CPSC 420 Lecture 7 : Today's announcements:

- Examlet 1 on Jan 27 in class. Closed book & no notes
- Reading: Maximum Flows & Minimum Cuts [Algorithms by Erickson Ch. 10]

Today's Plan

- Network Flow
  - Ford-Fulkerson algorithm
  - Augmenting paths

Network Flows

A **flow network** is a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a positive **capacity** $c(u, v)$ (non-edges have capacity 0).

G contains a **source** vertex s and a **sink** vertex t.

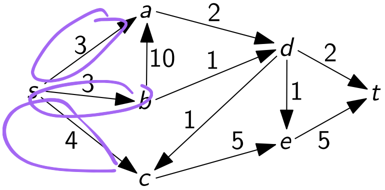
A **flow** is an assignment f of real numbers to edges of G:

1. For all u, v: $0 \le f(u, v) \le c(u, v)$ capacity constraint
2. For all $v \ne s, t$: $\sum_u f(u, v) = \sum_w f(v, w)$ flow conservation

The **size** (or **value**) of a flow is: $\text{size}(f) = \sum_{(s,v)\in E} f(s, v)$

Goal: Find flow with maximum size.

max $f(s,a)+f(s,b)+f(s,c)$



11 capacity constraints
11 non-neg "
5 flow conservation constraints

Network Flows [Harris & Ross 1955]

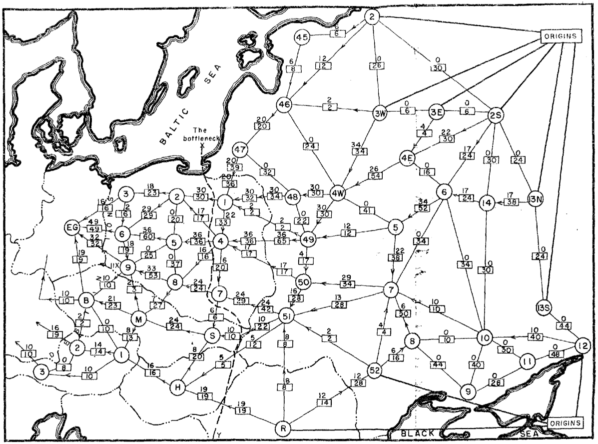


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Fig. 7 — Traffic pattern: entire network available

Legend:

International boundary

Railway operating division

Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / √1000's of tons each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

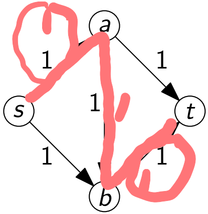
Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

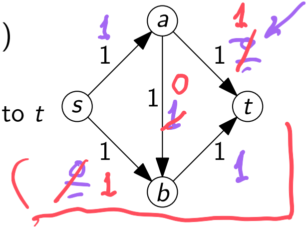
Note IIX at Division 9, Poland

Max Flow via Path Augmentation [Ford & Fulkerson 1962]

1. Start with zero flow (a feasible solution)
2. Repeat until impossible
   - Choose an **augmenting path** from s to t
   - Increase flow on this path as much as possible



Max Flow via Path Augmentation [Ford & Fulkerson 1962]



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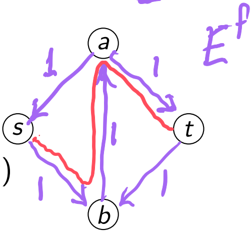
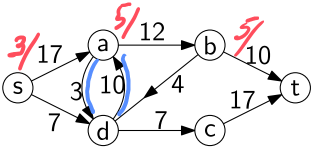
The **residual network** of flow network $G = (V, E)$ with flow f is $G^f = (V, E^f)$ where
$E^f = \{(u, v) | f(u, v) < c(u, v) \text{ or } f(v, u) > 0\}$
The **residual capacity** of an edge $(u, v) \in E^f$ is

$$c^f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } f(u, v) < c(u, v) \\ f(v, u) & \text{if } f(v, u) > 0 \end{cases}$$

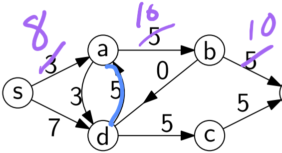


An **augmenting path** in G is an $s \rightsquigarrow t$ path in G^f

Residual Network Example

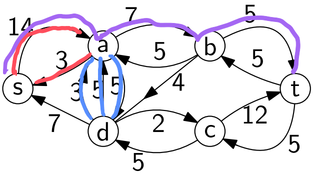


Flow network

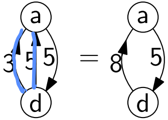


Flow

augment flow by 5



Residual network



Correctness of Ford & Fulkerson

source sink

A **cut** is a partition (S, T) of V such that $s \in S$ and $t \in T$. (Cut separates s from t.)
The **capacity** of cut (S, T) is $c(S, T) = \sum_{u \in S, v \in T} c(u, v)$
The **flow** across cut (S, T) is $f(S, T) = \sum_{u \in S, v \in T} f(u, v)$

Lemma

For any flow f and any cut (S, T), size$(f) = f(S, T) \leq c(S, T)$

Proof outline

1. $f(S, T) \leq c(S, T)$ [Capacity Constraint]
2. $f(S, T) = f(S - \{v\}, T + \{v\})$ [Flow Conservation]
3. $f(\{s\}, V - \{s\}) = \text{size}(f)$ [Definition]

Correctness of Ford & Fulkerson

$$\geq 0$$

1. $$\begin{aligned} f(S,T) &= \sum_{a \in S, b \in T} f(a,b) - f(b,a) \\ &\leq \sum_{a \in S, b \in T} f(a,b) \\ &\leq \sum_{a \in S, b \in T} c(a,b) = c(S,T) \end{aligned}$$

Correctness of Ford & Fulkerson

2. $$f(S-\{v\}, T+\{v\}) \qquad = \qquad f(S,T)$$

$$\text{difference} = \boxed{\sum_{\substack{a \in S-\{v\} \\ b \in T+\{v\}}} f(a,b) - f(b,a)} - \boxed{\sum_{a \in S, b \in T} f(a,b) - f(b,a)}$$

$$= \sum_{a \in S-\{v\}} (f(a,v) - f(v,a)) - \sum_{b \in T} (f(v,b) - f(b,v))$$

$$= \sum_{u \in V} f(u,v) - \sum_{w \in V} f(v,w) = 0$$

Repeat until $S = \{s\}$ implies $\text{size}(f) = f(S,T)$ $\square$

Correctness of Ford & Fulkerson

Theorem

If residual network G^f has no augmenting path then f is a max size flow.

Proof: Let $S = \{v | s \rightsquigarrow v \text{ in } G^f\}$. The sink $t \notin S$ since G^f has no augmenting path. Let $T = V - S$. Size of flow $f = f(S, T)$ equals $c(S, T)$ since $f(u, v) = c(u, v)$ for $u \in S, v \in T$. Size of any flow $\leq c(S, T)$ by Lemma. $\square$

Max-Flow Min-Cut Theorem

Size of max-flow f^* equals capacity of min capacity cut (S^*, T^*).

Proof: size(f^*) $\leq c(S^*, T^*)$ by Lemma

size(f^*) $= c(S, T)$ as defined by $S = \{v | s \rightsquigarrow v \text{ in } G^{f^*}\}$ and
$c(S, T) \geq c(S^*, T^*)$ since (S^*, T^*) is min capacity cut. $\square$