CPSC 420 Lecture 21: Today's announcements:

- HW3 is on Gradescope, due Mar 9, 23:59
- Examlet 3 on Mar 17 in class. Closed book & no notes
- Reading: Approximation Algorithms [Intro to Algs 4th Ed. by Cormen, Leiserson, Rivest, Stein Ch. 35]

Today's Plan

- Approximation algorithms
  - Vertex Cover ✓
  - List Scheduling ✓
  - ΔTSP
- Hardness of approximation

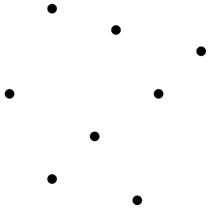
Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a,c) \leq w(a,b) + w(b,c)$ for all vertices a, b, c.

$O(n \lg n)$

using Delaunay Δtion



$\Theta(n^2 \lg n)$

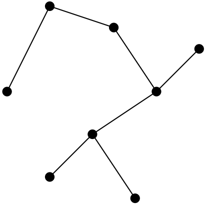
$= \Theta(E \log E)$

1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T.
3. Let E be Euler tour in $T \cup M$.
4. Eliminate repeated vertices in E to get approx TSP.

Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a,c) \leq w(a,b) + w(b,c)$ for all vertices a, b, c.



1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T.
3. Let E be Euler tour in $T \cup M$.
4. Eliminate repeated vertices in E to get approx TSP.

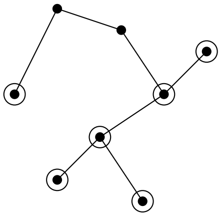
Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a,c) \le w(a,b) + w(b,c)$ for all vertices a, b, c.

min spanning tree T

odd degree vertices in T ○



$O(EV^2)$ Edmonds

$O(E\sqrt{V})$ Micali Vazirani

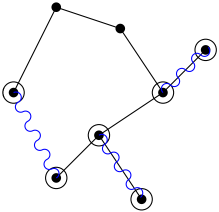
1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T. ⇒ even # of vertices to match
3. Let E be Euler tour in $T \cup M$.
4. Eliminate repeated vertices in E to get approx TSP.

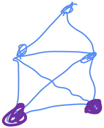
Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a, c) \leq w(a, b) + w(b, c)$ for all vertices a, b, c.

min spanning tree T
odd degree vertices in T ○
min weight matching M





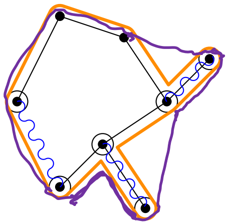
1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T.
3. Let E be Euler tour in $T \cup M$. $O(E)$
4. Eliminate repeated vertices in E to get approx TSP.

Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a,c) \leq w(a,b) + w(b,c)$ for all vertices a, b, c.

min spanning tree T
odd degree vertices in T ○
min weight matching M
Euler tour E



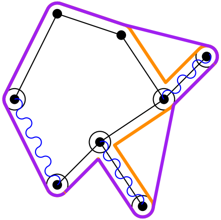
1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T.
3. Let E be Euler tour in $T \cup M$.
4. Eliminate repeated vertices in E to get approx TSP.

Approximating ΔTSP [Christofides '76]

MinTSP: Given graph G with positive weights on the edges, find a Hamiltonian cycle of minimum total weight

ΔTSP is MinTSP where the edge weights obey the triangle inequality: $w(a,c) \leq w(a,b) + w(b,c)$ for all vertices a, b, c.

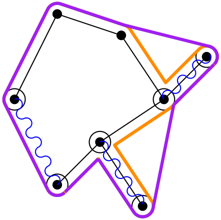
min spanning tree T
odd degree vertices in T ◯
min weight matching M
Euler tour E
approximate TSP P



1. Let T be a Minimum Spanning Tree of G.
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T.
3. Let E be Euler tour in $T \cup M$.
4. Eliminate repeated vertices in E to get approx TSP.

