Differential Topology
Coursework 1

Jack Kennedy

16.02.21

4. Show that there are no nowhere-vanishing fields on an even-dimensional sphere, S^{2m}.

Assume $\exists \xi : S^n \to TS^n : x \mapsto \xi(x) \in T_x S^n$ smooth s.t. $\xi(x) \neq 0 \, \forall x \in S^n$. We can include this into $\mathbb{R}^{n+1}$. We will create a homotopy on $\mathbb{R}^n$ so that we can use some vector calculus, and then show that this homotopy is invariant on S^n. Assume $\left\| i \circ \xi(x) \right\| = 1 \, \forall x \in S^n$, where $i : S^n \to \mathbb{R}^{n+1}$ is the usual inclusion. Define the homotopy

$$H : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : (t,x) \mapsto x \cos \pi t + i \circ \xi(x) \sin \pi t$$

We can note this is a combination of smooth maps and also $H(0,x) = x\cos(0) + \xi(x) \sin 0 = x$ and $H(1,x) = x \cos \pi + i \circ \xi(x) \sin \pi = -x$, so this is a smooth homotopy between the identity and antipodal map on $\mathbb{R}^{n+1}$. Moreover, we know that a tangent vector v to any point x will be perpendicular to x, i.e. $x.v = \sum x_i v_i = 0$. This is useful because if $y, z \in \mathbb{R}^{n+1}$ are two perpendicular vectors, then

$$\left\| y \cos \pi t + z \sin \pi t \right\|^2 = \sum y_i^2 \cos^2 \pi t + 2 y_i z_i \cos \pi t \sin \pi t + z_i^2 \sin^2 \pi t$$

$$= \cos^2 \pi t \sum y_i^2 + 2 \cos \pi t \sin \pi t \cancelto{0}{\sum y_i z_i} + \sin^2 \pi t \sum z_i^2$$

If we also say $y, z \in S^n$, i.e. $\sum y_i^2 = \sum z_i^2 = 1$, then

$$\left\| y \cos \pi t + z \sin \pi t \right\|^2 = \cos^2 \pi t + 0 + \sin^2 \pi t = 1$$

So for our homotopy (setting $y = x$, $z = i \circ \xi(x)$), we must have $\left\| H(t,x) \right\| = 1$ for all $(t,x) \in [0,1] \times S^n$. Therefore H maps from S^n to S^n, so it can be restricted to that and still be a smooth homotopy, i.e. $\mathrm{id} \sim_{H|_{[0,1]\times S^n}} -\mathrm{id}$, where $-\mathrm{id}$ is the antipodal map on S^n. Therefore the pullbacks of these maps are equal on cohomology. Fix a volume form $\omega \in \Omega^n(S^n)$. Since $\Omega^{n+1}(S^n) = 0$, this is closed and since S^n is compact, ω is compact. Therefore by Proposition 7.8[1]

$$\int_{S^n} \omega = \int_{S^n} \mathrm{id}^* \omega = \int_{S^n} (-\mathrm{id})^* \omega = (-1)^{n+1} \int_{S^n} \omega$$

with the $(-1)^{n+1}$ coming from the fact that S^n is in $\mathbb{R}^{n+1}$ ($(-1)^{n+1}$ is the determinant of the antipodal map on $\mathbb{R}^{n+1}$) and the linearity of the integral. Dividing by the non-zero and finite integral, we arrive at

$$1 = (-1)^{n+1}$$

[1] Let $f, g : \mathcal{M} \to \mathcal{N}$ be smooth maps of manifolds, and assume $f \sim g$. If $\omega \in \Omega^n_c(\mathcal{N})$ is closed, then $\int_{\mathcal{M}} f^* \omega = \int_{\mathcal{M}} g^* \omega$.

which is not a contradiction given n is not even, as required $\square$.

