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A constraint-following control for uncertain mechanical systems: given force coupled with constraint force

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Abstract A novel constraint-following control for uncertain mechanical systems is proposed. In mechanical systems, certain given forces may arise due to the constraint forces, which means the given forces are coupled with the constraint forces. By using the second-order form of the constraints, the given forces are decoupled explicitly. The uncertainty of the mechanical system is time-varying and bounded. But its bound is unknown. A series of adaptive parameters are invoked to estimate the bound information of the uncertainty in virtue of state feedback. Based on the estimated bound information, a robust control is designed to render the mechanical system an approximate constraint-following. The system performance under the control is guaranteed as uniform boundedness and uniform ultimate boundedness.

Keywords Mechanical system · Constraint-following · Robust control · Given force · Constraint force

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1 Introduction

In Lagrangian mechanics, the mechanical system is usually designed to follow the certain constraints. In order to meet the constraints, the appropriate constraint forces are needed. From the realization of the constraint forces, the constraint problem can be divided into two classes: passive and servo [1]. In passive constraint problem, the focus is to let the environment including the structure of the mechanical system to generate the required constraint forces. This aspect is along with the original Lagrange mechanics which has investigated in many classical literatures, such as [2–5]. The servo constraint problem, on the other hand, is far less studied in Lagrangian mechanics. In this case, a mechanical system, equipped with servo controls, is designed to follow a set of constraints (hence is called the constraint-following) by generating the required constraint forces through the servo controls. As a result, the generation of constraint forces is no longer a design work but a control problem. While, it is different from the traditional motion control problem in mechanical systems [6,7] and their bibliographies.

Constraint-following control, inspired by the servo constraint problem in analytical mechanics, has been one of the few research frontiers in mechanical system control. There are several developments of constraint-following control design. Most of them are precise model-based designs (e.g., [1] and [8]) and none of them refers to the coupling effect of constraint forces and given forces. Due to the limitation of the engineer's

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knowledge of the system parameters, disturbance and imperfectly known inputs, the uncertainty is inevitable in mechanical systems. Hence, the control of uncertain systems has been a subject of intense research interest over the past decades. Recently, some successful control designs of uncertain systems are proposed: such as a robust adaptive control is designed to solve the uncertainty and disturbance for the tracking control of the multi-agent systems [9]; a robust control approach is constructed to deal with the actuator faults [10]; the control designs are proposed based on transforming the uncertain systems as LVP systems [11,12]. The uncertainty considered in this paper includes uncertain parameters and external disturbance, which is (possibly fast) time varying and bounded. The exact bounds of those uncertain terms are unknown. An adaptive robust control is invoked to render the uncertain mechanical systems deterministic performance (uniform boundedness and uniform ultimate boundedness).

The coupling effect of given forces and constraint forces may introduce a difficulty in the constraint-following design. In Lagrangian mechanics, forces are divided into given (or impressed) and constraint [5]. In most practical cases, the given forces are irrelevant to the constraint forces and only determined by known physical laws (such as the gravity force), which could be directly used in the model-based constraint-following control designs [13,14]. However, certain given forces may depend on the constraint forces. In other words, the given forces are functions of the constraint forces. A typical example of this is the friction force, acting as a given force, is due to the normal force, which is a constraint force. In traditional Lagrangian mechanics framework, there is no systematic method to determine those forces. Most literatures on Lagrangian mechanics only address this issue by examples [2–5,15–17].

In recent applications of mechanical system such as robotics, in which the dynamics modeling is derived by the Lagrangian mechanics, several indirect methods have been applied to deal with this issue: such as the normal force is assumed to be a *constant* [18]; the estimator is constructed to estimate the friction force [19]. Those methods could 'help' to decouple the given forces and constraint forces. But the simplification or identification may lead to the loss of the model accuracy or the lag of the motion control. By taking the advantage of the *second-order* constraints, we propose a systematic method to decouple the given forces and constraint forces in constraint-following control.

There are three main contributions of this paper. First, by *second-order* constraints, we formulate the dynamics modeling of the mechanical system with the uncertainty in which the given forces and the constraint forces are decoupled and presented in the closed form. Second, based on the achieved dynamics modeling, a new adaptive robust control for the approximate constraint-following problem is designed. The uncertainty in the coupled given forces is especially addressed and solved. The control scheme regulates the system with the deterministic performance, including uniform boundedness and uniform ultimate boundedness. Third, a leakage type adaptive law with dead zone, which governs the evolution of a parameter vector related with the uncertainty bound, is proposed. The leakage term in the adaptive law is designed to decrease the control gain when the constraint-following error is small. The dead zone can simplify the calculation of the adaptive mechanism.

2 Mechanical system subject to constraints

Consider the mechanical system in the form of [13]

$$\begin{aligned} &M(q(t),\sigma(t),t)\ddot{q}(t)+C(q(t),\dot{q}(t),\sigma(t),t)\dot{q}(t)\\ &\quad+g(q(t),\sigma(t),t)+F(q(t),\dot{q}(t),\sigma(t),t)=\tau(t). \end{aligned} \tag{2.1}$$

Here $t \in \mathbf{R}$ is the independent variable, $q \in \mathbf{R}^n$ is the coordinate, $\dot{q} \in \mathbf{R}^n$ is the velocity, $\ddot{q} \in \mathbf{R}^n$ is the acceleration, $\sigma \in \Sigma \subset \mathbf{R}^p$ is the uncertain parameter, and $\tau \in \mathbf{R}^n$ is the control input. Here the $\Sigma \subset \mathbf{R}^p$ is compact but unknown, which stands for the possible bounding of Σ. Furthermore, $M(q,\sigma,t)$ is the inertial matrix, $C(q,\dot{q},\sigma,t)\dot{q}$ is the Coriolis/centrifugal force, $g(q,\sigma,t)$ is the gravitational force, and $F(q,\dot{q},\sigma,t)$ represents the external disturbances. The matrices/vector $M(q,\sigma,t)$, $C(q,\dot{q},\sigma,t)$, $g(q,\sigma,t)$ and $F(q,\dot{q},\sigma,t)$ are of appropriate dimensions. We assume that the functions $M(\cdot)$, $C(\cdot)$, $g(\cdot)$ and $F(\cdot)$ are continuous (this can be generalized to be Lebesgue measurable in t). In addition, the bounding set Σ is prescribed and compact.

Remark 1 The coordinate q can be selected based on the specifics of the problem and does not need to be the generalized coordinate [20]. The uncertainty parameter σ is a p-vector, which are composed of the uncertain parameters and disturbances of mechanical system. The dimension of the uncertainty parameter is decided by the number of the uncertain terms of system. Those uncertain terms belong to a compact set $\Sigma \subset \mathbf{R}^p$, which means each of them are bounded. As the existence of the uncertainty, the determinations of the inertial matrix M, the Coriolis/centrifugal force $C\dot{q}$, the gravitational force g and the external disturbances F may depend on not only the coordinate q and the velocity $\dot{q}$ but also the uncertainty parameter σ.

Suppose the following constraints:

$$\sum_{i=1}^{n} A_{li}(q,t)\dot{q}_i = c_l(q,t), \quad l = 1, \ldots, m, \tag{2.2}$$

where $\dot{q}_i$ is the ith component of $\dot{q}$, $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1, $m \leq n$. They are the *first-order* form of the constraints. The constraints may not be integrable and may be non-holonomic in general. The constraints can be put in the matrix form

$$A(q,t)\dot{q} = c(q,t), \tag{2.3}$$

where $A = [A_{li}]_{m\times n}$, $c = [c_1, c_2, \ldots, c_m]^{\mathrm{T}}$.

The constraints can be interpreted into two ways. First is the passive constraint which is generally assumed that the environment (which includes the structure of the machine under study) can automatically generate the required constraint force. Second is the servo constraint. That is, the system's control input supplies the required force so that the constraints are met.

By differentiating the constraints (2.2), we can convert the first-order form constraints into the *second-order* form constraints [20] as

$$\sum_{i=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t} A_{li}(q,t)\right)\dot{q}_i + \sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t} c_l(q,t), \tag{2.4}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} A_{li}(q,t) = \sum_{k=1}^{n} \frac{\partial A_{li}(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial A_{li}(q,t)}{\partial t}, \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} c_l(q,t) = \sum_{k=1}^{n} \frac{\partial c_l(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial c_l(q,t)}{\partial t}. \tag{2.6}$$

Let

$$b_l(q,\dot{q},t) := \frac{\mathrm{d}}{\mathrm{d}t} c_l(q,t) - \sum_{i=1}^{n} \left(\frac{\mathrm{d}}{\mathrm{d}t} A_{li}(q,t)\right)\dot{q}_i. \tag{2.7}$$

Then the (2.4) can be written as

$$\sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_i = b_l(q,\dot{q},t), \tag{2.8}$$

where $l = 1, \ldots, m$, or in the matrix form

$$A(q,t)\ddot{q} = b(q,\dot{q},t), \tag{2.9}$$

where $b = [b_1, b_2, \ldots, b_m]^{\mathrm{T}}$.

Remark 2 The second-order constraints in (2.9) are not the 'real' constraints of mechanical system, which are the derivatives of the physical constraints in (2.3) (the restrictions of the configuration or motion of mechanical system). The constraints in second-order form have the 'similar' mathematical appearances of the system dynamics. By taking the advantage of this mathematical conformity, the motion equation of the constrained mechanical system can be derived based on Gauss's principle or Lagrange's form of d'Alembert's principle. The resulting dynamics of system is explicit, Lagrange multiplier free and complete. There are various control problems, including stabilization, trajectory following, and optimality, and can be cast into this form [1].

3 Constraint force

We show the constraint force in mechanical systems when the uncertainty exists.

Assumption 1 For each $(q,t) \in \mathbf{R}^n \times \mathbf{R}$, $\sigma \in \Sigma$, $M(q,\sigma,t) > 0$.

Remark 3 The assumption of the positive definiteness of the inertia matrix will be vital in later developments. In the past, it is often believed to be true rather than an assumption. But the inertial matrix is not always positive definitive in all mechanical systems. There are some counter-examples listed in [21].

