Introduction to Statistical Modelling

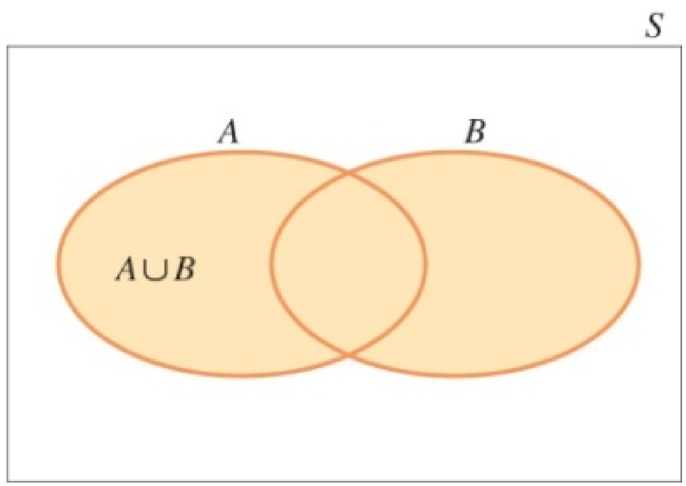
STAT2507A

Chapter 4 -2

Probability and Probability Distributions

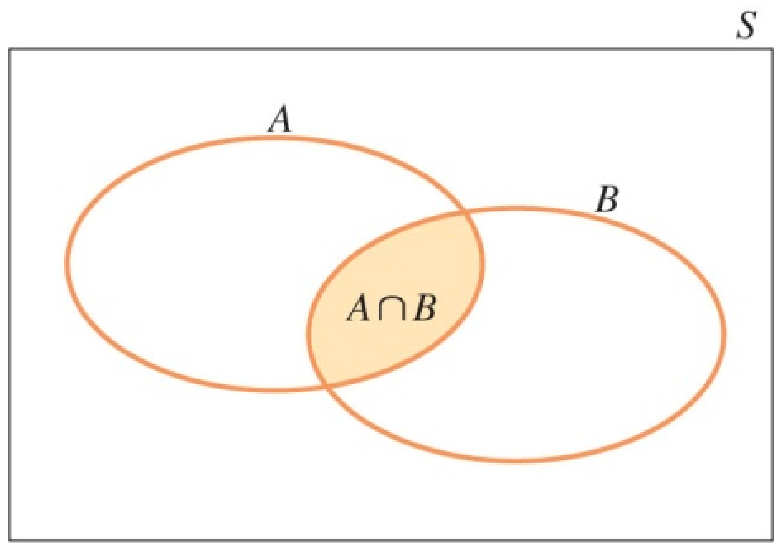
EVENT RELATIONS

- Event of interest as a combination of several other events
  - Union: The union of events A and B, denoted by $A \cup B$, is the event that either A or B or both occur.



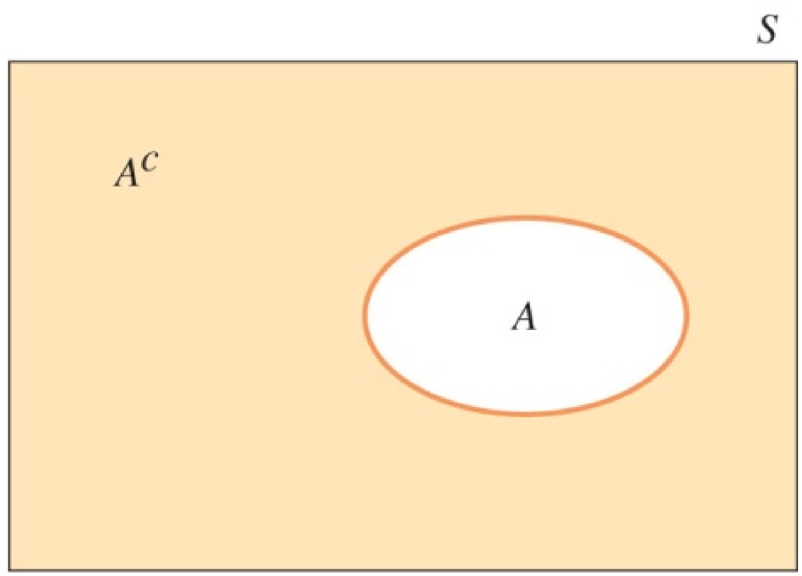
EVENT RELATIONS

- Intersection: The intersection of events A and B, denoted by $A \cap B$, is the event that both A and B occur.