7. (Cohomology of punctured manifolds) Let $\mathcal{M}$ be a connected manifold of dimension $n \geq 3$. For any $x \in \mathcal{M}$ and $0 \leq p \leq n-2$, show that the map $H^p(\mathcal{M}) \to H^p(\mathcal{M}\backslash\{x\})$ is an isomorphism.

Firstly, $\mathcal{M}$ connected implies that $H^0(\mathcal{M}) = \mathbb{R}$. Let (V, ϕ) be a chart around x. Then $V\backslash\{x\} \cong S^{n-1}$ as V is homeomorphic to some (WLOG) n-dimensional ball. For $n > 1$, S^{n-1} is connected, and hence so is $V\backslash\{x\}$. Unioning $V\backslash\{x\}$ with all other charts of $\mathcal{M}$ that do not contain x in the atlas for $\mathcal{M}$ gives a cover for $\mathcal{M}\backslash\{x\}$. Since $V\backslash\{x\}$ is connected and is connected to all these other charts, we must have that the union of all of them is connected i.e. $\mathcal{M}\backslash\{x\}$ is connected. I struggled to make this rigorous without being pedantic. The argument essentially is: $V\backslash\{x\}$ locally connected and $\mathcal{M}$ globally connected $\Rightarrow$ $\mathcal{M}\backslash\{x\}$ connected. Therefore $H^0(\mathcal{M}\backslash\{x\}) \cong \mathbb{R} \cong H^0(\mathcal{M})$.
Now let V be a contractible nbd of x, and let $U = \mathcal{M}\backslash V'$ where V' is a contractible nbd of x s.t. $V' \subseteq V$ (we could consider images of concentric balls under the same chart around x to make this rigorous). We now have U, V s.t. $U \cup V = \mathcal{M}$, $U \cap V \neq \varnothing$ and furthermore $U \sim \mathcal{M}\backslash\{x\}$, $V \sim \{x\}$ and $U \cap V \sim S^{n-1}$ (consider the radius of these balls going to 0 to get the homotopies). We know from lectures that $H^p(V) \cong H^p(\{x\}) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p \geq 0 \end{cases}$ and $H^p(U \cap V) \cong H^p(S^{n-1}) = \begin{cases} \mathbb{R}, & p = 0, n-1 \\ 0, & \text{o/w} \end{cases}$. Therefore we can construct the Meyer-Vietoris sequence as follows: (filling in the first row as we know it already)

$$\begin{tikzcd}
p=0: & 0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R} \arrow[dll] & \\
p=1: & & H^1(\mathcal{M}) \arrow[r] & H^1(\mathcal{M}\backslash\{x\})\oplus 0 \arrow[r] & 0 \arrow[dll] & \\
1<p\leq n-2 & & H^p(\mathcal{M}) \arrow[r] & H^p(\mathcal{M})\oplus 0 \arrow[r] & 0 \arrow[dll] & \\
p=n-1: & & H^{n-1}(\mathcal{M}) \arrow[r] & H^{n-1}(\mathcal{M}\backslash\{x\})\oplus 0 \arrow[r] & \mathbb{R} \arrow[dll] & \\
p=n: & & H^n(\mathcal{M}) \arrow[r] & H^n(\mathcal{M}\backslash\{x\})\oplus 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Pulling out the first two rows,

$$p = 0, 1: \quad 0 \to \mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R} \to H^1(\mathcal{M}) \to H^1(\mathcal{M}\backslash\{x\}) \oplus 0 \to 0 \to ...$$

which is something to which we can apply Lemma 12.3 (alternating sequence of ranks of groups in a LES sums to 0) i.e. if $H^1(\mathcal{M}) \cong \mathbb{R}^a$, $H^1(\mathcal{M}\backslash\{x\}) \cong \mathbb{R}^b$, then $1-2+1+a-b+0 = 0 \Rightarrow a = b$ i.e. $H^1(\mathcal{M}) \cong H^1(\mathcal{M}\backslash\{x\})$. For any other row between 2 and $n-2$, we have

$$0 \xrightarrow{h} H^p(\mathcal{M}) \xrightarrow{f} H^p(\mathcal{M}\backslash\{x\}) \xrightarrow{g} 0$$

