Example 2.64. *Let*

$$A\colon S^n \to S^n \qquad x \mapsto -x$$

be the antipodal map. We want to show that

$$\deg A = (-1)^{n+1}.$$

Let $i\colon S^n \to \mathbb{R}^{n+1}$ *be the inclusion and let*

$$\tilde{\omega} := x_1 dx_2 \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1}).$$

Let $\omega := i^*\tilde{\omega} \in \Omega^n(S^n)$. *By Stokes' Theorem (cf. Theorem 1.32), we have*

$$\int_{S^n} \omega = \int_{S^n} i^*\tilde{\omega} = \int_{D^{n+1}} d\tilde{\omega} = \int_{D^{n+1}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \neq 0$$

Let $\tilde{A}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be the extension of* A *to* $\mathbb{R}^{n+1}$. *Then* $\tilde{A} \circ i = i \circ A$ *and* $\tilde{A}^*\tilde{\omega} = (-1)^{n+1}\tilde{\omega}$. *Thus,*

$$A^*\omega = A^*i^*\tilde{\omega} = (i \circ A)^*\tilde{\omega} = (\tilde{A} \circ i)^*\tilde{\omega} = i^*\tilde{A}^*\tilde{\omega} = (-1)^{n+1} i^*\tilde{\omega} = (-1)^{n+1}\omega.$$

Thus,

$$\deg A \int_{S^n} \omega = \int_{S^n} A^*\omega = (-1)^{n+1} \int_{S^n} \omega,$$

and the claim follows.

3. Morse Theory

3.1. Introduction.

Definition 3.1. *Let* M *be a manifold of dimension* n *and let* $f\colon M \to \mathbb{R}$ *be a smooth function. A point* $x \in M$ *is called a* **critical point** *of* f *if* $Df_x = 0$*, or equivalently, given local coordinates* $x_1, \ldots, x_n$ *in a neighbourhood* U *of* x*, then*

$$\frac{\partial}{\partial x_i} f(x) = 0 \qquad \text{for } i = 1, \ldots, n$$

A **critical value** *is the image of a critical point.*

A critical point $x \in M$ *is called* **non-degenerate** *if the Hessian matrix*

$$H_f := \left(\frac{\partial^2}{\partial x_i \partial x_j} f \right)_{i,j=1,\ldots,n}$$

is invertible at x.

A **Morse function** *on* M *is a function* $f\colon M \to \mathbb{R}$ *such that all the critical points of* f *are non-degenerate.*

For any $h \in \mathbb{R}$*, we denote by* S^h *the* **sublevel** *set*

$$S^t := \{x \in M \mid f(x) \leq h\}.$$

Let $D_n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ denote the unit disc, so that $\partial D_n = S^{n-1}$ and $\operatorname{int}(D_n) = D_n \setminus \partial D_n$ is the open disc.

Definition 3.2. *An n-***cell** *(or cell of dimension n) is a topological space homeomorphic to the open disk* $\mathrm{int}(D_n)$*. A* **cell decomposition** *of a topological space M is a family $F = \{e_i\}_{i\in I}$ of pairwise disjoint subspaces of M such that*

- *each e_i is a cell;*
- *the union $\sqcup_I e_i = M$.*

If I is finite, then M is called a **finite cell decomposition***. The m-***skeleton** *of M is the subspace*

$$\mathrm{sk}_m(M) := \bigsqcup_{\dim e_i \leq m} e_i.$$

Example 3.3. *The circle S^1 can be thought as $S^1 = e_0 \sqcup e_1$ where e_0 is a point of S^1 and e_1 is a 1-cell.*

Notation 3.4. *Let M be a topological space and let $f_\partial \colon S^{n-1} \to M$ be a continuous map. We consider the quotient*

$$M \cup_{f_\partial} D_n := M \sqcup D_n / \sim$$

where $\sim$ is the relation given by

$$x \sim f_\partial(x) \qquad \text{for all } x \in \partial D_n.$$

We refer to $M \cup_{f_\partial} D_n$ as the space obtained from M by **attaching** *and n-cell and f_∂ is the attaching map.*

Example 3.5. *Let $T = S^1 \times S^1$ be the torus embedded in $\mathbb{R}^3$ and resting upright on the horizontal xy-plane. Consider the height function*

$$f \colon T \to \mathbb{R} \qquad (x, y, z) \mapsto z.$$

For each $h \in \mathbb{R}$, we consider the sublevel set

$$S^h := f^{-1}((-\infty, h]).$$

Fig 1

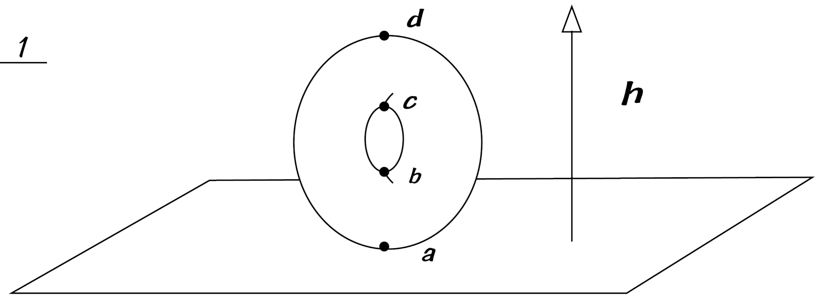
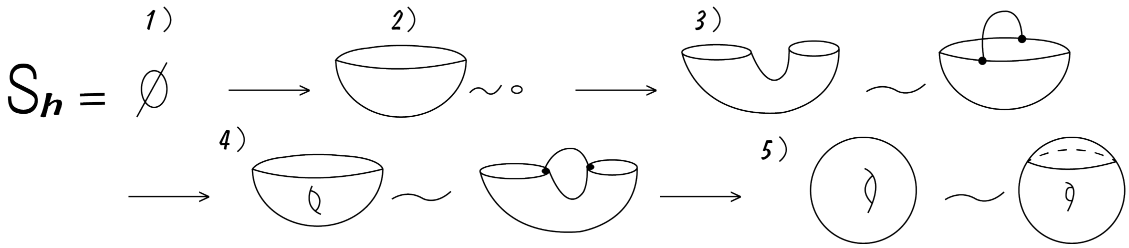


