \int -1 dt \\ \ln |y| &= -t + c_2 \\ e^{\ln |y|} &= e^{-t+c_2} \\ y &= e^{-t} e^{c_2} \end{aligned}$$

$$\text{Solution: } y(t) = c_2 e^{-t}$$

$$\text{IC: } y(0) = y_0 = c_2$$

$$\text{Final Solution: } y(t) = y_0 e^{-t}$$

- b) Determine which variable decays faster as $t \rightarrow +\infty$, and justify your answer using the explicit solutions.

As $t \rightarrow +\infty$, we can see both solutions $x(t) = x_0 e^{-5t}$ and $y(t) = y_0 e^{-t}$ approach zero. However, the solution for $x(t)$ will decay faster than $y(t)$ because $x(t)$ has a larger negative exponent compared to $y(t) = y_0 e^{-t}$, which means that $x(t)$ will approach zero faster than $y(t)$ as time increases.

- c) Show that for any solution with $x_0 \neq 0$

$$\lim_{t \rightarrow +\infty} \frac{y(t)}{x(t)} = \infty$$

Given our solutions $x(t) = x_0 e^{-5t}$ and $y(t) = y_0 e^{-t}$, we can compute the limit:

$$\lim_{t \rightarrow +\infty} \frac{y(t)}{x(t)} = \lim_{t \rightarrow +\infty} \frac{y_0 e^{-t}}{x_0 e^{-5t}} = \lim_{t \rightarrow +\infty} \frac{y_0}{x_0} e^{4t}$$

Since e^{4t} increases to infinity as $t \rightarrow +\infty$, then the whole limit will also approach infinity. The only condition we need is that $x_0 \neq 0$ to ensure that the denominator does not equal zero, which would make the limit undefined.

- d) Conclude that, except for the trajectory along the x-axis, all other solution curves asymptotically converge to the x-axis as $t \rightarrow +\infty$.

$$\lim_{t \rightarrow +\infty} x_0 e^{-5t} = 0 \qquad \lim_{t \rightarrow +\infty} y_0 e^{-t} = 0$$

Since we see that both solutions converge to 0, this implies that the slopes also converge to horizontal lines, thus they become parallel to the x-axis.

- e) Conclude that, except for the trajectory along the x-axis, all other solution curves asymptotically converge to the y-axis as $t \rightarrow -\infty$.

$$\lim_{t \rightarrow -\infty} x_0 e^{-5t} = \infty \qquad \lim_{t \rightarrow -\infty} y_0 e^{-t} = \infty$$

Since we see that both solutions converge to ∞ , this implies that the slopes also converge to vertical lines, thus they become parallel to the y-axis.

Question 3:

Show that for a linear, homogeneous system of ODE with $a, b, c, d \neq 0$, where x' and y' are not scalar multiples of each other, the only equilibrium point is $(0, 0)$.

$$x' = ax + by, \quad y' = cx + dy$$

First, let's find the determinate of this system, since we are basically talking about linear independence. Let a matrix A , be the coefficient matrix of this system.

We want then force $\det(A) \neq 0$ so x' and y' are not scalar multiples of each other

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \quad (1)$$

Now, let's try to find the equilibrium points of this system. We set $x' = 0$ and $y' = 0$ and solve for x and y .

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} ax + \quad = -by \\ cx + dy = 0 \end{cases} \rightarrow \begin{cases} x + \quad = -\frac{by}{a} \\ cx + dy = 0 \end{cases}$$

Now let's substitute the first equation into the second equation to solve for y .

$$0 = cx + dy = c \left(-\frac{by}{a} \right) + dy = -\left(\frac{bc}{a} \right) y + dy = y \left(d - \frac{bc}{a} \right) \quad (2)$$

This is where things get interesting, since we're given $a, b, c, d \neq 0$ and $ad - bc \neq 0$, then we can see we have two solutions for this equation $y = 0$ or $d - \frac{bc}{a} = 0$.

$$0 = d - \frac{bc}{a} \implies ad - bc = 0$$

But we are given that $ad - bc \neq 0$ from equation (1), so the only valid solution for equation (2) is $y = 0$. Now we can substitute this back to solve for x .

$$x = -\frac{by}{a} = -\frac{b \cdot 0}{a} = 0$$

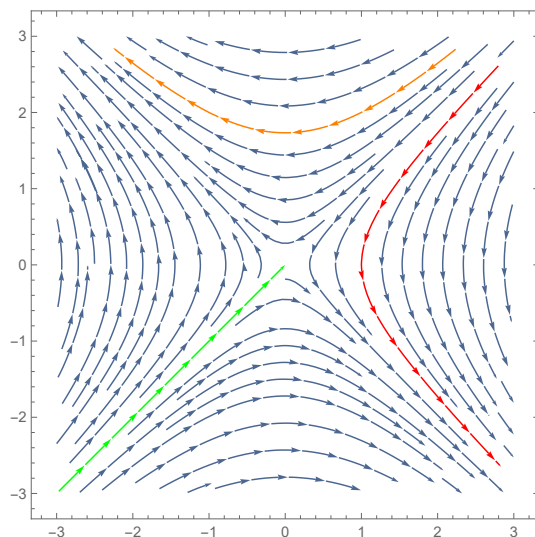
Therefore, the only equilibrium point is $(0, 0)$.

