Capacity Analysis of Coprime Communication

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Overview

Introduction

The study of capacity of analog Gaussian channels and capacity-achieving transmission strategies was pioneered by Shannon, whose work focused on capacity of channels sampled at or above twice the channel bandwidth.

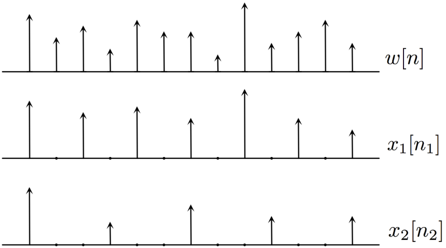
However, in practice, the Nyquist rate may be **excessive** for perfect reconstruction of signals that the transmission channel possess certain structures known a priori. On the other hand, the **hardware** and **power limitations** may preclude sampling at the Nyquist rate for a wideband communication system.

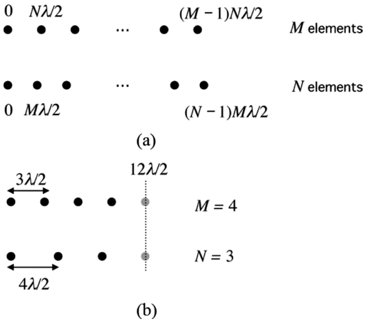
This motivates the exploration of the effects of sub-Nyquist sampling upon the capacity of an analog Gaussian channel, and the capacity limits that result from general sampling methods.

Related works

- When the channel or signal structure is unknown, the blind sub-Nyquist sampling approaches have be proposed to exploit the structure of various classes of input signals based on sampling with modulation and filter banks [Mishali, 2010].
- At the transmitter side, although MRC maximizes the combiner SNR for a MISO channel, it is suboptimal for the joint optimization problem compared with selection combining [Goldsmith, 2005].
- The capacity with sampling rate under filter-banks with/without modulation are not monotonously increasing, which indicates that more sophisticated sampling techniques are necessary to maximize the capacity [Chen, 2013].

Preliminaries: Revisit Coprime





Theorem 1: Channel capacity

[4, Th. 8.5.1] For the channel with a power constraint P and noise PSD $N(f)$, assume that $|H(f)|^2/N(f)$ is bounded and integrable, and that either $\int_{-\infty}^{+\infty} N(f)df < +\infty$ or that $N(f)$ is white. Then the channel capacity is given by

$$\mathbb{C} = \frac{1}{2}\int_{-\infty}^{+\infty} \log^+ \left(\nu \frac{|H(f)|^2}{N(f)}\right) df, \tag{1}$$

where ν is the water-filling for additive Gaussian channel and satisfying

$$\int_{-\infty}^{+\infty} \left[\nu - \frac{N(f)}{|H(f)|^2}\right]^+ df = P. \tag{2}$$

Theorem 2: Landau Sampling Rate

[5, Th. 1 and Lemma 4] Given a fixed set S of N-frequency bands whose total finite length, including both the negative and positive frequencies, denoted by $m(S)$, the space $\mathscr{B}(S)$ of all signals of finite energy with frequencies contained only in S, and Λ is a sampling set from $\mathscr{B}(S)$, All signals $f(t) \in \mathscr{B}(S)$ can be uniquely determined by their samples $\{f(t_n) | t_n \in \Lambda\}$ only if the lower Beurling density $D^-(\Lambda)$ satisfying

$$\begin{aligned} D^-(\Lambda) &= \lim_{r\to\infty} \inf \frac{n^-(r)}{r} \\ &\geq \lim_{r\to\infty} \left[\frac{m(S)}{2\pi} - \frac{A \log^+(r) - B}{r} \right] \end{aligned} \tag{3}$$

with constant A and B relevant with the choices of S and Λ and independent of the sampling interval r.

Basics of Toeplitz Matrix

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(n-1)} \\ t_1 & t_0 & t{-1} & \dots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \dots & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \dots & t_0 \end{bmatrix} \tag{4}$$

- Toeplitz matrices are closely related with Fourier series, and has the property of asymptotical commutation.
- A Toeplitz matrix can be briefly described as $T_n = [t_{i-j}; i, j = 0, 1, \dots, n-1]$, which is uniquely defined by the sequence $\{t_k\}$.