Fig 2



The critical values for f are a, b, c and d as in the picture. We have

1) *If $h < a$ then $S_1 := S^h = \emptyset$;*
2) *if $h \in (a, b)$ then $S_2 := S^h$ is homotopic equivalent to a point, i.e. a 0-cell.*
3) *if $h \in (b, c)$ then $S_3 := S^h$ is a cylinder, which is homotopic equivalent to the space obtained attaching a 1-cell to S_2.*
4) *If $h \in (c, d)$ then $S_4 := S^h$ is homotopic equivalent to the space obtained attaching a 1-cell to S_3.*
5) *If $h > d$ then $S_5 := S^h = T$ and it can be obtained by attaching a 2-cell to S_4.*

Definition 3.6. *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. For any non-degenerate critical point $x \in M$, let $\mathrm{Eig}^- H_f(x)$ be the space spanned by eigenvectors with negative eigenvalues for the Hessian of f at x. The* **index** *of f at x is the dimension of $\mathrm{Eig}^- H_f(x)$.*

We omit the proof of the following:

Lemma 3.7 (Morse Lemma)**.** *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. Let $x_0 \in X$ be a non-degenerate critical point of index λ.*

Then there exist coordinates $x_1, \dots, x_n$ locally around x such that $x_0 = (0, \dots, 0)$ and

$$f(x) = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2.$$

Note that, in particular, Morse Lemma implies that non-degenerate critical points of a smooth function $f\colon M \to \mathbb{R}$ are isolated.

Example 3.8. *Let $f\colon T \to \mathbb{R}$ be as in the previous example. Then the critical points a, b, c and d have index $0, 1, 1$ and 2 respectively.*

3.2. **CW complexes.**

Definition 3.9. *A topological space M is said to have a* **CW-structure** *if there are subspaces*

$$M^{(0)} \subseteq M^{(1)} \subseteq \cdots \subseteq M = \bigcup_{n \in \mathbb{Z}^+} M^{(n)}$$

such that

(1) *$M^{(0)}$ is a discrete set of points;*
(2) *$M^{(n+1)}$ is obtained from $M^{(n)}$ by attaching $(n+1)$-cells for all $n \geq 0$;*
(3) *a subset $V \subset M$ is closed if and only if $V \cap M^{(n)}$ is closed for all $n \geq 0$.*

Such a topological space is called a **CW** *complex and the subspace $M^{(n)}$ is the n-***skeleton** *of M.*

A **finite complex** *is a CW complex with only finitely many cells.*

Note that, for a finite complex, condition (3) is redundant.

An attaching map $f_\partial\colon S^{n-1} \to M^{(n-1)}$ extends to a map $f\colon D_n \to M^{(n)}$ called the **characteristic map**. The image of D_n under f is called a **closed cell** and the image of $\mathrm{int}(D_n)$ under f is called an **open cell**. Note that the open cell is open in $M^{(n)}$ but not necessarily in M.

A **subcomplex** S of M is a closed subspace of M which is a union of cells. It is clear that S is then itself a CW-complex.

Example 3.10.

(1) *We can think of $\mathbb{R}^n$ as a CW complex obtained as the union of n-cubes whose vertices have integer coefficients, so that the 0-cells are the the integer points, the 1-cells are the edges, etc...*
(2) *The sphere S^n is a CW-complex consisting of a point, as a 0-cell and a n-cell.*
(3) *If $n \neq 4$, any manifold of dimension n is a CW complex. It is still unknown if manifolds of dimension 4 all admit a CW complex structure.*

Proposition 3.11. *Let M be a CW complex. Then*

(1) *if $K \subset M$ is a compact subset, then K is contained in a finite union of open cells;*
(2) *the closure of every cell of M is contained in a finite subcomplex of M.*

Proof. We first prove (1). Let $K \subset M$ be a compact subset. We want to show that K only intersects finitely many cells of M. Assume by contradiction that there is an infinite sequence of points $S = \{x_j\} \subset K$ all lying in distinct cells. We claim that $S \cap M^{(n)}$ is closed and discrete for all $n \geq 0$. We proceed by induction on n. For $n = 0$, this follows from the fact that $M^{(0)}$ is closed and discrete. Assume now that $S \cap M^{(n)}$ is closed and discrete. Then, if $\{e_i\}_I$ are the $(n+1)$-cells, then the open cell corresponding to e_i contains at most one $x_j \in S$. Thus $S \cap (\cup_i e_i)$ is closed and discrete. It follows that $S \cap M^{(n+1)}$ is closed and discrete, as claimed. Since $S \subset K$, it follows that S is finite, a contradiction.

We now prove (2). To this end, we proceed by induction on the dimension n of the cell. For $n = 0$, the result is clear. Assume now that the result is true for any m-cell with $m < n$ and let e_n be an n-cell. In particular, the border K of e_n is the image of S^{n-1} and it is compact. Hence, it is contained in a finite union of open cells of dimension smaller than n by (1). By induction, each of these cells is contained in a finite subcomplex. The union of these subcomplexes is a finite subcomplex containing K. Hence attaching e_n results in a finite subcomplex containing e_n. □

Corollary 3.12. *Let M be a CW complex. Then any compact subset of M is contained in a finite subcomplex.*

Proof. Since a finite union of finite subcomplex is again a finite subcomplex, the result follows immediately from the previous Proposition. □

3.3. **CW-structure associated to a Morse function.** We begin by recalling some basic notions from differential geometry, which we will need later.

Definition 3.13. *Let M be a manifold. A* **flow** *on M is a smooth one-parameter group of diffeomorphisms $\phi_t\colon M \to M$, i.e. a smooth map*

$$\phi\colon \mathbb{R} \times M \to M$$

such that, if $\phi_t(x) := \phi(t, x)$, then

(1) $\phi_0 = \mathrm{Id}_M$,
(2) *for each $t \in \mathbb{R}$, the function ϕ_t is a diffeomorphism, and*
(3) *for each $t, s \in \mathbb{R}$, we have $\phi_{t+s} = \phi_t \circ \phi_s$.*

In particular, if we fix $x \in M$, the map $\gamma_x := \phi(\cdot, x)\colon \mathbb{R} \to M$ is called a **flow line** *(or* **integral curve***).*

For any $x \in M$, the flow line $\phi(\cdot, x)$ passes through x and the tangent vector

$$\frac{d}{dt}\gamma_x(0) \in T_x M$$

is called **velocity vector***.*

Note that a flow induces a vector field on the manifold, i.e. a section of the tangent bundle. Conversely, we have:

Lemma 3.14. *Let M be a manifold and let X a smooth compactly supported vector field on M.*

Then X generates a unique one-parameter group of diffeomorphisms $\phi_t\colon M \to M$ such that, for all $x \in M$, we have

$$X \circ \gamma_x(t) = \frac{d}{dt}\gamma_x(t).$$

It can be shown that two distinct flow lines are disjoint. Thus, the manifold M decomposes into a disjoint union of flow lines.