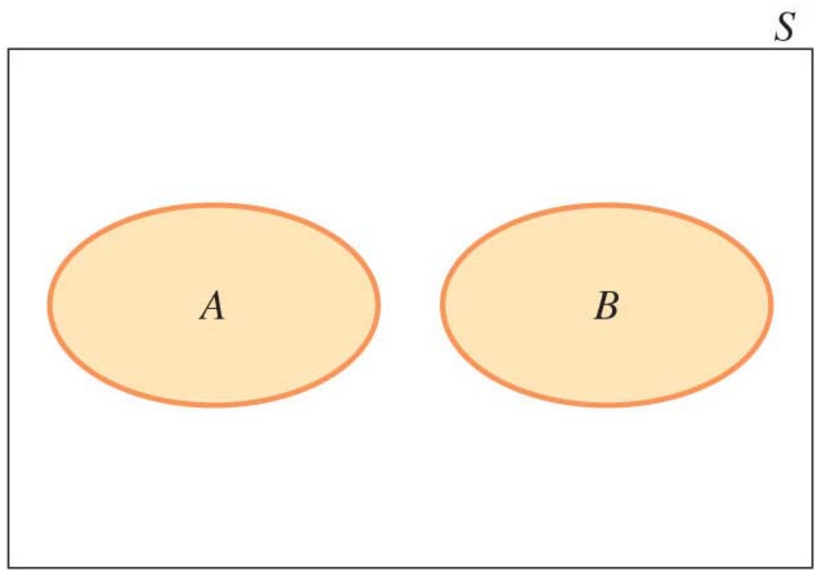
EVENT RELATIONS

- Complement: The complement of an event A, denoted by A^c (A'), is the event that A does not occur.



EVENT RELATION

- If two events A and B are **mutually exclusive**, then $\mathbf{P(A \cap B) = 0}$



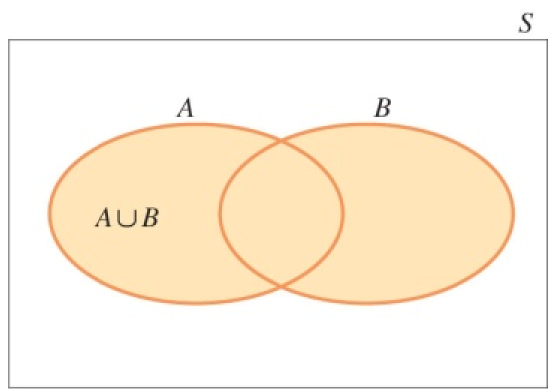
EXERCISE

4.13) Select a student from the classroom and record his/her **hair colour** and **gender.** Describe A^c , $B \cap C$, $B \cup C$.

- **A:** student has brown hair
- **B:** student is female
- **C:** student is male

CALCULATING PROBABILITIES

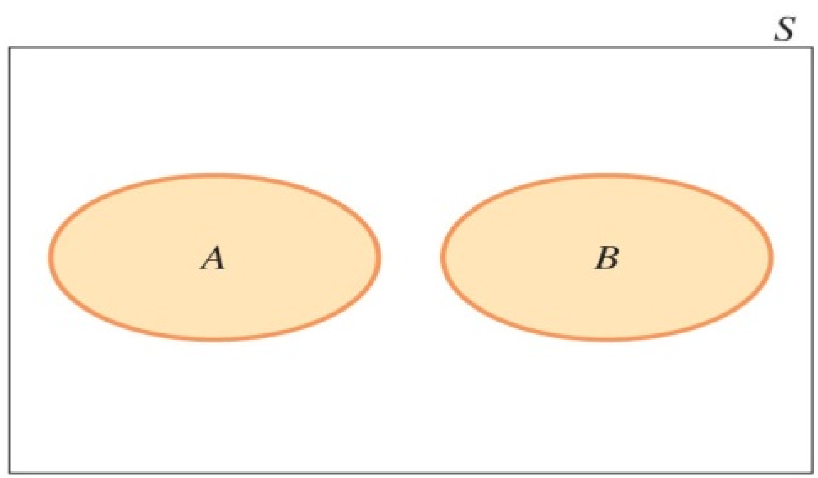
- Addition Rule (Rule that deals with union of events): Given two events A and B, the probability of their union, A∪B



