CPSC 420 Lecture 8 : Today's announcements:

- Examlet 1 on Jan 27 in class. Closed book & no notes
- Reading: Maximum Flows & Minimum Cuts [Algorithms by Erickson Ch. 10]

Today's Plan

- Network Flow
  - Ford-Fulkerson algorithm
  - Augmenting paths
  - Max-Flow Min-Cut Theorem
  - Edmonds-Karp algorithm

Network Flows

A **flow network** is a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a positive **capacity** $c(u, v)$ (non-edges have capacity 0).

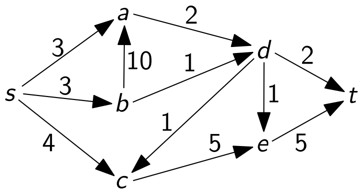
G contains a **source** vertex s and a **sink** vertex t.

A **flow** is an assignment f of real numbers to edges of G:

1. For all u, v: $0 \leq f(u, v) \leq c(u, v)$ capacity constraint
2. For all $v \neq s, t$: $\sum_u f(u, v) = \sum_w f(v, w)$ flow conservation

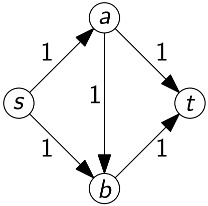
The **size** (or **value**) of a flow is: $\text{size}(f) = \sum_{(s,v) \in E} f(s, v)$

Goal: Find flow with maximum size.



Max Flow via Path Augmentation [Ford & Fulkerson 1962]

1. Start with zero flow (a feasible solution)
2. Repeat until impossible
   - Choose an **augmenting path** from s to t
   - Increase flow on this path as much as possible



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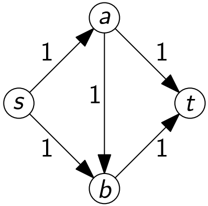
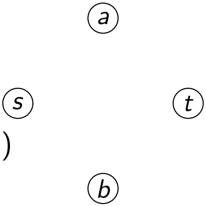


The **residual network** of flow network $G = (V, E)$ with flow f is $G^f = (V, E^f)$ where
$E^f = \{(u, v) | f(u, v) < c(u, v) \text{ or } f(v, u) > 0\}$
The **residual capacity** of an edge $(u, v) \in E^f$ is

$$c^f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } f(u, v) < c(u, v) \\ f(v, u) & \text{if } f(v, u) > 0 \end{cases}$$



An **augmenting path** in G is an $s \rightsquigarrow t$ path in G^f

Correctness of Ford & Fulkerson

A **cut** is a partition (S, T) of V such that $s \in S$ and $t \in T$. (Cut separates s from t.)
The **capacity** of cut (S, T) is $c(S, T) = \sum_{u \in S, v \in T} c(u, v)$
The **flow** across cut (S, T) is $f(S, T) = \sum_{u \in S, v \in T} f(u, v) - f(v, u)$

Lemma

For any flow f and any cut (S, T), size$(f) = f(S, T) \leq c(S, T)$

Proof outline

1. $f(S, T) \leq c(S, T)$ [Capacity Constraint]
2. $\forall v \neq s, f(S, T) = f(S - \{v\}, T + \{v\})$ [Flow Conservation]
3. $f(\{s\}, V - \{s\}) = \text{size}(f)$ [Definition]

Correctness of Ford & Fulkerson

2. $f(S - \{v\}, T + \{v\}) \quad = \quad f(S, T)$

$$\text{difference} = \boxed{\sum_{\substack{a \in S - \{v\} \\ b \in T + \{v\}}} f(a, b) - f(b, a)} - \boxed{\sum_{a \in S, b \in T} f(a, b) - f(b, a)}$$

b = v

a = v

this doesn't cancel

$$= \sum_{a \in S - \{v\}} (f(a, v) - f(v, a)) - \sum_{b \in T} (f(v, b) - f(b, v))$$

$$= \sum_{u \in V} f(u, v) - \sum_{w \in V} f(v, w) = 0$$

exactly flow conservation

Repeat until $S = \{s\}$ implies $\text{size}(f) = f(S, T)$ $\square$

def of size of flow

Correctness of Ford & Fulkerson

1 Theorem: *If residual network G^f has no augmenting path then f is a max size flow.* Correctness

Proof: Let $S = \{v | s \rightsquigarrow v \text{ in } G^f\}$. The sink $t \notin S$ since G^f has no augmenting path. Let $T = V - S$. Size of flow $f = f(S, T)$ equals $c(S, T)$ since $f(u, v) = c(u, v)$ for $u \in S, v \in T$. Size of any flow $\leq c(S, T)$ by Lemma. $\square$

2 Max-Flow Min-Cut Theorem

Size of max-flow f^* equals capacity of min capacity cut (S^*, T^*).

Proof: $\text{size}(f^*) \leq c(S^*, T^*)$ by Lemma

$\text{size}(f^*) = c(S, T)$ where $\boxed{S = \{v | s \rightsquigarrow v \text{ in } G^{f^*}\}}$ and

$c(S, T) \geq c(S^*, T^*)$ since (S^*, T^*) is min capacity cut. $\square$

3 Integrality Theorem

If all capacities are integers then there is a max-flow with integer flow on every edge.

Proof: Ford-Fulkerson augments by an integral amount. $\square$

Running time of Ford & Fulkerson

For flow networks (V, E) with integral capacities:

Running time $O(|E|\text{size}(f^*))$

Running time of Ford & Fulkerson

For flow networks (V, E) with integral capacities:

Running time $O(|E|\text{size}(f^*))$

For flow networks (V, E) with irrational capacities:

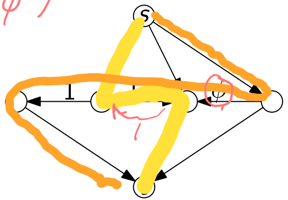
Running time ∞

res. cap.

$(1, 0, 1)$ $(\phi^k, 0, \phi^{k+1})$

$(\phi^2, \phi^2, 0)$

(ϕ^2, ϕ, ϕ^3)



$\phi = \frac{\sqrt{5}-1}{2} < 1 \qquad 1 - \phi = \phi^2$

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Running time of Ford & Fulkerson

For flow networks (V, E) with integral capacities:
Running time $O(|E|\text{size}(f^*))$

For flow networks (V, E) with irrational capacities:
Running time ∞

Edmonds-Karp
Shortest augmenting path: Running time $O(|V||E|^2)$
regardless of capacities

Running time of Ford & Fulkerson

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Beyond Ford-Fulkerson

[Orlin 2012]	$O(\|V\|\|E\|)$
[Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva 2022]	$O(\|E\|^{1+o(1)})$