Definition 3.15. *A* **Riemannian metric** *g on a manifold M is a family of positive definite inner products $g_x\colon T_xM \times T_xM \to \mathbb{R}$, with $x \in M$, such that for any vector field X and Y, we have that the function*

$$M \to \mathbb{R} \qquad x \mapsto g_x(X_x, Y_x)$$

is smooth.

A manifold M with a Riemannian metric g is called a **Riemannian manifold** and it is denoted (M, g). It can be shown that any manifold admits a Riemannian metric. Since a positive definite inner product on a vector space V defines an isomorphism of V with its dual V^*, a Riemannian metric defines an isomorphism between the tangent and the cotangent bundle of M. In particular, we can define:

Definition 3.16. *Let (M, g) be a Riemannian manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. The* **gradient vector field** *of f with respect to g is the unique smooth vector field ∇f such that for any vector field X on M, we have*

$$g(\nabla f, X) = Df(X) = X(f),$$

(recall that $X(f)$ denotes the directional derivates of f along the vector field X).

In particular, we have

$$\|\nabla f\|^2 = g(\nabla f, \nabla f) = Df(\nabla f)$$

Note that ∇f vanishes exactly at the critical points of f. Moreover, we can check that is always orthogonal to the level sets $f^{-1}(c)$ for all $c \in \mathbb{R}$.

Lemma 3.14 implies that there exists a local flow $\phi_t\colon M \to M$ generated by $-\nabla f$, i.e. such that, for all $x \in M$, if as above we denote $\gamma_x(t) = \phi_t(x)$, then

$$\frac{d}{dt}\gamma_x(0) = (-\nabla f) \circ \gamma_x(0) \qquad \gamma_x(0) = x.$$

The flow ϕ_t is called the **gradient flow** of f and the curves $\gamma_t(x)$ are called the **gradient flow lines** (or just **gradient lines**).

Proposition 3.17. *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be a smooth function.*
Then f decreases along its gradient lines.

Proof. Let $\gamma_x(t)$ be the gradient line at x. Then

$$\begin{aligned}
\frac{d}{dt}f(\gamma_x(t)) &= Df_{\gamma_x(t)}(\frac{d}{dt}\gamma_x(t)) \\
&= Df_{\gamma_x(t)}((-\nabla f)(\gamma_x(t))) \\
&= -\|\nabla f(\gamma_x(t))\|^2 \leq 0.
\end{aligned}$$

Thus, the claim follows. □

Proposition 3.18. *Let M be a compact manifold and let $f\colon M \to \mathbb{R}$ be a Morse function.*

Then, for every $x \in M$, the gradient flow line $\gamma_x(t)$ begins and ends at a critical point, i.e. the limits

$$\lim_{t\to-\infty}\gamma_x(t) \qquad \text{and} \qquad \lim_{t\to\infty}\gamma_x(t)$$

both exist and are critical points.

Proof. We first prove that if these limits exist then they are critical points. Since M is compact, Lemma 3.14 implies that the gradient flow line $\gamma_x(t)$ is defined for all $t \in \mathbb{R}$. Moreover, for every $x \in M$, the image of $f(\gamma_x(t))$ is a bounded set in $\mathbb{R}$. Thus, as in the proof of Proposition 3.17, since $f(\gamma_x(t))$ is decreasing, we have

$$-\lim_{t\to\pm\infty}\|\nabla f(\gamma_x(t)\|^2 = \lim_{t\to\pm\infty}\frac{d}{dt}f(\gamma_x(t)) = 0, \tag{4}$$

and the claim follows.

Thus, we just need to check that the limits exist. Let U be the union of small disjoint open balls around the critical points (recall that, by Lemma 3.7 the critical points of a Morse function are discrete). Since M is compact, it follows that $M \setminus U$ is also compact. Therefore, for all $x \in M$, since there are no critical points inside $M \setminus U$, the function $\|\nabla f(\gamma_x(t))\|^2$ is bounded from below by a positive constant. By (4), it follows that for all t sufficiently large, $\gamma_x(t) \in U$. Since the balls are disjoints and $\gamma_x(t)$ is continuous, there exists a critical point $x_0 \in M$ such that for any open ball around x_0, $\gamma_x(t)$ is in that ball for sufficiently large t and therefore $\lim_{t\to\infty}\gamma_x(t) = x_0$. Similarly $\lim_{t\to-\infty}\gamma_x(t)$ exists and it is a critical point. ∎

Definition 3.19. *Let X be a topological space and let $S \subset X$ be a subspace. A* **deformation retraction of X onto S** *is a continuous map $F\colon X \times [0,1] \to X$ such that for every $x \in X$ and $s \in S$, we have*

$$F(x,0) = x \qquad F(x,1) \in S \qquad \text{and } F(s,1) = s$$

In particular, S is homotopy equivalent to M.

Theorem 3.20 (First Fundamental Theorem of Morse Theory)**.** *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Let $a < b$ and suppose that $f^{-1}([a,b])$ is compact and contains no critical points. Let*

$$S_h := f^{-1}((-\infty,h)) = \{x \in M \mid f(x) \le h\}.$$

Then S_a is diffeomorphic to S_b and S_b is a deformation retract of S_b.

The idea is to let the level set $f^{-1}(b)$ flow down to the level set $f^{-1}(a)$ along the gradient flow lines orthogonal to the level sets of f.

Proof. By Lemma 3.7, the critical points of f are isolated. Thus, since $f^{-1}([a,b])$ does not contain any critical points, it follows that for every $\epsilon > 0$ small enough, $f^{-1}([a-\epsilon, b+\epsilon])$ does not contain any critical points either. Define a smooth function ρ such that

$$\rho = \begin{cases} \frac{1}{\|\nabla f\|^2}, & \text{on } f^{-1}[a,b] \\ 0, & \text{on } M \setminus f^{-1}([a-\varepsilon, b+\varepsilon]) \end{cases}.$$

Then the support of ρ is compact and contained in $f^{-1}([a-\epsilon, b+\epsilon])$. Fix a Riemannian metric g on M and define the vector field

$$X(x) = -\rho(x)\nabla f(x)$$

for all $x \in M$. Thus, by Lemma 3.14, it generates a flow ϕ_t on $f^{-1}([a-\epsilon, b+\epsilon])$. Let $\gamma_x(t) := \phi_t(x)$. Then

$$\begin{aligned} \frac{d}{dt} f(\gamma_x(t)) &= Df_{\gamma_x(t)}(\frac{d}{dt}\gamma_x(t)) \\ &= g(\nabla f(\gamma_x(t)), \frac{d}{dt}\gamma_x(t)) \\ &= g(\nabla f(\gamma_x(t)), -\rho\nabla f(\gamma_x(t)) \\ &= -\rho(\gamma_x(t))\|\nabla f(\gamma_x(t))\|^2. \end{aligned}$$

Thus, it follows from the definition of ρ that

$$\frac{d}{dt} f(\gamma_x(t)) = -1$$

for all t such that $\gamma_x(t) \in f^{-1}([a,b])$. Moreover, since $\rho \geq 0$, we have that

$$\frac{d}{dt} f(\gamma_x(t)) \leq 0$$

for all $t \in \mathbb{R}$. Thus $f(\gamma_x(t))$ is decreasing.

By the fundamental theorem of calculus, we have that if $f(\gamma_x(s)) \in [a,b]$ for all $s \in [0,t]$ then

$$f(\gamma_x(t)) - f(\gamma_x(0)) = \int_0^t \frac{d}{ds} f(\gamma_x(s)) ds = -t.$$

Since $\gamma_x(0) = x$, we have

$$f(\gamma_x(t)) = f(x) - t.$$

Thus, by taking $t = b - a$, we obtain:

(1) If $f(x) \leq b$ then $f(\phi_{b-a}(x)) \leq a$.
(2) if $f(x) > b$ then $f(\phi_{b-a}(x)) > a$.

(1) implies that ϕ_{b-a} maps S^b to S^a and, similarly, (2) implies that ϕ_{a-b} maps S^a to S^b. Thus, it follows that ϕ_{b-a} is a diffeomorphism between S^b and S^a. For any $t \in [0,1]$ define

$F_t\colon S^b \to S^a$ such that

$$F_t(x) = \begin{cases} x & \text{if } f(x) \le a \\ \phi_{t(f(x)-a)}(x) & \text{if } a \le f(x) \le b. \end{cases}$$

Then, it is easy to check that F is a deformation retract and the theorem follows. □

Corollary 3.21 (Reeb's Theorem)**.** *Let M be a compact manifold of dimension n without boundary and let $f\colon M \to \mathbb{R}$ be a Morse function admitting only two critical points.*

Then M is homeomorphic to a sphere S^n.

Proof. Since M is compact, f admits a maximum at $x_{\max}$ and a minimum at $x_{\min}$ and these are the two critical points. Let $h_{\max} := f(x_{\max})$ and $h_{\min} := f(x_{\min})$. Then $\lambda(x_{\max}) = n$ and $\lambda(x_{\min}) = 0$. Morse Lemma (cf Lemma 3.7) implies that there exist local coordinates $x_1, \dots, x_n$ in an open neighbourhood $U_{\min}$ of $x_{\min}$ such that

$$f = h_{\min} + \sum_{i=1}^{n} x_i^2.$$

For some $a > h_{\min}$ sufficiently close to h_{min}, we have

$$f^{-1}\left([h_{\min}, a]\right) = \left\{(x_1, \dots, x_n) \,|x_1^2 + \cdots + x_n^2 \le a - h_{\min}\right\}.$$

Hence $f^{-1}([h_{\min}, a]) = f^{-1}((-\infty, a]) = S^a$ is a closed n-cell D_n^-. Similarly, for $b < h_{\max}$, we have that $f^{-1}([b, h_{\max}])$ is a closed n-cell D_n^+. By Theorem 3.20, S^a is diffeomorphic to S^b, since there are no critical points between a and b. Hence $M = f^{-1}([h_{\min}, h_{\max}])$ is the union of two n-cells attached at their common boundary $S^{n-1} = \partial D_n^+ = \partial D_n^-$.

Now we need to construct an homeomorphism between M and S^n. The (closed) northern hemisphere H^+ is diffeomorphic to D_n^+ and the (closed) southern hemisphere H^- to D_n^-. The only problem is that the two n-cells are not necessarily glued by the identity map, but by a homeomorphism $f\colon \partial H^- \to \partial H^+$. Let $\phi_\pm\colon H^\pm \to D_n^\pm$ be the two homeomorphisms and $E = H^+ \cap H^-$ the equator so that $\phi_-|_E = \phi_+|_E \circ f$. We claim that $f\colon E \to E$ extends to a homeomorphism $F\colon H^+ \to H^+$ such that $F|_E = f$. Assuming the claim, we can define a homeomorphism $\tilde{\phi}\colon S^n \to M$ as follows: we have

$$\tilde{\phi}|_{H^+} = \phi_+ \circ F \qquad \text{and} \qquad \tilde{\phi}_{H^-} = \phi_-.$$

This is well-defined since

$$(\phi_+ \circ F)|_E = \phi_+|_E \circ f = \phi_-|_E.$$

It is continuous and bijective since it is restricted to each of $D_n^\pm$. Thus, it is a homeomorphism.

It remains to prove the claim. Since H^- is homeomorphic to the unit n-ball D_n it is enough to show that every homeomorphism $g\colon \partial D_n \to \partial D_n$ extends to a homeomorphism $G\colon D_n \to D_n$. This is done via the *Alexander trick*, using concentric spheres: for $v \in \partial D$, the unit $(n-1)$-sphere, considered as a vector in $\mathbb{R}^n$, we define $G(tv) = tg(v)$ for all

$0 \leq t \leq 1$. This is continuous and the same argument shows that g^{-1} extends to G^{-1}. Thus, it is a homeomorphism, as claimed. $\square$

Theorem 3.22 (Second Fundamental Theorem of Morse Theory)**.** *Let M be a manifold of dimension n and let $f\colon M \to \mathbb{R}$ be a smooth function with a non-degenerate critical point $x_0 \in M$ of index λ. Let $c = f(x_0)$ and assume that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and does not contain any critical point of f other than x_0, for some $\epsilon > 0$.*

Then, if ϵ is sufficiently small, $S^{c+\epsilon}$ is homotopy equivalent to $S^{c-\epsilon}$ with a λ-cell attached.

Proof. The difficult part of the Theorem is to prove it for critical points of index $\lambda \in \{1, \dots, n-1\}$. Otherwise x_0 is either a minimum or a maximum of the function f. If it is a minimum, then the Theorem follows from the fact that a n-disc is homotopy equivalent to a point. If it is a maximum, then we can use the same method as in the proof of Reeb's Theorem (cf. Corollary 3.21).

