Higher-Order Functions

Class outline:

- Iteration example
- Designing functions
- Generalization
- Higher-order functions
- Lambda expressions
- Conditional expressions

Iteration example

Virahaṅka-Fibonacci numbers

Discovered by Virahanka in India, 600-800 AD, later re-discovered in Western mathematics and commonly known as Fibonacci numbers.

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`0    1    1    2    3    5    8    13 + 21 = 34   …`

Virahanka's question

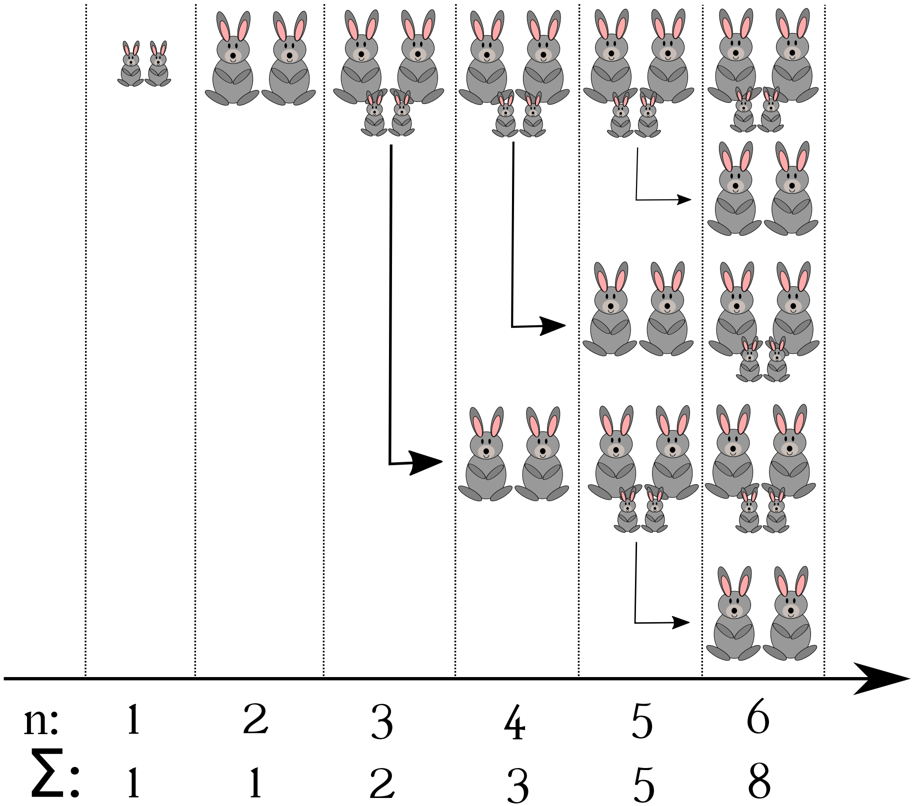
How many poetic meters exist for a total duration?

S = short syllable, L = long syllable

Duration	Meters	Total
1	S	1
2	SS, L	2
3	SSS, SL, LS	3
4	SSSS, SSL, SLS, LSS, LL	5
5	SSSSS, SSSL, SSLS, SLSS, SLL, LLS, LSL, LSSS	8

Fibonacci's question

How many pairs of rabbits can be bred after N months?