Basics of Toeplitz Matrix

The $n \times n$ Toeplitz matrix generated by Fourier coefficients is

$$T_n(f) = \left[\int_0^{2\pi} f(\lambda) e^{j(i-k)\lambda} \frac{d\lambda}{2\pi}; \quad k, i = 0, 1, \ldots, n-1 \right]. \tag{5}$$

Besides, let U_n be the circulant matrix of T_n by filling in the upper right and lower left corners with the appropriate entries. In particular, U_n consists of cyclic shifts of $[c_0^{(n)}, \ldots, c_{n-1}^{(n)}]$ where

$$c_k^{(n)} = \begin{cases} t_{-k} & k = 0, 1, \ldots, m \\ t_{n-k} & k = n-m, \ldots, n-1. \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Similarly, the Fourier coefficients of the circulant matrix $U_n(f)$ is also well defined

$$U_n(f) = \left[\frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{2\pi j}{n}\right) e^{\left(\frac{2\pi ijk}{n}\right)}; i, k = 0, 1, \ldots, n-1 \right]. \tag{7}$$

Definition: Asymptotic Equivalence of Toeplitz Matrix

[6] Two $n \times n$ matrices $\{A_n\}$ and $\{B_n\}$ are said to be asymptotically equivalent and be abbreviated as $A_n \sim B_n$ if

- A_n and B_n are uniformly bounded for any integer n and a constant Ψ independent of n:

$$||A_n||_2, ||B_n||_2 \leq \Psi < +\infty \tag{8}$$

- The determinant of the differnece between A_n and B_n is approximately zero as $n \to \infty$.

$$\lim_{n\to\infty} |A_n - B_n| = 0 \tag{9}$$

Lemma 1: Eighenvalues of Toeplitz Matrices

[6, Lemma 1] Suppose $A_n \sim B_n$ with eighenvalues $\{\alpha_{n,k}\}$ and $\{\beta_{n,k}\}$, respectively. Let $g(x)$ be an arbitrary continuous function. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\alpha_{n,k} - \beta_{n,k}) = 0 \tag{10}$$

and hence if either limit exists individually,

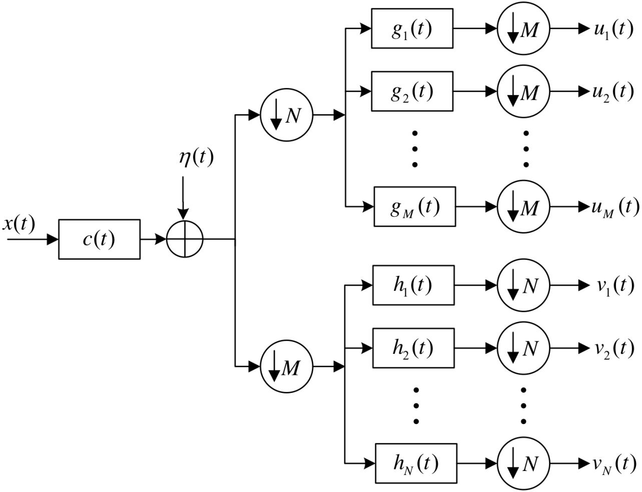
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\alpha_{n,k}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\beta_{n,k}) \tag{11}$$

Lemma 2: Operations of Toeplitz Matrices

[6, Lemma 7 and 10]

- Suppose a banded Toeplitz matrix $T_n = \{t_{i-j}; i, j = 0, 1, \ldots, n-1\}$, which satisfies that $\{t_k\}$ is absolutely summable, and the Fourier serieris $f(\lambda)$ related to T_n is positive and T_n is Hermitian. Then we have
$$T_n(f) \sim U_n(f) \tag{12}$$
If $f(\lambda)$ is real and there exists a constant $\varepsilon > 0$ such that $\inf f \geq \varepsilon$, then
$$T_n(f)^{-1} \sim U_n(f)^{-1} = U_n(1/f) \sim T_n(1/f) \tag{13}$$
- Suppose $\boldsymbol{A}_n \sim \boldsymbol{B}_n$ and $\boldsymbol{C}_n \sim \boldsymbol{D}_n$, then $\boldsymbol{A}_n \boldsymbol{C}_n \sim \boldsymbol{B}_n \boldsymbol{D}_n$.

System Modeling



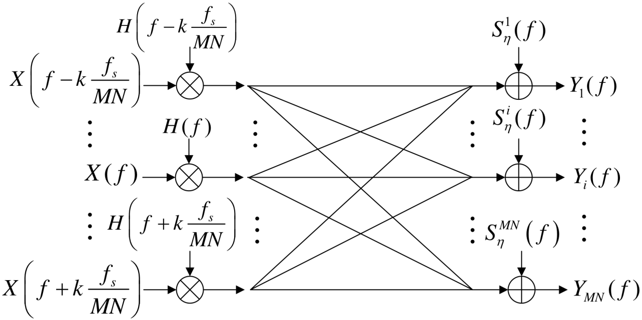
Matrices of Impulse Responses

$$\boldsymbol{G}_m = [\dots \;\; G_{m,-1} \;\; G_{m,0} \;\; G_{m,1} \;\; \dots] \quad 1 \leq m \leq M \tag{14}$$