Therefore $0 \cong \operatorname{Im} h \cong \operatorname{Ker} f$ and $H^p(\mathcal{M})/\operatorname{Ker} f \cong \operatorname{Im} f \cong \operatorname{Ker} g \cong H^p(\mathcal{M}\backslash\{x\}) \Rightarrow H^p(\mathcal{M})/0 \cong H^p(\mathcal{M}) \cong H^p(\mathcal{M}\backslash\{x\})$, as required $\square$.
Before we leave, we note that we cannot generalise this result with the same method to $p = n-1$, n as

$H^p(S^{n-1}) \neq 0$ there. However, taking out those rows we see:

$$0 \to H^{n-1}(\mathcal{M}) \to H^{n-1}(\mathcal{M}\backslash\{x\}) \to \mathbb{R} \to H^n(\mathcal{M}) \to H^n(\mathcal{M}\backslash\{x\}) \to 0$$

Applying Lemma 12.3 again, we get

$$b_{n-1}(\mathcal{M}) - b_{n-1}(\mathcal{M}\backslash\{x\}) + 1 - b_n(\mathcal{M}) + b_n(\mathcal{M}\backslash\{x\}) = 0$$

which we can rewrite as

$$(-1)^{n-1}b_{n-1}(\mathcal{M}\backslash\{x\}) + (-1)^n b_n(\mathcal{M}\backslash\{x\}) = (-1)^{n-1}b_{n-1}(\mathcal{M}) + (-1)^n b_n(\mathcal{M}) + (-1)^{n-1}$$

which very neatly gives us

$$\chi(\mathcal{M}\backslash\{x\}) = \sum_{p=0}^{n}(-1)^p b_p(\mathcal{M}\backslash\{x\}) = \sum_{p=0}^{n-2}(-1)^p b_p(\mathcal{M}\backslash\{x\}) + \sum_{p=n-1}^{n}(-1)^p b_p(\mathcal{M}\backslash\{x\})$$

$$= \sum_{p=0}^{n-2}(-1)^p b_p(\mathcal{M}) + \sum_{p=n-1}^{n}(-1)^p b_p(\mathcal{M}) + (-1)^{n-1}$$

with the first bit coming from the proof in exercise 7 and the second bit from what we just showed. Therefore,

$$\chi(\mathcal{M}\backslash\{x\}) = \chi(\mathcal{M}) + (-1)^{n-1} \tag{†}$$

which will be very useful in the next question.

9. The Euler characteristic of a manifold $\mathcal{M}^n$ is the alternating sum of its Betti numbers (if finite): $\chi(M) = \sum_{i=0}^{n}(-1)^i b_i(\mathcal{M})$.

(1) Let U and V be open subsets of a manifold $\mathcal{M}$. Show that (if the terms are well-defined)

$$\chi(U) + \chi(V) = \chi(U \cup V) + \chi(U \cap V).$$

(2) Let $\mathcal{M}_1$, $\mathcal{M}_2$ be manifolds of dimension n. For $i = 1, 2$ let $U_i \subset \mathcal{M}_i$ be an open set taken sufficiently small that it is contained in a coordinate patch, and diffeomorphic to a ball in $\mathbb{R}^n$. Then $\mathcal{M}_i' := \mathcal{M}_i \backslash U_i$ is a manifold with boundary diffeomorphic to S^{n-1}. We define the *connect sum* $\mathcal{M}_1 \# \mathcal{M}_2$ to be the manifold obtained by attaching the manifolds M_i' together along their boundaries via a diffeomorphism. Show that

$$\chi(\mathcal{M}_1 \# \mathcal{M}_2) = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2) - \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$$