Question 4:

Consider the coupled ODE system

$$x' = -y, \quad y' = -x$$

- a) Use Mathematica to plot the phase portrait of this system.



- b) Follow the outline below to prove that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$ where C is a constant.
- i) Show that the system of ODEs implies $x \frac{dx}{dt} - y \frac{dy}{dt} = 0$.

$$x' = -y, \quad y' = -x$$

$$\frac{x'}{y} = -1 \quad \frac{y'}{x} = -1$$

$$\frac{x'}{y} = \frac{y'}{x}$$

$$xx' = yy'$$

$$x \frac{dx}{dt} - y \frac{dy}{dt} = 0$$

- ii) Use the chain rule to show that $\frac{1}{2} \frac{d}{dt}(x^2) = x \frac{dx}{dt}$.

$$\frac{1}{2} \frac{d}{dt}(x(t))^2 = \frac{1}{2} (2x(t)) \frac{dx}{dt} = x(t) \frac{dx}{dt}$$

iii) Substitute the identity for both variables in part ii) into EQ(1)

$$x \frac{dx}{dt} - y \frac{dy}{dt} = 0$$

$$\frac{1}{2} \frac{d}{dt}(x^2) - \frac{1}{2} \frac{d}{dt}(y^2) = 0$$

iv) Integrate both sides of the equation and use the fundamental theorem of calculus to get the results

$$\frac{d}{dt} \left(\frac{x^2}{2} - \frac{y^2}{2} \right) = 0$$

$$\int \frac{d}{dt} \left(\frac{x^2}{2} - \frac{y^2}{2} \right) dt = \int 0 dt$$

$$\frac{x^2}{2} - \frac{y^2}{2} = C$$

$$x^2 - y^2 = C$$

Question 5:

In this problem we consider the same coupled ODE system as in the previous problem

$$x' = -y, y' = -x$$

We want to decouple and solve this analytically. So please do the following:

- a) Introduce new variables u and v defined by $u = x + y$ and $v = x - y$.
Solve for the variables u, v in terms of x and y .

$$u + v = (x + y) + (x - y) \quad u - v = (x + y) - (x - y)$$

$$u + v = 2x \quad u - v = 2y$$

$$\frac{u + v}{2} = x \quad \frac{u - v}{2} = y$$

- b) Rewrite the ODE system in terms of u and v using the previous part We want to differentiate u and v with respect to t to get u' and v' . Then substitute the original ODEs for x' and y' to get the new ODEs in terms of u and v .

$$u' = x' + y' = -y - x = -(x + y) = -u$$

$$v' = x' - y' = -y + x = (x - y) = v$$

c) Solve for $u(t)$ and $v(t)$ starting from an arbitrary initial condition (u_0, v_0) .

$u'(t) = -u$	$v'(t) = -v$
$\frac{du}{dt} = -u$	$\frac{dv}{dt} = v$
$\int \frac{1}{u} du = \int -1 dt$	$\int \frac{1}{v} dv = \int 1 dt$
$\ln u = -t + c_1$	$\ln v = t + c_2$
$e^{\ln u } = e^{-t+c_1}$	$e^{\ln v } = e^{t+c_2}$
$u = e^{-t} e^{c_1}$	$v = e^t e^{c_2}$
Solution: $u(t) = c_1 e^{-t}$	Solution: $v(t) = c_2 e^t$
IC: $u(0) = u_0 = c_1$	IC: $v(0) = v_0 = c_2$
Final Solution: $u(t) = u_0 e^{-t}$	Final Solution: $v(t) = v_0 e^t$

d) Finally, using the answer above, write the general solution in terms of $x(t)$ and $y(t)$. Make sure to also write starting from the initial conditions (x_0, y_0) . We can note that if $t = 0$ implies $u(0) = u_0$ and $v(0) = v_0$ then we can also conclude that we can obtain the initial conditions of u and v in terms of the initial conditions of x and y .

$$u(0) = x(0) + y(0) = x_0 + y_0 \qquad v(0) = x(0) - y(0) = x_0 - y_0$$

Now, substitute these modified initial conditions into our solutions to get:

$x(t) = \frac{1}{2} [u(t) + v(t)]$	$y(t) = \frac{1}{2} [u(t) - v(t)]$
$= \frac{1}{2} [u_0 e^{-t} + v_0 e^t]$	$= \frac{1}{2} [u_0 e^{-t} - v_0 e^t]$
$= \frac{1}{2} [(x_0 + y_0) e^{-t} + (x_0 - y_0) e^t]$	$= \frac{1}{2} [(x_0 + y_0) e^{-t} - (x_0 - y_0) e^t]$
$= \frac{1}{2} [x_0 (e^{-t} + e^t) + y_0 (e^{-t} - e^t)]$	$= \frac{1}{2} [x_0 (e^{-t} - e^t) + y_0 (e^{-t} + e^t)]$
$= x_0 \cosh(t) - y_0 \sinh(t)$	$= -x_0 \sinh(t) + y_0 \cosh(t)$