Christofides Alg. is a 3/2-approximation for ΔTSP

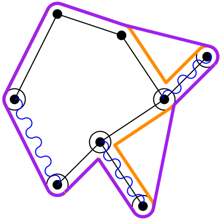
min spanning tree T
odd degree vertices in T ○
min weight matching M
Euler tour E
approximate TSP P



1. Let T be a Minimum Spanning Tree of G. $\quad |T| \leq |\mathrm{TSP}(G)|$

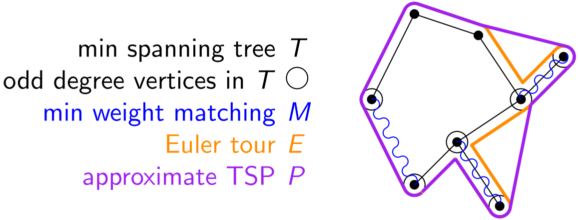
Christofides Alg. is a 3/2-approximation for ΔTSP

min spanning tree T
odd degree vertices in T ○
min weight matching M
Euler tour E
approximate TSP P



1. Let T be a Minimum Spanning Tree of G. $|T| \leq |\mathrm{TSP}(G)|$
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T. $|M| \leq \frac{1}{2}|\mathrm{TSP}(G)|$

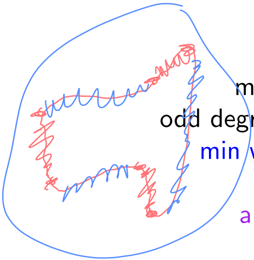
Christofides Alg. is a 3/2-approximation for ΔTSP



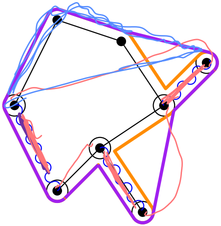
1. Let T be a Minimum Spanning Tree of G. $|T| \leq |\mathrm{TSP}(G)|$
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T. $|M| \leq \frac{1}{2}|\mathrm{TSP}(G)|$
3. Let E be Euler tour in $T \cup M$. $|E| \leq |T| + |M| \leq \frac{3}{2}|\mathrm{TSP}(G)|$

Christofides Alg. is a 3/2-approximation for ΔTSP

10^{-36}



min spanning tree T
odd degree vertices in T ◯
min weight matching M
Euler tour E
approximate TSP P

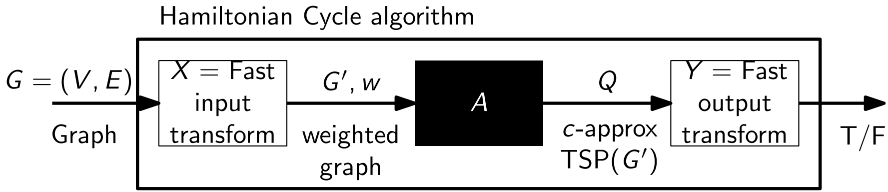


1. Let T be a Minimum Spanning Tree of G. $|T| \leq |\text{TSP}(G)|$
2. Let M be a min length complete matching in the complete graph on odd degree vertices of T. $|M| \leq \frac{1}{2}|\text{TSP}(G)|$
3. Let E be Euler tour in $T \cup M$. $|E| \leq |T| + |M| \leq \frac{3}{2}|\text{TSP}(G)|$
4. Eliminate repeated vertices in E to get approx TSP. $|P| \leq |E|$

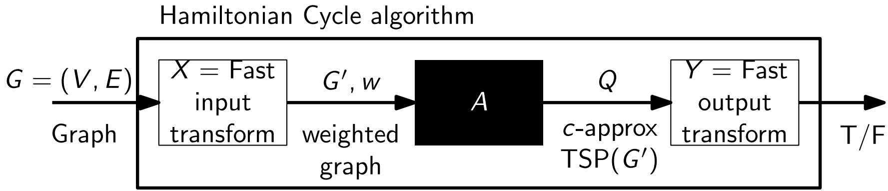
Hardness of Approximation

Claim: General TSP is NP-hard to approximate within a constant factor.

Proof: Suppose A is a polytime c-approximation algorithm for general TSP. We show how to use A to solve Ham. Cycle (an NP-hard decision problem) in polytime.



Hardness of Approximation



Transform X: G' is complete graph on V.

$$w(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ c|V| + 1 & \text{otherwise} \end{cases}$$

Transform Y: If $|Q| \le c|V|$ then output "T" else output "F"

Why does this work?

Edges not in the original graph are so costly that there is a **gap** between cost of tour if G contains Ham Cycle (cost $= |V|$) versus when it does not (cost $> c|V|$). A c-approx. alg is sensitive enough to detect this difference.