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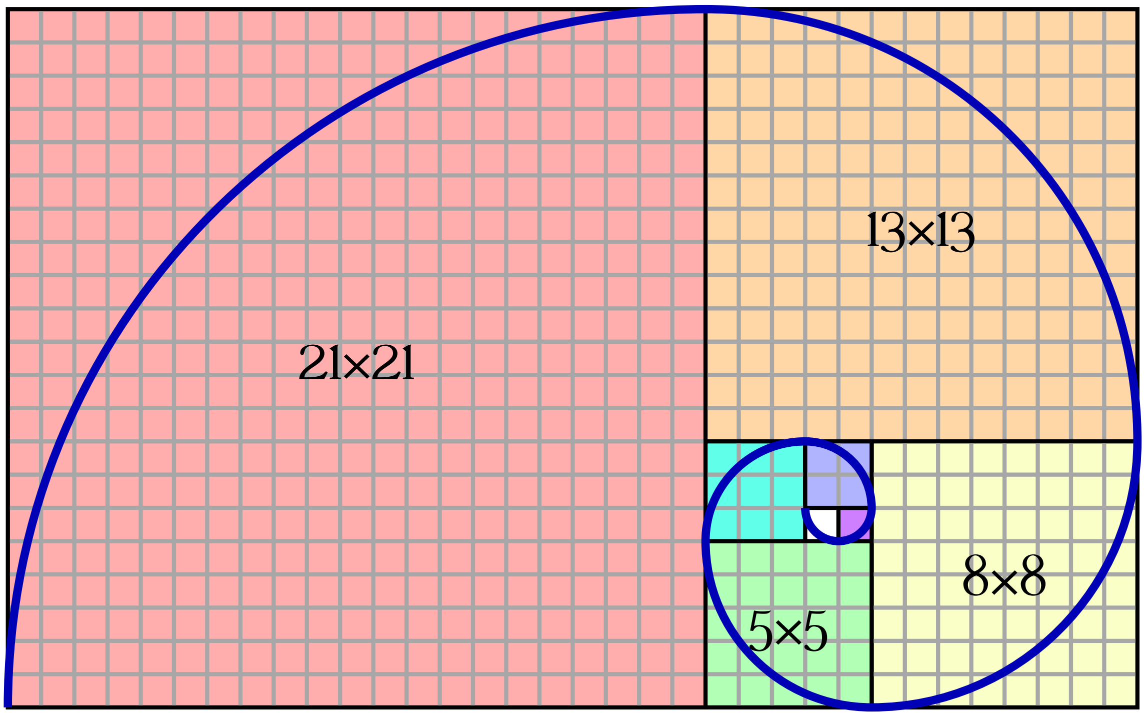
Virahanka-Fibonacci number generation

VF	0	1	1	2	3	5	8	13	21	34	55	...
N	0	1	2	3	4	5	6	7	8	9	10	...

```python
def vf_number(n):
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> vf_number(2)
    1
    >>> vf_number(6)
    8
    """
    prev = 0  # First Fibonacci number
    curr = 1  # Second Fibonacci number
    k = 1
    while k < n:
        (prev, curr) = (curr, prev + curr)
        k += 1
    return curr
```

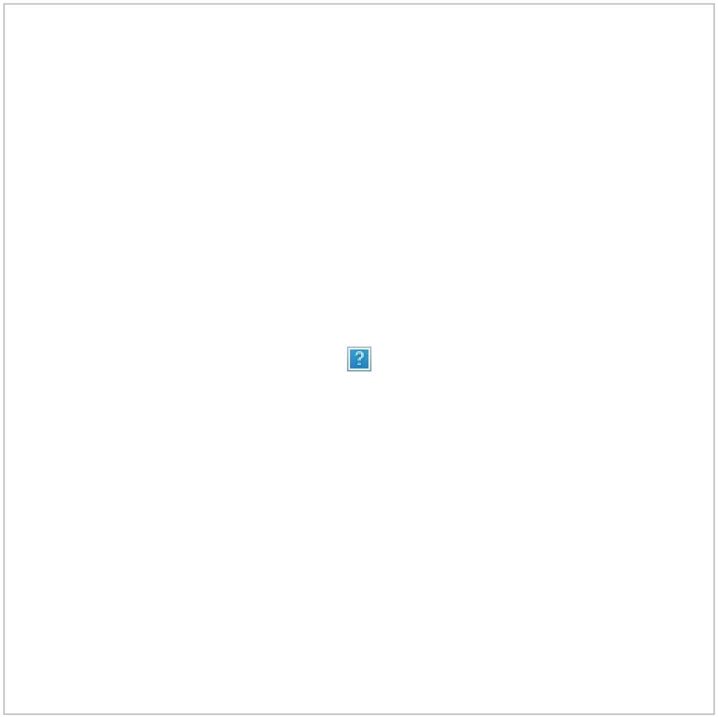
Golden spiral

The Golden spiral can be approximated by Virahanka-Fibonacci numbers.