Assumption 2 The constraint $A(q,t)\ddot{q} = b(q,\dot{q},t)$ is consistent; that is,

$$A^{+}(q,t)A(q,t)b(q,\dot{q},t) = b(q,\dot{q},t). \tag{3.1}$$

Remark 4 The consistency of the constraint suggests that there exists at least one solution $\ddot{q}$ of the constraint $A(q,t)\ddot{q} = b(q,\dot{q},t)$. The consistency part in (3.1) is equivalent to assuming $Rank\,[A(q,t)] \geq 1$

Lemma 1 *Consider the system* (2.1) *and constraints* (2.9)*. Subject to Assumptions* 1 *and* 2*, by the ideal constraints, the constraint force with uncertainty is*

$$\begin{aligned} F^{c}(q,\dot{q},\sigma,t) = M^{1/2}&(q,\sigma,t) \\ &\times (A(q,t)M^{-1/2} \\ &\times (q,\sigma,t))^{+}\,[b(q,\dot{q},t) \\ &+ A(q,t)M^{-1}(q,\sigma,t) \\ &\times (C(q,\dot{q},\sigma,t)\dot{q} + g(q,\sigma,t)) \\ &+F(q,\dot{q},\sigma,t)]\,, \end{aligned} \tag{3.2}$$

where '+' is the Moore–Penrose generalized inverse [4]*. The constraint force obeys the Lagrange's form of d'Alembert's principle and renders the system to meet the constraint.*

Proof See [22]. □

Remark 5 The Lagrange's form of d'Alembert's principle renders the constraint force to be the one with minimum norm, out of all possible alternative forces which can also meet (2.9). Note that the $F^{c} \in \mathcal{R}(A)$ (in d'Alembert's principle the virtual displacement $\delta q \in \mathcal{N}(A)$, in addition $\mathcal{R}(A) \perp \mathcal{N}(A)$).

Lemma 1 shows the strategy the Nature will undertake to meet the constraint. The constraint force is model-based. That is, it is based on the exact model information. If the servo constraint is ideal, we can directly use the constraint force F^{c} in the control design.

4 Given force coupled with constraint force

The given forces can be divided into two subclasses. The first class $F^{I} \in \mathbf{R}^{n}$ is unrelated to the constraints, which can be partially determined by constitutive equations, such as the Coriolis/centrifugal force $C(q,\dot{q},\sigma,t)\dot{q}$, the gravitational force $g(q,\sigma,t)$ and the external disturbances $F(q,\dot{q},\sigma,t)$ (always considered as unknown) in the motion equation of the mechanical system (2.1). The second class $F^{\mathrm{II}} \in \mathbf{R}^{n}$ depends on the constraints, which may arise only due to the constraint forces. In other words, they are coupled. The most well-known example is that friction forces are due to normal forces. The given force F^{II} can be represented by

$$F^{\mathrm{II}} = F^{\mathrm{II}}\left(F^{c}\right) \tag{4.1}$$

where F^{c} is the constraint force in (3.2), $F^{\mathrm{II}}(\cdot) : \mathbf{R}^{n} \to \mathbf{R}^{n}$ is such that $F^{\mathrm{II}}(0) = 0$.

Assumption 3 Suppose the mapping $F^{\mathrm{II}}(\cdot)$ is known and determined by the physical laws. For any $(q,\dot{q},t) \in \mathbf{R}^{n} \times \mathbf{R}^{n} \times \mathbf{R}$, $\sigma \in \Sigma$, we can calculate the given force $F^{\mathrm{II}}(q,\dot{q},\sigma,t)$ through $F^{c}(q,\dot{q},\sigma,t)$ by the known mapping $F^{\mathrm{II}}(\cdot)$.

The given force mapping $F^{\mathrm{II}}(\cdot)$ is usually determined by the known physical laws which can be achieved experimentally. Taking Coulomb friction as an example, it is the product of the friction coefficient and the normal force, in a direction opposite to the velocity of the object at the contact point. The friction force acts as a given force, and the normal force is a constraint force.

Lemma 2 *Suppose the system* (2.1) *and constraints* (2.9)*. Subject to Assumptions* 1*,* 2 *and* 3*, by the non-ideal constraints, the decoupled constraint force F_{d}^{c} and given force $F_{\mathrm{d}}^{\mathrm{II}}$with uncertainty, are*

$$\begin{aligned} F_{\mathrm{d}}^{c}(q,\dot{q},\sigma,t) = M^{1/2}&(q,\sigma,t) \\ &\times \left(A(q,t)M^{-1/2}(q,\sigma,t)\right)^{+} \\ &\times \Big[b(q,\dot{q},t) + A(q,t)M^{-1}(q,\sigma,t) \\ &\times F^{I}(q,\dot{q},\sigma,t)\Big], \end{aligned} \tag{4.2}$$

$$\begin{aligned} F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t) = M^{1/2}&(q,\sigma,t) \\ &\times \Big[I - \left(A(q,t)M^{-1/2}(q,\sigma,t)\right)^{+} \\ &\times \left(A(q,t)M^{-1/2}(q,\sigma,t)\right)\Big] \\ &\times M^{-1/2}(q,\sigma,t)F^{\mathrm{II}}(q,\dot{q},\sigma,t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} F^{I}(q,\dot{q},\sigma,t) = C(q,\dot{q},&\sigma,t)\dot{q} + g(q,\sigma,t)) \\ &+F(q,\dot{q},\sigma,t). \end{aligned} \tag{4.4}$$

Proof See [23] and [24]. □

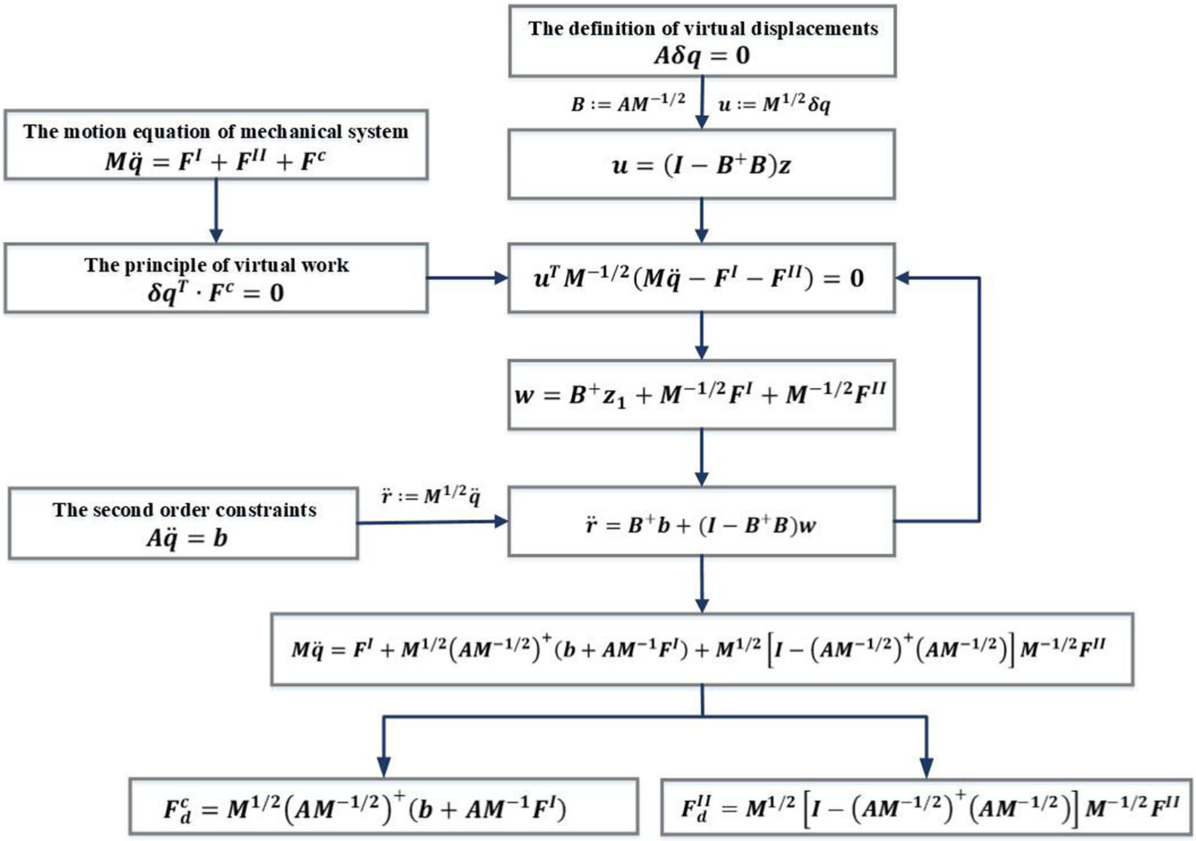


Fig. 1 The illustration diagram of the decoupling procedure

Remark 6 As constraints (2.9) are nonideal, the given force F^{II} may arise with the constraint force F^{c}. In Lemma 1, the constraint force F^{c} is defined as a function of all the given forces (the first class given force F^{I} and the second class given force F^{II}). Therefore the constraint force F^{c} and the given force F^{II} are coupled. In constraint-following control, the constraint forces are used to design the control. Then the following control may be crippled by this coupling effect. In this article, by virtue of the second-order constraints and the fundamental laws of mechanical system (such as, virtual displacements, the principle of virtual work), the given force and constraint force are decoupled. The details of the decoupling procedure is illustrated by a diagram in Fig. 1, where $w \in \mathbf{R}^n$, $z \in \mathbf{R}^n$ and $z_1 \in \mathbf{R}^n$ are arbitrary vectors. In Fig. 1, arguments of all functions are omitted for simplification when no confusions are likely to arise. The decoupled constraint force $F_{\mathrm{d}}^{\mathrm{c}}$ in (4.2) is no longer relevant with the given force F^{II}. Then the determination of the decoupled given force $F_{\mathrm{d}}^{\mathrm{II}}$ in (4.3) can be achieved by using the decoupled constraint force $F_{\mathrm{d}}^{\mathrm{c}}$.

Remark 7 Consider $\Sigma_1 = \{$uncertain parameters$|$associated with $M(q, \sigma, t)$, $C(q, \dot{q}, \sigma, t)$, $g(q, \sigma, t)$, $F(q, \dot{q}, \sigma, t)\}$, $\Sigma_2 = \{$uncertain parameters$|$associated with $F^{\mathrm{II}}(q, \dot{q}, \sigma, t)\}$. Therefore σ in the decoupled given force $F_{\mathrm{d}}^{\mathrm{II}}(q, \dot{q}, \sigma, t)$ is such that $\sigma \in \Sigma_1 \bigcup \Sigma_2$.

The decoupled constraint force $F_{\mathrm{d}}^{\mathrm{c}}$ and given force $F_{\mathrm{d}}^{\mathrm{II}}$ are in the explicit form and suitable for the control design, which will be discussed in Sect. 5. In servo constraint problem, if there the uncertainty is known, we can design the control input $\tau = F_{\mathrm{d}}^{\mathrm{c}} - F_{\mathrm{d}}^{\mathrm{II}}$ to drive the system to meet the constraints (2.9).

5 Adaptive robust control design

Taking the uncertainty into account, We now design the adaptive robust control τ of mechanical system (4.2). Decompose the M, C, g, F and $F_{\mathrm{d}}^{\mathrm{II}}$ as follows:

$$M(q,\sigma,t) = \bar{M}(q,t) + \Delta M(q,\sigma,t), \tag{5.1}$$

$$C(q,\dot{q},\sigma,t) = \bar{C}(q,\dot{q},t) + \Delta C(q,\dot{q},\sigma,t), \tag{5.2}$$

$$g(q,\sigma,t) = \bar{g}(q,t) + \Delta g(q,\sigma,t), \tag{5.3}$$

$$F(q,\dot{q},\sigma,t) = \bar{F}(q,\dot{q},t) + \Delta F(q,\dot{q},\sigma,t), \tag{5.4}$$

$$F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t) = \bar{F}_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},t) + \Delta F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t). \tag{5.5}$$

Here $\bar{M}$, $\bar{C}$, $\bar{g}$, $\bar{F}$ and $\bar{F}_{\mathrm{d}}^{\mathrm{II}}$ denote the 'norm' portions with $\bar{M} > 0$ (this is always feasible since it is the designer's decision), while ΔM, ΔC, Δg, ΔF and $\Delta F_{\mathrm{d}}^{\mathrm{II}}$ are the uncertain portions. The functions $\bar{M}(\cdot)$, $\Delta M(\cdot)$, $\bar{C}(\cdot)$, $\Delta C(\cdot)$, $\bar{g}(\cdot)$, $\Delta g(\cdot)$, $\bar{F}(\cdot)$, $\Delta F(\cdot)$, $\bar{F}_{\mathrm{d}}^{\mathrm{II}}(\cdot)$ and $\Delta F_{\mathrm{d}}^{\mathrm{II}}(\cdot)$ are all continuous. We can choose the nominal part of $F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t)$ as

$$\begin{aligned}\bar{F}_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},t) = {}& \bar{M}^{1/2}(q,t) \\ & \times \left[I - \left(A(q,t)\bar{M}^{-1/2}(q,t)\right)^{+} \right. \\ & \left. \times \left(A(q,t)\bar{M}^{-1/2}(q,t)\right) \right] \\ & \times \bar{M}^{-1/2}(q,t)\bar{F}^{\mathrm{II}}\left(\bar{F}_{\mathrm{d}}^{\mathrm{c}}(q,\dot{q},t)\right),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\bar{F}_{\mathrm{d}}^{\mathrm{c}}(q,\dot{q},t) = {}& \bar{M}^{1/2}(q,t)\left(A(q,t)\bar{M}^{-1/2}(q,t)\right)^{+} \\ & \times \left[b(q,\dot{q},t) + A(q,t)\bar{M}^{-1}(q,t) \right. \\ & \left. \times \left(\bar{C}(q,\dot{q},t) + \bar{g}(q,t) + \bar{F}(q,\dot{q},t)\right) \right],\end{aligned} \tag{5.7}$$

$\bar{F}^{\mathrm{II}}$ is the given force model without uncertain parameters.

Let

$$D(q,t) := \bar{M}^{-1}(q,t), \tag{5.8}$$

$$\Delta D(q,t) := M^{-1}(q,\sigma,t) - \bar{M}^{-1}(q,t), \tag{5.9}$$

$$E(q,\sigma,t) := \bar{M}(q,t)M^{-1}(q,\sigma,t) - I, \tag{5.10}$$

Assumption 4 There exists a matrix $\Omega(q,t) \in \mathbf{R}^{r\times m}$ such that for each $(q,t) \in \mathbf{R}^n \times \mathbf{R}$, the matrix $(\Omega(q,t)A(q,t))(\Omega(q,t)A(q,t))^{\mathrm{T}}$ is invertible.

Remark 8 Assumption 2 indicates that the $Rank(A) \geq 1$, which means that there exists at least a matrix $\Omega(q,t) \in \mathbf{R}^{1\times m}$ such that $Rank(\Omega A) = 1$, that is, $\Omega A \in \mathbf{R}^{1\times n}$ is of full rank. Hence, $(\Omega(q,t)A(q,t))(\Omega(q,t)A(q,t))^{\mathrm{T}}$ is invertible. Assumption 4 can reduce to the special case that A is full rank in [12]. In this special case, let $\Omega = I$, so that the matrix AA^{T} is invertible.

Assumption 5 Under the provision of Assumption 4, for given $P \in \mathbf{R}^{r\times r}$, $P > 0$, let

$$\begin{aligned}W(q,\sigma,t) := {}& P(\Omega A)DE(q,\sigma,t)\bar{M}(\Omega A)^{\mathrm{T}} \\ & \times \left[(\Omega A)(\Omega A)^{\mathrm{T}}\right]^{-1} P^{-1}.\end{aligned} \tag{5.11}$$

There exists a (possibly unknown) parameter $\rho_{\mathrm{E}} \in \mathbf{R}$ such that for all $(q,t) \in \mathbf{R}^n \times \mathbf{R}$,

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m(W(q,\sigma,t) + W^{\mathrm{T}}(q,\sigma,t)) \geq \rho_{\mathrm{E}} > -1. \tag{5.12}$$

Remark 9 The constant ρ_{E} is unknown since the uncertainty bound Σ is unknown. In the special case that $M = \bar{M}$ (i.e., no uncertainty), $E = 0$, $W = 0$ and hence one can choose $\rho_{\mathrm{E}} = 0$. Thus by continuity, this assumption imposes the effect of uncertainty on the possible deviation of M from $\bar{M}$ to be within a certain threshold. We also stress that this threshold is unidirectional (that is, it is not bounded in one direction).

Assumption 6 *(1)* There exists an unknown vector $\alpha \in (0,\infty)^k$ and a known function $\Pi(\cdot) : \mathbf{R}^p \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that for all $(q,\dot{q},t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, $\sigma \in \Sigma$,

$$\begin{aligned}\Pi(\alpha,q,\dot{q},t) \geq {}& (1+\rho_{\mathrm{E}})^{-1}\max_{\sigma\in\Sigma}\Big\| P\Omega(q,t)A(q,t) \\ & \times [D(q,t)(-\Delta C(q,\dot{q},\sigma,t)\dot{q} \\ & - \Delta g(q,\sigma,t) - \Delta F(q,\dot{q},\sigma,t) \\ & + \Delta F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t)) \\ & + \Delta D(q,\sigma,t)(-C(q,\dot{q},\sigma,t)\dot{q} \\ & - g(q,\sigma,t) - F(q,\dot{q},\sigma,t) + p_1 \\ & + p_2 + F_{\mathrm{d}}^{\mathrm{II}}(q,\dot{q},\sigma,t))]\Big\|.\end{aligned} \tag{5.13}$$

(2) For each $(q,\dot{q},t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, the function $\Pi(\cdot,q,\dot{q},t)$ is (i) C^1, (ii) concave; that is, for any $\alpha_{1,2}$,

$$\begin{aligned}& \Pi(\alpha_1,q,\dot{q},t) - \Pi(\alpha_2,q,\dot{q},t) \\ & \leq \frac{\partial\Pi}{\partial\alpha}(\alpha_2,q,\dot{q},t)(\alpha_1-\alpha_2),\end{aligned} \tag{5.14}$$

and (iii) non-decreasing with respect to each component of its argument α.

Let $\beta(q, \dot{q}, t) := A(q, t)\dot{q} - c(q, t)$. Consider the adaptive law with the dead zone :

$$\dot{\hat{\alpha}} = \begin{cases} if \quad \|\Omega\beta(q, \dot{q}, t)\| \left\| \frac{\partial \Pi}{\partial \alpha}(\hat{\alpha}, q, \dot{q}, t) \right\| > \epsilon, \\ \kappa \left[k_1 \frac{\partial \Pi^{\mathrm{T}}}{\partial \alpha}(\hat{\alpha}, q, \dot{q}, t) \|\Omega\beta\| - \left(k_2 e^{-\|\Omega\beta\|} + k_3 \right) \hat{\alpha} \right]; & (5.15) \\ if \quad \|\Omega\beta(q, \dot{q}, t)\| \left\| \frac{\partial \Pi}{\partial \alpha}(\hat{\alpha}, q, \dot{q}, t) \right\| \le \epsilon, \\ -\kappa \left(k_2 e^{-\|\Omega\beta\|} + k_3 \right) \hat{\alpha}. & (5.16) \end{cases}$$

With $\hat{\alpha}_i(t_0) > 0$, where $\hat{\alpha}_i$ stands for the ith component of the vector $\hat{\alpha}$, $i = 1, \ldots, k$. In addition, $k_1, k_2, k_3 \in \mathbf{R}^{k \times k}$, each entry of k_1, k_2 and k_3 is nonnegative, $\kappa \in \mathbf{R}$, $\kappa > 0$, $\epsilon \in \mathbf{R}$, $\epsilon > 0$.

We now propose the adaptive control law as follows:

$$\begin{aligned} \tau(t) = &\, p_1(q(t), \dot{q}(t), t) + p_2(q(t), \dot{q}(t), t) \\ &+ p_3(\hat{\alpha}(t), q(t), \dot{q}(t), t), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} p_1(q(t), \dot{q}(t), t) &= \bar{M}^{1/2}(q, t) \\ &\times \left(A(q, t)\bar{M}^{-1/2}(q, t) \right)^{+} [b(q, \dot{q}, t) \\ &+ A(q, t)\bar{M}^{-1}(q, t) \times \left(\bar{C}(q, \dot{q}, t) + \bar{g}(q, t) + \bar{F}(q, \dot{q}, t) \right)] \\ &- \bar{M}^{1/2}(q, t) \left[I - \left(A(q, t)\bar{M}^{-1/2}(q, t) \right)^{+} (A(q, t) \right. \\ &\left. \times \bar{M}^{-1/2}(q, t)) \right] \bar{M}^{-1/2}(q, t) \bar{F}^{\mathrm{II}}(\bar{F}_{\mathrm{d}}^{\mathrm{c}}(q, \dot{q}, t)), \end{aligned} \tag{5.18}$$

$$\begin{aligned} p_2(q(t), \dot{q}(t), t) &= -\bar{M}(q, t)(\Omega A(q, t))^{\mathrm{T}} \\ &\times \left[(\Omega A(q, t))(\Omega A(q, t))^{\mathrm{T}} \right]^{-1} P^{-1} \\ &\times (\kappa \Omega\beta(q, \dot{q}, t) + \dot{\Omega}\beta(q, \dot{q}, t)), \end{aligned} \tag{5.19}$$

$$\begin{aligned} p_3(\hat{\alpha}(t), q(t), \dot{q}(t), t) &= -\bar{M}(q, t)(\Omega A(q, t))^{\mathrm{T}} \\ &\times \left[(\Omega A(q, t))(\Omega A(q, t))^{\mathrm{T}} \right]^{-1} P^{-1} \\ &\times \gamma(\hat{\alpha}, q, \dot{q}, t)\mu(\hat{\alpha}, q, \dot{q}, t)\Pi(\hat{\alpha}, q, \dot{q}, t), \end{aligned} \tag{5.20}$$

Here

$$\gamma(\hat{\alpha}, q, \dot{q}, t) = \begin{cases} \frac{1}{\|\mu(\hat{\alpha}, q, \dot{q}, t)\|}, & \text{if} \quad \|\mu(\hat{\alpha}, q, \dot{q}, t)\| > \xi, \\ \frac{1}{\xi}, & \text{if} \quad \|\mu(\hat{\alpha}, q, \dot{q}, t)\| \le \xi, \end{cases} \tag{5.21}$$

$$\mu(\hat{\alpha}, q, \dot{q}, t) = \Omega\beta(q, \dot{q}, t)\Pi(\hat{\alpha}, q, \dot{q}, t). \tag{5.22}$$

Remark 10 The adaptive law (5.15)–(5.16) is a combined dead zone type and leakage type. The first term on the right-hand side of (5.15) stops its action when the norm of $\bar{\beta}(\partial\Pi/\partial\alpha)$ enters a zone with size ϵ. The second term on the right-hand side of (5.15) is the leakage term. Note that if the initial condition $\hat{\alpha}_i(t_0)$ is selected to be strictly positive, then $\hat{\alpha}_i(t) > 0$ for all i and $t \ge t_0$. This is since the first term on the right-hand side of (5.15) is always nonnegative and the second term alone will render an exponentially decaying (to zero) solution from above (hence this alone is always positive). The dead zone portion (that is, the second part of (5.15)) is actually an option. We may choose (5.15) alone for all q, $\dot{q}$, $\hat{\alpha}$ and t. If, however, we choose to combine (5.15) with (5.16), then as the adaptive law is simplified after entering the dead zone, the system performance may degrade as a pay-off, comparing to without invoking the dead zone at all.

