

Introduction

Understanding the dynamics between predator and prey populations is a fundamental aspect of ecological research. In this study, we investigate how changes in prey population size influence the dynamics of predator populations within an insulated environment with limited resources. By simulating this scenario using Python, we aim to explore the intricate relationship between predator and prey populations in a controlled environment. This research question is crucial for gaining insights into the mechanisms that govern predator-prey interactions and can have significant implications for ecosystem management and conservation efforts. Through our experimental simulation, we seek to shed light on the impact of prey population fluctuations on predator populations and contribute to the broader understanding of ecological dynamics.

Research Question

How do prey population affect predator population under a insulated environment with limited resources?

Background Research

The Lotka-Volterra equation, also known as the predator-prey model, is a foundational concept in ecological and mathematical biology that describes the dynamics between two species: one as a predator and the other as its prey (Sternberg, 2009). [4] The model, which was separately developed by Vito Volterra in 1926 and Alfred J. Lotka in 1925, shows how the populations of the predator and prey affect one another over time through a system of first-order non-linear differential equations. American biophysicist Lotka first utilized these equations to describe chemical reactions, whereas Italian mathematician Volterra concentrated on ecological interactions, especially to explain the oscillatory population shifts seen in fishing data. Since then, the equations have been used extensively in the study of ecological systems to illustrate ideas like population cycles, stability, and how changes in the environment affect interactions between different species. The Lotka-Volterra model offers a vital theoretical framework for comprehending intricate biological interactions despite its idealized assumptions and simplicity, and it has sparked a great deal of developments and applications in a wide range of scientific domains.

Lotka investigates the existence of rhythmic effects in chemical reactions, pointing out that previous research frequently discovered that these oscillations were mild and fleeting. Contrary to earlier theories, Lotka discovers via his own research the circumstances in which undamped, permanent oscillations can occur. He models the evolution of systems made up of different kinds of matter using differential equations, taking into account both biological entities and elements of the inorganic environment.

Prediction

The population of prey and predator will be in a cyclical mismatch relationship. The number of prey increases and decreases in a period, and the number of predators follows the same period but with a different phase. This is because periodic phenomena play an important role in nature, where permanent oscillations often occur in natural systems(Chasnov, 2022). [1]

Model Establishment

Assumption

Few assumptions will have to be made to ensure accuracy of the model:

- The natural system consists of only two species: Prey and Predator
- The natural system is closed
- Predator's only food source is prey
- The natural system have sufficient resource to support Prey and Predator
- Mortality rate and population growth rate are constant
- Probability of a success Predation is constant
- Prey's ability to feed predator is constant

Variable Description

Variable	Unit	Description
t	month	Time
x	N/A	Population of Prey
y	N/A	Population of Predator
r	N/A	Population growth rate of Prey
a	N/A	Probability of a success Predation
d	N/A	Mortality rate of Prey
b	N/A	Prey ability to feed predator

Table 1: Variable Table

Lotka-Volterra Equation

The independent population growth for prey can be denoted by differential equation

$$\dot{x} = rx$$

After introducing Predator to natural system, Predator allows the prey growth rate to decrease, where the decrease is proportional to Predator population (y)

$$r \rightarrow r - ay$$

The independent population growth for predator can be denoted by differential equation

$$\dot{y} = -dy$$

After introducing Prey to natural system, Prey allows the mortality rate of Predator to decrease, where the decrease is proportional to Prey population (x)

$$d \rightarrow d - bx$$

Therefore, we could derive the Lotka-Volterra Equation (Lotka, 1920) [2]:

$$\dot{x} = (r - ay)x \quad (1)$$

$$\dot{y} = -(d - bx)y \quad (2)$$

Model Solving

This set of differential equation have no analytic solution, thus we use python to simulate a numerical solution.

The simulation runs under the following conditions:

$$r = 1$$

$$a = 0.1$$

$$d = 0.5$$

$$b = 0.02$$

The initial condition is given as:

$$x(t = 0) = 25$$

$$y(t = 0) = 2$$

Method applied to simulate numerical solution is first-order differential approximation with a step length of 0.001

$$f(x + dx) \approx f(x) + f'(x)dx$$

Result Analysis

The simulation results are shown below:

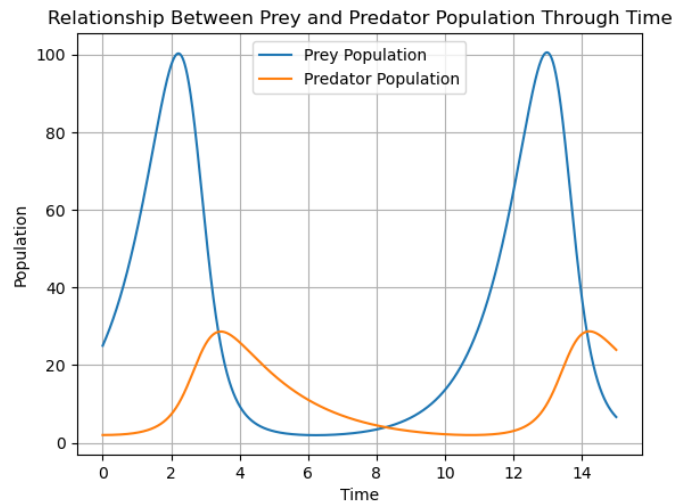


Figure 1: Population of Prey and Predator plotted with respect to time

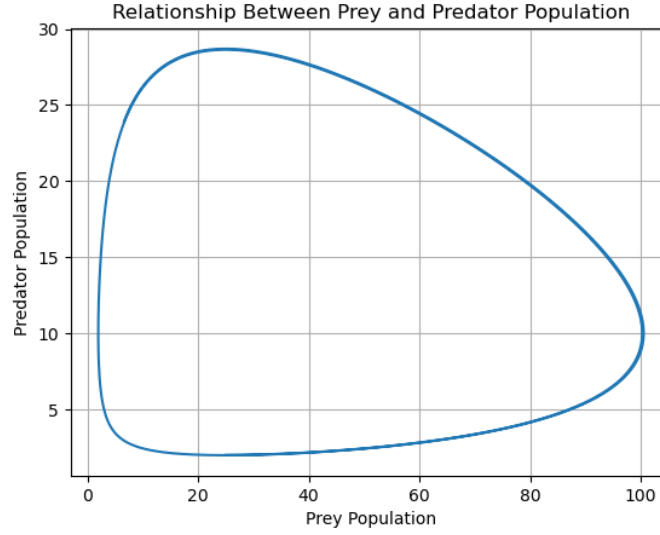


Figure 2: Population of Prey plotted with respect to Predator

As can be obtained from Figure 1, the period of the two functions is around 10.8. Using the numerical integration method, it can be obtained that the average values in a period for Prey is 25 and for Predator is 10, respectively. As can be seen from Figure 2, the relationship between x and y is in the form of a closed function which can be solved mathematically as an implicit function by eliminating t . [3] Divide the two differential equation to get:

$$\frac{dx}{dy} = \frac{x(r - ay)}{y(-d + bx)}$$

Separating the variables and integrating both sides gives:

$$-d\ln(x) + bx = r\ln(y) - ay + c_1$$

Simplify, and we get the numerical relationship between population of Predator and Prey.

This is defined as the phase trajectory of the system of differential equations.

$$(x^d e^{-bx})(y^r e^{-ay}) = c \quad (3)$$

Where constant "c" is determined by initial condition. In our condition, $c \approx 4.97$

Observing both two figure we find that the Predator and Prey populations does not reach an equilibrium point. But from the differential equation itself, we can derive an equilibrium point with zero rate of change. Obviously, when $x = \frac{d}{b}$ and $y = \frac{r}{a}$, the system reaches equilibrium.

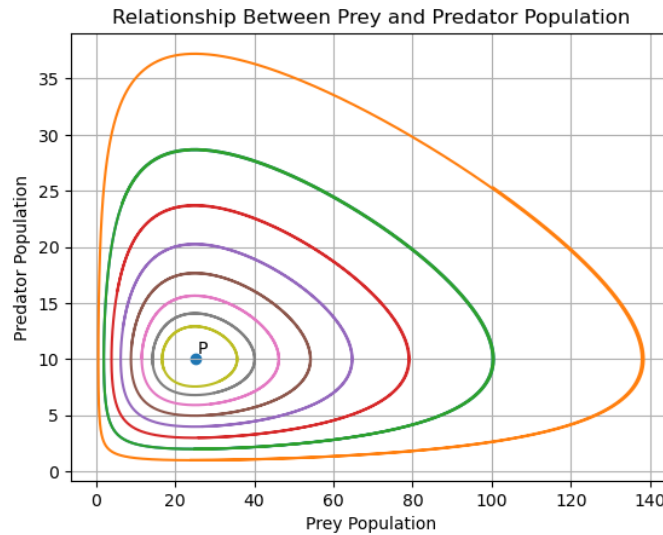


Figure 3: Prey-Predator graph simulated with different initial condition

If we plot this ordered-pair into graph and simulate a few more graphs with different initial condition as shown in Figure 3, we find that point p is the center of this family of phase trajectories. Thus, in any initial value that does not satisfy the unique equilibrium condition, the phase trajectory is a closed path motion around point p .

Model Evaluation

The model has several limitations:

- The phase trajectory is a closed curve and the structure lacks stability. After changing the initial value it enters another curve from which there is no recovery. In nature, ecosystems with periodic equilibrium are structurally stable and have internal constraints to restore the system to its original state after deviating from its trajectory.
- Modeling assumptions are too limited to accommodate more complex food webs and ecological relationships in nature.
- Model does not consider competition within species populations.
- Model does not consider environmental carrying capacity.
- Model does not consider the predator's predation preference to the extent that it prevents the prey population from reproducing efficiently when it consumes too much of a single-sex prey.
- Models does not consider changes in genetic frequency caused by selective pressure which might effect population survival or predatory capacity.

Conclusion

By simulating the numerical solution of Lotka-Volterra Equation, we conclude that the prey and predator population sizes show periodic fluctuations over time. Both fluctuate in similar cycles, but with some

phase differences. Therefore, the periodic staggered fluctuation relationship predicted at the beginning of the paper is verified by the model. If the time variable t is eliminated by dividing the two equations, the prey and predator populations are mutually explicit, resulting in a closed curve centered on the equilibrium point. The curve carries a pending parameter determined by the initial conditions. Different initial conditions can determine different closed curves, but since the equilibrium point remains the same, the center remains the same.

AI Usage Clarification

Clarification: In preparation of this work, the author used Chat-GPT 3.5 to create the text content. After using this tool, the author reviewed and edited the content as needed.

Input Prompt 1: write me a paragraph about background research on The Lotka - Volterra equation

Output 1:

The Lotka-Volterra equation, also known as the predator-prey model, is a foundational concept in ecological and mathematical biology that describes the dynamics between two species: one as a predator and the other as its prey. Developed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926, the model uses a set of first-order, non-linear differential equations to illustrate how the population sizes of the predator and prey influence each other over time. Lotka, an American biophysicist, applied these equations to chemical reactions initially, while Volterra, an Italian mathematician, focused on ecological interactions, particularly to explain oscillatory population changes observed in fishery data. The equations have since become instrumental in studying ecological systems, illustrating concepts such as population cycles, stability, and the impact of environmental changes on species interactions. Despite its simplicity and idealized assumptions, the Lotka-Volterra model provides a crucial theoretical framework for understanding complex biological interactions and has inspired numerous extensions and applications across various scientific fields.

Input Prompt 2: write me a literature review upon this eassay, make it short Output 2:

Alfred J. Lotka's 1920 paper, "Analytical Note on Certain Rhythmic Relations in Organic Systems," explores periodic phenomena in organic and inorganic systems through the lens of mathematical modeling. This literature review summarizes Lotka's approach, findings, and contributions to the field.

Background and Context Lotka builds on previous work by researchers such as McEwen and Michael (1919), Reed and Holland (1919), and Robertson (1913), who explored various biological and chemical dynamics using statistical and analytical methods. Lotka acknowledges the support of Dr. G.F. McEwen, highlighting the collaborative nature of his research.

Key Concepts and Methodology Lotka examines the occurrence of rhythmic effects in chemical reactions, noting that earlier studies often found these oscillations to be damped and transient. However, through his own investigations, Lotka identifies conditions under which undamped, permanent oscillations can arise, contrary to previous assumptions.