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

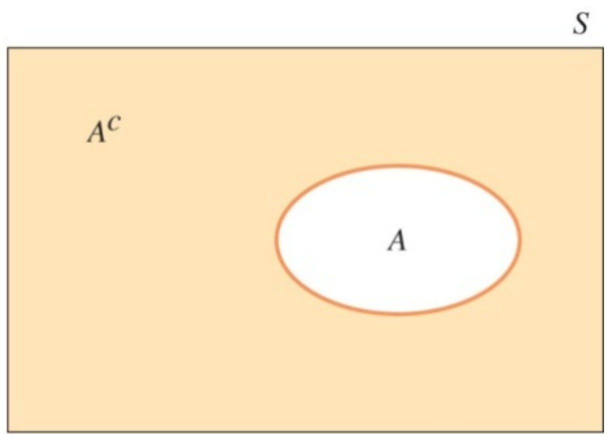
CALCULATING PROBABILITIES

- When two events A and B are mutually exclusive or disjoint, if A occurs, B cannot and vice versa => $P(A \cap B) = 0$ => addition rule simplifies to $P(A \cup B) = P(A) + P(B)$



CALCULATING PROBABILITIES

- Rule for complement: A and A^c are mutually exclusive and $A \cup A^c = S$



$$P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$$

EXERCISE

4.14) Suppose that there were 120 students in the classroom, and that they could be classified as follows:

	Brown Hair	Not Brown Hair	
Female	30	30	60
Male	20	40	60
	50	70	120

Define event: A – Brown hair; B – Female; C – Male with brown Hair; D – Female with brown Hair; E – Male

Find $P(A \cup B)$, $P(C \cup D)$, $P(E)$

INDEPENDENCE AND CONDITIONAL PROBABILITY

- The rules for calculating $P(A \cap B)$ depends on the idea of **independent** and **dependent** events
- Two events, A and B, are said to be independent if and only if the probability that event A occurs does not change, depending on whether or not event B has occurred

Independence and Conditional Probability

- The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if } P(B) \neq 0$$

EXAMPLE

4.15) Toss a fair coin twice and define: A – head on second toss, B- head on first toss. What are $P(A|B)$ and $P(A|B^c)$?

4.16) A bowl contains five M&Ms®, two red and three blue. Randomly select two candies, and define: A – Second candy is red, B – First candy is blue. What are $P(A|B)$ and $P(A|B^c)$?

Definition of Independence

- We can redefine independence in terms of conditional probabilities:
  - Two events are independent if and only if $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

    Otherwise, they are dependent.

DEFINITION OF INDEPENDENCE

- The following rule (Multiplication Rule) can be used to calculate the intersection of two events depending on whether they are independent or not
  - For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)P(B \text{ given that A occured}) = P(A)P(B \mid A)$$

  - If the events A and B are independent, then the probability that both A and B occur is

$$P(A \cap B) = P(A)P(B)$$

EXERCISE

4.17) In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?

EXERCISE

4.18) Suppose we have additional information for the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high-risk female?

EXERCISE

4.19) Suppose that five good fuses and two defective one have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of defective fuses in the first two tests?