Theorem 1 *Let* $\delta(t) := \left[\beta^{\mathrm{T}}(q(t), \dot{q}(t), t)\Omega^{\mathrm{T}}(q(t), t), (\hat{\alpha}(t) - \alpha)^{\mathrm{T}} \right]^{\mathrm{T}} \in \mathbf{R}^{m+k}$. *Subject to Assumptions 1–6, consider the system (2.1) and constraints (2.9). The control design (5.17) renders the combined controlled system and the adaptive law (5.15)–(5.16) the following performance:*

(i) Uniform boundedness: For any $\gamma > 0$ *with* $\|\delta(t_0)\| \le \gamma$, *there exists a* $d(\gamma) > 0$ *such that* $\|\delta(t)\| \le d(\gamma)$ *for all* $t \ge t_0$;

(ii) Uniform ultimate boundedness: For any $\gamma > 0$ *with* $\|\delta(t_0)\| \le \gamma$, *there is a* $\underline{d} > 0$ *such that for any* $\bar{d} > \underline{d}$, *there exists a finite time* $T(\bar{d}, \gamma) < \infty$ *such that* $\|\delta(t)\| \le \bar{d}$ *for all* $t \ge t_0 + T(\bar{d}, \gamma)$.

Proof Let the Lyapunov function candidate as

$$\begin{aligned} V(\Omega\beta, \hat{\alpha} - \alpha) = &\, (\Omega\beta)^{\mathrm{T}} P (\Omega\beta) \\ &+ (1 + \rho_{\mathrm{E}})(\hat{\alpha} - \alpha)^{\mathrm{T}} (\kappa k_1)^{-1} (\hat{\alpha} - \alpha). \end{aligned} \tag{5.23}$$

Remark 11 The Lyapunov function in (5.23) is a legitimate candidate which can be used to verify the system performance (uniform boundedness and uniform ultimate boundedness). The first part of the right-hand side of (5.23) is chosen to investigate the error of the constraint following under the control design in (5.17). The second part of the right-hand side of (5.23) is chosen to detect the difference between the adaptive parameter $\hat{\alpha}$ and the parameter α of the uncertainty bound function $\Pi(\cdot)$ in (5.13).

With a given uncertainty $\sigma(\cdot)$ and corresponding trajectory $q(\cdot)$, $\dot{q}(\cdot)$ and $\hat{\alpha}(\cdot)$ of the controlled system, the derivative of V is evaluated as:

$$\begin{aligned} \dot{V} = &\, 2(\Omega\beta)^{\mathrm{T}} P (\dot{\Omega}\beta) + 2(\Omega\beta)^{\mathrm{T}} P (\Omega\dot{\beta}) \\ &+ 2(1 + \rho_E)(\hat{\alpha} - \alpha)(\kappa k_1)^{-1} \dot{\hat{\alpha}}. \end{aligned} \tag{5.24}$$

In the proof, for simplicity, arguments of functions are omitted when no confusions are likely to arise, except for a few critical ones.

We can analyze each item of Eq. (5.24) separately. First,

$$
\begin{aligned}
&2(\Omega\beta)^{\mathrm{T}}P(\Omega\dot{\beta})\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega(A\ddot{q}-b)\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega\\
&\qquad\times\Big[AM^{-1}\Big(-C\dot{q}-g-F+p_1+p_2\\
&\qquad+p_3+F_{\mathrm{d}}^{\mathrm{II}}\Big)-b\Big]\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega[A(D+\Delta D)(-\bar{C}\dot{q}-\bar{g}-\bar{F}\\
&\qquad-\Delta C\dot{q}-\Delta g-\Delta F+p_1+p_2+p_3\\
&\qquad+\bar{F}_{\mathrm{d}}^{\mathrm{II}}+\Delta F_{\mathrm{d}}^{\mathrm{II}})-b]\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega\left\{A[D(-\bar{C}\dot{q}-\bar{g}-\bar{F}\right.\\
&\qquad+p_1+p_2+\bar{F}_{\mathrm{d}}^{\mathrm{II}})\\
&\qquad+D(-\Delta C\dot{q}-\Delta g-\Delta F+\Delta F_{\mathrm{d}}^{\mathrm{II}})\\
&\qquad+\Delta D\left(-C\dot{q}-g-F+p_1+p_2+F_{\mathrm{d}}^{\mathrm{II}}\right)\\
&\qquad\left.+(D+\Delta D)p_3]-b\right\}.
\end{aligned}
\tag{5.25}
$$

Considering the design of p_1, we have

$$
A\Big[D(-\bar{C}\dot{q}-\bar{g}-\bar{F}+p_1+\bar{F}_{\mathrm{d}}^{\mathrm{II}})-b\Big]=0. \tag{5.26}
$$

By the equation of (5.19), performing matrix cancelation yields

$$
\begin{aligned}
&2(\Omega\beta)^{\mathrm{T}}P\Omega ADp_2+2(\Omega\beta)^{\mathrm{T}}P(\dot{\Omega}\beta)\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega AD\Big\{-\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times[(\Omega A)[(\Omega A)^{\mathrm{T}}]^{-1}P^{-1}\\
&\qquad\times(\kappa\Omega\beta+\dot{\Omega}\beta)\Big\}+2(\Omega\beta)^{\mathrm{T}}P(\dot{\Omega}\beta)\\
&\quad= -2\kappa(\Omega\beta)^{\mathrm{T}}\Omega\beta\\
&\quad= -2\kappa\|\Omega\beta\|^2.
\end{aligned}
\tag{5.27}
$$

Then

$$
\begin{aligned}
&2(\Omega\beta)^{\mathrm{T}}P\Omega A[\Delta D(-C\dot{q}-g-F\\
&\qquad+p_1+p_2+F_{\mathrm{d}}^{\mathrm{II}})\\
&\qquad+D(-\Delta C\dot{q}-\Delta g-\Delta F+\Delta F_{\mathrm{d}}^{\mathrm{II}})]\\
&\quad\le 2\|\Omega\beta\|\,\Big\|P\Omega A\Big[\Delta D\Big(-C\dot{q}-g-F\\
&\qquad+p_1+p_2+F_{\mathrm{d}}^{\mathrm{II}}\Big)\\
&\qquad+D\left(-\Delta C\dot{q}-\Delta g-\Delta F+\Delta F_{\mathrm{d}}^{\mathrm{II}}\right)\Big]\Big\|\\
&\quad\le 2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t).
\end{aligned}
\tag{5.28}
$$

By Eq. (5.20) and with $\Delta D = DE$,

$$
\begin{aligned}
&2(\Omega\beta)^{\mathrm{T}}P\Omega A(D+\Delta D)p_3\\
&\quad= 2(\Omega\beta)^{\mathrm{T}}P\Omega A(D+\Delta D)\Big\{-\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times\Big[(\Omega A)(\Omega A)^{\mathrm{T}}\Big]^{-1}P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\Big\}\\
&\quad= -2(\Omega\beta)^{\mathrm{T}}P\Omega D(1+E)\Big\{\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times\Big[(\Omega A)(\Omega A)^{\mathrm{T}}\Big]^{-1}P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\Big\}\\
&\quad= -2(\Omega\beta)^{\mathrm{T}}D\bar{M}(\Omega A)^{\mathrm{T}}\Big[(\Omega A)(\Omega A)^{\mathrm{T}}\Big]^{-1}\\
&\qquad\times P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\\
&\qquad-2(\Omega\beta)^{\mathrm{T}}P\Omega AM^{-1}\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times\Big[(\Omega A)(\Omega A)^{\mathrm{T}}\Big]^{-1}P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t).
\end{aligned}
\tag{5.29}
$$

Recalling that $\mu=\Omega\beta\Pi(\hat{\alpha},q,\dot{q},t)$, we have

$$
\begin{aligned}
&-2(\Omega\beta)^{\mathrm{T}}D\bar{M}(\Omega A)^{\mathrm{T}}[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}\\
&\qquad\times P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\\
&\quad= -2(\Omega\beta)^{\mathrm{T}}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\\
&\quad= -2\gamma\|\mu\|^2.
\end{aligned}
\tag{5.30}
$$

Adopting the Rayleigh's principle [7] and Assumption 5, we have

$$
\begin{aligned}
&-2(\Omega\beta)^{\mathrm{T}}P\Omega AM^{-1}\bar{M}(\Omega A)^{\mathrm{T}}[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}\\
&\qquad\times P^{-1}\gamma\mu\Pi(\hat{\alpha},q,\dot{q},t)\\
&\quad= -2\mu^{\mathrm{T}}\Big\{P\Omega ADE\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}P^{-1}\gamma\Big\}\mu\\
&\quad= -2\gamma\frac{1}{2}\mu^{\mathrm{T}}\Big\{P\Omega ADE\bar{M}(\Omega A)^{\mathrm{T}}\\
&\qquad\times[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}P^{-1}\\
&\qquad+P^{-T}[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-T}
\end{aligned}
$$

$$\times(\Omega A)\bar{M}E^{\mathrm{T}}D(\Omega A)^{\mathrm{T}}P^{\mathrm{T}}\Big\}$$
$$\leq -2\gamma\mu^{\mathrm{T}}\frac{1}{2}(W+W^{\mathrm{T}})\mu$$
$$\leq -2\gamma\rho_{\mathrm{E}}\|\mu\|^2. \tag{5.31}$$

So we can calculate Eq. (5.29) as

$$2(\Omega\beta)^{\mathrm{T}}P\Omega A(D+\Delta D)p_3 \leq -2\gamma(1+\rho_{\mathrm{E}})\|\mu\|^2. \tag{5.32}$$

The derivative of the Lyapunov function is

$$\begin{aligned}\dot{V} &= 2(\Omega\beta)^{\mathrm{T}}P(\dot{\Omega}\beta)+2(\Omega\beta)^{\mathrm{T}}P(\Omega\dot{\beta})\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}\\ &\leq -2\kappa\|\Omega\beta\|^2+2(1+\rho_{\mathrm{E}})\\ &\quad\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\gamma\|\mu\|^2\right)\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}.\end{aligned} \tag{5.33}$$

Considering the dead zone condition of the adaptive law in (5.15)–(5.16) and the robust gain design condition in (5.21), we will develop (5.33) under the four possible combinations of the inequalities.

