CPSC 420 Lecture 6 : Today's announcements:

- Examlet 1 on Jan 27 in class.
- Reading: Linear Programming [Intro to Algs 4th Ed. by Cormen, Leiserson, Rivest, Stein Ch. 29-29.2, pg 817]
- Reading: Maximum Flows & Minimum Cuts [Algorithms by Erickson Ch. 10]

Today's Plan

- Linear Programming
- Network Flow

Linear Programming

General Problem We are given a set of variables. We want to assign real values to them so that:

1. They satisfy given linear equations and/or inequalities.
2. They maximize (or minimize) a given linear "objective" function.

$$\max \; x_1 + 6x_2$$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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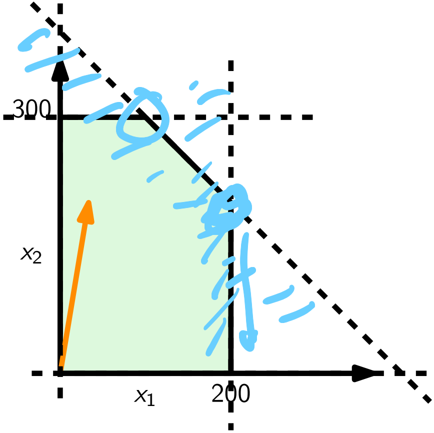

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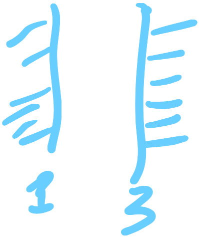
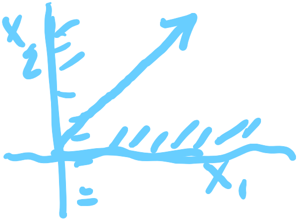
Linear Programming



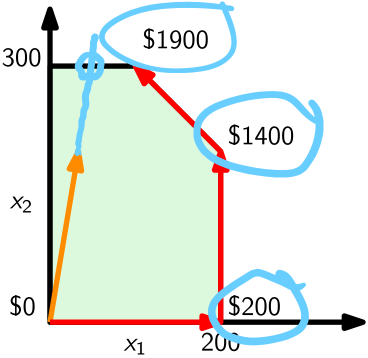
Optimum point exists unless

- **infeasible** The feasible set is empty.
  E.g. $x_1 < 1$ and $x_1 > 3$
- **unbounded** The feasible set is unbounded (in the direction of optimization).
  E.g. $\max\ (x_1 + x_2)$ where $x_1, x_2 \geq 0$.



Simplex algorithm Dantzig 1947

1. Start at a vertex v of the feasible set
2. While there is a neighbor v' of v with better objective value
3. $\qquad v = v'$



More Chocolate

	profit	demand
Box 1	\$1	$\leq$ 200 boxes/day
Box 2	\$6	$\leq$ 300 boxes/day
Box 3	\$13	unlimited

We can produce $\leq$ 400 boxes/day. Box 3 uses three times the filling as Box 2 and total filling is enough for 600 Box 2's per day.

$$\max \quad x_1 + 6x_2 + 13x_3$$

$$x_1 + x_2 + x_3 \leq 400$$

$$x_2 + 3x_3 \leq 600$$

$$x_3 \geq 0$$

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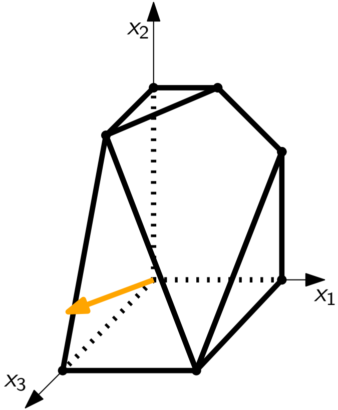
More Chocolate



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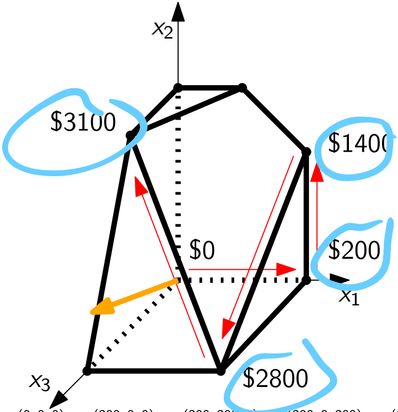
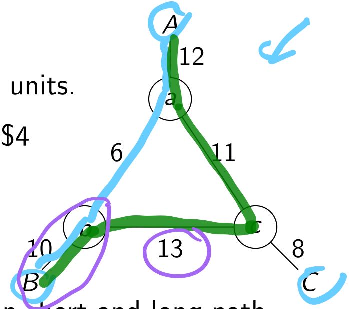


(0, 0, 0) → (200, 0, 0) → (200, 200, 0) → (200, 0, 200) → (0, 300, 100)

$$\max \; x_1 + 6x_2 + 13x_3$$

$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 + x_3 \le 400$$
$$x_2 + 3x_3 \le 600$$
$$x_1, x_2, x_3 \ge 0$$

Bandwidth allocation



Need to connect each pair of users with ≥ 2 units.

