CPSC 420 Lecture 11: Today's announcements:

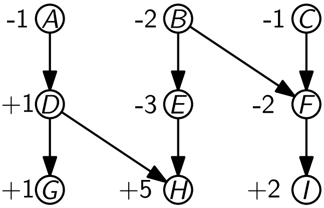
- HW2 is on Gradescope, due Feb 9, 23:59
- Examlet 2 on Feb 17 in class. Closed book & no notes
- Reading: Ch.10 Maximum Flows & Minimum Cuts
  Ch.11 Applications of Flows and Cuts
  Ch.H Linear Programming [all by Erickson]

Today's Plan

- Network Flow
  - Open Pit Mining
- Linear programming duality

Open Pit Mining

Imagine the earth is a lattice of cubes.
Every cube has a value (think “gold” minus “cost” to process)
Constraint: must remove some cubes before others (think cave-in)
Input: Directed acyclic graph $G = (V, E)$. $V =$ set of cubes
$E = \{(u, v) | u$ must be removed before $v\}$
$w(v) =$ value of cube v



Find most profitable set of cubes to process but obey constraints.

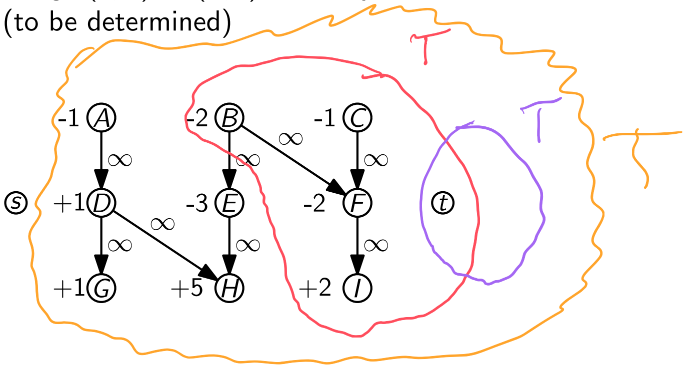
Maximum Value Initial Set

Convert the vertex-valued directed graph $G = (V, E)$ into a flow network so that

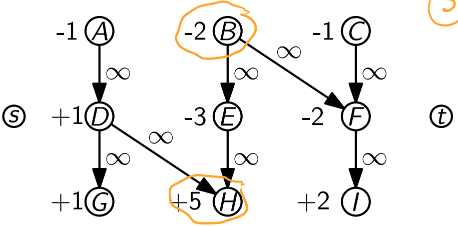
A. Any **finite capacity cut** corresponds to an **initial set**.

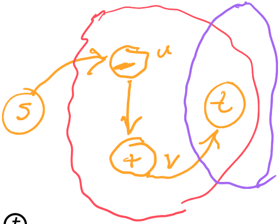
B. A **min capacity cut** corresponds to a max value initial set.

1. Add source s and sink t
2. Set capacity $c(u, v) = \infty$ for $(u, v) \in E$
3. Create an edge (s, v) or (v, t) for every $v \in V$ with finite capacity (to be determined)



Maximum Value Initial Set





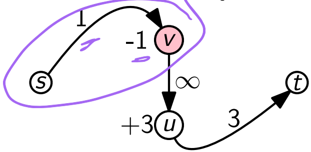
Claim: In this network any finite capacity cut (S, T) defines an initial set $T - t$ (and vice-versa).

Proof: If cut (S, T) has finite capacity then no original edges are directed into T from S so $T - t$ is an initial set.
If $U \subseteq V$ is initial then cut $(S = (V - U) + s, T = U + t)$ has finite capacity. Only edges $(s, u) | u \in T$ and $(v, t) | v \in S$ cross the cut from S to T (and they have finite capacity). $\square$

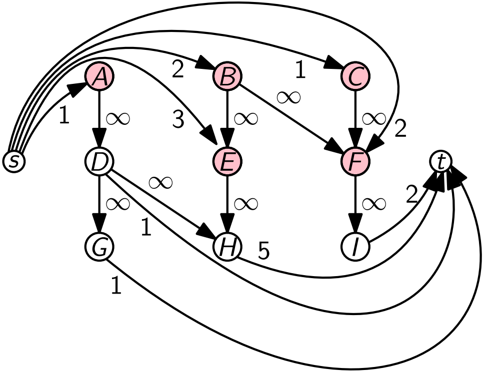
How do we set capacities on the edges from s and edges to t so minimum capacity cut yields maximum value initial set?