POINTS ON MUTUALLY EXCLUSIVE AND INDEPENDENCE

- Two events are mutually exclusive or disjoint => they cannot both happen when the experiment is performed.
- Once the event B has occurred, A can not occur => $P(A|B) = 0$ or vice versa. The occurrence of event B certainly affects the probability that event A can occur => mutually exclusive events must be dependent

Points on Mutually Exclusive and Independence

- When two events are mutually exclusive or disjoint, $P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B)$$

- When two events are independent,

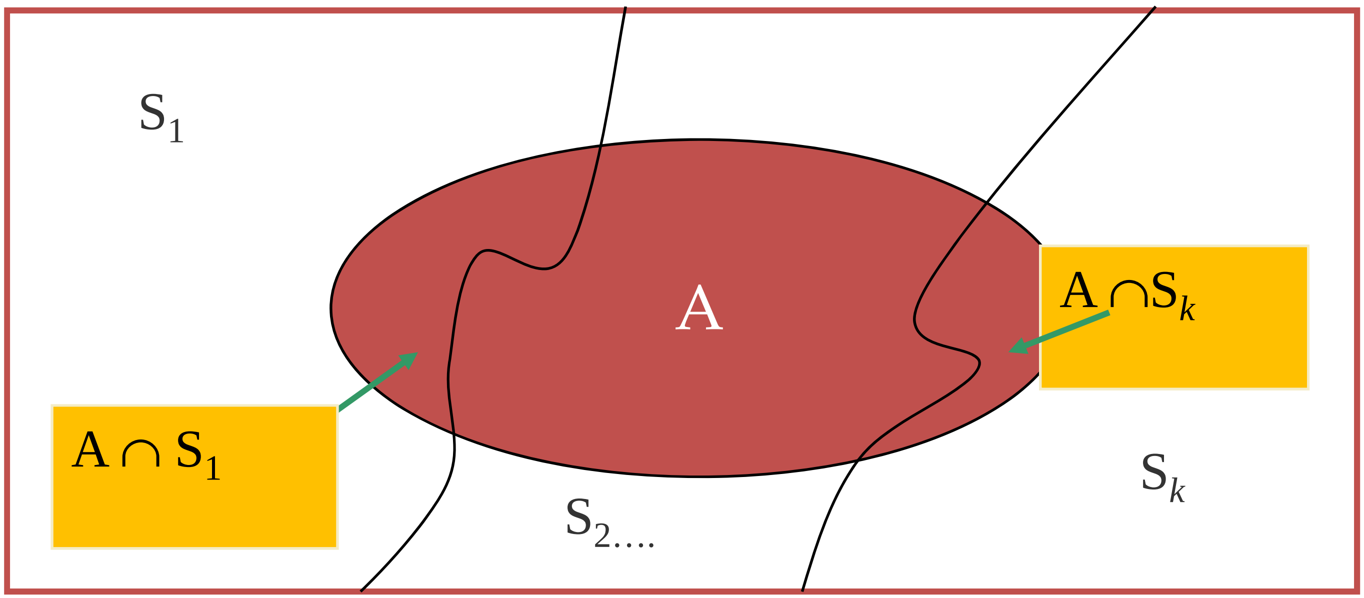
$$P(A \cap B) = P(A)P(B) \Rightarrow P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

THE LAW OF TOTAL PROBABILITY

- Let S_1, S_2 , S_3, ..., S_k be mutually exclusive and exhaustive events
  - Mutually exclusive – One only one must happen
  - Exhaustive events - All these events make up the entire sample space. $S = S_1 \cup S_2 \cdots \cap S_{k-1} \cup S_k$

The Law of Total Probability

$$A = (A \cap S_1) \cup (A \cap S_2) \cup \cdots \cup (A \cap S_{k-1}) \cup (A \cap S_k)$$



THE LAW OF TOTAL PROBABILITY CONT'D

- As mutually exclusive events

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \cdots$$
$$+ P(A \cap S_{k-1}) + P(A \cap S_k)$$

$$P(A) = P(S_1)P(A \mid S_1) + P(S_2)P(A \mid S_2) + \cdots$$
$$+ P(S_{k-1})P(A \mid S_{k-1}) + P(S_k)P(A \mid S_k)$$

- $P(S_1)$, $P(S_2)$, …, $P(S_{k-1})$, $P(S_k)$ are prior probabilities

BAYES' RULE

Let S_1, S_2, … S_k represent k mutually exclusive and exhaustive subpopulations with prior probabilities $P(S_1)$, $P(S_2)$, …., $P(S_k)$. If an event A occurs, the posterior probability of S_i, given that A occurred is

$$P(S_i \mid A) = \frac{P(S_i)P(A \mid S_i)}{\sum_{j=1}^{k} P(S_j)P(A \mid S_j)}, \; i = 1, 2, \cdots, \mathrm{k}$$

EXERCISE

4.20) From example 4.17, we know that 49% of the population are female. Of the female patients 8% are high risk for heart attack, while 12% of the male patients are high risk. A single person is selected at random and found to be high risk. What is the probability that it is a male?