Go bears!

The Golden spiral is found everywhere in nature…



Designing Functions

Describing Functions

```
def square(x):
    """Returns the square of X."""
    return x * x
```

Aspect	**Example**
A function's **domain** is the set of all inputs it might possibly take as arguments.	`x` is a number
A function's **range** is the set of output values it might possibly return.	`square` returns a non-negative real number
A pure function's **behavior** is the relationship it creates between input and output.	`square` returns the square of `x`

Designing a function

Give each function exactly one job, but make it apply to many related situations.

```
round(1.23)       # 1
round(1.23, 0)    # 1
round(1.23, 1)    # 1.2
round(1.23, 5)    # 1.23
```

Don't Repeat Yourself (DRY): Implement a process just once, execute it many times.

Generalization

Generalizing patterns with arguments

Geometric shapes have similar area formulas.

Shape			
Area	$1 * r^2$	$\pi * r^2$	$\frac{3\sqrt{3}}{2} * r^2$

A non-generalized approach

```python
from math import pi, sqrt

def area_square(r):
    return r * r

def area_circle(r):
    return r * r * pi

def area_hexagon(r):
    return r * r * (3 * sqrt(3) / 2)
```

How can we generalize the common structure?

Generalized area function

```
from math import pi, sqrt

def area(r, shape_constant):
    """Return the area of a shape from length measurement R."""
    if r < 0:
        return 0
    return r * r * shape_constant

def area_square(r):
    return area(r, 1)

def area_circle(r):
    return area(r, pi)

def area_hexagon(r):
    return area(r, 3 * sqrt(3) / 2)
```

Higher-order functions

What are higher-order functions?

A function that either:

- Takes another function as an argument
- Returns a function as its result

All other functions are considered first-order functions.

Generalizing over computational processes

$$\sum_{k=1}^{5} \boxed{k} = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=1}^{5} \boxed{k^3} = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$\sum_{k=1}^{5} \boxed{\frac{8}{(4k-3)\cdot(4k-1)}} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323}$$

The common structure among functions may be a computational process, not just a number.

Functions as arguments

```
def cube(k):
    return k ** 3

def summation(n, term):
    """Sum the first N terms of a sequence.
    >>> summation(5, cube)
    225
    """
    total = 0
    k = 1
    while k <= n:
        total = total + term(k)
        k = k + 1
    return total
```

Functions as return values

Locally defined functions

Functions defined within other function bodies are bound to names in a local frame.

```
def make_adder(n):
    """Return a function that takes one argument k
       and returns k + n.
    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

Call expressions as operator expressions

$$\underbrace{\texttt{make\_adder(1)}}_{\text{Operator}}\texttt{(}\underbrace{\quad\texttt{2}\quad}_{\text{Operand}}\texttt{)}$$

Call expressions as operator expressions

```
make_adder(1) (      2      )
```

Operator Operand

```
make_adder(1)
```

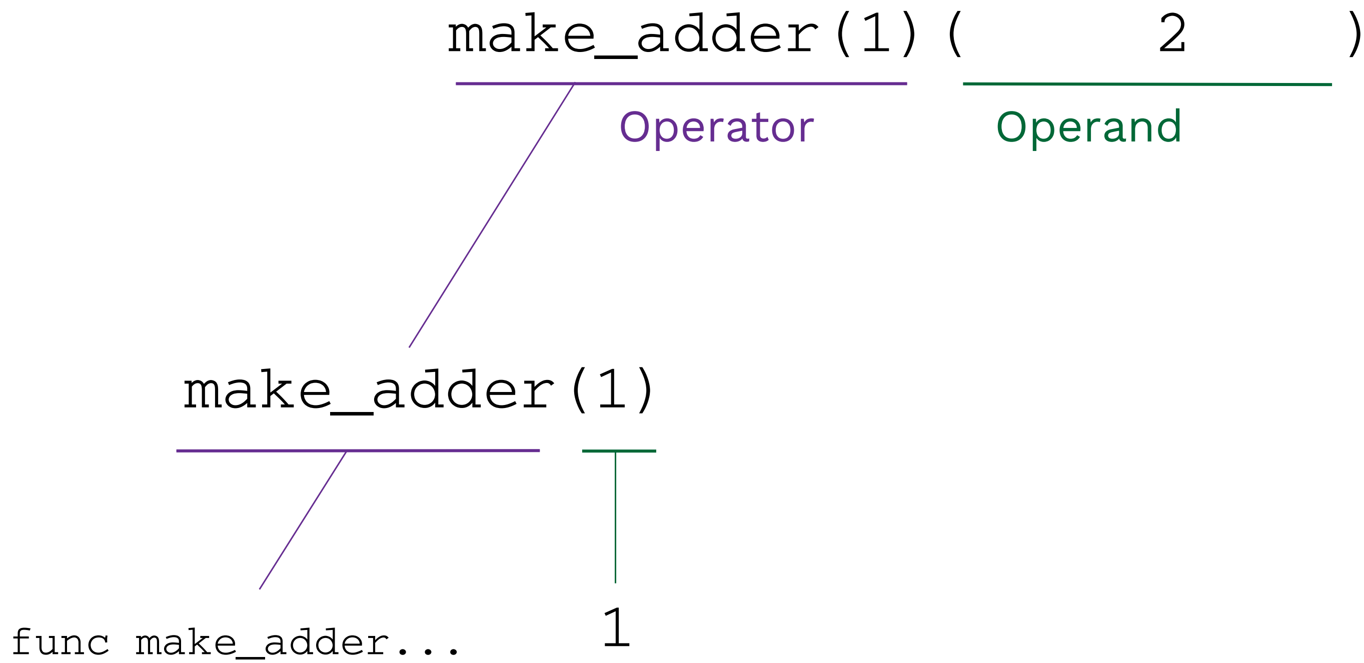
Call expressions as operator expressions

$$\underbrace{\texttt{make\_adder(1)}}_{\text{Operator}}\texttt{(}\underbrace{\texttt{2}}_{\text{Operand}}\texttt{)}$$

$$\underbrace{\texttt{make\_adder}}\texttt{(}\underbrace{\texttt{1}}\texttt{)}$$

`func make_adder...`

Call expressions as operator expressions



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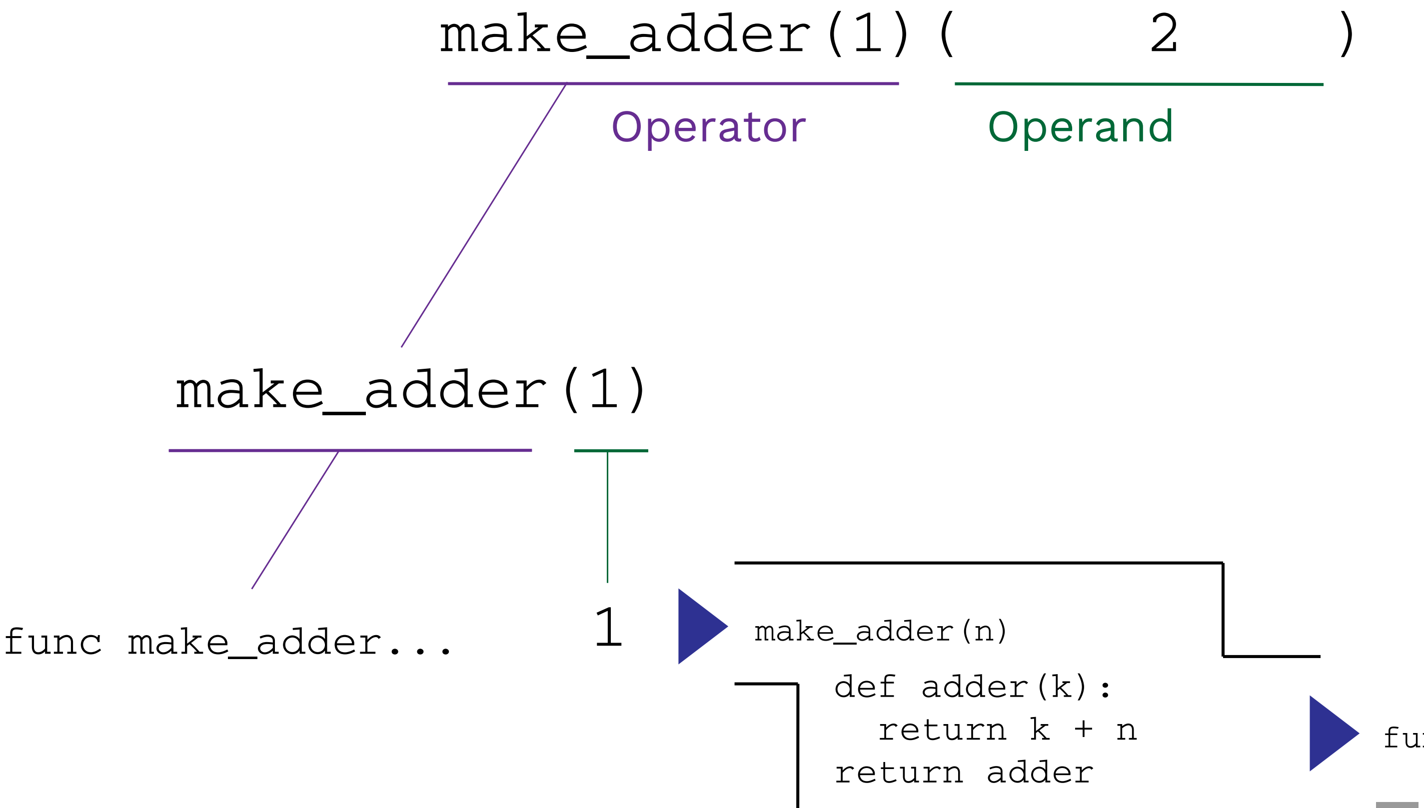

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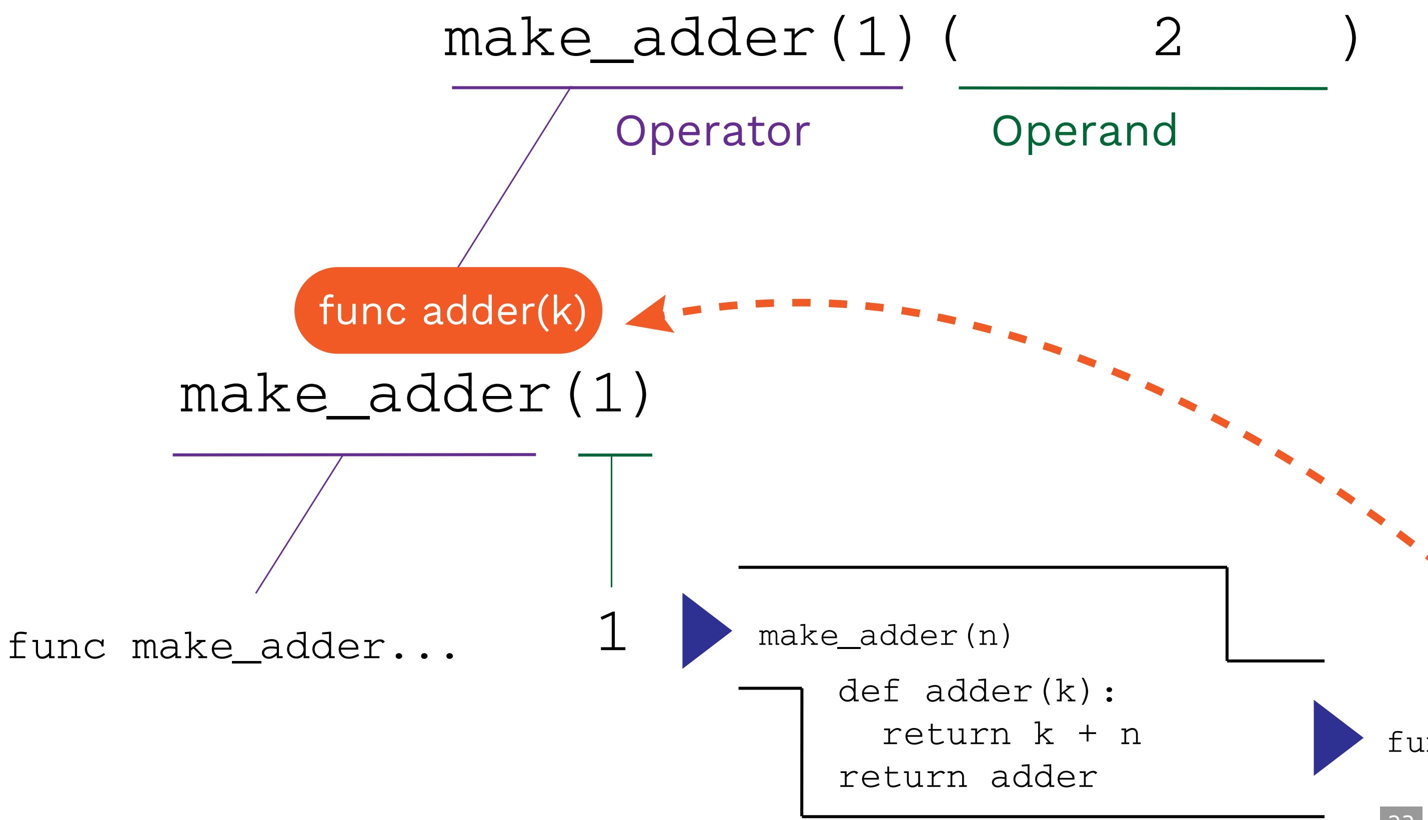

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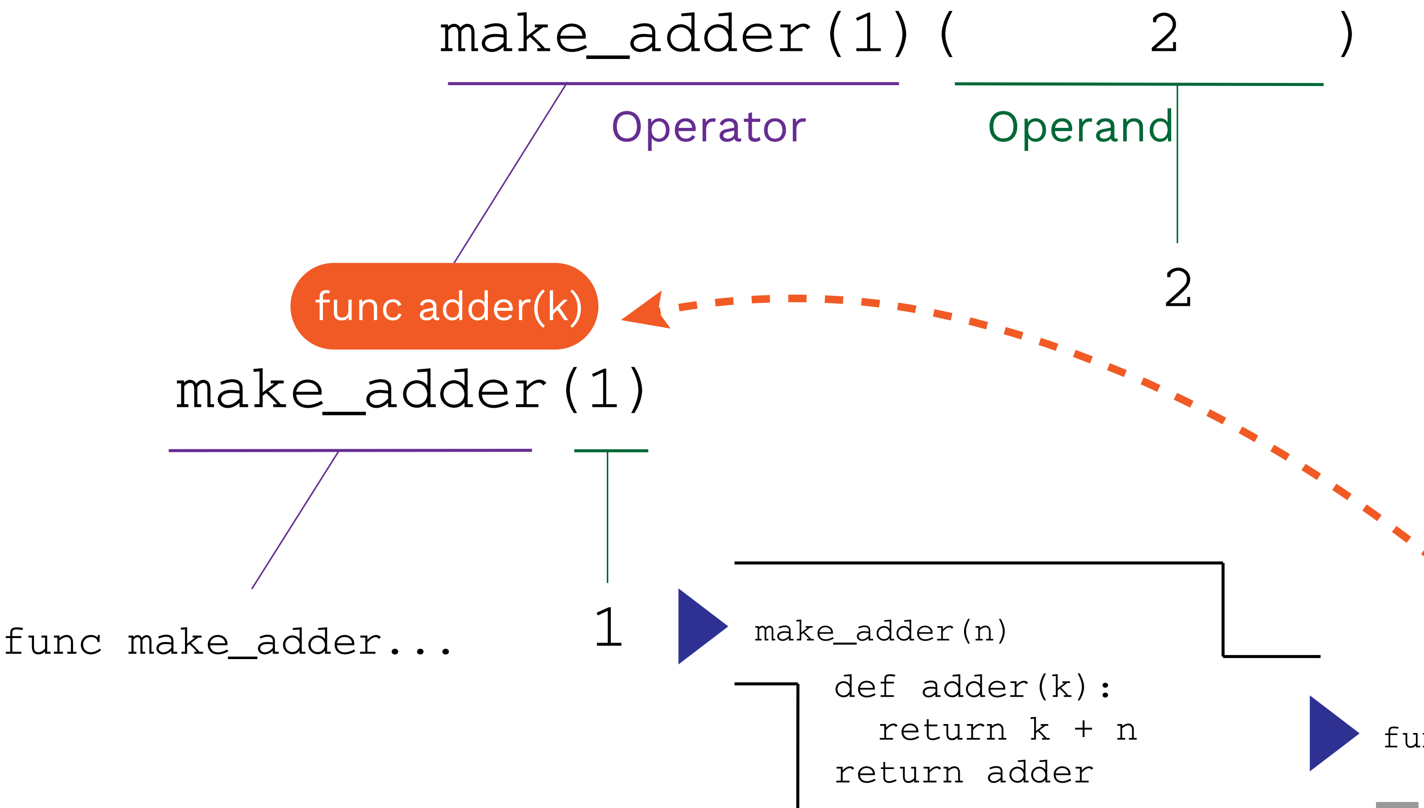

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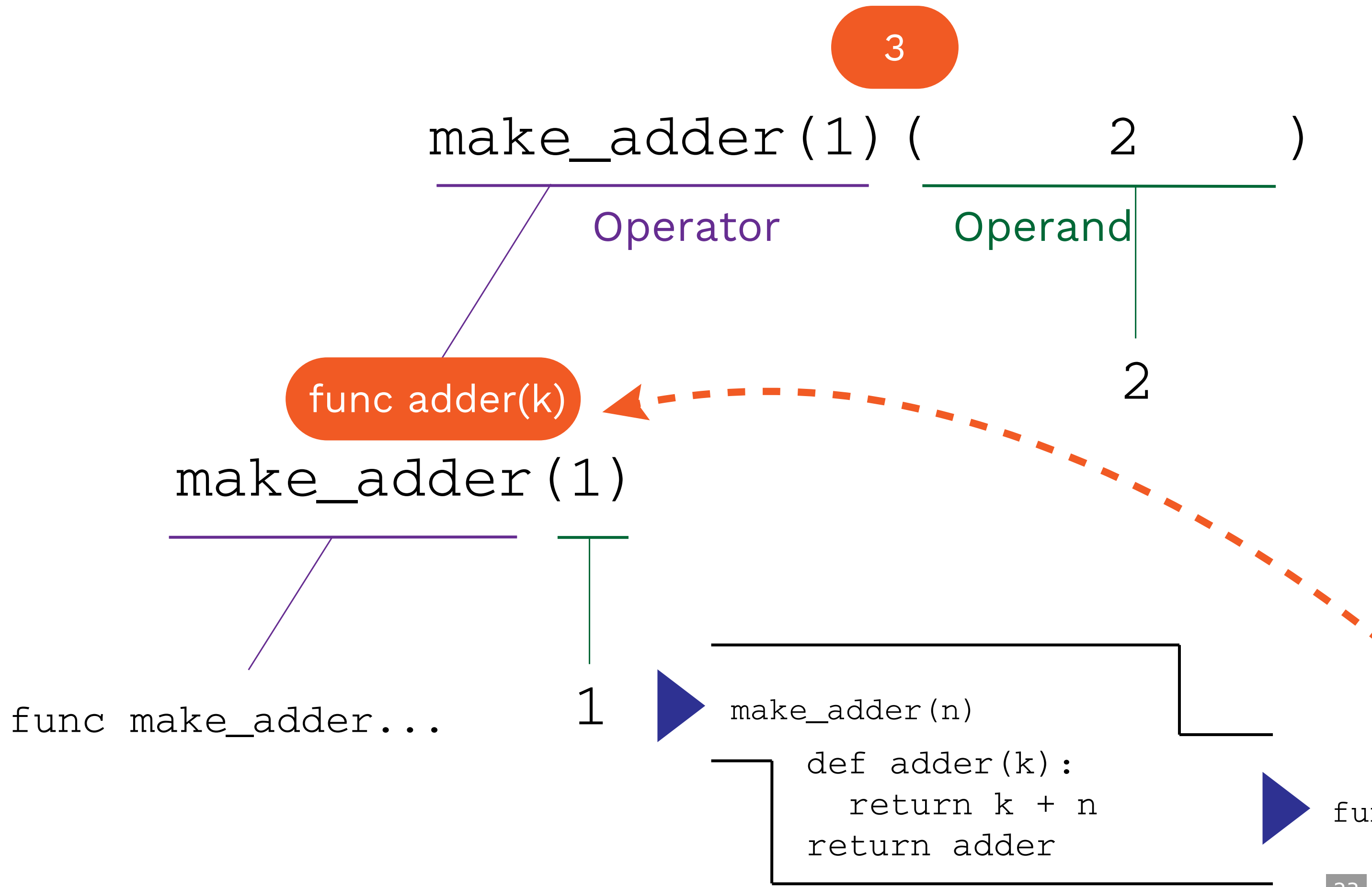

Lambda expressions

Lambda syntax

A **lambda expression** is a simple function definition that evaluates to a function.

The syntax:

```
lambda <parameters>: <expression>
```

A function that takes in `parameters` and returns the result of `expression`.

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A lambda version of the `square` function:

```
square = lambda x: x * x
```

A function that takes in parameter `x` and returns the result of `x * x`.

Lambda syntax tips

A lambda expression does **not** contain return statements or any statements at all.

Incorrect:

```
square = lambda x: return x * x
```

Correct:

```
square = lambda x: x * x
```

Def statements vs. Lambda expressions

```
def square(x):
    return x * x
```

vs

```
square = lambda x: x * x
```


Both create a function with the same domain, range, and behavior.

Both bind that function to the name square.

Only the `def` statement gives the function an **intrinsic name**, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).

Lambda as argument

It's convenient to use a lambda expression when you are passing in a simple function as an argument to another function.

Instead of...

```
def cube(k):
    return k ** 3

summation(5, cube)
```

We can use a lambda:

```
summation(5, lambda k: k ** 3)
```

Conditional expressions

Conditional expressions

A conditional expression has the form:

```
<consequent> if <predicate> else <alternative>
```

Evaluation rule:

- Evaluate the <predicate> expression.
- If it's a true value, the value of the whole expression is the value of the <consequent>.
- Otherwise, the value of the whole expression is the value of the <alternative>.

Lambdas with conditionals

This is invalid syntax:

```
lambda x: if x > 0: x else: 0
```

Conditional expressions to the rescue!

```
lambda x: x if x > 0 else 0
```