(1) Any chart of a manifold is a submanifold and hence a manifold in its own right. We will take $U \cup V$ to be a manifold and use Meyer-Vietoris on it, and apply Lemma 12.3 to the LES we get from it. Assume for now $U \cap V \neq \varnothing$. So we have the LES:

$$0 \to H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \cdots \to H^n(U \cup V) \to H^n(U) \oplus H^n(V) \to H^n(U \cap V) \to 0$$

Let $x_p \coloneqq \dim H^p(U \cup V) = b_p(U \cup V)$, $y_p \coloneqq \dim H^p(U) \oplus H^p(V)$, $z_p \coloneqq H^p(U \cap V) = b_p(U \cap V)$. We note that $y_p = \dim H^p(U) + \dim H^p(V) = b_p(U) + b_p(V)$. Applying Lemma 12.3, we must have:

$$x_0 - y_0 + z_0 - x_1 + y_1 - z_1 + x_2 - \cdots + (-1)^n(x_n - y_n + z_n) = \sum_{p=0}^{n}(-1)^p(x_p - y_p + z_p) = 0$$

Rearranging gives,

$$\sum_{p=0}^{n}(-1)^p x_p + \sum_{p=0}^{n}(-1)^p z_p = \sum_{p=0}^{n}(-1)^p y_p$$

$$\sum_{p=0}^{n}(-1)^p b_p(U \cup V) + \sum_{p=0}^{n}(-1)^p b_p(U \cap V) = \sum_{p=0}^{n}(-1)^p b_p(U) + \sum_{p=0}^{n}(-1)^p b_p(V)$$

i.e.

$$\chi(U \cup V) + \chi(U \cap V) = \chi(U) + \chi(V)$$

If $U \cap V = \varnothing$, then $U \cup V \cong U \sqcup V$. We have not proved this for cohomology yet, but from my algebraic topology course we have that $H_p(U \sqcup V) = H_p(U) \oplus H_p(V)$ and since Betti numbers can be defined from cohomology or homology and give the same value, we can say $b_p(U \sqcup V) = b_p(U) + b_p(V)$. Summing alternatively over this equality gives the result $\square$.
(2) This result falls out from the previous. Let $U = \mathcal{M}_1'$ and $V = \mathcal{M}_2'$, both viewed as open subsets of $\mathcal{M}_1 \# \mathcal{M}_2$. Then $U \cup V = \mathcal{M}_1 \# \mathcal{M}_2$ and $U \cap V \cong S^{n-1}$. Therefore

$$\chi(\mathcal{M}_1 \# \mathcal{M}_2) + \chi(S^{n-1}) = \chi(\mathcal{M}_1') + \chi(\mathcal{M}_2') \tag{1}$$

Now note that $\mathcal{M}_i' \cong \mathcal{M}_i \backslash \{x_i\}$, since we can retract the set U_i to a point x_i within because it is diffeomorphic to a ball. Therefore $\chi(\mathcal{M}_i') = \chi(\mathcal{M}_i) + (-1)^{n-1}$ for $i = 1, 2$, using (†). $\chi(S^{n-1}) = 1 + (-1)^{n-1}$ because $H^p(S^{n-1}) = \begin{cases} \mathbb{R}, & p = 0, n-1 \\ 0, & \text{o/w} \end{cases}$. Plugging all of this into (1) gives,

$$\chi(\mathcal{M}_1 \# \mathcal{M}_2) + 1 + (-1)^{n-1} = \chi(\mathcal{M}_1) + (-1)^{n-1} + \chi(\mathcal{M}_2) + (-1)^{n-1}$$

Simplifying to

$$\chi(\mathcal{M}_1 \# \mathcal{M}_2) = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2) - 1 + (-1)^{n-1} = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2) - \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$$

as required $\square$.

This was a fun coursework, thank you. It did feel much more algebraic than differential, but perhaps this is just what the course is like.