Per unit AB pays \$3, BC pays \$2, AC pays \$4

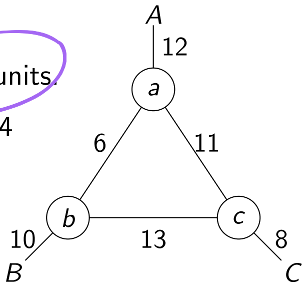
Two paths satisfy each connection:
a short path: $A \to a \to b \to B$ for AB
a long path: $A \to a \to c \to b \to B$ for AB

Let x_{AB} and x'_{AB} be AB bandwidth routed on short and long path

$$\max\ 3(x_{AB} + x'_{AB}) + 2(x_{BC} + x'_{BC}) + 4(x_{AC} + x'_{AC})$$

$$x_{AB} + x'_{AB} + x_{BC} + x'_{BC} \leq 10$$

Bandwidth allocation



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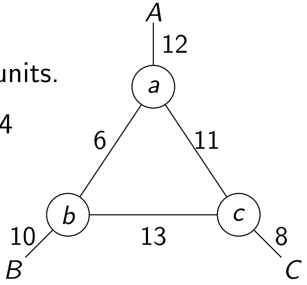
Let x_{AB} and x'_{AB} be AB bandwidth routed on short and long path

$$\begin{aligned}
\max\ & 3(x_{AB} + x'_{AB}) + 2(x_{BC} + x'_{BC}) + 4(x_{AC} + x'_{AC}) \\
x_{AB} + x'_{AB} + x_{BC} + x'_{BC} &\leq 10 \\
x_{AB} + x'_{AB} + x_{AC} + x'_{AC} &\leq 12 \\
x_{AB} + x'_{AC} + x'_{BC} &\leq 6 \\
x_{AB} + x'_{AB} &\geq 2 \\
&\vdots
\end{aligned}$$

Bandwidth allocation



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Per unit AB pays \$3, BC pays \$2, AC pays \$4

Two paths satisfy each connection:
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a long path: $A \rightarrow a \rightarrow c \rightarrow b \rightarrow B$ for AB

Let x_{AB} and x'_{AB} be AB bandwidth routed on short and long path

$$\begin{aligned}
\max\ 3(x_{AB} + x'_{AB}) + 2(x_{BC} + x'_{BC}) + 4(x_{AC} + x'_{AC}) \\
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x_{AB} + x'_{AC} + x'_{BC} &\leq 6 \\
x_{AB} + x'_{AB} &\geq 2 \\
&\vdots
\end{aligned}$$

Solution:

x_{AB}	x'_{AB}	x_{BC}	x'_{BC}	x_{AC}	x'_{AC}
0	7	1.5	1.5	0.5	4.5

Network Flows

A **flow network** is a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a positive **capacity** $c(u, v)$ (non-edges have capacity 0).

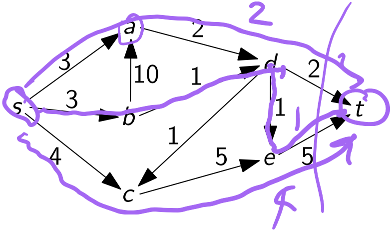
G contains a **source** vertex s and a **sink** vertex t.

A **flow** is an assignment f of real numbers to edges of G:

1. $0 \le f(u, v) \le c(u, v)$ capacity constraints
2. For all $v \ne s, t$: $\sum_u f(u, v) = \sum_w f(v, w)$ flow conservation

The **size** (or **value**) of a flow is: $\text{size}(f) = \sum_{(s,v) \in E} f(s, v)$

Goal: Find flow with maximum size.



Network Flows [Harris & Ross 1955]

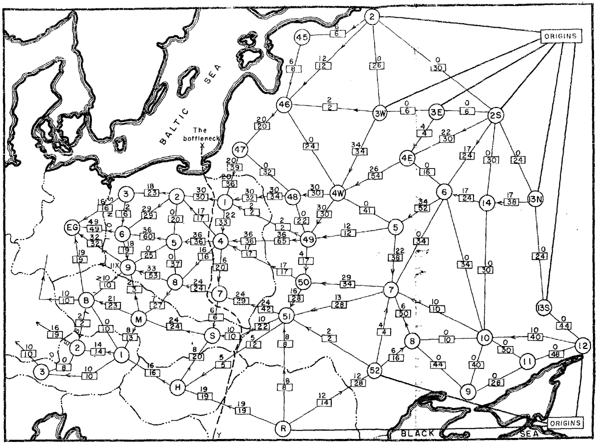


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Fig. 7 — Traffic pattern: entire network available

Legend:

International boundary

Railway operating division

Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / 1000's of tons each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland