Introduction to Integration

Ray Zhu

December 24, 2024

1 What is Integration

Imagine you have a curve, and you want to find the **Area under** it. This is where integration comes in. It helps calculate that area precisely. But before we jump to integration, let's first look at a simpler idea: breaking the area into smaller pieces.

2 Riemann Sum: The Building Block

A quadratic function $f(x) = 100 - x^2$ is given. You want to find the area under the function from $x = 0$ to $x = 10$. This smooth shape cannot be computed by area formula from regular geometric shapes. Thus, we estimate this area.

2.1 Step 1: Divide the Area into Rectangles

Divide the interval $[0, 10]$ into n equal parts. Each part has a width:

$$\Delta x = \frac{10 - 0}{n}$$

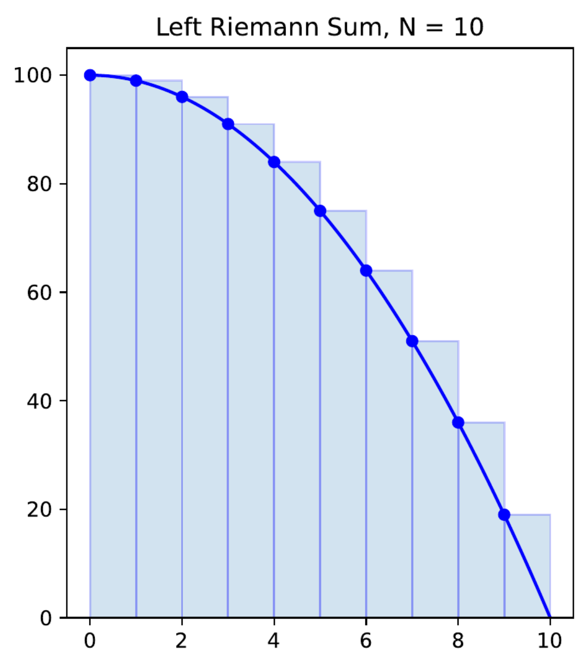

Figure 1: f(x) divided into 10 rectangles

For this example we let $n = 10$ as shown in Figure 1. Over each sub-interval, draw a rectangle. The height of the rectangle is determined by the value of $f(x)$ at some point in that sub-interval.

2.2 Step 2: Calculate the Area of Each Rectangle

The area of a rectangle is:

$$Area = Height \times Width$$

Here, height is $f(x_i)$, and the width is Δx. Notice that since Δx is constant, the value of x_i follows a arithmetic sequence $x_i = a + i\Delta x$. Thus i denotes the index in arithmetic sequence and a denotes the first value in sequence.

2.3 Step 3: Add Up All the Rectangles

The total area is approximately the sum of all rectangle areas:

$$TotalArea \approx f(x_0)\Delta x + f(x_1)\Delta x + ... + f(x_{n-1}))\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$TotalArea \approx 1 * 100 + 1 * 99 + 1 * 96... + 1 * 19 = 715$$

This sum is called a **Riemann Sum**.

3 More Riemann Sum Approximations

Notice that the rectangle in Figure 1 have its left vertex fall on $f(x)$. We can observe a clearly overestimated total area. This is called a **Left Riemann Sum** Thus, if we change the intersection point of rectangle and $f(x)$, we could get different Riemann Sums:

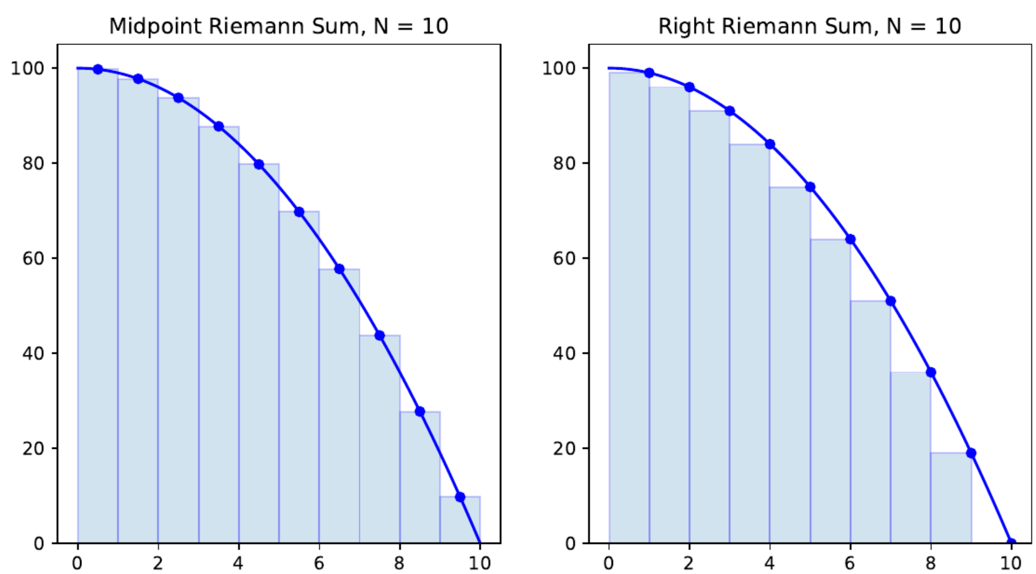

Figure 2: f(x) divided into 10 rectangles with intersection in the right and middle

Figure 2 demonstrates **Midpoint Riemann Sum** and **Right Riemann Sum** of $f(x)$. The estimation value evidently decreases from the Left Riemann Sum of 715 to Midpoint Riemann Sum of 667.5 then to the Right Riemann Sum of 615.

