CPSC 420 Lecture 24: Today's announcements:

- Examlet 3 on Mar 17 in class. Closed book & no notes
- Reading: Randomized Algorithms [by Motwani and Raghavan]

Today's Plan

- Online Algorithms
  - Hiring problem ✓
  - Page replacement
  - List Update
  - Experts

Any Marking Algorithm is k-competitive

Marking Algorithm MARK

0. Start with all k pages in cache unmarked
1. On page request p
2.   if p not in cache then
3.   evict any unmarked page
            (if no unmarked page, first unmark all k pages)
5.   bring p into cache
6.   mark p

	A	B	C	D	B	A	B	C	D	A	B
MARK	•A	•A	•A	•D	•D	•D	•D	•C	•C	•C	•B
	Y	•B	•B	B	•B	•B	•B	B	•D	•D	D
	Z	Z	•C	C	C	•A	•A	A	A	•A	A
	*	*	*	*		*		*	*		*
FIFO	A	B	C	D	D	A	B	C	D	A	B
	Y	A	B	C	C	D	A	B	C	D	A
	Z	Y	A	B	B	C	D	A	B	C	D
	*	*	*	*		*	*	*	*	*	*

k-competitive

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```

Proof.

Partition $p_1, p_2, \ldots, p_n$ into **phases**, a maximal subsequence with k distinct pages. (The first starts with p_1.) Assume p_1 is not in cache. MARK faults $\leq k$ times per phase.
OPT must have the first page p_i of a phase in cache at the beginning of a phase. Since the remainder of the phase plus the first page of the next phase consists of k different pages (different from p_i), OPT must fault at least once during these requests.
$\Rightarrow$ OPT faults $\geq$ #phases $- 1$ times. $\square$

Randomized online algorithm

Online Hide and Seek

Mouse hides in one of m hiding spots.
Cat looks in one spot each time step.
If Cat finds Mouse, Mouse runs to another spot.

Cost = #times Mouse moves
OPT = min #times future-knowing Mouse must move

1	2	3	4

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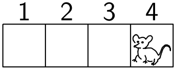

OPT(1 2 3 4 1 2 3 4) = 2

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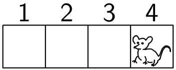

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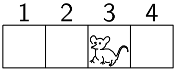

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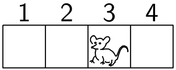

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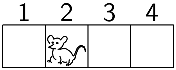

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Randomized online algorithm

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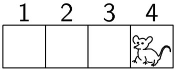

$$\mathrm{OPT}(1\ 2\ 3\ 4\ 1\ 2\ 3\ 4) = 2$$

$$\mathrm{OPT}(1\ 2\ 1\ 1\ 3\ 4\ 1\ 2) = 1$$

Randomized Marking Mouse

If Mouse follows a deterministic strategy, there is a sequence S of Cat probes that causes

$$\text{MouseCost}(S) \geq (m-1)\text{OPT}(S)$$

$$\text{Paging}\ \boxed{\begin{aligned} m-1 &= \text{cache size} \\ m &= \text{ different pages} \\ \text{Mouse} &= \text{page not in cache} \\ \text{Cat probes} &= \text{page requests} \\ \text{Must move} &= \text{page fault} \end{aligned}}$$

Randomized Marking Mouse (RMM)

- Start at random spot
- If Cat probes a spot, mark it
- If Cat probes Mouse's spot,
  Mouse moves to random unmarked spot
- If Mouse is at last unmarked spot, clear marks [phase ends]

Randomized Marking Mouse performance

Claim: $E[\text{RMMCost}(S)] \le O(\log m)\text{OPT}(S)$

Proof: Initially, RMM is equally likely to be at any of the m spots.

1st probe finds Mouse with probability $1/m$.

Whether Mouse is found or not, Mouse is at each of the $m-1$ unmarked spots with prob. $1/(m-1)$.

2nd probe (to unmarked spot) finds Mouse with prob $1/(m-1)$.

Mouse is at each of the $m-2$ unmarked spots with prob. $1/(m-2)$. Etc.

Let $X_i = \begin{cases} 1 & \text{if Mouse found on } i\text{th probe to unmarked spot} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E[\#\text{times found per phase}] &= E[X_1 + X_2 + \cdots + X_m] \\ &\le \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{1} = O(\log m) \end{aligned}$$

OPT moves once per phase. $\square$

Is Totally Random Mouse (TRM) better?

TRM runs to a random spot if found.

Consider the Methodical Cat (MC):

- Probe spots 1, 2, 3, ... until Mouse found
- Repeat

What does the OPT mouse do?

Is Totally Random Mouse (TRM) better?

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- Repeat

What does the OPT mouse do? Hide in spot m

$E[\#\text{times RM found before MC probes } m] =$
$E[\#\text{rolls of } m\text{-sided dice before } m] = m$

$\Rightarrow$ RM is m-competitive.

Random Marking Mouse is best

Claim: Any Mouse A has $E[A(S)] \in \Omega(\log m)\mathrm{OPT}(S)$

Proof:

Idea: Show that a Cat exists that will cause $E[A(S)] \in \Omega(\log m)$ regardless of the Mouse.

Random Cat (RC) probes a random spot with each probe. RC finds Mouse with prob. $\frac{1}{m}$ no matter what Mouse does.
$\Rightarrow E[A(S)]$ after t probes is $\frac{t}{m}$.

How many RC probes until RC examines every spot?

Coupon Collector Problem $\Rightarrow \Theta(m \log m)$

So OPT Mouse (that knows RC's probes) moves once in sequence S of $\Omega(m \log m)$ probes, while Mouse A moves $E[A(S)] \in \frac{\Omega(m \log m)}{m} = \Omega(\log m)$ times. $\square$