CPSC 420 Lecture 23: Today's announcements:

- Examlet 3 on Mar 17 in class. Closed book & no notes
- Reading: Randomized Algorithms [by Motwani and Raghavan]

Today's Plan

- Online Algorithms
  - Hiring problem ✓
  - Page replacement
  - List Update
  - Experts

Online Algorithms

For input sequence $p_1, p_2, \ldots, p_n$ an **online algorithm** must produce an output given $p_1, p_2, \ldots, p_i$ for each i **without seeing** $p_{i+1}, \ldots, p_n$.

Page replacement in a cache

$p_1, \ldots, p_n$ is a sequence of **page requests** made by a program.

k is cache size (in pages).

At the ith page request, p_i, the cache contains some k pages. If p_i is **not** in cache (**page fault**) some page must be evicted from cache to make room for p_i, then p_i is added to the cache.

The cost of a page replacement algorithm A on sequence $p_1, p_2, \ldots, p_n$ is

$$f_A(p_1, p_2, \ldots, p_n) = \# \text{ faults } A \text{ has on } p_1, p_2, \ldots, p_n$$

Online algorithm must decide what page to evict without knowing future requests.

Some paging algorithms

- **LRU** (Least Recently Used) Evict page whose most recent request occurred furthest in the past.
- **FIFO** (First In First Out) Evict page that has been in cache the longest.
- **LFU** (Least Frequently Used) Evict page that has been requested least often.
- **OPT** (Optimal) Evict best* page knowing the future requests.

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* **next** request is furthest in the future

Paging algorithms: Example

page requests — time

	A	B	C	D	B	A	B	C	D	A	B	requests
LRU	A	B	C	D	D	D	D	C	C	C	B	
	Y	A	B	C	C	A	A	A	D	D	D	
	Z	Y	A	B	B	B	B	B	B	A	A	
	*	*	*	*		*		*	*	*	*	9 faults ✓
FIFO	A	B	C	D	D	A	B	C	D	A	B	
	Y	A	B	C	C	D	A	B	C	D	A	
	Z	Y	A	B	B	C	D	A	B	C	D	
	*	*	*	*		*	*	*	*	*	*	10 faults ✓
LFU	A	B	C	D	D	D	D	C	C	A	A	
	Y	A	B	C	C	A	A	A	D	D	D	
	Z	Y	A	B	B	B	B	B	B	B	B	
	*	*	*	*		*		*	*	*		8 faults ✓
OPT	A	B	C	D	D	D	D	D	D	D	B	
	Y	A	B	B	B	B	B	C	C	C	C	
	Z	Y	A	A	A	A	A	A	A	A	A	
	*	*	*	*				*			*	6 faults

Evaluating online algorithms

Which algorithm is best?

Worst case cost

$$n = \max_{p_1..p_n} f_{\mathrm{LRU}}(p_1..p_n) = \max_{p_1..p_n} f_{\mathrm{FIFO}}(p_1..p_n) = \max_{p_1..p_n} f_{\mathrm{LFU}}(p_1..p_n)$$

Every online algorithm can be made to fault on every request.

Average case cost

$$E[f_X(p_1..p_n)] = n(1 - k/m)$$

where $m = \#$ possible pages

Competitive analysis

How does online alg compare to best offline alg?

An online algorithm A is c-**competitive** if for all $p_1, p_2, \ldots, p_n$,

$$f_A(p_1..p_n) \leq c \cdot f_{\mathrm{OPT}}(p_1..p_n) + b$$

for some constant b.

Lower Bound on Competitive Factor

Claim: If A is a deterministic online alg for paging then $c \geq k$.
Proof: Idea: Find a sequence that is bad for A but good for OPT.
Suppose both A and OPT start with pages $1, 2, \ldots, k$ in cache.

k+1 one additional page

Request page $a_1 = k + 1$

A evicts some page, call it a_2

Request page a_2

A evicts some page, call it a_3

Request page a_3

etc...

$\Rightarrow$ Online alg A faults on every request.

How many different pages are there? k+1

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    A evicts some page, call it a_3
Request page a_3
    etc...

$\Rightarrow$ Online alg A faults on every request.

How many different pages are there? $k + 1$

Lower Bound on Competitive Factor

How often does OPT fault?

	a_1 a_2 a_3 ... a_j a_{j+1} ...	requests
OPT	* $\qquad\qquad\qquad\qquad$ *	faults

Sub-claim: OPT faults at most once in k successive page requests from this sequence

Proof.
OPT evicts the page p in cache that is requested furthest in the future. Since there are only $k+1$ different pages, the next k pages requested can be kept in cache. □

$\Rightarrow f_A(a_1, a_2, \ldots, a_n) \geq k \cdot f_{\text{OPT}}(a_1, a_2, \ldots, a_n)$ □

LRU is k-competitive

Theorem

LRU is k-competitive

Proof.

Let $p_1, p_2, \ldots, p_n$ be **any** sequence of page requests. Partition this sequence into contiguous subsequences (**phases**) such that LRU faults on the first page of the phase and the phase contains exactly k different pages (or $\leq k$ in the last phase).

OPT faults ≥ 1

$k=3$	A	C	D	C	C	B	C	A	D	A	A	requests
LRU	*	~~?~~ *	~~?~~ *			*		*	*			faults
OPT	?	?	?			*			*			faults
	phase 1					ph 2			ph 3			

$$\frac{k \ \#phases}{\#phases - 1}$$

LRU faults $\leq k$ times per phase.

OPT must have the first page of a phase in cache at the beginning of a phase. Since the remainder of the phase plus the first page of the next phase consists of k different pages, OPT must fault at least once during these requests.$\Rightarrow$ OPT faults $\geq$ #phases $- 1$

Any Marking Algorithm is k-competitive

Marking Algorithm MARK

```
0. Start with all k pages in cache unmarked
1. On page request p
2.      if p not in cache then
3.           evict any unmarked page
             (if no unmarked page, first unmark all k pages)
5.           bring p into cache
6.      mark p
```

	A	B	C	D	B	A	B	C	D	A	B
MARK	•A	•A	•A	•D	•D	•D	•D	•C	•C	•C	•B
	Y	•B	•B	B	•B	•B	•B	~~B~~	•D	•D	D
	Z	Z	•C	C	C	•A	•A	A	A	•A	A
	*	*	*	*		*		*	*		*
LRU	A	B	C	D	D	D	D	C	C	C	B
	Y	A	B	C	C	A	A	~~A~~	D	D	D
	Z	Y	A	B	B	B	B	B	B	A	A
	*	*	*	*		*		*	*	*	*

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5.   bring p into cache
6.   mark p

Proof.

Partition $p_1, p_2, \ldots, p_n$ into **phases**, a maximal subsequence with k distinct pages. (The first starts with p_1.) Assume p_1 is not in cache. MARK faults $\leq k$ times per phase.

OPT must have the first page p_i of a phase in cache at the beginning of a phase. Since the remainder of the phase plus the first page of the next phase consists of k different pages (different from p_i), OPT must fault at least once during these requests.

$\Rightarrow$ OPT faults $\geq$ #phases $- 1$ times. $\square$

Randomized online algorithm

Online Hide and Seek

Mouse hides in one of m hiding places.
Cat looks in one hiding place each time step.
If Cat finds Mouse, Mouse runs to another place.

Cost = #times Mouse moves
OPT = min #times future-knowing Mouse must move

1	2	3	4

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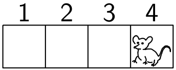

OPT(1 2 3 4 1 2 3 4) = 2

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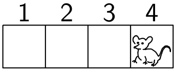

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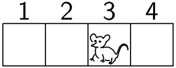

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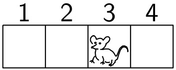

$\text{OPT}(1\ 2\ 3\ 4\ 1\ 2\ 3\ 4) = 2$

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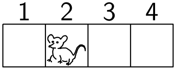

OPT(1 2 3 4 1 2 3 4) = 2

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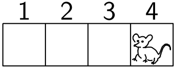

$$\mathrm{OPT}(1\ 2\ 3\ 4\ 1\ 2\ 3\ 4) = 2$$

$$\mathrm{OPT}(1\ 2\ 1\ 1\ 3\ 4\ 1\ 2) = 1$$