5.1 Case I

If $\|\Omega\beta(q,\dot{q},t)\|\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\| > \epsilon$ and $\|\mu(\hat{\alpha}, q,\dot{q},t)\| > \xi$, we have

$$\begin{aligned}\dot{V} &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\\ &\quad\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\frac{1}{\|\mu\|}\|\mu\|^2\right)\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}\\ &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\\ &\quad\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\|\Omega\beta\|\Pi(\hat{\alpha},q,\dot{q},t)\right)\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}},\end{aligned} \tag{5.34}$$

where $\|\mu\| = \|\Omega\beta\|\Pi(\hat{\alpha},q,\dot{q},t)$.

Then

$$\begin{aligned}\dot{V} &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\\ &\quad\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\|\Omega\beta\|\Pi(\hat{\alpha},q,\dot{q},t)\right)\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}\\ &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\ &\quad+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}.\end{aligned} \tag{5.35}$$

As the $\|\Omega\beta(q,\dot{q},t)\|\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\| > \epsilon$, let us substitute $\dot{\hat{\alpha}}$ into Eq. (5.35) and get

$$\begin{aligned}\dot{V} &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\ &\quad+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\dot{\hat{\alpha}}\\ &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\ &\quad+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\ &\quad\times\left[k_1\frac{\partial\Pi^{\mathrm{T}}}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\|\Omega\beta\|-\left(k_2e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\right]\\ &\leq -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\ &\quad+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\ &\quad+2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}\frac{\partial\Pi^{\mathrm{T}}}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\|\Omega\beta\|\\ &\quad-2(1+\rho_E)(\hat{\alpha}-\alpha)^{\mathrm{T}}k_1^{-1}\left(k_2e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\ &\leq -2\kappa\|\Omega\beta\|^2-2(1+\rho_E)\\ &\quad\times(\hat{\alpha}-\alpha)^{\mathrm{T}}k_1^{-1}\left(k_2e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\ &\leq -2\kappa\|\Omega\beta\|^2-2(1+\rho_E)\\ &\quad\times(\hat{\alpha}-\alpha)^{\mathrm{T}}k_1^{-1}\left(k_2e^{-\|\Omega\beta\|}+k_3\right)(\hat{\alpha}\overbrace{-\alpha+\alpha}^{=0})\\ &\leq 2\kappa\|\Omega\beta\|^2-2(1+\rho_{\mathrm{E}})\\ &\quad\times\left(k_1^{-1}k_2e^{-\|\Omega\beta\|}+k_1^{-1}k_3\right)\\ &\quad\times\left(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|\right)\\ &\leq 2\kappa\|\Omega\beta\|^2-2(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}\\ &\quad\times\left(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|\right)\\ &\quad-2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\left(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|\right).\end{aligned} \tag{5.36}$$

We can simplify the second part of Eq. (5.34) as

$$\begin{aligned}&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\ &\leq 2(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}\frac{\|\alpha\|^2}{4}\end{aligned}$$

$$\leq \frac{1}{2}(1+\rho_{\mathrm{E}})k_1^{-1}k_2 e^{-\|\Omega\beta\|}\|\alpha\|^2. \tag{5.37}$$

The last part of Eq. (5.34) could be deducted as

$$\begin{aligned}
&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_3(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\
&\quad\leq -2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\hat{\alpha}-\alpha\|^2\\
&\qquad+(1+\rho_{\mathrm{E}})k_1^{-1}k_3(\|\alpha\|^2+\|\hat{\alpha}-\alpha\|^2)\\
&\quad\leq -(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\hat{\alpha}-\alpha\|^2\\
&\qquad+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2.
\end{aligned} \tag{5.38}$$

With Eqs. (5.37) and (5.38), we may get

$$\begin{aligned}
\dot{V} \leq& -2\kappa\|\Omega\beta\|^2+\frac{1}{2}(1+\rho_{\mathrm{E}})k_1^{-1}k_2 e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&-(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\hat{\alpha}-\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2.
\end{aligned} \tag{5.39}$$

As the $\|\delta\|^2=\|\Omega\beta\|^2+\|\hat{\alpha}-\alpha\|^2$, we have

$$\dot{V} \leq -\rho\|\delta\|^2+\theta, \tag{5.40}$$

where

$$\rho := \min\left\{2\kappa, (1+\rho_{\mathrm{E}})k_1^{-1}k_2\right\}, \tag{5.41}$$

$$\begin{aligned}
\theta :=& \frac{1}{2}(1+\rho_{\mathrm{E}})k_1^{-1}k_2 e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2.
\end{aligned} \tag{5.42}$$

5.2 Case II

If $\|\Omega\beta(q,\dot{q},t)\|\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\| \leq \epsilon$ and $\|\mu(\hat{\alpha}, q,\dot{q},t)\| > \xi$, we may substitute the new adaptive law in (5.16) into Eq. (5.35) and get

$$\begin{aligned}
\dot{V} \leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\
&+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\
&-2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left(k_2 e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\
\leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\\
&\times\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\|\|(\alpha-\hat{\alpha})\|\\
&-2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left(k_2 e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\
\leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\epsilon\|(\alpha-\hat{\alpha})\|\\
&-2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left(k_2 e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}.
\end{aligned} \tag{5.43}$$

Since $\|\delta\|^2=\|\Omega\beta\|^2+\|\alpha-\hat{\alpha}\|^2$, we may have $\|\alpha-\hat{\alpha}\|\leq\|\delta\|$. Combining Eqs. (5.37) and (5.38), we can get

$$\begin{aligned}
\dot{V} \leq& -2\kappa\|\Omega\beta\|^2+2(1+\rho_{\mathrm{E}})\epsilon\|\delta\|\\
&+\frac{1}{2}(1+\rho_{\mathrm{E}})k_1^{-1}k_2 e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&-(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\hat{\alpha}-\alpha\|^2.
\end{aligned} \tag{5.44}$$

By Eqs. (5.41) and (5.42), we have

$$\dot{V} \leq -\rho\|\delta\|^2+\psi\|\delta\|+\theta, \tag{5.45}$$

where $\psi := 2(1+\rho_{\mathrm{E}})$.

5.3 Case III

If $\|\Omega\beta(q,\dot{q},t)\|\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\| > \epsilon$ and $\|\mu(\hat{\alpha}, q,\dot{q},t)\| \leq \xi$, we can rewrite Eq. (5.33) as

$$\begin{aligned}
\dot{V} \leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\\
&\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\gamma\|\mu\|^2\right)\\
&+2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left[k_1\frac{\partial\Pi^{\mathrm{T}}}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\|\Omega\beta\|\right.\\
&\left.-(k_2 e^{-\|\Omega\beta\|}+k_3)\hat{\alpha}\right]\\
\leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\\
&\times\left(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\|\Omega\beta\|\Pi(\hat{\alpha},q,\dot{q},t)\right.\\
&\left.+\|\Omega\beta\|\Pi(\hat{\alpha},q,\dot{q},t)-\gamma\|\mu\|^2\right)\\
&+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\hat{\alpha}-\alpha)\\
&-2(1+\rho_{\mathrm{E}})(\hat{\alpha}-\alpha)^{\mathrm{T}}\left(k_1^{-1}k_2 e^{-\|\Omega\beta\|}+k_1^{-1}k_3\right)\hat{\alpha}\\
\leq& -2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}\\
&\times(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})+2(1+\rho_{\mathrm{E}})\frac{\|\mu\|}{4\gamma}
\end{aligned}$$

$$
\begin{aligned}
&+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial \Pi}{\partial \alpha}(\hat{\alpha},q,\dot{q},t)(\hat{\alpha}-\alpha)\\
&-2(1+\rho_{\mathrm{E}})(\hat{\alpha}-\alpha)^{\mathrm{T}}\left(k_1^{-1}k_2e^{-\|\Omega\beta\|}+k_1^{-1}k_3\right)\hat{\alpha}\\
\le\ &-2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)+\frac{(1+\rho_{\mathrm{E}})\|\mu\|}{2\gamma}\\
&-2(1+\rho_{\mathrm{E}})(\hat{\alpha}-\alpha)^{\mathrm{T}}\left(k_1^{-1}k_2e^{-\|\Omega\beta\|}+k_1^{-1}k_3\right)\hat{\alpha}\\
\le\ &-2\kappa\|\Omega\beta\|^2+\frac{(1+\rho_{\mathrm{E}})\|\mu\|}{2\gamma}\\
&-2(1+\rho_{\mathrm{E}})\left(k_1^{-1}k_2e^{-\|\Omega\beta\|}+k_1^{-1}k_3\right)\\
&\times(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\
\le\ &-2\kappa\|\Omega\beta\|^2+\frac{(1+\rho_{\mathrm{E}})\|\mu\|}{2\gamma}\\
&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}\\
&\times(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\
&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_3(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|),
\end{aligned}
\tag{5.46}
$$

Combining Eqs. (5.37) and (5.38), we have

$$
\begin{aligned}
\dot{V}\le\ &-2\kappa\|\Omega\beta\|^2+\frac{(1+\rho_{\mathrm{E}})\|\mu\|}{2\gamma}\\
&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}\\
&\times(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\
&-2(1+\rho_{\mathrm{E}})k_1^{-1}k_3(\|\hat{\alpha}-\alpha\|^2-\|\alpha\|\|\hat{\alpha}-\alpha\|)\\
\le\ &-2\kappa\|\Omega\beta\|^2+\frac{(1+\rho_{\mathrm{E}})\|\mu\|}{2\gamma}\\
&+\frac{1}{2}(1+\rho_{\mathrm{E}})k_1^{-1}k_2e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&-(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\hat{\alpha}-\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2\\
\le\ &-2\kappa\|\Omega\beta\|^2+\frac{(1+\rho_{\mathrm{E}})\xi}{2\gamma}\\
&+\frac{(1+\rho_{\mathrm{E}})}{2}k_1^{-1}k_2e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2\\
\le\ &-\rho\|\delta\|^2+\underline{\theta},
\end{aligned}
\tag{5.47}
$$

where

$$
\begin{aligned}
\underline{\theta}:=\ &\frac{(1+\rho_{\mathrm{E}})\xi}{2\gamma}+\frac{(1+\rho_{\mathrm{E}})}{2}k_1^{-1}k_2e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2.
\end{aligned}
\tag{5.48}
$$