Thus, we will assume that $\lambda \in \{1, \dots, n-1\}$. By Morse Lemma (cf. Lemma 3.7) there exist a neighbourhood U of x_0 and coordinates $x_1, \dots, x_n$ on U such that $x_0 = (0, \dots, 0)$ and

$$f(x) = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2. \tag{5}$$

Let $\epsilon > 0$ small enough so that U contains the closed ball

$$B_{\sqrt{2\epsilon}} = \{(x_1, \dots, x_n) \mid \sum x_i^2 \leq 2\epsilon\}.$$

Define the λ-cell e_λ by

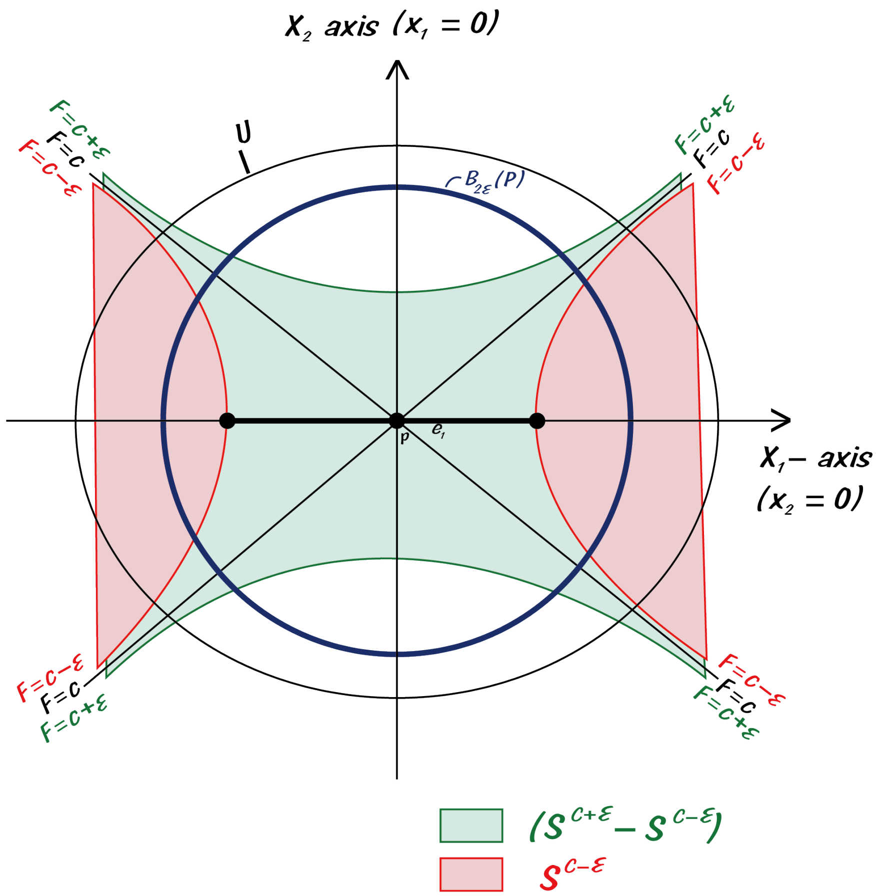
$$e_\lambda := \left\{(x_1, \dots, x_n) \,|x_1^2 + \cdots + x_\lambda^2 \leq \varepsilon, x_{\lambda+1} = \cdots = x_n = 0\right\}.$$

We want to show that $S^{c-\epsilon} \cup e_\lambda$ is a deformation retract of $S^{c+\epsilon}$.

In order to illustrate the method, we first consider the case $n = 2$ and $\lambda = 1$. We have

- The preimage of c in U
$$f^{-1}(c) \cap U = \{x \in U \mid f = c\} = \{(x_1, x_2) \mid x_1^2 = x_2^2\}$$
is the union of two lines.
- The level set
$$S^{c-\epsilon} = \{x_1, x_2) \mid x_1^2 - x_2^2 \geq \epsilon\}$$
is represented by the area in red.
- If
$$S^{c+\epsilon} = \{x_1, x_2) \mid x_1^2 - x_2^2 \geq -\epsilon\}$$
then $S^{c+\epsilon} \setminus S^{c-\epsilon}$ is represented by the area in green.

Fig 1



We define the 1-cell in U by

$$e_1 = \{(x_1, x_2) \mid x_1^2 \leq \epsilon, x_2 = 0\}.$$

Then e_1 is attached to the boundary of $S^{c-\epsilon}$ in $(\sqrt{\epsilon}, 0)$ and $(-\sqrt{\epsilon}, 0)$.

The same construction works for higher dimensional manifold and index $\lambda \in \{1, \dots, n-1\}$. Let

$$\xi = \sum_{i=1}^{\lambda} x_i^2 \qquad \text{and} \qquad \eta = \sum_{i=\lambda+1}^{n} x_i^2.$$

We have that

$$f^{-1}(c) \cap U = \{x_1, \dots, x_n \mid \xi = \eta\}$$

is a double cone with vertex in x_0. We also have

$$S^{c+\varepsilon} = \{(x_1, \dots, x_n) \,|\xi - \eta \geq -\varepsilon\}$$
$$S^{c-\varepsilon} = \{(x_1, \dots, x_n) \,|\xi - \eta \geq \varepsilon\}$$

and we define

$$e_\lambda := \{(x_1, \ldots, x_n) \mid \xi \leq \epsilon, x_{\lambda+1} = \ldots, x_n = 0\}. \tag{6}$$

As in the previous case, e_λ is attached to the boundary of $S^{c-\epsilon}$.

We now define a function g which coincides with the function f outside U, it is slightly smaller than f on U, and it has the same critical point of f. To this end, let $\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a smooth function such that

$$\mu(t) \begin{cases} > \epsilon & \text{if } t = 0 \\ = 0 & \text{if } t \geq 2\epsilon \end{cases}$$

and

$$-1 < \mu'(t) \leq 0 \qquad \text{for all } t \geq 0.$$

We define

$$g := \begin{cases} f & \text{outside } U \\ f - \mu(\xi + 2\eta) & \text{inside } U. \end{cases}$$

Note that $g \leq f$ and by (5), we have that if $x \in U$, then

$$g(x) = c - \xi + \eta - \mu(\xi + 2\eta). \tag{7}$$

Finally, since $\mu = 0$ if $r \geq 2\epsilon$, we have that $g = f$ outside the ellipsoid

$$E = \{(x_1, \ldots, x_n) \mid \xi + 2\eta \leq 2\epsilon\} \subset B_{\sqrt{2\epsilon}} \subset U$$

If $y \in E$ then

$$\eta(y) - \xi(y) \leq \epsilon - \frac{3}{2}\xi(y) \leq \epsilon.$$

Thus, $E \subset S^{c+\epsilon}$. We are going to prove the following properties:

(i) $g^{-1}((-\infty, c+\epsilon]) = f^{-1}((-\infty, c+\epsilon]) = S^{c+\epsilon}$.
(ii) f and g have the same critical points.
(iii) $g^{-1}((-\infty, c-\epsilon])$ is a deformation retract of $S^{c+\epsilon}$.
(iv) $S^{c-\epsilon} \cup e_\lambda$ is a deformation retract of $g^{-1}((-\infty, c-\epsilon])$.

