CPSC 420 Lecture 5 : Today's announcements:

- HW1 available on Gradescope, due Jan 19, 23:59
- Examlet 1 on Jan 27 in class.
- Reading: Linear Programming [Intro to Algs 4th Ed. by Cormen, Leiserson, Rivest, Stein Ch. 29-29.2, pg 817]
- Reading: Maximum Flows & Minimum Cuts [Algorithms by Erickson Ch. 10]

Today's Plan

- Voronoi diagrams
- Linear Programming

Voronoi Diagram using Divide and Conquer [Shamos & Hoey '75]

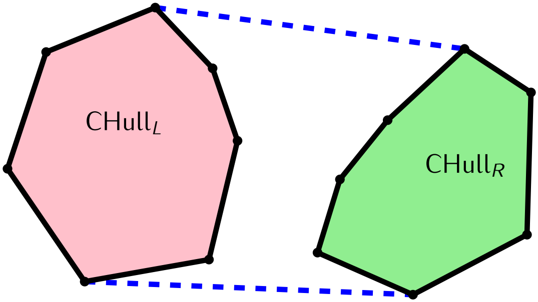
This algorithm finds both VorD and CH of S.

0. Sort sites by x-coord

VorD+CHull(S)

1. if $|S| = 1$ return VorD $= \emptyset$ and CHull $= [s_1]$
2. if $|S| = 2$ return VorD $= \perp$ -bisector of $\overline{s_1 s_2}$ and CHull $= [s_1 s_2]$
3. Recursively find VorD$_L$ and CHull$_L$ of $s_1, \ldots, s_{n/2}$
4. Recursively find VorD$_R$ and CHull$_R$ of $s_{n/2+1}, \ldots, s_n$
5. Merge CHull$_L$ and CHull$_R$ to get CHull
6. Stitch together VorD$_L$ and VorD$_R$ to get VorD
7. Return VorD and CHull

Merge CHull_L and CHull_R

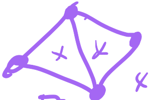


See your Homework 1 for a (more general) $O(n)$ time solution.
The two blue lines are the upper and lower bridges.

Stitch together VorD_L and VorD_R

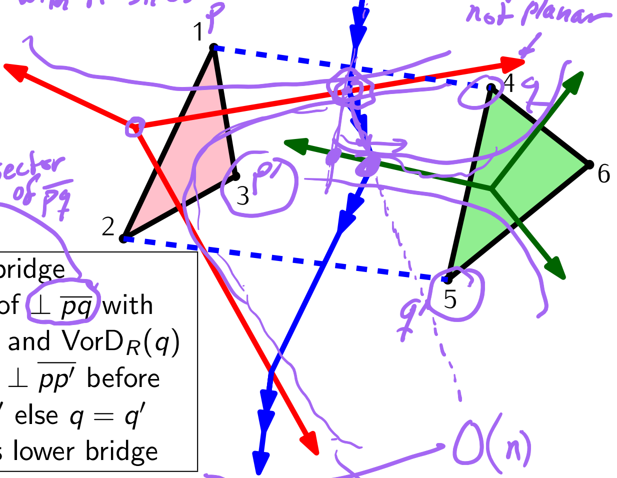
\# Vor edges in VorD with n sites $\leq 3(n-2)$

not planar



planar

perp. bisector of $\overline{pq}$



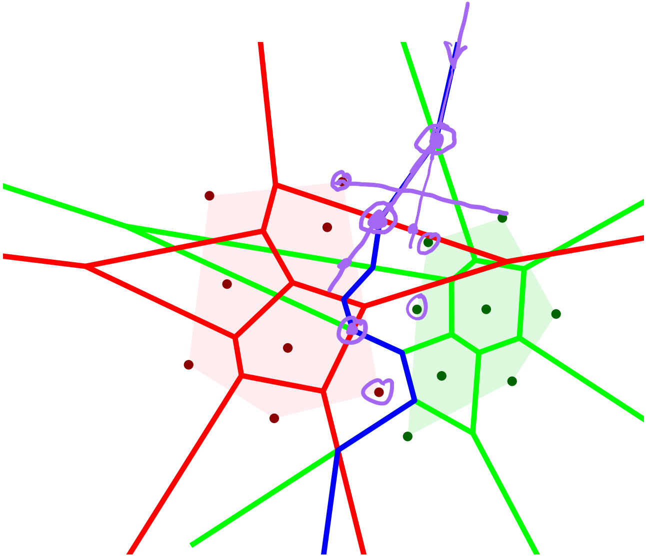
1. Let $\overline{pq}$ be upper bridge
2. Find intersection of $\perp \overline{pq}$ with edge of $\mathrm{VorD}_L(p)$ and $\mathrm{VorD}_R(q)$
3. If $\perp \overline{pq}$ intersects $\perp \overline{pp'}$ before $\perp \overline{qq'}$ then $p = p'$ else $q = q'$
4. Repeat until $\overline{pq}$ is lower bridge

$O(n)$

Note: Step 2 can be done by scanning edges of $\mathrm{VorD}_L(p)$ cw and $\mathrm{VorD}_R(q)$ ccw (no backtracking).

