# Efficiency

# Class outline:

- Exponentiation
- Orders of Growth
- Memoization

# Exponentiation

# Exponentiation approach #1

Based on this recursive definition:

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{(n-1)} & \text{otherwise} \end{cases}$$

```
def exp(b, n):
  if n == 0:
    return 1
  else:
    return b * exp(b, n-1)
```

How many calls are required to calculate `exp(2, 16)`?

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How many calls are required to calculate `exp(2, 16)`?

Can we do better?

# Exponentiation approach #2

Based on this alternate definition:

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{(n-1)} & \text{if } n \text{ is odd} \end{cases}$$

```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

square = lambda x: x * x
```

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```

How many calls are required to calculate `exp(2, 16)`?

Some algorithms are more efficient than others!

# Orders of Growth

# Common orders of growth

One way to describe the efficiency of an algorithm according to its **order of growth**, the effect of increasing the size of input on the number of steps required.

| **Order of growth** | **Description** |
| --- | --- |
| Constant growth | Always takes same number of steps, regardless of input size. |
| Logarithmic growth | Number of steps increases proportionally to the logarithm of the input size. |
| Linear growth | Number of steps increases in direct proportion to the input size. |
| Quadratic growth | Number of steps increases in proportion to the square of the input size. |
| Exponential growth | Number of steps increases faster than a polynomial function of the input size. |

# Adding to the front of linked list

```
def insert_front(linked_list, new_val):
    """Inserts NEW_VAL in front of LINKED_LIST,returning new linked list.
    >>> ll = Link(1, Link(3, Link(5)))
    >>> insert_front(ll, 0)
    Link(0, Link(1, Link(3, Link(5))))
    """
    return Link(new_val, linked_list)
```

How many operations will this require for increasing lengths of the list?

| List size | Operations |
|---|---|
| 1 | |
| 10 | |
| 100 | |
| 1000 | |

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|---|---|
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|---|---|
| 1 | 1 |
| 10 | 1 |
| 100 | 1 |
| 1000 | |

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    """
    return Link(new_val, linked_list)
```

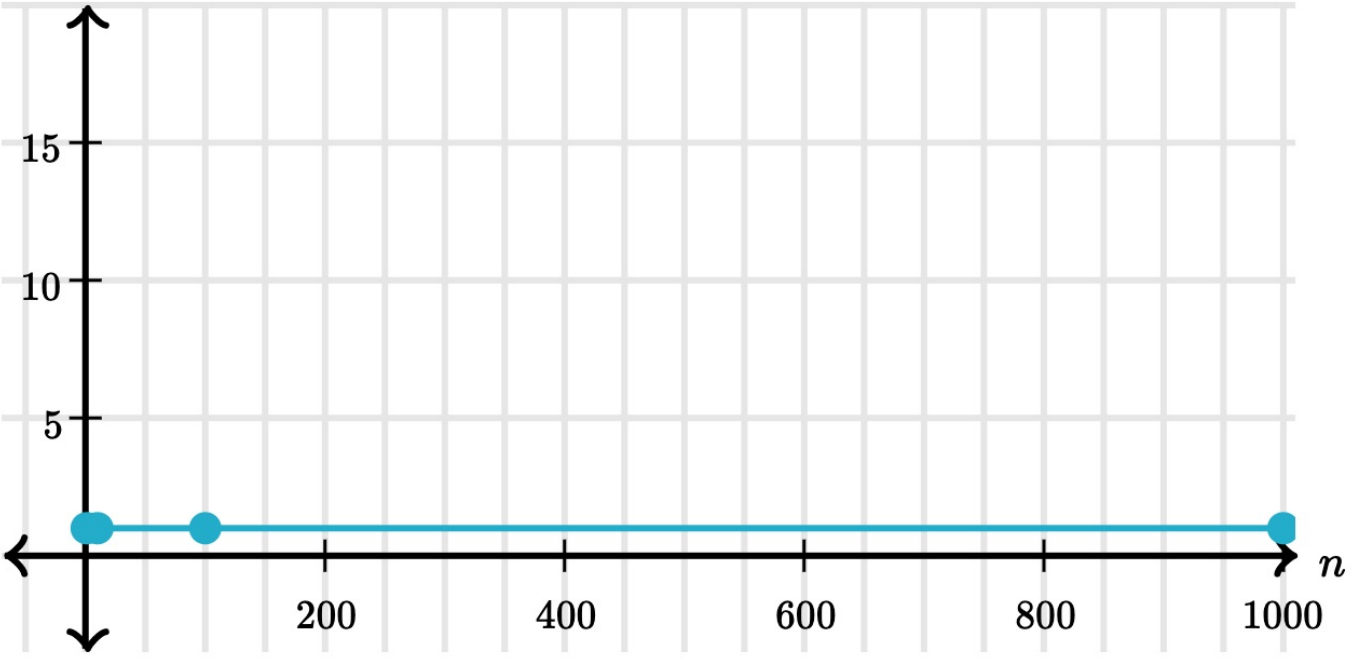
How many operations will this require for increasing lengths of the list?

| List size | Operations |
|---|---|
| 1 | 1 |
| 10 | 1 |
| 100 | 1 |
| 1000 | 1 |

# Constant time

An algorithm that takes **constant time**, always makes a fixed number of operations regardless of the input size.

| List size | Operations |
|---|---|
| 1 | 1 |
| 10 | 1 |
| 100 | 1 |
| 1000 | 1 |


15
10
5
200
400
600
800
1000
n

# Fast exponentiation