5.4 Case IV

If $\|\Omega\beta(q,\dot{q},t)\|\left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)\right\|\le\epsilon$ and $\|\mu(\hat{\alpha},q,\dot{q},t)\|\le\xi$, we have

$$
\begin{aligned}
\dot{V}\le\ &-2\kappa(\Omega\beta)^{\mathrm{T}}(\Omega\beta)\\
&+2(1+\rho_{\mathrm{E}})(\|\Omega\beta\|\Pi(\alpha,q,\dot{q},t)-\gamma\|\mu\|^2)\\
&-2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left(k_2e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\
\le\ &-2\kappa\|\Omega\beta\|^2\\
&+2(1+\rho_{\mathrm{E}})\|\Omega\beta\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q},t)(\alpha-\hat{\alpha})\\
&+2(1+\rho_{\mathrm{E}})\frac{\|\mu\|}{4\gamma}\\
&-2(1+\rho_{E})(\hat{\alpha}-\alpha)^{\mathrm{T}}(\kappa k_1)^{-1}\kappa\\
&\times\left(k_2e^{-\|\Omega\beta\|}+k_3\right)\hat{\alpha}\\
\le\ &-2\kappa\|\Omega\beta\|^2+2(1+\rho_{\mathrm{E}})\epsilon\|\hat{\alpha}-\alpha\|\\
&+2(1+\rho_{\mathrm{E}})\frac{\xi}{4\gamma}\\
&+\frac{(1+\rho_{\mathrm{E}})}{2}k_1^{-1}k_2e^{-\|\Omega\beta\|}\|\alpha\|^2\\
&+2(1+\rho_{\mathrm{E}})k_1^{-1}k_3\|\alpha\|^2\\
\le\ &-\rho\|\delta\|^2+\psi\|\delta\|+\underline{\theta}.
\end{aligned}
\tag{5.49}
$$

According to the above results of the derivative of Lyapunov function, we could rewrite the results in a standard form as

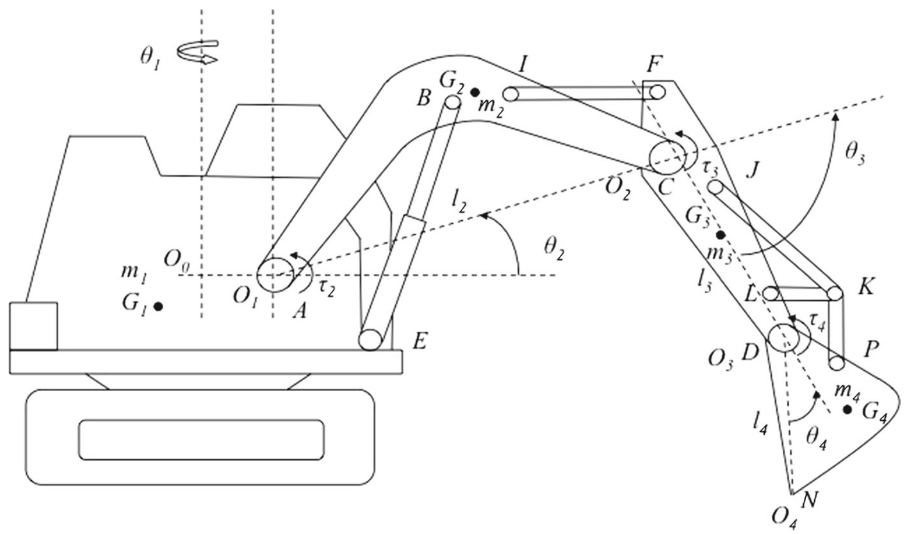
$$
\dot{V}\le-\lambda_1\|\delta\|^2+\lambda_2\|\delta\|+\lambda_3,
\tag{5.50}
$$

where $\lambda_1=\rho$, $\lambda_2=\psi$ or $\lambda_2=0$ and $\lambda_3=\theta$ or $\lambda_3=\underline{\theta}$.

Upon invoking the standard arguments as in Chen [25], we can conclude the uniform boundedness with

$$
d(r)=\begin{cases}\sqrt{\frac{\gamma_2}{\gamma_1}}R, & \text{if}\quad r\le R,\\ \sqrt{\frac{\gamma_2}{\gamma_1}}r, & \text{if}\quad r>R,\end{cases}
\tag{5.51}
$$

Fig. 2 The side view of excavator



$$R=\frac{1}{2\lambda_1}\left(\lambda_2+\sqrt{\lambda_2^2+4\lambda_1\lambda_3}\right), \tag{5.52}$$

where $\gamma_1=\min\left\{\lambda_m(P), (\kappa k_1)^{-1}(1+\rho_{\mathrm{E}})\right\}$ and $\gamma_2=\max\left\{\lambda_m(P), (\kappa k_1)^{-1}(1+\rho_{\mathrm{E}})\right\}$.

Furthermore, uniform ultimate boundedness also follows with

$$\underline{d}=\sqrt{\frac{\gamma_2}{\gamma_1}}R, \tag{5.53}$$

$$T(\overline{d}, r)=\begin{cases}0, & \text{if } \ r\le \overline{d}\sqrt{\frac{\gamma_2}{\gamma_1}},\\ \frac{\gamma_2 r^2-(\gamma_1^2/\gamma_2)\overline{d}^2}{\lambda_1\overline{d}^2(\gamma_1/\gamma_2)-\lambda_2\overline{d}(\gamma_1/\gamma_2)-\lambda_3}, & \text{otherwise.}\end{cases} \tag{5.54}$$

□

With the uncertainty in presence and no restrictions on the initial condition, it is only reasonable to expect approximate constraint following, which is shown in the sense that $\beta\neq 0$ in any finite time. In the special case when there is no uncertainty, i.e., $\Delta D\equiv 0$, $\Delta C\equiv 0$,$\Delta g\equiv 0$, $\Delta F\equiv 0$ and $\Delta F_{\mathrm{d}}^{\mathrm{II}}\equiv 0$, one may choose $\Pi(\alpha, q, \dot{q}, t)=0$ and hence $p_3=0$. This means $\tau=p_1+p_2$. If, in addition, we choose $p_2=0$, then $\dot{V}=0$. This means if $\beta=0$ initially (i.e., the constraint is met initially), then $\beta=0$ for all $t\ge t_0$. This special case falls into Theorem 2, the perfect constraint-following case.

6 An illustrative example

Consider an excavator shown in Fig. 2 during the digging operation as an example. Let the coordinate be $\theta=[\theta_1, \theta_2, \theta_3, \theta_4]^{\mathrm{T}}$, the velocity be $\dot{\theta}=[\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4]^{\mathrm{T}}$ and the acceleration be $\ddot{\theta}=[\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3, \ddot{\theta}_4]^{\mathrm{T}}$. The motion equation of the excavator is represented in [26] as

$$\begin{aligned}&M'(\theta,\sigma)\ddot{\theta}+C'(\theta,\dot{\theta},\sigma)\dot{\theta}+G'(\theta,\sigma)+B'(\dot{\theta})\\ &\quad=\Gamma'\tau(\theta,\dot{\theta})-F_{iL}(\theta,\dot{\theta},\sigma)+F_{nL}(\theta,\dot{\theta},\sigma),\end{aligned} \tag{6.1}$$

where $M'(\theta,\sigma)$ is the inertia matrix, $C'(\theta,\dot{\theta},\sigma)\dot{\theta}$ is determined by the Coriolis and centripetal effects, $G'(\theta,\sigma)$ is the gravity force, $B'(\dot{\theta})$ is friction force of joint shafts, Γ is the input matrix, vector $\tau=[\tau_1, \tau_2, \tau_3, \tau_4]^{\mathrm{T}}$ specifies the input torques acting on the joint shafts, $F_{iL}(\theta,\dot{\theta},\sigma)$ and $F_{nL}(\theta,\dot{\theta},\sigma)$ are the loading torques determined by the interaction of soil and bucket during the digging operation.

Denote the following parameters of the ith link ($i=1,2,3,4$; $j=2,3,4$):

m_i-mass of the ith link, the center of gravity is c_i;
l_i-length of the ith link;
$L_{O_iG_j}$-length from the ith joint to the gravity center of jth link.
I_i-inertial moment of the ith link at c_i.

The expressions of $M'(\theta)$, $C'(\theta,\dot{\theta})\dot{\theta}$ and $G'(\theta)$ are as follows:

$$M'(\theta,\sigma)=\begin{bmatrix}M_{11} & M_{12} & M_{13} & M_{14}\\ M_{21} & M_{22} & M_{23} & M_{24}\\ M_{31} & M_{32} & M_{33} & M_{34}\\ M_{41} & M_{42} & M_{43} & M_{44}\end{bmatrix},$$

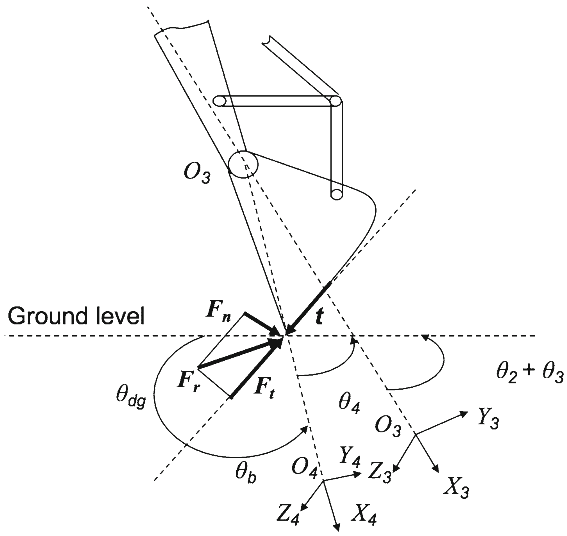


Fig. 3 The angle and force of the excavator bucket during digging operation

$$C'(\theta,\dot{\theta},\sigma)\dot{\theta}=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14}\\ C_{21} & C_{22} & C_{23} & C_{24}\\ C_{31} & C_{32} & C_{33} & C_{34}\\ C_{41} & C_{42} & C_{43} & C_{44}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\end{bmatrix},$$

$$G'(\theta,\sigma)=\begin{bmatrix}G_1\\ G_2\\ G_3\\ G_4\end{bmatrix}. \tag{6.2}$$

The expressions of $G'(\theta,\sigma)$ and Γ' are omitted for brevity. The friction matrix $B'(\dot{\theta})$ is

$$B'(\dot{\theta})=[B_1\dot{\theta}_1,\, B_2\dot{\theta}_2,\, B_3\dot{\theta}_3,\, B_4\dot{\theta}_4]^{\mathrm{T}}, \tag{6.3}$$

where B_i $(i=1,2,3,4.)$ is the viscous friction coefficient of the ith joint shaft.