RANDOM VARIABLES

- A variable X is a random variable, if the value that it assumes, corresponding to the outcome of an experiment, is a chance or random event.
- Random variable can be discrete or continuous
- Whether X is a discrete or continuous depend on the values that X can assume.

RANDOM VARIABLES

- Examples:
  - X = SAT score for a randomly selected student
  - X = number of people in a room at a randomly selected time of day
  - X = number on the upper face of a randomly tossed die
- The probability distribution for a discrete random variable x resembles the relative frequency distributions we constructed in Chapter 1.

EXERCISE

4.21) Toss two fair coins and define X= number of heads. Find the possible values of X and associated probabilities, create probability table and probability histogram.

PROBABILITY DISTRIBUTIONS FOR DISCRETE

- The distribution give this information about X
  - What values of X occurred
  - How often each value of X occurred
- Probability is defined as the limiting value of the relative frequency as the experiment is repeated over and over again.

Probability Distributions for Discrete

- Probability distribution for a random variable X is defined as the relative frequency distribution constructed for the entire population of measurements
- Definition: The **probability distribution** for a discrete random variable is a formula, table, or graph that gives the possible values of X, and the probability p(x) associated with each value of x.

$$0 \le p(x) \le 1, \ \sum \mathrm{p(x)} = 1$$

PROBABILITY DISTRIBUTIONS

- Shape:
  - Symmetric, Skewed, mound-shaped
- Outliers
  - unusual or unlikely measurements
- Centre and spread:
  - mean and standard deviation

PROBABILITY DISTRIBUTIONS

- The mean and standard deviation
  - Let X be a discrete random variable with probability distribution p(x)
  - Mean: $\mu = \sum xp(x)$
  - Variance: $\sigma^2 = \sum (x-\mu)^2 p(x)$
  - Standard deviation: $\sigma = \sqrt{\sigma^2}$

EXERCISE

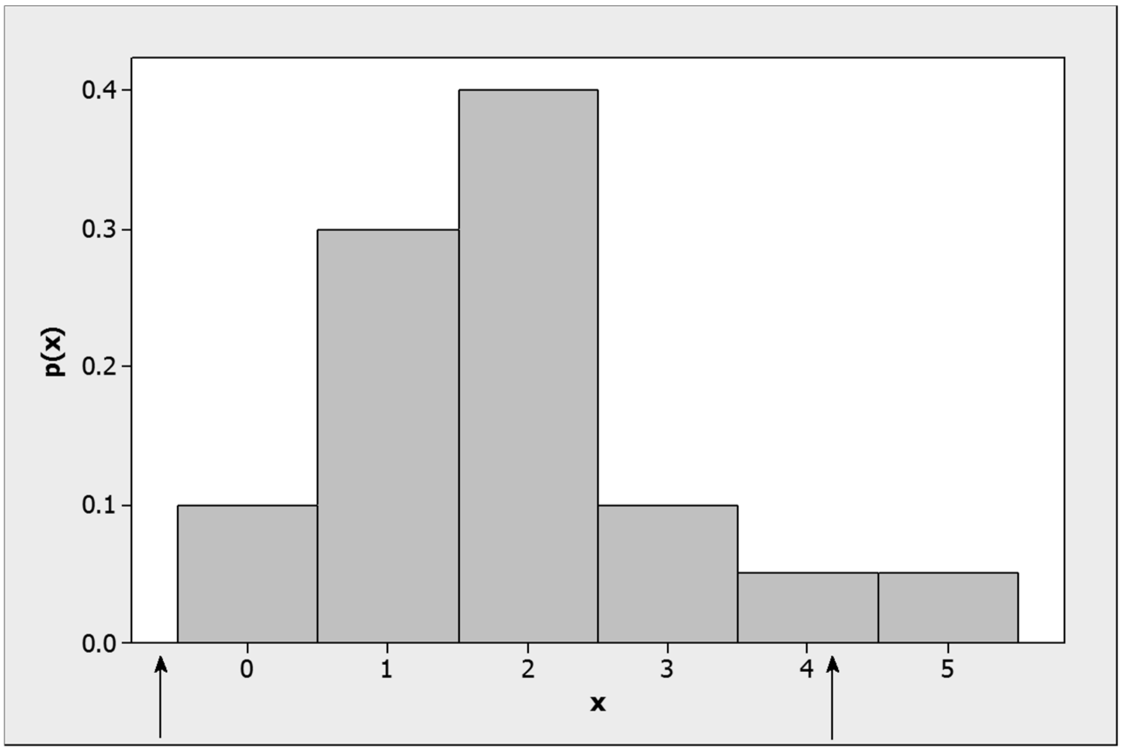
4.22) A random variable X can assume five values: 0, 1, 2, 3, 4, 5. A portion of the probability distribution is shown here