$$G_{m,0} = [g_m^0, \underbrace{0, \dots, 0}_{\text{N-1}}, g_m^N, 0, \dots, 0, g_m^{\frac{L-1}{M}}]^T \tag{15}$$

$$\boldsymbol{G}_m = \begin{bmatrix} \dots & g_m^0 & 0 & \dots & 0 & g_m^{-N} & \dots \\ \dots & 0 & g_m^0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & \dots & g_m^0 & 0 & \dots \\ \dots & g_m^N & 0 & \dots & 0 & g_m^0 & \dots \\ \dots & 0 & g_m^N & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & g_m^{\frac{L-1}{M}} & 0 & \dots & 0 & g_m^{\frac{L-N-1}{M}} & \dots \end{bmatrix}. \tag{16}$$

Matrices of Impulse Responses



Let $\tilde{T}_s$ be the actual sampling time and T_s be the equivalent sampling time, their relation is $\tilde{T}_s = MNT_s$. Suppose the overall time of signal receiving is $T = L\tilde{T}_s$ and $\tilde{T}_s = \beta\xi$ with integers L and β.

Equivalent Filter Impulse Responses

We also define the *virtual* channel impulse response $\tilde{c}(t)$ in terms of *equivalent* filter impulse response $s(t)$. For the ith branch, it is

$$\tilde{c}_i(t) = c(t) * s_i(t) \quad 1 \le i \le MN$$
$$\tilde{\boldsymbol{c}}_i^l = \left[\tilde{c}_i(\alpha \tilde{T}_s), \tilde{c}_i(\alpha \tilde{T}_s - \xi), \dots, \tilde{c}_i(\alpha \tilde{T}_s - (\beta - 1)\xi)\right],$$

where $0 \le \alpha \le L$. Specifically, the matrix representations are

$$\tilde{\boldsymbol{C}}_i = \begin{bmatrix} \tilde{\boldsymbol{c}}_i^0 & \tilde{\boldsymbol{c}}_i^{-1} & \dots & \tilde{\boldsymbol{c}}_i^{-L+1} \\ \tilde{\boldsymbol{c}}_i^1 & \tilde{\boldsymbol{c}}_i^0 & \dots & \tilde{\boldsymbol{c}}_i^{-L+2} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{\boldsymbol{c}}_i^{L-1} & \tilde{\boldsymbol{c}}_i^{L-2} & \dots & \tilde{\boldsymbol{c}}_i^0 \end{bmatrix} \tag{17}$$

$$\boldsymbol{S}_i = \begin{bmatrix} s_i^{L-1} & \dots & s_i^0 & s_i^{-1} & \dots & s_i^{-L+1} \\ s_i^{-L+1} & \dots & s_i^1 & s_i^0 & \dots & s_i^{-L+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_i^{-1} & \dots & s_i^{L-1} & s_i^{L-2} & \dots & s_i^0 \end{bmatrix} \tag{18}$$

Equivalent Filter Impulse Responses

Considering the virtual filter impulse responses $\boldsymbol{S}$ in (18) and any $1 \leq i \leq j \leq MN$, we have

$$(\boldsymbol{S}_u \boldsymbol{S}_v^*)_{i,j} = (\boldsymbol{S}_u \boldsymbol{S}_v^*)_{i,j}^* = \sum_{t=-\infty}^{\infty} s_u^{j-i+t} (s_v^t)^* \tag{19}$$

Hence, the Hermitian matrix $\tilde{\boldsymbol{S}} := \boldsymbol{S}_u \boldsymbol{S}_v^*$ is still Toeplitz, which is used to relate the matrix of interest in the capacity $(\boldsymbol{G}_M \boldsymbol{H}_N^*)^{-1/2}$ with the asymptotically equivalent circulant matrix $\boldsymbol{U}$ defined in (7), which is able to perserve the Toeplitz property as being taken the inverse square root $(\boldsymbol{U})^{-1/2}$.

Theorem 3: Asymptotic Equivalence for Coprime Filter-banks

If there exists some constant $\varepsilon > 0$ such that for all $f \in \left[-\frac{fs}{2MN}, \frac{fs}{2MN}\right]$,

$$\sum_{l \in \mathbb{Z}} |G(f - lf_s)H(f - lf_s)| \geq \varepsilon > 0 \tag{20}$$

holds, then $(\boldsymbol{U})^{-1/2} \sim (\boldsymbol{G}_M \boldsymbol{H}_N^*)^{-1/2}$.