We first show (i). Since $g \leq f$, it is clear that

$$f^{-1}((-\infty, c+\epsilon]) \subset g^{-1}((-\infty, c+\epsilon]).$$

Assume that $f(x) > c + \epsilon$. Then, since $E \subset S^{c+\epsilon}$, we have that $g(x) = f(x)$. Thus, $g(x) > c + \epsilon$ and (i) follows.

We now prove (ii). Outside U, we have that $f = g$ and there is nothing to prove. By (7), inside of U, we have

$$\begin{aligned} \tfrac{\partial g}{\partial \xi} &= -1 - \mu'(\xi + 2\eta) < 0 \\ \tfrac{\partial g}{\partial \eta} &= 1 - 2\mu'(\xi + 2\eta) > 1 \end{aligned}$$

where the two inequalities come from the fact that, by construction, $-1 < \mu' \leq 0$. Thus,

$$dg = \frac{\partial g}{\partial \xi} d\xi + \frac{\partial g}{\partial \eta} d\eta = 0$$

if and only if

$$d\xi = d\eta = 0$$

which is satisfied if and only if $x_1 = \cdots = x_n = 0$, i.e. $x = x_0$. Thus, (ii) follows.

We now prove (iii). By (ii), x_0 is the only critical point of g and by (7), we have

$$g(x_0) = c - \mu(0) < c - \epsilon.$$

Thus $x_0 \notin g^{-1}([c-\epsilon, c+\epsilon])$ and $g^{-1}([c-\epsilon, c+\epsilon])$ does not contain any critical point. Thus, (iii) follows by the First Fundamental Theorem of Morse Theory (cf. Theorem 3.20) and by (i).

We finally show (iv). Let H denote the closure of $g^{-1}((-\infty, c+\epsilon]) \setminus S^{c-\epsilon}$, then

$$g^{-1}((-\infty, c-\epsilon]) = S^{c-\epsilon} \cup H.$$

We first show that $e_\lambda \subset H$. Indeed, since $\frac{\partial g}{\partial \xi} < 0$ for all $x \in e_\lambda$, we have $g(x) \leq g(x_0)$. Moreover $\xi(x_0) = 0 \leq \xi(x)$, hence $g(x) < c - \epsilon$ and $x \in g^{-1}((-\infty, c-\epsilon])$. On the other hand, by (5) and (6)., we have

$$f(x) = c - \xi + \eta > c - \epsilon$$

for all x in the interior of e_λ. Thus, $x \notin S^{c-\epsilon}$ and the claim follows.

In order to construct the deformation retraction, we consider three cases.

<u>Case I:</u> Assume $\xi < \epsilon$.

Define the deformation retract by

$$r_t\colon\ (x_1, \ldots x_n) \mapsto (x_1, \ldots, x_\lambda, tx_{\lambda+1}, \ldots tx_n)\,.$$

Thus, $r_1 = \mathrm{Id}$ and r_0 maps everything to e_λ. Moreover, since $\frac{\partial g}{\partial \eta} < 0$, each r_t maps $g^{-1}((-\infty, c-\epsilon])$ into itself.

<u>Case II:</u> Assume $\epsilon \leq \xi \leq \eta + \epsilon$.