He uses differential equations to model the evolution of systems comprising various species of matter, considering both biological organisms and components of the inorganic environment.

Reference

- [1] Jeffrey R. Chasnov. 1.4: The lotka-volterra predator-prey model, 01 2022.
- [2] Alfred J. Lotka. Analytical note on certain rhythmic relations in organic systems. *Proceedings of the National Academy of Sciences*, 6:410–415, 06 1920.
- [3] Joseph M. Mahaffy and San Diego State University. Lotka-volterra models, 2010.
- [4] Shlomo Sternberg. Lecture 15 lotka-volterra, 04 2009.

Appendix

```
import matplotlib.pyplot as plt
import numpy as np
import math

#  $dx/dt = (r - ay)x$  prey, Changes in the size of prey populations result from growth
#  $dy/dt = -(d-bx)y$  predator, Changes in the size of the predator community is the growth
x0 = 25
y0 = 2
tstep = 0.001
t0 = 0
r = 1 #prey population growth rate
a = 0.1 #measure predator ability to predate on prey
d = 0.5 #predator mortality rate
b = 0.02 #measure prey ability to feed predator
cons = math.pow(x0, d) * math.exp(-b * x0) * math.pow(y0, r) * math.exp(-a * y0)
print(cons)
x = []
y = []
t = []

#Simulation by First-order differential approximation
def simulationX (x, y, r, a, tstep):
    return x + (r * x - a * x * y) * tstep

def simulationY (x, y, b, d, tstep):
    return y + (-d * y + b * x * y) * tstep

def main (x0, y0, tstep, t0, r, a, b, d, target):
    t = t0
    x = x0
    y = y0
    xlist = [x0]
    ylist = [y0]
    tlist = [t0]
    while t < target:
        x, y = simulationX(x, y, r, a, tstep), simulationY(x, y, b, d, tstep)
        xlist.append(x)
        ylist.append(y)
        tlist.append(t)
```



```

        t += tstep
    return xlist , ylist , tlist

#Calculate period
def findPeriod (vlist , tlist):
    count = 0
    mlist = []
    for j in tlist:
        if count == 0:
            count += 1
        elif vlist[count - 1] < vlist[count] and vlist[count + 1] < vlist[count]:
            mlist.append(j)
            count += 1
        elif count + 1 == len(vlist):
            diff = [mlist[i+1]-mlist[i] for i in range(len(mlist)-1)]
            print (diff)
            return diff[0]
    count += 1

#Calculate Average
def findAverage (period , vlist):
    period /= 0.001
    period = round(period)
    sum = 0
    count = 0
    for i in vlist:
        if count == period:
            return sum/period
        else:
            sum += i
            count += 1

#plot fig 2
x, y, t= main (x0, y0, tstep, t0, r, a, b, d, target = 15)
plt.plot(x,y)
plt.xlabel("Prey-Population")
plt.ylabel("Predator-Population")
plt.title("Relationship-Between-Prey-and-Predator-Population")
plt.grid()
plt.show()

```

```

#plot fig 1
plt.plot(t,x, label="Prey - Population")
plt.plot(t,y, label="Predator - Population")
plt.legend()
plt.xlabel("Time")
plt.ylabel("Population")
plt.title("Relationship - Between - Prey - and - Predator - Population - Through - Time")
plt.grid()
plt.show()

#find period
xperiod = findPeriod(x, t)
yperiod = findPeriod(y, t)

#find average
print(findAverage(xperiod, x))
print(findAverage(yperiod, y))

#plot multiple phase trajectory
x, y, t= main(x0, y0, timestep, t0, r, a, b, d, target = 15)
plt.plot(x,y)
for i in range(-1,7):
    x, y, t= main (x0+i, y0+i, timestep, t0, r, a, b, d, target = 15)
    plt.plot(x,y)
plt.scatter(d/b, r/a)
plt.annotate("P", xy = (d/b, r/a), xytext = (d/b+0.5, r/a+0.5))
plt.xlabel("Prey - Population")
plt.ylabel("Predator - Population")
plt.title("Relationship - Between - Prey - and - Predator - Population")
plt.grid()
plt.show()

```