Differential Topology
Coursework 2

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All theorem numbering will be according to my notes, which you can find here.

4. Show that there is an orientation-reversing diffeomorphism $\mathbb{CP}^n \to \mathbb{CP}^n$ if and only if n is odd.

Let $\mathcal{M} = \mathbb{CP}^n$. $\mathcal{M}$ compact gives that $H_c^{2n}(\mathcal{M}) \cong H^{2n}(\mathcal{M}) \cong \mathbb{R}$, so the pullback f^* will be giving multiplication by the same number $(\deg f)$ on both groups. We know from lectures that $H^2(\mathcal{M}) \cong \mathbb{R}$, so up to scaling there is only one equivalence class $[\omega]$. Let $\omega^n := \omega \wedge \ldots \wedge \omega$, n times. We know from lectures that $\omega^n \neq 0$. Since it is a top degree form, it must be closed. If ω^n was exact, $\exists \eta$ s.t. $d\eta = \omega^n$. Therefore $\int_{\mathcal{M}} d\eta = \int_{\partial \mathbb{CP}^n} \eta = \int_{\varnothing} \eta = 0$, using Stokes' Theorem and the fact that $\mathbb{CP}^n$ has no boundary. However, we also know from lectures (proof of Prop 18.3) that $\int_{\mathcal{M}} \omega^n \neq 0$, so we have a contradiction and so ω^n is not exact. Therefore $[\omega^n] \neq 0$, meaning that up to scaling it is the only equivalence class in $H^{2n}(\mathcal{M})$. This essentially says that the top cohomological group is generated by the nth power of the generator of the second[1]. This does not cause us any troubles with non-surjectivity because ω is a complex form.
Since both $H^2(\mathcal{M})$ and $H^{2n}(\mathcal{M})$ are isomorphic to $\mathbb{R}$, f^* will be a multiplication by some real number on both. If $f^*\omega = c\omega$ with $c \in \mathbb{R}\backslash\{0\}$, then $c^n\omega^n = (c\omega)^n = (f^*\omega)^n = f^*\omega \wedge \ldots \wedge f^*\omega = f^*(\omega^n)$, giving $c^n[\omega^n] = \deg f \cdot [\omega^n]$. From Prop 17.2.2, if f is orientation reversing then $\deg f = -1$, so $c^n = -1$. If $n = 2k$, then we arrive at a contradiction because c is a real number, and so such a diffeomorphism cannot exist. If n odd then this proof does not work.
Instead define $f : \mathbb{CP}^n \to \mathbb{CP}^n : [z_0 : \ldots : z_n] \mapsto [\bar{z_0} : \ldots : \bar{z_n}]$. For $n = 1$, this would just be the antipodal map but in general it sends dz_i to $d\bar{z_i}$ and vice versa. It is clearly a homeomorphism. If $\omega = dz_0 \wedge d\bar{z_0} \wedge dz_1 \wedge d\bar{z_1} \wedge \ldots \wedge dz_n \wedge d\bar{z_n}$, then $f^*\omega = d\bar{z_0} \wedge dz_0 \wedge \ldots \wedge d\bar{z_n} \wedge dz_n = (-dz_0 \wedge d\bar{z_0}) \wedge (-dz_n \wedge d\bar{z_n}) = (-1)^n dz_0 \wedge d\bar{z_0} \wedge \ldots \wedge dz_n \wedge d\bar{z_n} = (-1)^n\omega = -\omega$ for n odd. So Df is orientation reversing for odd n as it has a negative determinant.

5. Let $\mathcal{M}$ be an oriented manifold of dimension n and let $f : S^n \to \mathcal{M}$ be a smooth morphism. Assume that $H^p(\mathcal{M}) \neq 0$ for some $1 \leq p \leq n-1$. Show that the degree of f is zero.

Assume $\mathcal{M}$ is non-compact. Since f is continuous and S^n is compact then the image will be compact in $\mathcal{M}$, and so f cannot be surjective as it cannot cover the whole space. From an earlier question on the problem sheet, non-surjective functions have zero degree.
Now assume $\mathcal{M}$ is compact. We can use the isomorphisms between compactly supported and de Rham cohomology for compact manifolds, along with Poincaré duality to say that $H^p(\mathcal{M}) \neq 0 \implies H_c^{n-p}(\mathcal{M}) \neq 0$. So there exists representatives $\omega \in H_c^{n-p}(\mathcal{M})$ and $\eta \in H^p(\mathcal{M})$, giving $\omega \wedge \eta \in H^n(\mathcal{M}) = H_c^n(\mathcal{M})$. Now by

[1] basis might be a better word but it makes more intuitive sense to say generator for me

the definition of degree and Stokes' Theorem:

$$\deg f \int_{\mathcal{M}} \omega \wedge \eta = \int_{S^n} f^*(\omega \wedge \eta) = \int_{D_{n+1}} df^*(\omega \wedge \eta) = \int_{D_{n+1}} f^* \left(d\omega \wedge \eta + (-1)^{n-p} \omega \wedge d\eta\right)$$

and the fact that ω and η must be closed gives,

$$\deg f \int_{\mathcal{M}} \omega \wedge \eta = 0$$

This does not yet give the degree as 0, as the integral of $\omega \wedge \eta$ could be 0. However, since $\mathcal{M}$ is orientable there must exist a volume form on $\mathcal{M}$. This volume form cannot be exact as the integral of exact forms are zero and all top forms are closed, so we can take this volume form as being a representative of the top cohomology class. We can then 'factor' this volume form using Poincaré duality as above (as $H^n(\mathcal{M})$ is one dimensional, we just need to rescale the $\omega \wedge \eta$ above to get any of the volume forms). Once we do this, we can conclude the integral of $\omega \wedge \eta$ above is nonzero, and hence the degree of f must be zero.