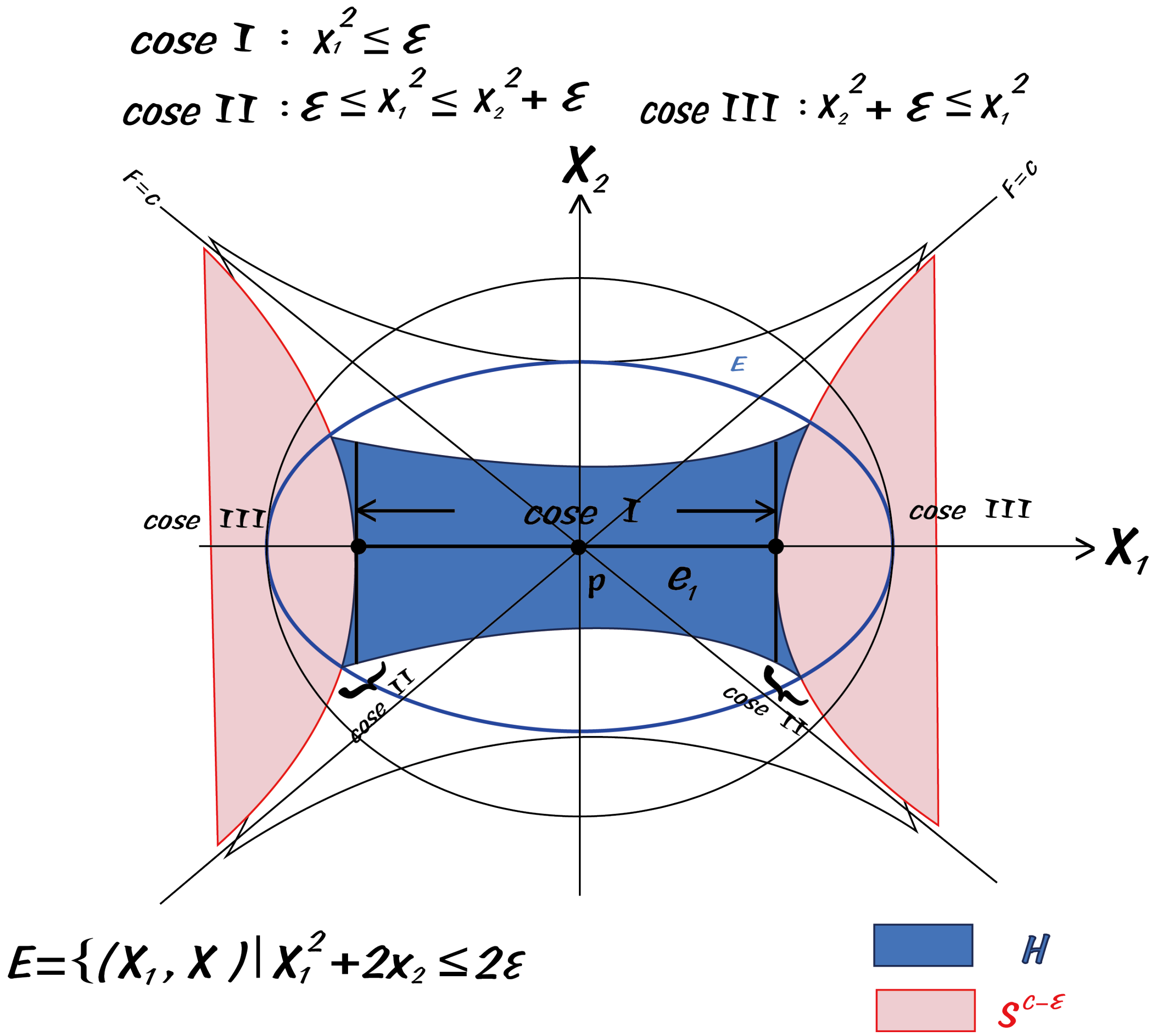
Define the deformation retract by

$$r_t\colon\ (x_1, \ldots x_n) \mapsto (x_1, \ldots, x_\lambda, l_t x_{\lambda+1}, \ldots l_t x_n)$$

where

$$l_t := t + (1-t)\sqrt{\frac{\xi - \epsilon}{\eta}}.$$

Then it follows easily that r_t is continuous. As in the previous case, we have $r_1 = \mathrm{Id}$ and r_0 maps everything to $S^{c-\epsilon}$, since $(x_1, \ldots, x_\lambda, l_0 x_{\lambda+1}, \ldots l_0 x_n)$ is the level set $f^{-1}(c-\epsilon)$. Moreover, as in Case I, r_t maps $g^{-1}((-\infty, c-\epsilon])$ to itself.



Case III: We finally assume $\eta + \epsilon \leq \xi$.
This is exactly the set $S^{c-\epsilon}$. Thus, we choose $r_t = \mathrm{Id}$.

Note that Case I, coincides with Case II when $\xi = \epsilon$ and Case II coincides with Case III when $\xi = \eta + \epsilon$. Thus, we have shown (iv).

By (iii) and (iv), it follows that $S^{c-\epsilon} \cup e_\lambda$ is a deformation retract of $S^{c+\epsilon}$. Thus, the Theorem follows. □

Remark 3.23. *We can generalise the proof of the previous Theorem to the case of k critical points $x_1, \ldots, x_k$ such that $f(x_1) = \cdots = f(x_k) = c$, with indices $\lambda_1, \ldots, \lambda_k$ respectively. In this case, the sublevel set S^{c+k} is homotopically equivalent to $S^{c-\epsilon} \cup_{f_1} D_{\lambda_1} \cup \cdots \cup_{f_k} D_k$ for some attaching maps $f_1, \ldots, f_k$.*

We can finally state the main Theorem of this Section.

Theorem 3.24. *Let M be a manifold and let $f\colon M \to \mathbb{R}$ be a Morse function such that the sublevel sets S^t are compact for all $t \in \mathbb{R}$.*

Then M is homotopy equivalent to a CW complex with one λ-cell for each critical point of index λ.

Before we proceed with the proof of the Theorem, we need to recall few facts from Topology.

Definition 3.25. *A continuos map $f\colon X \to Y$ between CW complexes is* **cellular** *if it maps skeletons to skeletons (cf. Definition 3.9), i.e. $f(X^{(n)}) \subset Y^{(n)}$ for all $n \geq 0$.*

Theorem 3.26 (Cellular Approximation)**.** [2] *Let X and Y be CW-complexes and $S \subset X$ be a subcomplex. Assume that $f\colon X \to Y$ is a continuous morphism such that $f|_S$ is cellular.*

Then f is homotopic to a cellular map $\tilde{f}\colon X \to Y$ such that $\tilde{f}|_S = f|_S$.

The idea of the proof is to prove the claim on the skeleton $M^{(n)}$ by proceeding by induction on n.

Theorem 3.27 (Whitehead)**.** [3] *Let X be a topological space and let*

$$\phi_0, \phi_1 \colon \partial D_k \to X$$

be homotopic attaching maps.

Then the identity maps of X extends to an homotopy equivalence

$$H\colon X \cup_{f_0} D_k \to X \cup_{f_1} D_k$$

In particular, the Theorem implies that the homotopy of a space $X \cup_f D_k$ does not depend on the way the cell D_k is attached if the attaching maps are homotopically equivalent. Similarly, the following theorem implies that the homotopy of a space $X \cup_f D_k$ does not depend on the homotopy class of X.

Theorem 3.28 (Hilton)**.** [4] *Let X be a topological space and let $f\colon \partial D_k \to X$ an attaching map.*

Then any homotopy equivalence $h\colon X \to Y$ extends to an homotopy equilvance

$$H\colon X \cup_f D_k \to X \cup_{h\circ f} D_k.$$

We can now proceed with the proof of Theorem 3.24.

Proof of Theorem 3.24. Let $c_0, c_1, \ldots$ be the critical values of f. Since the sublevel set S^t is compact for all $t \in \mathbb{R}$, there are no accumulation points at the critical values. Moreover, S^t is empty, if $t < c_0$.

[2] E.g. see [Hat02, Theorem 4.8]
[3] E.g. see [Hat02, Theorem 4.6]
[4] E.g. See [Mil63, Lemma 3.7]

We now prove by induction that for each i there exists $\epsilon_i > 0$ such that $S^{c_i+\epsilon_i}$ is homotopy equivalent to a CW complex with one λ-cell for each critical point x of index λ and such that $f(x) \leq c_i$. If $\epsilon_0 > 0$ is sufficiently small, then $S^{c_0+\epsilon_0}$ is just the disjoint union of contractible subsets containing all the points $x \in M$ such that $f(x) = c_0$ and such that, for each such point, the index is zero. Thus, the claim follows.