The reaction force F_{r} on the edge of the bucket in Fig. 3, generated by the interaction of the soil and the bucket, is determined by

$$F_{\mathrm{r}}=F_{i\mathrm{r}}+F_{n\mathrm{r}}, \tag{6.4}$$

$$F_{i\mathrm{r}}:=k_p\left[k_s bh+\zeta\left(1+\frac{\mu_{\mathrm{s}}}{\mu_{\mathrm{b}}}\right)bh\sum_i \Delta x_i\right], \tag{6.5}$$

$$F_{n\mathrm{r}}:=k_p\mu N, \tag{6.6}$$

where k_p and k_s are specific resistances in cutting silty clay; b and h are width and thickness of the cut slice of soil, respectively; μ_{s} and μ_{b} are volumes of the prism of soil and the bucket, respectively; and Δx is the increment along the horizontal axis (in meters); μ is the coefficient of friction between the bucket and the soil; N is the pressure force of the bucket with the soil; ζ is the coefficient of resistance experienced in filling the bucket during the movement of the prism of soil (hence Assumption 3). Therefore, the loading torque F_{iL} generated by $F_{i\mathrm{r}}$ can be given by

$$F_{iL}=\begin{bmatrix} F_{it}\\ l_2\left(F_{it}\sin(\theta_2-\theta_{dg})-F_{in}\cos(\theta_2-\theta_{dg})\right)\\ l_3\left(F_{it}\sin(\theta_2+\theta_3-\theta_{dg})-F_{in}\cos(\theta_2+\theta_3-\theta_{dg})\right)\\ l_4\left(-F_{it}\sin\theta_b+F_{in}\cos\theta_b\right)\end{bmatrix}, \tag{6.7}$$

where $F_{it}=F_{i\mathrm{r}}\cos(\theta_{dg}-0.1)$ and $F_{in}=F_{i\mathrm{r}}\sin(0.1)$.

The pressure force N in (6.20) is the component force of the constraint forces in the normal direction of the constrained surface, which can be formulated as

$$N=\|(JJ^{\mathrm{T}})^{-1}JF^{\mathrm{c}}\|, \tag{6.8}$$

where $F^{\mathrm{c}}\in\mathbf{R}^3$ is the constraint force and the J is the Jacobian matrix of the system. Then, according to Theorem 1, the loading torque F_{nL} determined by the reaction force component $F_{n\mathrm{r}}$ can be calculated as

$$F_{nL}=M^{1/2}\left[I-\left(AM^{-1/2}\right)^{+}\left(AM^{-1/2}\right)\right]M^{-1/2}F_{f\mathrm{r}}, \tag{6.9}$$

$$F_{f\mathrm{r}}=-k_p\mu\frac{Nt}{\|t\|}, \tag{6.10}$$

where $F_{f\mathrm{r}}\in\mathbf{R}^3$ is the $F_{n\mathrm{r}}$ in the work space , $t\in\mathbf{R}^3$ is the tangential vector of the digging point and can be calculated as

$$t=\frac{J\dot{q}}{\|J\dot{q}\|},\quad J\dot{q}\neq 0. \tag{6.11}$$

Influenced by the unknown terrain in the digging operation, there are several parameters of the system considered as uncertainty: the mass of the excavator bucket including the cutting soil m_4 and the thickness of the cutting slice of the soil h. The uncertain parameters can be expressed as: $m_4=\bar{m}_4+\Delta m_4(t)$ and $h=\bar{h}+\Delta h(t)$, where $\bar{m}_4$ and $\bar{h}$ are the constant nominal values which are selected to be strictly positive. Δm_4 and Δh are the time-varying uncertain portions. So the uncertainty parameter is chosen as $\sigma=[\Delta m_4,\Delta h]^{\mathrm{T}}$. During the digging operation, θ_1 is the constant. Then the elements M_{1i}, M_{i1}, C_{1i}, C_{i1}, Γ_{1i}, G_1, B_1, F_{iL1} and F_{nL1} are omitted in this paper. In the following deduction, the general coordinates vector considered in this paper is $q = [\theta_2, \theta_3, \theta_4]^{\mathrm{T}}$. We can rewrite the motion equation of the excavator as

$$\begin{aligned} &M(q,\sigma)\ddot{q} + C(q,\dot{q},\sigma)\dot{q} + G(q,\sigma) + B(\dot{q}) \\ &\quad = \Gamma\tau(q,\dot{q}) - F_{iL}(q,\dot{q},\sigma) + F_{nL}(q,\dot{q},\sigma), \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} M(q,\sigma) &= \begin{bmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{bmatrix}, \\ C(q,\dot{q},\sigma)\dot{q} &= \begin{bmatrix} C_{22} & C_{23} & C_{24} \\ C_{32} & C_{33} & C_{34} \\ C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix}, \\ G(q,\sigma) &= \begin{bmatrix} G_2 \\ G_3 \\ G_4 \end{bmatrix}, \quad B(\dot{q}) = \begin{bmatrix} B_2\dot{\theta}_2 \\ B_3\dot{\theta}_3 \\ B_4\dot{\theta}_4 \end{bmatrix}. \end{aligned} \tag{6.13}$$

Suppose the excavator dig on a constrained surface at certain velocity. The constraints of the system are

$$\begin{cases} \dot{\theta}_2 - \dot{\theta}_3 + \dot{\theta}_4 = 0, \\ \dot{\theta}_4 = \theta_2\dot{\theta}_3 + \dot{\theta}_2\theta_3. \end{cases} \tag{6.14}$$

We could derivative the (6.14) and get the second-order constraints in the matrix form of $A(q)\ddot{q} = b(q,\dot{q})$, where

$$A(q) = \begin{bmatrix} 1 & -1 & 1 \\ \theta_3 & \theta_2 & -1 \end{bmatrix}, \tag{6.15}$$

$$b(q,\dot{q}) = \begin{bmatrix} 0 \\ 2\dot{\theta}_2\dot{\theta}_3 \end{bmatrix}. \tag{6.16}$$

The matrix $A(q)$ is not full rank (notice that the Assumption 2 is met). By the Assumption 4, we could choose the $\Omega = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, so that $\Omega A(q)$ is full rank and $(\Omega A(q))(\Omega A(q))^{\mathrm{T}}$ is invertible.

For the control design, we decompose the M, C, G and F_L as $\bar{M}$, ΔM, $\bar{C}$, ΔC, $\bar{G}$, ΔG, $\bar{F}_L$ and ΔF_L. Assumption 1 is verified by the chosen $\bar{M}$. Note that $\Delta M = 0$; thus, $\rho_{\mathrm{E}} = 0$ (hence Assumption 5).

By (5.15)–(5.16), the adaptive law is designed as

$$\dot{\hat{\alpha}} = \begin{cases} \kappa\left[k_1 \frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q})\|\Omega\beta\| - \left(k_2 e^{-\|\Omega\beta\|} + k_3\right)\hat{\alpha}\right], & if\,\|\Omega\beta(q,\dot{q})\| \left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q})\right\| > \epsilon, \\ -\kappa\left(k_2 e^{-\|\Omega\beta\|} + k_3\right)\hat{\alpha}, & if\,\|\Omega\beta(q,\dot{q})\| \left\|\frac{\partial\Pi}{\partial\alpha}(\hat{\alpha},q,\dot{q})\right\| \le \epsilon. \end{cases} \tag{6.17}$$

Therefore, according to Theorem 1, we design the control p_1 as

$$\begin{aligned} p_1 = &\overbrace{\bar{M}^{1/2}(A\bar{M}^{-1/2})^{+}\left[b + A\bar{M}^{-1}(\bar{C}\dot{q} + \bar{G} + B - F_{iL})\right])}^{\text{constraint force } \bar{F}^{\mathrm{c}}} \\ &- \overbrace{\bar{M}^{1/2}\left[I - (A\bar{M}^{-1/2})^{+}(A\bar{M}^{-1/2})\right]\bar{M}^{-1/2}F_{f\mathrm{r}}}^{\text{given force } \bar{F}_{nL}}, \end{aligned} \tag{6.18}$$

where $\bar{F}^{\mathrm{c}}$ is the nominal portion of the constraint force, $\bar{F}_{nL}$ is the nominal part of the given force that is decoupled with the constraint force N. Note that the terms in M, $C\dot{q}$ and G are either constant, trigonometric in positions, quadratic in velocities; Assumption 5 is met by choosing

$$\begin{aligned} \Pi(\alpha,q,\dot{q}) &= \alpha_1\|\dot{q}\|^2 + \alpha_2\|\dot{q}\| + \alpha_3 \\ &\le \alpha(\|\dot{q}\|^2 + \|\dot{q}\| + 1), \end{aligned} \tag{6.19}$$

where $\alpha = \max\{\alpha_1,\alpha_2,\alpha_3\}$. According to Eqs. (5.18) and (5.19), we can get

$$p_2 = -\bar{M}(\Omega A)^{\mathrm{T}}[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}P^{-1}(\kappa\Omega\beta + \dot{\Omega}\beta), \tag{6.20}$$

$$p_3 = -\bar{M}(\Omega A)^{\mathrm{T}}[(\Omega A)(\Omega A)^{\mathrm{T}}]^{-1}P^{-1}\gamma\mu\Pi. \tag{6.21}$$

For the simulation of the excavator system in MATLAB, the numerical values of the parameters used in simulation are as follows [26]: $m_2 = 1566\,\mathrm{kg}$, $m_3 = 735\,\mathrm{kg}$, $m_4 = 432\,\mathrm{kg}$, $I_2 = 14250.6\,\mathrm{kg\,m^2}$, $I_3 = 727.7\,\mathrm{kg\,m^2}$, $I_4 = 224.6\,\mathrm{kg\,m^2}$, $l_2 = 5.16\,\mathrm{m}$, $l_3 = 2.59\,\mathrm{m}$, $l_4 = 1.33\,\mathrm{m}$, $L_{O_1G_2} = 0.05\,\mathrm{m}$, $L_{O_2G_3} = 0.64\,\mathrm{m}$, $L_{O_3G_4} = 0.65\,\mathrm{m}$, $k_p = 1.05$, $k_s = 5500$, $\mu = 0.1$, $\zeta = 5500\,\mathrm{kg\,cm^{-2}\,s^{-2}}$, $\mu_{\mathrm{s}} = 1921.8\,\mathrm{kg\,m^{-3}}$, $\mu_{\mathrm{b}} = 0.58\,\mathrm{m^3}$, $\bar{h} = 0.2\,\mathrm{m}$, $\Delta m_4 = 0.1\bar{m}_4\sin(t)$ and $\Delta h = 0.2\bar{h}\cos(t)$. Suppose the initial condition of the system is $q_0 = [\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}]^{\mathrm{T}}$, $\dot{q}_0 = [0.1, 0.2, -0.2]^{\mathrm{T}}$ and $\ddot{q}_0 = [1.4, 1.0, -2.5]^{\mathrm{T}}$. We choose the control parameters as follows: $\kappa = 0.2$, $k_1 = 400$, $k_2 = 0.5$, $k_3 = 0.01$, $\xi = 0.1$ and $\epsilon = 0.1$. For comparison, we also adopt a linear–quadratic regulator (LQR), which is well known to possess robustness to uncertainty in coefficients of the system.