Formulae Definitions for Theorem 4

$\boldsymbol{F}_s$ and $\boldsymbol{F}_c$ are both defined in the Fourier domain. $\boldsymbol{F}_s$ is an infinite matrix of mn rows and infinite many columns, and $\boldsymbol{F}_c$ is a diagonal infinite matrix, such that for i $(1 \leq i \leq \beta)$ and every integer l

$$(\boldsymbol{F}_s)_{i,l} = S_i\left(f - \frac{lf_s}{MN}\right)\sqrt{S_\eta\left(f - \frac{lf_s}{MN}\right)} \tag{21}$$

$$(\boldsymbol{F}_c)_{i,l} = C\left(f - \frac{lf_s}{MN}\right)\Bigg/\sqrt{S_\eta\left(f - \frac{lf_s}{MN}\right)} \tag{22}$$

Theorem 4: Capacity of Coprime Filter-banks

Assume that $c(t)$ and $s_i(t)$ $(1 \leq i \leq MN)$ are all continuous, bounded and absolutely Riemann integrable, and $c_\eta(t) = \mathscr{F}^{-1}\left(\frac{C(f)}{\sqrt{S_\eta(f)}}\right)$ satisfies $c_\eta(t) = o(t^{-\varepsilon})$ for some constant $\varepsilon > 1$, and that $\boldsymbol{F}_s(f)$ is right-invertible for every f. Define $\tilde{\boldsymbol{F}}_s \triangleq (\boldsymbol{F}_s\boldsymbol{F}_s^*)^{-\frac{1}{2}}\boldsymbol{F}_s$. The capacity $\mathbb{C}(f_s)$ of the sampled channel with a power constraint P is given as

$$\mathbb{C}(f_s) = \frac{1}{2}\int_{-\frac{f_s}{2MN}}^{\frac{f_s}{2MN}} \sum_{i=1}^{nMN} \log^+\left(\nu\lambda_i\left(\tilde{\boldsymbol{F}}_s\boldsymbol{F}_c\boldsymbol{F}_c^*\tilde{\boldsymbol{F}}_s^*\right)\right) df \tag{23}$$

where the ν is satisfying

$$\int_{-\frac{f_s}{2MN}}^{\frac{f_s}{2MN}} \sum_{i=1}^{nMN} \left(\nu - \frac{1}{\lambda_i\left(\tilde{\boldsymbol{F}}_s\boldsymbol{F}_c\boldsymbol{F}_c^*\tilde{\boldsymbol{F}}_s^*\right)}\right)^+ df = P \tag{24}$$

Corollary: Existence of the Maximum Capacity

For each aliased set $\left\{f - \frac{if_s}{MN} | i \in \mathbb{Z}\right\}$ and each k $(1 \leq k \leq MN)$, there exists an integer l such that $\frac{|C(f-\frac{lf_s}{MN})|^2}{S_\eta(f-\frac{lf_s}{MN})}$ is euqal to the kth largest element in $\left\{\frac{|C(f-\frac{lf_s}{MN})|^2}{S_\eta(f-\frac{lf_s}{MN})} | i \in \mathbb{Z}\right\}$. The maximum capacity is

$$\mathbb{C}(f_s) = \frac{1}{2} \int_{-\frac{f_s}{2MN}}^{\frac{f_s}{2MN}} \sum_{i=1}^{nMN} \log^{+}\left(\nu \lambda_i \left(\boldsymbol{F}_c \boldsymbol{F}_c^*\right)\right) df \tag{25}$$

under the conditions that...

Corollary: Existence of the Maximum Capacity (cont.)

the frequency response of the kth filter of the filter bank is given by

$$S_i\left(f - \frac{lf_s}{MN}\right) = \begin{cases} 1, & \frac{|C(f-\frac{lf_s}{MN})|^2}{S_\eta(f-\frac{lf_s}{MN})} = \lambda_i\left(\boldsymbol{F}_c\boldsymbol{F}_c^*\right) \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

and the corresponding water filling scheme for ν is

$$\int_{-\frac{f_s}{2MN}}^{\frac{f_s}{2MN}} \sum_{i=1}^{nMN} \left(\nu - \frac{1}{\lambda_i\left(\boldsymbol{F}_c\boldsymbol{F}_c^*\right)}\right)^+ df = P \tag{27}$$

Conclusions

- The work illuminates a connection between MIMO channel capacity and capacity using Coprime sampling strategy.
- The capacity optimizing sampling structures are shown to extract the frequency components with highest SNRs from each aliased set, and hence suppress aliasing and out-of-band noise.
- Given the same number of filter-banks, the achievable rate of the channel is closer to the upper bound of capacity via Coprime sampling than uniform sampling.

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Thanks!