4 From Approximated Sum to Precise Area

As you can observe, the total sum of area could vary by taking different Riemann Sum. To acquire more accurate value of area under $f(x)$, consider increasing n, the number of rectangles used to approximate area. Imagine a process of cutting $f(x)$ approaches infinity. When $n \to \infty$, the width of each rectangle $\Delta x \to 0$. The area of $f(x)$ could be approximated more precisely.

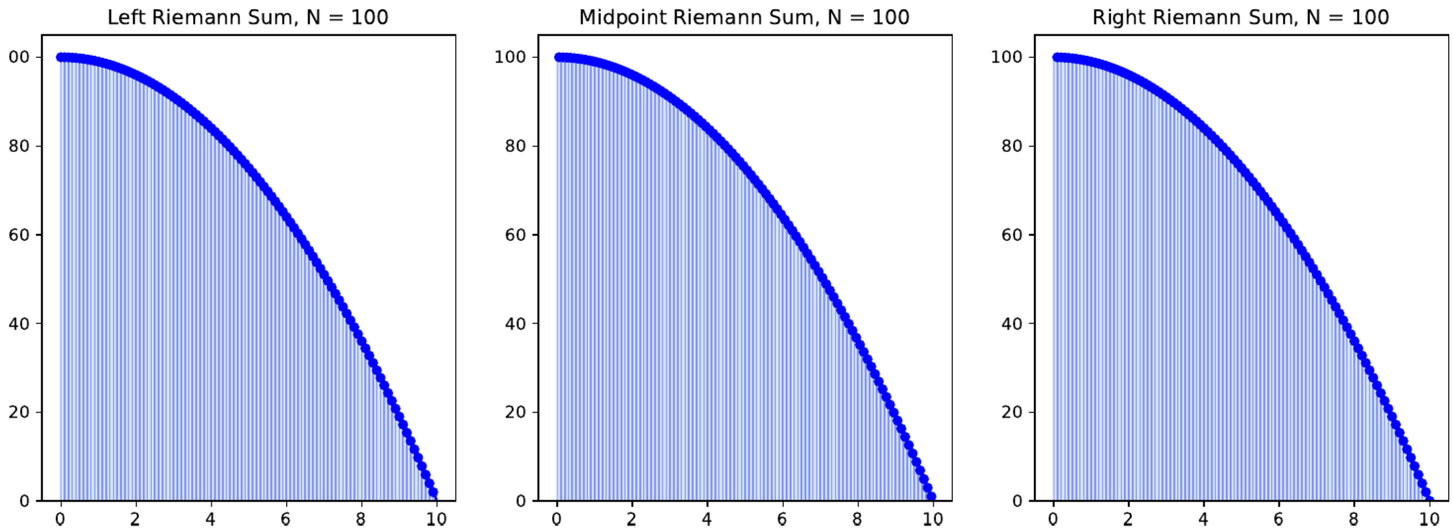

Figure 3: f(x) divided into 100 rectangles

Figure 3 demonstrates the Left, Midpoint and Right Riemann Sum with 100 rectangles. They have a estimated area of 671.65, 666.675, and 661.65. The difference between 3 different approach decreases as n increases. Back to the imagination where $n \to \infty$, 3 value will eventually be the same. Take Right Riemann Sum as example. To find the limit value, first evaluate the sum of this equation: (note that $\Delta x = \frac{10-0}{n}$ and $x_i = a + i\Delta x$)

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n}(100 - (0 + i \times \frac{10}{n})^2)\frac{10}{n} \\ &= \sum_{i=1}^{n} 100 \times \frac{10}{n} - \sum_{i=1}^{n}(i^2 \times \frac{100}{n^3}) \\ &= 1000 - \frac{100}{n^3}\sum_{i=1}^{n} i^2\end{aligned}$$

The sum of the squares $1^2 + 2^2 + 3^2 + 4^2 ... + n^2$ could be computed by $\frac{n(n-1)(2n-1)}{6}$. This formula could be proved by high school algebra thus will not be discussed here.

$$
\begin{aligned}
1000 - \frac{100}{n^3}\sum_{i=1}^{n} i^2 &= 1000 - \frac{100}{n^3}(\frac{n(n-1)(2n-1)}{6}) \\
&= \frac{2000}{3} - \frac{500}{n} - \frac{500}{3n^2}
\end{aligned}
$$

Imagine as $n \to \infty$, the fraction of $\frac{500}{n}$ and $\frac{500}{3n^2}$ will approach 0. Thus this leaves us to a precised area under $f(x)$ as $\frac{2000}{3}$.
At this point, you have acquired the basic idea of Integration: **Area = The limit of Riemann Sum as n approach infinity.** The equation of $\sum_{i=1}^{n} f(x_i)\Delta x$ as $n \to \infty$ could be denoted in another way:

$$
\int_0^{10} f(x)dx
$$

4.1 Further Extension

Now you have learned how to find the integral of $f(x) = 100 - x^2$ over x range of $[0, 10]$, try the prove that result is same when computing limit of **Midpoint Riemann Sum** and **Left Riemann Sum**.