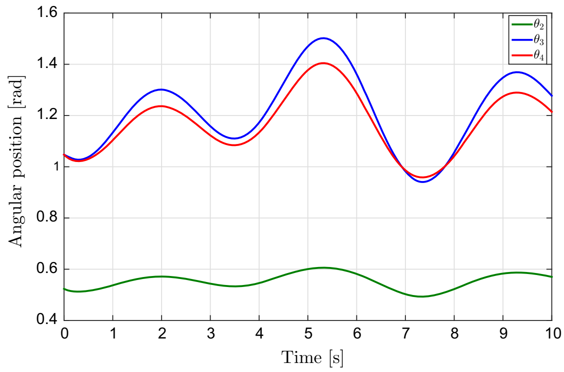


Fig. 4 The angular position time history

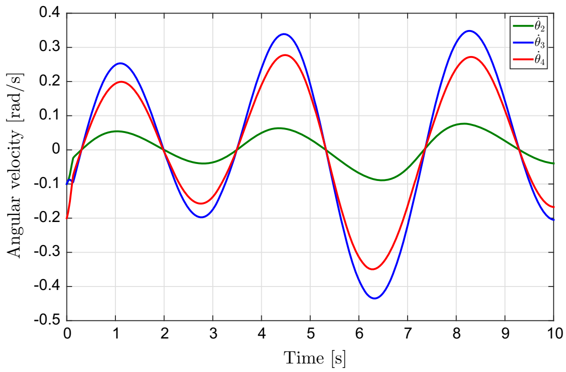


Fig. 5 The angular velocity time history

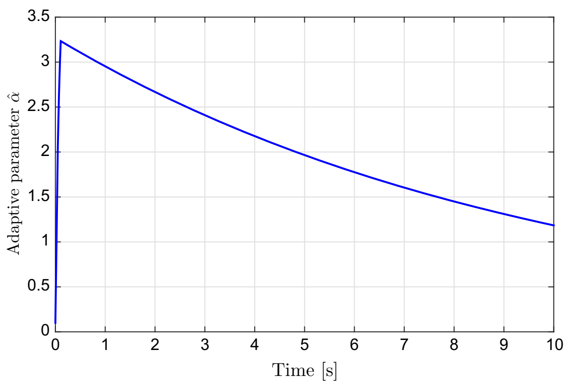


Fig. 6 The adaptive parameter $\hat{\alpha}$ time history

Figures 4 and 5 demonstrate the angular position and velocity time histories. Figure 6 shows the adaptive parameter $\hat{\alpha}$ time history, which increases in the beginning 0.1 s and decreases in the rest simulation time. Figure 7 displays the constraint-following error $\|\Omega\beta\|$ simulated in three cases: the control is p_1; the control is $p_1 + p_2$; the control is $p_1 + p_2 + p_3$. The error $\|\Omega\beta\|$

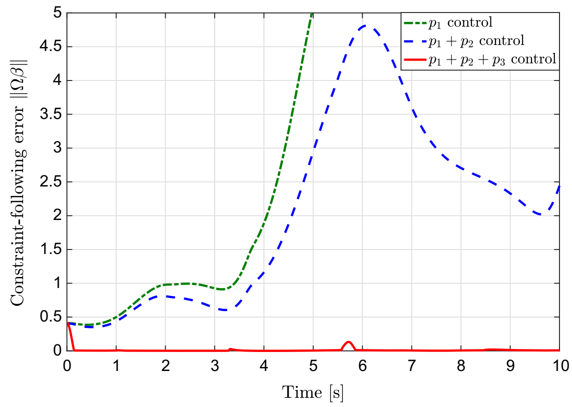


Fig. 7 The comparisons of the constraint-following error $\|\Omega\beta\|$

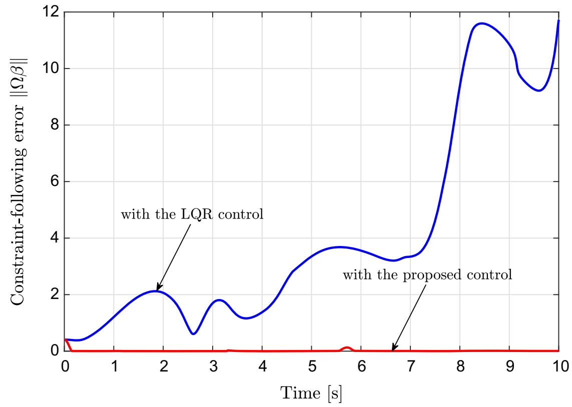


Fig. 8 The comparisons of the constraint-following error $\|\Omega\beta\|$ with different controls

with p_1 control could not converge in the simulation time. The error $\|\Omega\beta\|$ with $p_1 + p_2$ control is bounded but remains larger than 0.5 after 1.2 s. The error $\|\Omega\beta\|$ with $p_1 + p_2 + p_3$ control starts from 0.4 and enters a very small zone around 0 after 0.2 s. In Fig. 8, the constraint-following error $\|\Omega\beta\|$ under the proposed control quickly settles to a very small zone around 0 in less than 0.2 s. In the LQR control, the constraint-following error $\|\Omega\beta\|$ diverges from the initial error 0.4 and reaches almost 12 in the simulation time. Figures 9 and 10 show the time histories of the control torques τ_2, τ_3 and τ_4. Figure 11 demonstrates the time histories of constraint force $\|\bar{F}^c\|$ and the decoupled given force $\|\bar{F}_{nL}\|$. Let $\Delta m_4 := \max_t \|\Delta m_4\|$, $\delta h := \max_t \|\Delta h\|$ and $\hat{\alpha}_{\max} := \max_t \|\hat{\alpha}\|$. Figure 12 shows the effect of uncertainty bounds (Δm_4 and Δh) on $\hat{\alpha}_{\max}$.

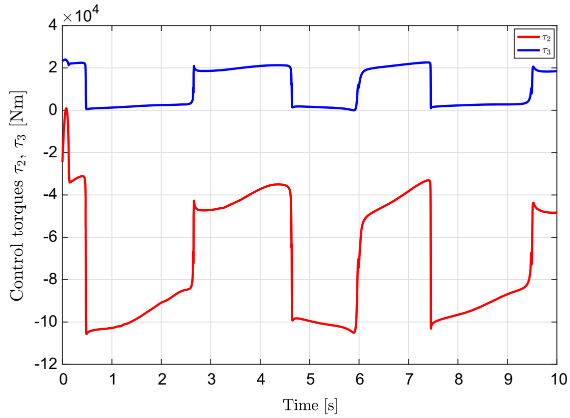


Fig. 9 The control input torques τ_2 and τ_3 time histories

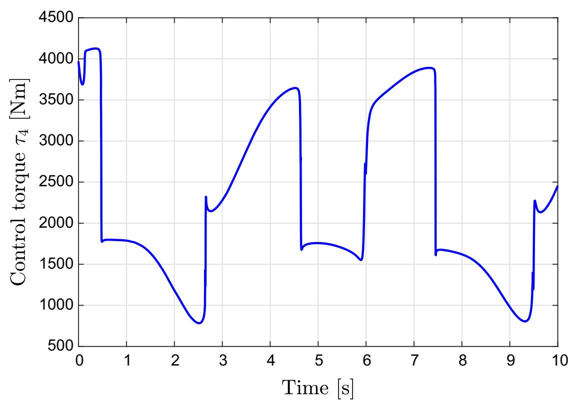


Fig. 10 The control input torque τ_4 time history

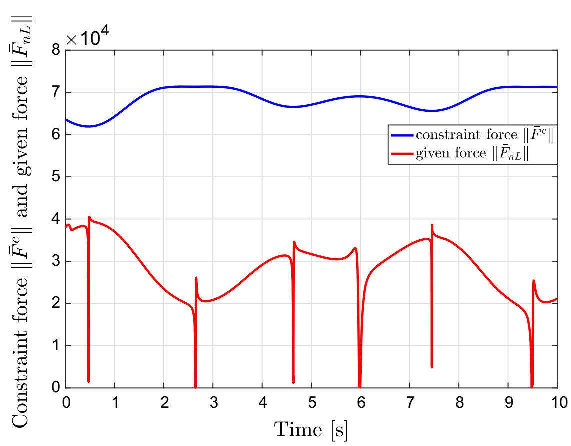


Fig. 11 The constraint force $\|\bar{F}^c\|$ and given force $\|\bar{F}_{nL}\|$ time histories

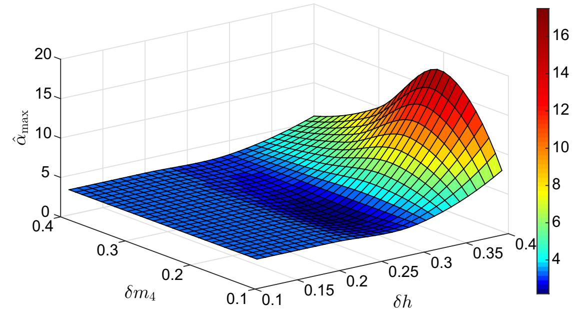


Fig. 12 The maximum of adaptive parameter ($\hat{\alpha}_{\max}$) with respect to δm_4 and δh

7 Conclusion

We have proposed a novel adaptive robust control design for the uncertain mechanical systems to approximately follow a set of constraints. The coupling problem of given force and constraint force is resolved by using the second-order constraints. Those decoupled forces are explicit, Lagrange multiplier free and complete, which are used as a feedforward control in the following constraint-following control. The uncertainty of mechanical system considered in the paper includes unknown parameters and external disturbances. The bounds of all uncertain terms are assumed to be unknown but exist. The effects of all uncertain terms are packaged into a function which upper bound can be estimated by an adaptive mechanism with leakage term and dead zone. Based on the estimated bound information of the uncertainty, a robust control is proposed to regulate the constraint-following system with certain performances such as uniform boundedness and uniform ultimate boundedness. There are several interesting findings that could be further studied in the future: the decoupling method is general and could be implemented in the friction compensation control of mechanical system; by the mathematical conformity of second-order constraints and system dynamics, we realize that the motion control may influence the regulation of the force, which will bring an insight for the force control of mechanical system; if the limitation of the amplitude and rate of actuator is imposed as in [27], the control design with the control input constraints will be investigated in the future.

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