7. a) Let $f \in \mathbb{C}[x]$ be a degree d polynomial. Show that the degree of the smooth map $f : \mathbb{C} \to \mathbb{C}$ equals d. Deduce the fundamental theorem of algebra.

Define $H : [0,1] \times \mathbb{C} \to \mathbb{C} : (t,z) \mapsto z^d + t(a_{d-1}z^{d-1} + \ldots + a_1 z + a_0)$. This is clearly a continuous homotopy between f and $g(z) = z^d$. Moreover, we can note that because $H_t(z)$ is a polynomial for any t, it fixes the point at infinity. Therefore any bounded set has a bounded set as its preimage (otherwise infinity would be mapped to by a bounded set), and any closed set has a closed set as its preimage (polynomials are continuous). By identifying $\mathbb{C}$ with $\mathbb{R}^2$ and using Heine-Borel then, any compact set must have a compact preimage under H_t. Since $t \in [0,1]$ and $H_t(z)$ is continuous in t, we can extend this properness to all of $[0,1] \times \mathbb{C}$ Therefore H is a proper homotopy, and so $\deg f = \deg g$. 1 is a regular value of g, and we know that it has d complex roots of unity $\{e^{\frac{2\pi i k}{d}}\}_{k=0}^{d}$ i.e. d preimages. By the inverse function theorem, there is an open neighbourhood U_k around each of these complex roots of unity (small enough that none intersect) and a neighborhood V around 1 such that g restricted to each individual neighbourhood is a diffeomorphism. Furthermore, since g is a holomorpsim, it is orientation preserving and so signdet $Df_{e^{\frac{2k\pi i}{d}}} = +1$. Let $\omega \in H_c^2(\mathbb{C})$ be s.t. $\int_V \omega = 1$ and supp $\omega \subset V$. Therefore $\int_{\mathbb{C}} g^*\omega = \sum_{k=1}^{d} \int_{U_k} g|_{U_i}^* \omega = \sum_{k=1}^{d} \text{sign}(\det Df_{e^{\frac{2k\pi i}{d}}}) = \sum_{k=1}^{d} 1 = d$. Therefore $\deg f = \deg g = d$ and the differential and algebraic concepts of degree align. Now say that f has m roots and 0 is a regular value i.e. $f^{-1}(0) = \{x_1, \ldots, x_m\}$. By the formula, $\int_{\mathbb{C}} f^*\omega = \sum_{k=1}^{m} \text{signdet}(Df_{x_k})$. We know the summand is 1 as f is a holomorphism and so each summand is $+1$, giving that $\deg f = m$, from which we conclude that f must have $m = d$ roots. If 0 is not a regular value, we count with multiplicity the preimage set.

7. b) Let $f, g \in \mathbb{C}[x]$ be coprime. Show that the degree of the meromorphic map $f/g : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the maximum of the degrees of the polynomials f and g.
Let $q(z) = \frac{f(z)}{g(z)}$. If $\deg f \geq \deg g$, then $q(z) = a \implies f(z) - ag(z) = 0$, which is a $\deg f$ polynomial and so by fundamental theorem of algebra must have $\deg f$ roots. Similarly if $\deg f \leq \deg g$, then $q(z)$ has $\deg g$ roots. So $q(z) = a$ has $n = \max\{\deg f, \deg g\}$ solutions, or the fibre of each point in $\mathbb{C}$ is of size n (counted without multiplicity if we assume its a regular point). We can also note that because f/g is meromorphic, it is still holomorphic on punctured neighbourhoods of its poles, so is still orientation preserving. Therefore,

using the formula again $\int_{\mathbb{CP}} \left(\frac{f}{g}\right)^* \omega = \sum_{k=1}^{n} \operatorname{sign} \det D\left(\frac{f}{g}\right)_{x_k} = \sum_{k=1}^{n} 1$, giving $\deg \frac{f}{g} = n$.
I see how shaky this is. I was considering trying to homogenize the function on $\mathbb{CP}$ by defining a polynomial on it by $p([z:w]) = [\sum_{k=1}^{\deg f} a_k z^k w^{n-k} : \sum_{k=1}^{\deg g} b_k z^k w^{n-k}]$, where $f(z) = \sum_{k=1}^{\deg f} a_k z^k$ and $g(z) = \sum_{k=1}^{\deg g} b_k z^k$. I think this would then have full rank anywhere and so we could do the above argument and then restrict to $w = 1$ to retrieve the function, but I'm not sure how to make any of this rigorous or show it has full rank.

Thank you for taking the time to read this.