```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

square = lambda x: x * x
```

How many operations will this require for increasing values of n?

| N | Operations |
|---|---|
| 0 | |
| 8 | |
| 16 | |
| 1024 | |

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| 0 | 1 |
| 8 | |
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| 1024 | |

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| N | Operations |
|---|---|
| 0 | 1 |
| 8 | 5 |
| 16 | |
| 1024 | |

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How many operations will this require for increasing values of n?

| N | Operations |
|---|---|
| 0 | 1 |
| 8 | 5 |
| 16 | 6 |
| 1024 | |

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```

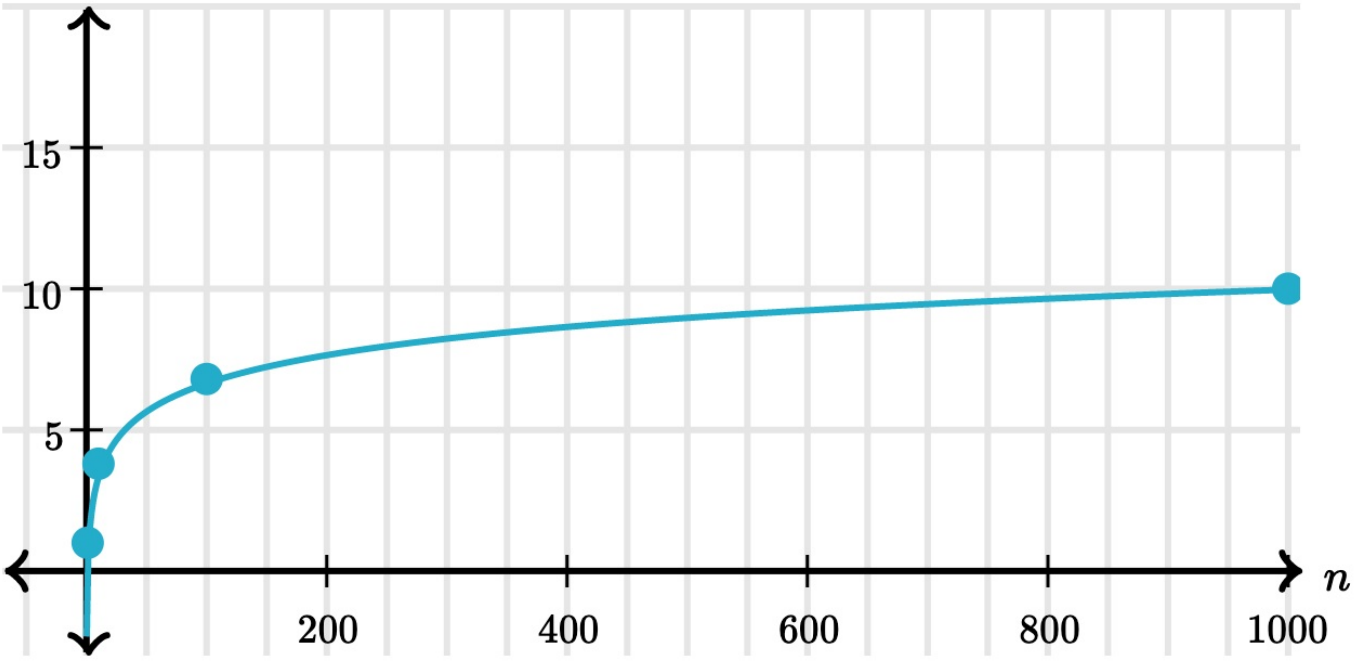
How many operations will this require for increasing values of n?

| N | Operations |
|---|---|
| 0 | 1 |
| 8 | 5 |
| 16 | 6 |
| 1024 | 12 |

# Logarithmic time

When an algorithm takes **logarithmic time**, the time that it takes increases proportionally to the logarithm of the input size.

| N | Operations |
|---|---|
| 0 | 1 |
| 8 | 5 |
| 16 | 6 |
| 1024 | 12 |


15
10
5
200
400
600
800
1000
n

# Finding value in a linked list