Euler $V - E + F = 2$

Example from Computational Geometry by Preparate & Shamos



Linear Programming

Manufacturing chocolate

	profit	demand
Box 1	\$1	$\leq$ 200 boxes/day
Box 2	\$6	$\leq$ 300 boxes/day

We can produce $\leq$ 400 boxes/day. How many boxes of each type?
x_1 is # Box 1, x_2 is # Box 2

$$\max \quad x_1 + 6x_2$$

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1 , x_2 \geq 0$$

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Linear Programming

General Problem We are given a set of variables. We want to assign real values to them so that:

1. They satisfy given linear equations and/or inequalities.
2. They maximize (or minimize) a given linear "objective" function.

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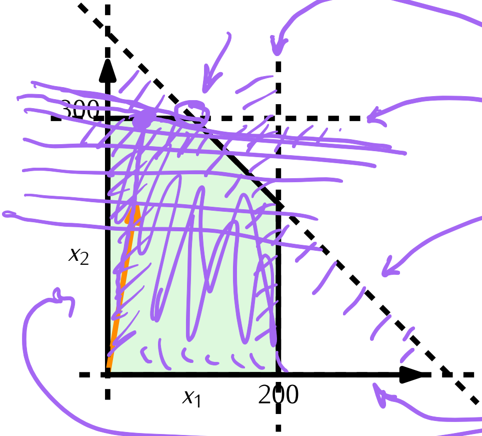

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$$x_1, x_2 \geq 0$$

Linear Programming

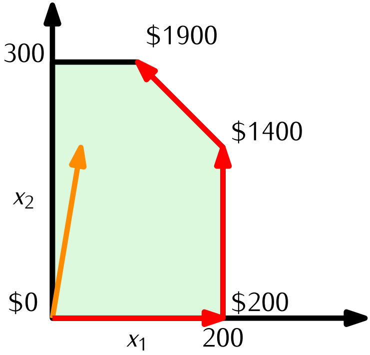
Optimum point exists unless

infeasible: The feasible set is empty.
E.g. $x_1 < 1$ and $x_1 > 3$

unbounded: The feasible set is unbounded (in the direction of optimization).
E.g. $\max\ (x_1 + x_2)$ where $x_1, x_2 \geq 0$.

Simplex algorithm Dantzig 1947

1. Start at a vertex v of the feasible set
2. While there is a neighbor v' of v with better objective value
3. $\quad v = v'$



More Chocolate

	profit	demand
Box 1	\$1	$\leq$ 200 boxes/day
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Box 3	\$13	unlimited

We can produce $\leq$ 400 boxes/day. Box 3 uses three times the filling as Box 2 and total filling is enough for 600 Box 2's per day.

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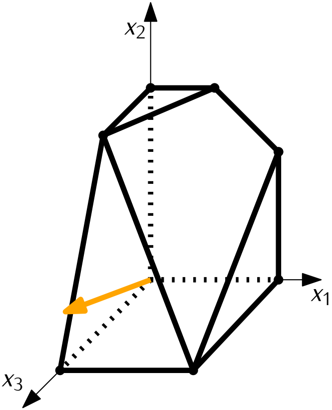
$$\max \; x_1 + 6x_2 + 13x_3$$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 + x_3 &\leq 400 \\ x_2 + 3x_3 &\leq 600 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

More Chocolate



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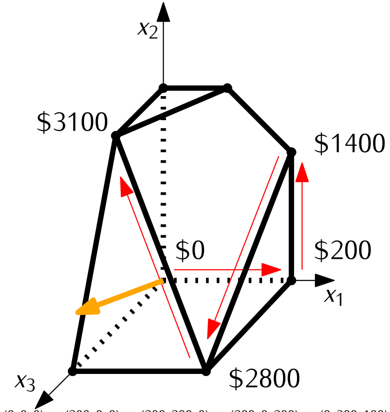

$(0,0,0) \to (200,0,0) \to (200,200,0) \to (200,0,200) \to (0,300,100)$

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