Maximum Value Initial Set

Idea If $w(u) > 0$, increase cut capacity if we **don't** take u ($u \notin T$).
If $w(v) < 0$, increase cut capacity if we **do** take v ($v \in T$).



Final flow network



Maximum Value Initial Set

initial set

For any initial set U, the capacity of the corresponding cut $(S = (V - U) + s, \boxed{T = U + t})$ is

$$c(S, T) = \sum_{\substack{u \notin U \\ w(u) > 0}} w(u) + \sum_{\substack{v \in U \\ w(v) < 0}} -w(v)$$

$$\text{profit} = \sum_{\substack{u \in U \\ w(u) > 0}} w(u) + \sum_{\substack{v \in U \\ w(v) < 0}} w(v)$$

$\mathcal{U}$

To maximize profit, minimize (over cuts (S, T), which define U)

$$\sum_{\substack{u \in V \\ w(u) > 0}} w(u) - \text{profit} = \boxed{\sum_{\substack{u \notin U \\ w(u) > 0}} w(u) - \sum_{\substack{v \in U \\ w(v) < 0}} w(v)} = c(S, T)$$

↑ constant over choices of $\mathcal{U}$

↑ cap(S,T)

Linear Programming Duality

$$\begin{aligned} \max\ x_1 + 6x_2 & \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution
$x_1 = 100, x_2 = 300$
with value 1900.

How can we verify this? Can we show $x_1 + 6x_2 \leq 1900$ for all feasible x_1, x_2?

Multiply inequalities by 0, 5, 1 and add them together:

$$\begin{aligned} 0\times \quad & (x_1 \leq 200) \\ +5\times \quad & (x_2 \leq 300) \\ +1\times \quad & (x_1 + x_2 \leq 400) \\ = \quad & x_1 + 6x_2 \leq 1900 \end{aligned}$$

0

$5x_2 \leq 1500$

How did I determine these coefficients?

Linear Programming Duality

Multiply inequalities by y_1, y_2, y_3 and add them together:

$$
\begin{aligned}
y_1 \times \quad & & (x_1 \le 200) \\
+y_2 \times \quad & & (x_2 \le 300) \\
+y_3 \times \quad & & (x_1 + x_2 \le 400) \\
= \quad & \underbrace{(y_1 + y_3)}_{\geq 1} x_1 + \underbrace{(y_2 + y_3)}_{\geq 6} x_2 & \le 200y_1 + 300y_2 + 400y_3
\end{aligned}
$$

1. y_i's must be non-negative (or inequality flips).

$$y_1, y_2, y_3 \ge 0$$

2. $200y_1 + 300y_2 + 400y_3$ is an upper bound on $x_1 + 6x_2$ if

$$y_1 + y_3 \ge 1 \text{ and } y_2 + y_3 \ge 6$$

3. Choose the y_i's to obtain the best upper bound.

$$\text{minimize } 200y_1 + 300y_2 + 400y_3$$

Linear Programming Duality

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

$$\begin{aligned} \text{for all } x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned} \qquad \begin{aligned} \text{for all } y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 6 \\ \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

The objective value of any feasible solution of the dual LP is an upper bound on the objective value of any feasible solution of the primal LP.

Duality Theorem If LP has bounded optimum then so does its dual and the two optimum values are the same.

Duality in general

Primal

$$\max c_1x_1 + \cdots + c_nx_n$$
$$a_{i,1}x_1 + \cdots + a_{i,n}x_n \leq b_i$$
$$x_j \geq 0$$
$$i = 1 \ldots m \quad j = 1 \ldots n$$

Dual

$$\min b_1y_1 + \cdots + b_my_m$$
$$a_{1,j}y_1 + \cdots + a_{m,j}y_m \geq c_j$$
$$y_i \geq 0$$
$$i = 1 \ldots m \quad j = 1 \ldots n$$