```
def find_in_link(ll, value):
    """Return true if linked list LL contains VALUE.
    >>> find_in_link(Link(3, Link(4, Link(5))), 4)
    True
    >>> find_in_link(Link(3, Link(4, Link(5))), 7)
    False
    """
    if ll is Link.empty:
        return False
    elif ll.first == value:
        return True
    return find_in_link(ll.rest, value)
```

How many operations will this require for increasing lengths of the list? Consider both the **best case** and **worst case**.

| List size | Best case: Operations | Worst case: Operations |
|---|---|---|
| 1 | | |
| 10 | | |
| 100 | | |
| 1000 | | |

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| 1000 | | |

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|---|---|---|
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| 10 | 1 | 10 |
| 100 | | |
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| 1000 | | |

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| 1000 | 1 | |

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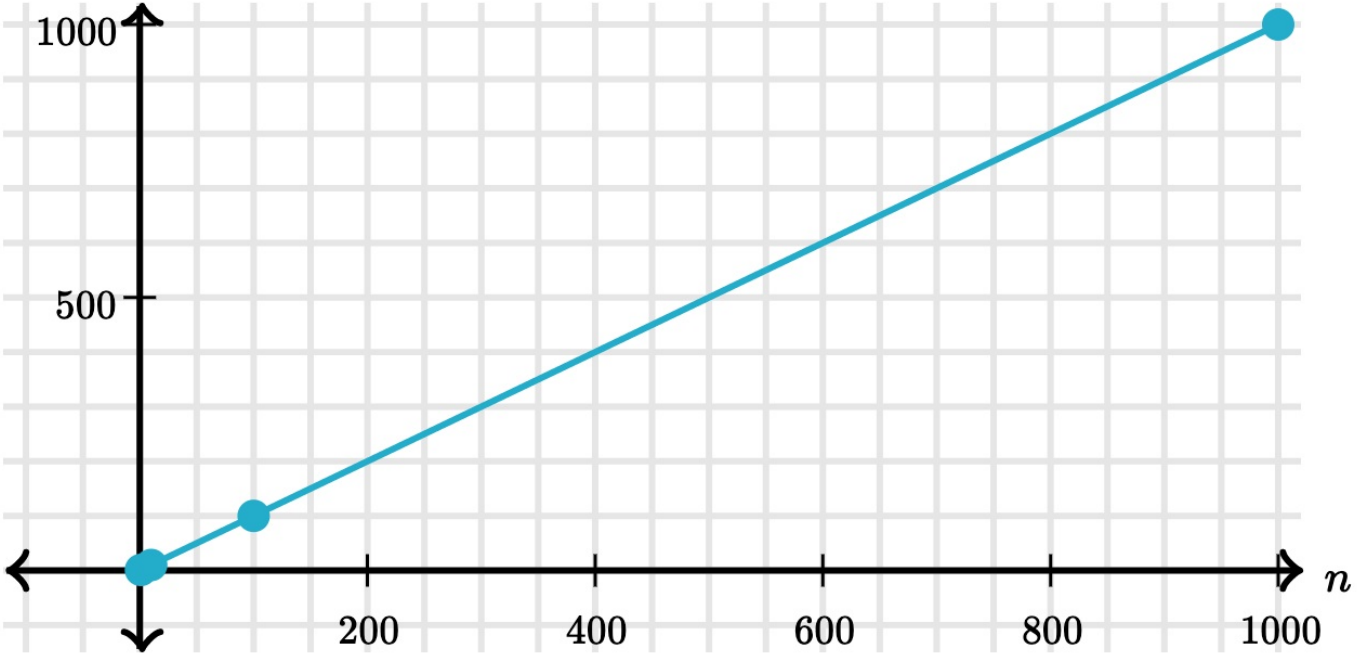
| List size | Best case: Operations | Worst case: Operations |
|---|---|---|
| 1 | 1 | 1 |
| 10 | 1 | 10 |
| 100 | 1 | 100 |
| 1000 | 1 | 1000 |

# Linear time

When an algorithm takes **linear time**, its number of operations increases in direct proportion to the input size.

| List size | Worst case: Operations |
|---|---|
| 1 | 1 |
| 10 | 10 |
| 100 | 100 |
| 100 | 1000 |


1000
500
200
400
600
800
1000
n

# Counting overlapping items in lists

```
def overlap(a, b):
    """
    >>> overlap([3, 5, 7, 6], [4, 5, 6, 5])
    3
    """
    count = 0
    for item in a:
        for other in b:
            if item == other:
                count += 1
    return count
```

| | 3 | 5 | 6 | 7 |
|---|---|---|---|---|
| 4 | | | | |
| 5 | | + | | |
| 6 | | | | + |
| 5 | | + | | |

How many operations are required for increasing lengths of the lists?

| List size | Operations |
|---|---|

| |
|---|
| 1 |
| 10 |
| 100 |
| 1000 |

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|---|---|---|---|---|
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| 6 | | | | + |
| 5 | | + | | |

How many operations are required for increasing lengths of the lists?

| List size | Operations |
|---|---|

| | |
|---|---|
| 1 | 1 |
| 10 | |
| 100 | |
| 1000 | |

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|---|---|---|---|---|
| 4 | | | | |
| 5 | | + | | |
| 6 | | | | + |
| 5 | | + | | |

How many operations are required for increasing lengths of the lists?

| List size | Operations |
|---|---|

| | |
|---|---|
| 1 | 1 |
| 10 | 100 |
| 100 | |
| 1000 | |

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|---|---|---|---|---|
| 4 | | | | |
| 5 | | + | | |
| 6 | | | | + |
| 5 | | + | | |

How many operations are required for increasing lengths of the lists?

| List size | Operations |
|---|---|

| | |
|---|---|
| 1 | 1 |
| 10 | 100 |
| 100 | 10000 |
| 1000 | |

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```

| | 3 | 5 | 6 | 7 |
|---|---|---|---|---|
| 4 | | | | |
| 5 | | + | | |
| 6 | | | | + |
| 5 | | + | | |

How many operations are required for increasing lengths of the lists?

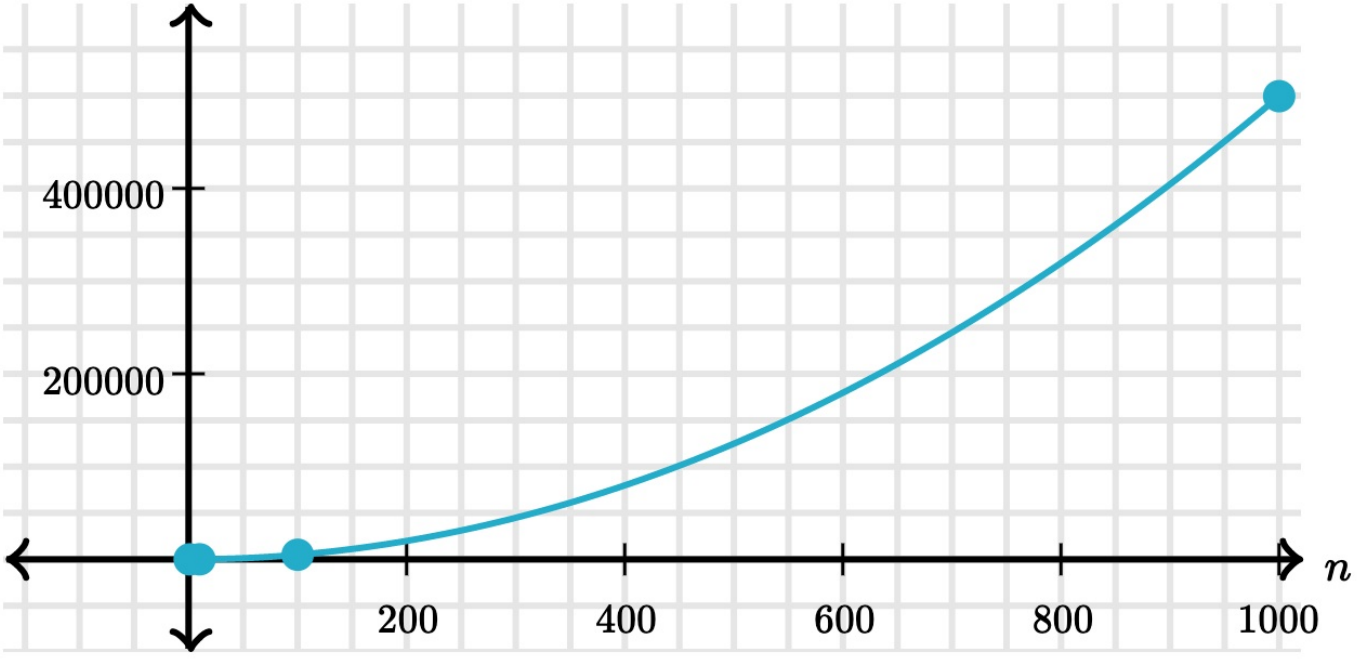
| List size | Operations |
|---|---|

| | |
|---|---|
| 1 | 1 |
| 10 | 100 |
| 100 | 10000 |
| 1000 | 1000000 |

# Quadratic time

When an algorithm grows in **quadratic time**, its steps increase in proportion to square of the input size.

| List size | Operations |
|---|---|
| 1 | 1 |
| 10 | 100 |
| 100 | 10000 |
| 1000 | 1000000 |


400000
200000
200
400
600
800
1000
n

# Recursive Virahanka-Fibonacci

```
def virfib(n):
  if n == 0:
    return 0
  elif n == 1:
    return 1
  else:
    return virfib(n-2) + virfib(n-1)
```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 7 | |
| 8 | |
| 20 | |

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```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | |
| 3 | |
| 4 | |
| 7 | |
| 8 | |
| 20 | |

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```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | |
| 4 | |
| 7 | |
| 8 | |
| 20 | |

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```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | |
| 7 | |
| 8 | |
| 20 | |

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```
def virfib(n):
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```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 9 |
| 7 | |
| 8 | |
| 20 | |

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```
def virfib(n):
  if n == 0:
    return 0
  elif n == 1:
    return 1
  else:
    return virfib(n-2) + virfib(n-1)
```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 9 |
| 7 | 41 |
| 8 | |
| 20 | |

# Recursive Virahanka-Fibonacci

```
def virfib(n):
  if n == 0:
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```

How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 9 |
| 7 | 41 |
| 8 | 67 |
| 20 | |

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  elif n == 1:
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```

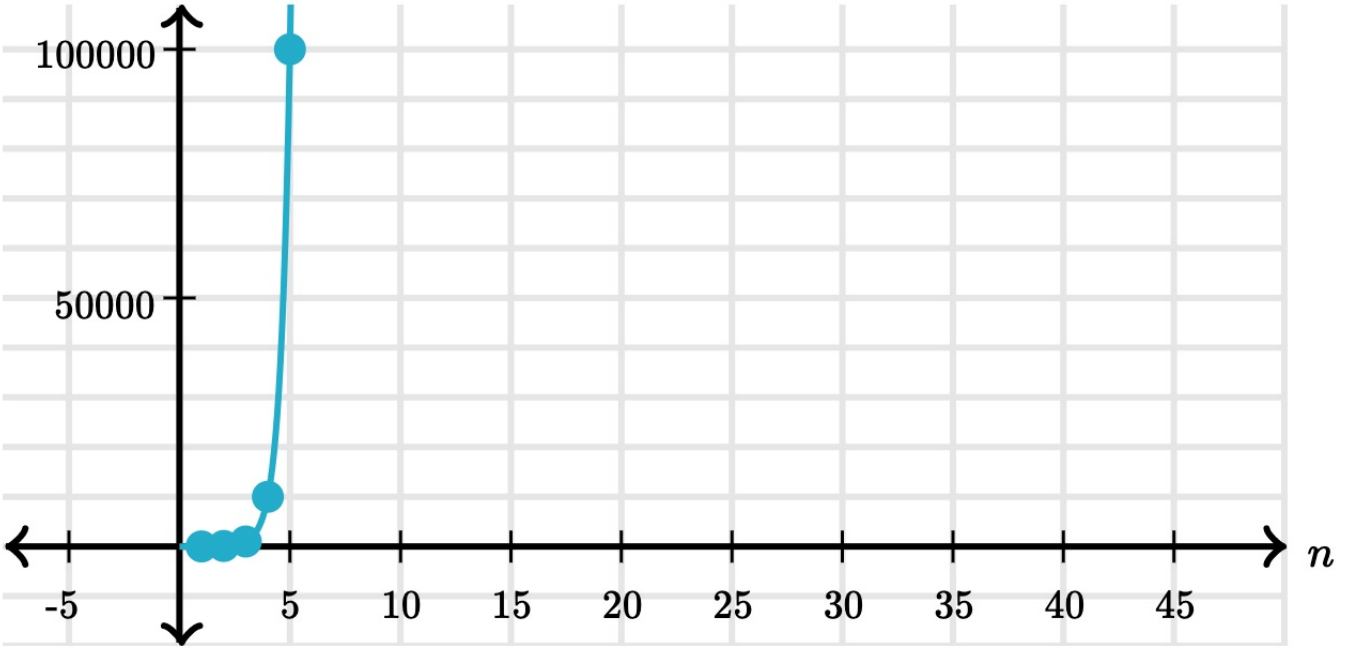
How many operations are required for increasing values of n?

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 9 |
| 7 | 41 |
| 8 | 67 |
| 20 | 21891 |

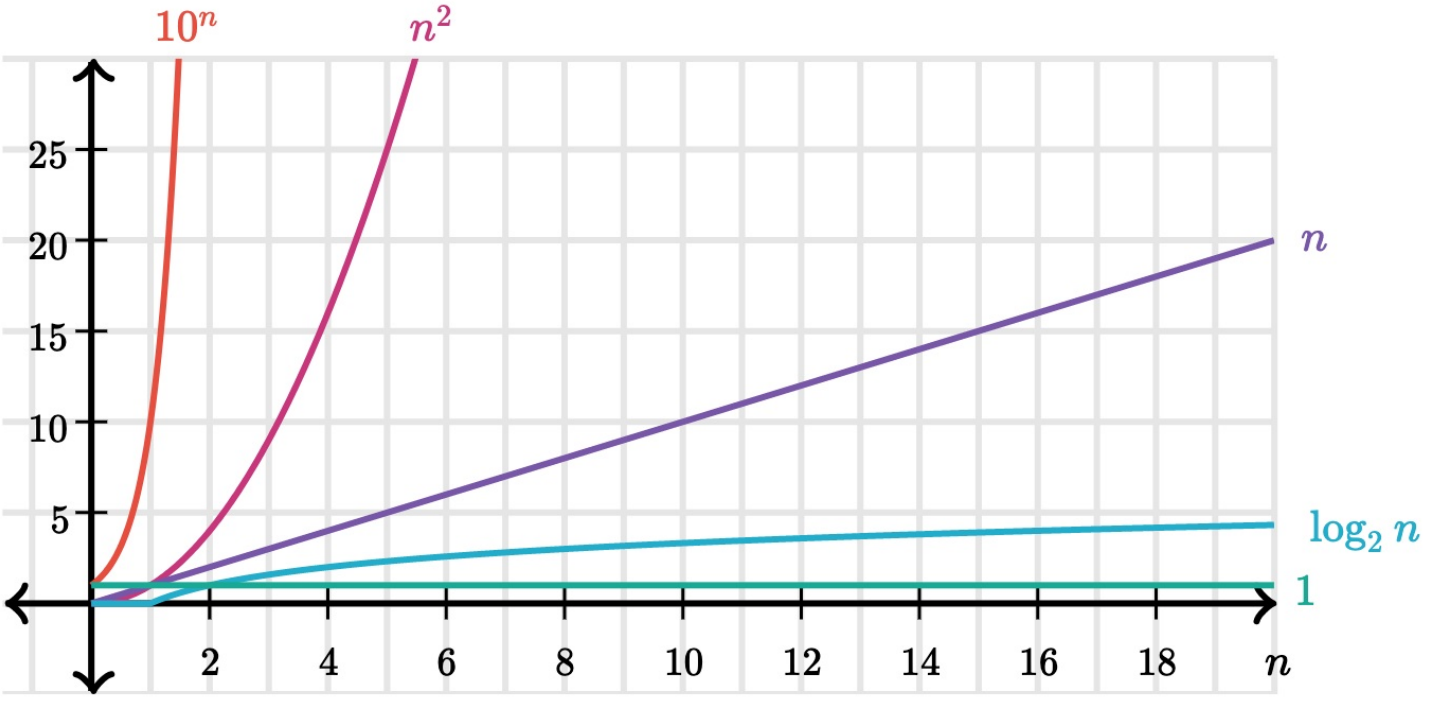
# Exponential time

When an algorithm grows in **exponential time**, its number of steps increases faster than a polynomial function of the input size.

| N | Operations |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 9 |
| 7 | 41 |
| 8 | 67 |
| 20 | 21891 |


100000
50000
-5
5
10
15
20
25
30
35
40
45
n

# Comparing orders of growth



# Big O/Big Theta Notation

A formal notation for describing the efficiency of an algorithm, using asymptotic analysis.

| Order of growth | Example | Big Theta | Big O |
|---|---|---|---|
| Exponential growth | recursive `virfib` | | |
| Quadratic growth | `overlap` | | |
| Linear growth | `find_in_link` | | |
| Logarithmic growth | `exp_fast` | | |
| Constant growth | `add_to_front` | | |

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| Constant growth | `add_to_front` | $\Theta(1)$ | |

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| Constant growth | `add_to_front` | $\Theta(1)$ | $O(1)$ |

# Space

# Space and environments

The space needed for a program depends on the environments in use.

At any moment there is a set of **active environments**.

Values and frames in active environments consume memory.

Memory that is used for other values and frames can be recycled.

Active environments:

- Environments for any function calls currently being evaluated.
- Parent environments of functions named in active environments.

# Active environments in PythonTutor

```
def virfib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return virfib(n-2) + virfib(n-1)
```

View in PythonTutor

Make sure to select "don't display exited functions".

# Visualization of space consumption

# Memoization

# Memoization

**Memoization** is a strategy to reduce redundant computation by remembering the results of previous function calls in a "memo".

# A memoization HOF

```
def memo(f):
    cache = {}
    def memoized(n):
        if n not in cache:
            cache[n] = f(n)
        return cache[n]
    return memoized
```

# Memoizing Virahanka-Fibonacci

| n | **Original** | **Memoized** |
|---|---|---|
| 5 | 15 | 9 |
| 6 | 25 | 11 |
| 7 | 41 | 13 |
| 8 | 67 | 15 |
| 9 | 109 | 17 |
| 10 | 177 | 19 |

Video visualization