x	0	1	2	3	4	5
P(x)	0.1	0.3	0.4	0.1	?	0.05

a. Find p(4)
b. Construct a probability histogram to describe p(x)
c. Find μ, σ^2, and σ.
d. Locate the interval $\mu \pm 2\sigma$ on the x-axis of the histogram. What is the probability that X will fall into this interval?
e. If you were to select a very large number of values of X from the population, would most fall into the interval $\mu \pm 2\sigma$? Explain

EXERCISE

- Probability histogram shown here



REMARKS

- $\mu = E(X) = \sum_{all\ x} xp(x)$
- $\sigma^2 = Var(X) = \sum_{all\ x} (x - \mu)^2 p(x)$
- If random variable $Y = aX + b$
  - $E(Y) = aE(X) + b = a\mu + b$
  - $Var(Y) = a^2 Var(X) = a^2\sigma^2$

SUMMARY

- Experiments and the Sample Space
  - Experiments, events, mutually exclusive events, simple events
  - The sample space
  - Venn diagrams, tree diagrams, probability tables

SUMMARY

- Probabilities
  - Relative frequency definition of probability
  - Properties of probabilities
    - Each probability lies between 0 and 1
    - Sum of all simple-event probabilities equals 1
  - P(A), the sum of the probabilities for all simple events in A

SUMMARY

- Counting Rules
  - Mn Rule;
  - Extended mn Rule
  - Permutations:

$$P_r^n = \frac{n!}{(n-r)!}$$

  - Combinations:

$$C_r^n = \frac{n!}{r!(n-r)!}$$

SUMMARY

- Event Relations
  - Unions and Intersections
  - Events
    - Disjoint or mutually Exclusive: $P(A \cap B) = 0$
    - Complementary: $P(A^c) = 1 - P(A)$
  - Conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) \neq 0$$

  - Independent and dependent events:

$$P(A \cap B) = P(A)P(B) \qquad P(A \cap B) = P(A)P(B \mid A)$$

SUMMARY

- Event relation cont'd
  - Additive Rule of Probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  - Multiplicative Rule of Probability:

$$P(A \cap B) = P(A)P(B \mid A)$$

  - Law of Total Probability

$$P(A) = P(S_1)P(A \mid S_1) + P(S_2)P(A \mid S_2) + \cdots + P(S_k)P(A \mid S_k)$$

  - Bayes' Rule

$$P(S_i \mid A) = \frac{P(S_i)P(A \mid S_i)}{\sum_{j=1}^{k} P(S_j)P(A \mid S_j)}$$

SUMMARY

- Discrete Random Variables and Probability Distributions
  - Random variables, discrete and continuous
  - Properties of probability distributions

$$0 \le p(x) \le 1, \ \sum p(x) = 1$$

  - Mean or expected value of a discrete random variable

$$\mu = \sum xp(x)$$

  - Variance and standard deviation of a discrete random variable

$$\sigma^2 = \sum \left(x - \mu\right)^2 p(x) \qquad \sigma = \sqrt{\sigma^2}$$