We now assume that $i > 0$. By induction, there exists ϵ_{i-1} such that there is a homotopy equivalence $S^{c_{i-1}+\epsilon_{i-1}} \to X_{i-1}$, where X_{i-1} is a CW complex. Let $x_1, \dots, x_k \in M$ be the critical points such that $f(x_1) = \dots = f(x_k) = c_i$ and with indices $\lambda_1, \dots, \lambda_k$ respectively. By the second fundamental Theorem of Morse theory (cf. Theorem 3.22 and Remark 3.23), there exists $\epsilon_i > 0$ such that $S^{c_i+\epsilon_i}$ has the homotopy type of

$$S^{c_i-\epsilon_i} \cup_{f_1} D_{\lambda_1} \cup \dots \cup_{f_k} D_k$$

for some attaching maps $f_1, \dots, f_k$. Thus, for each k, we have a morphism

$$\partial D_{\lambda_k} \xrightarrow{f_k} S^{c_i-\varepsilon} \xrightarrow{g_{i-1}} S^{c_{i-1}+\epsilon_{i-1}} \xrightarrow{h_{i-1}} X_{i-1},$$

where the existence of the second morphism follows from the first fundamental Theorem of Morse theory (cf. Theorem 3.20).

By the Cellular Approximation Theorem (cf. Theorem 3.26), the induced morphism $f_k \circ g_{i-1} \circ h_{i-1}$ is homotopic equivalent to a cellular map $\phi_k \colon \partial D_{\lambda_k} \to X_{i-1}^{(\lambda_k)}$.

By Whitehead Theorem and Hilton Theorem (cf. Theorem 3.27 and 3.28), it follows that the CW complex

$$X_i := X_{i-1} \cup_{\psi_1} D^{\lambda_1} \cup_{\psi_2} \dots \cup_{\psi_k} D^{\lambda_k}$$

is homotopy equivalent to $S^{c_i+\epsilon_i}$, i.e. to

$$S^{c_i-\epsilon_i} \cup_{f_1} D_{\lambda_1} \cup \dots \cup_{f_k} D_{\lambda_k}.$$

Thus, the Theorem follows. □

3.4. **Morse homology.** Let (M, g) be a Riemannian manifold and let $f \colon M \to \mathbb{R}$ be a Morse function.

For every $x \in M$, let $\gamma_x(t)$ be the gradient flow line with respect to f starting at x. For any critical point $x \in M$, we may define the subsets

$$W^s(x) := \{x \in M \mid \lim_{t\to\infty} \gamma_x(t) = x\}$$

$$W^u(x) := \{x \in M \mid \lim_{t\to-\infty} \gamma_x(t) = x\}$$

called the **stable** and **unstable** manifold of c respectively. In addition, the sets $W^s(x)$ (resp. $W^u(x)$) are pairwise disjoint. Thus, by Proposition 3.18, it follows that they define two partitions of M:

$$M = \bigsqcup_c W^s(x) = \bigsqcup_c W^u(x).$$

It can be shown moreover that if λ is the index of the critical point c then $W^s(x)$ and $W^u(x)$ are diffeomorphic to open discs of dimension $n-\lambda$ and λ respectively. More precisely, we have:

Theorem 3.29 (Stable Manifold Theorem)**.** [5] *Let (M,g) be a Riemannian manifold, let $f\colon M \to \mathbb{R}$ be a Morse function and $x \in M$ be a critical point of f.*

Then the tangent space of M at x can be decomposed as follows:

$$T_xM = T_xW^u(x) \oplus T_xW^s(x).$$

Moreover, $W^s(x)$ and $W^u(x)$ are smooth submanifolds diffeomorphic to open discs with

$$\dim W^s(x) = \lambda \qquad \textit{and} \qquad \dim W^u(x) = n - \lambda.$$

Example 3.30. *Let $M = S^n$ and consider the height function $f\colon M \to \mathbb{R}$. The only critical points of f are the points S and N corresponding to the minimum and maximum of f respectively. Then*

$$W^u(N) = S^n \setminus \{S\} \qquad \textit{and} \qquad W^s(N) = \{N\}.$$

Similarly,

$$W^u(S) = S \qquad \textit{and} \qquad W^s(N) = S^n \setminus \{S\}.$$

Definition 3.31. *Let M be a manifold and let K, L be submanifolds of M. Then K and L are said to be* **transverse** *if for every point $x \in K \cap L$, we have*

$$T_xK + T_xL = T_xM,$$

i.e. every vector in T_xM can be written as a sum of a vector in T_xK and a vector in T_xL.

Proposition 3.32. *If $K, L \subset M$ are transverse embedded submanifolds then $K \cap L$ is also an embedded submanifold of dimension* $\dim K + \dim L - \dim M$.

Proof. Let $k = \dim M - \dim K$ be the codimension of K and let $x \in K \cap L$. Then there exists a neighbourhood U of x in M such that $K \cap U = f^{-1}(0)$ where $f\colon U \to \mathbb{R}^k$ is a smooth function and $0 \in \mathbb{R}^k$ is a regular value for f. Similarly if $l = \dim M - \dim L$ is the codimension of L, then there exists a smooth function $g\colon U \to \mathbb{R}^l$ such that $L \cap U = g^{-1}(0)$ and $0 \in \mathbb{R}^l$ is a regular value for g.

Then $K \cap L \cap U = F^{-1}(0)$, where $F = (f,g)\colon U \to \mathbb{R}^{k+l}$. Moreover, $0 \in \mathbb{R}^{k+l}$ is a regular value for F since

$$\mathrm{Ker} DF_x = \mathrm{Ker} Df_x \cap \mathrm{Ker} Dg_x = T_xL \cap T_xL$$

has codimension $k+l$ by the transversality condition. □

Definition 3.33. *Let (M,g) be a Riemannian manifold and let $f\colon M \to \mathbb{R}$ be a Morse function. Then f is said to be* **Morse-Smale** *if for all critical points $x, y \in M$ for f, the manifolds $W^u(x)$ and $W^s(y)$ are transverse.*

Note that, the definition depends on the metric g.

[5] See [BH04, Theorem 4.2].

Proposition 3.34. *Let (M,g) be a compact Riemannian manifold of dimension n and let $f\colon M\to\mathbb{R}$ be a Morse-Smale function. Let $x,y\in M$ be critical points for f of index λ_x,λ_y respectively and such that*

$$N := W^u(y)\cap W^s(x)\neq\emptyset.$$

Then N is an embedded submanifold of dimension $\lambda_y-\lambda_x$.

Proof. By Proposition 3.32 and Theorem 3.29, we have that N is an embedded manifold of dimension

$$\dim N=\dim W^u(y)+\dim W^s(x)-n=\lambda_y+n-\lambda_x-n=\lambda_y-\lambda_x.$$

Thus, the claim follows. □

Notation 3.35. *In the same set-up as in Proposition 3.34, we denote*

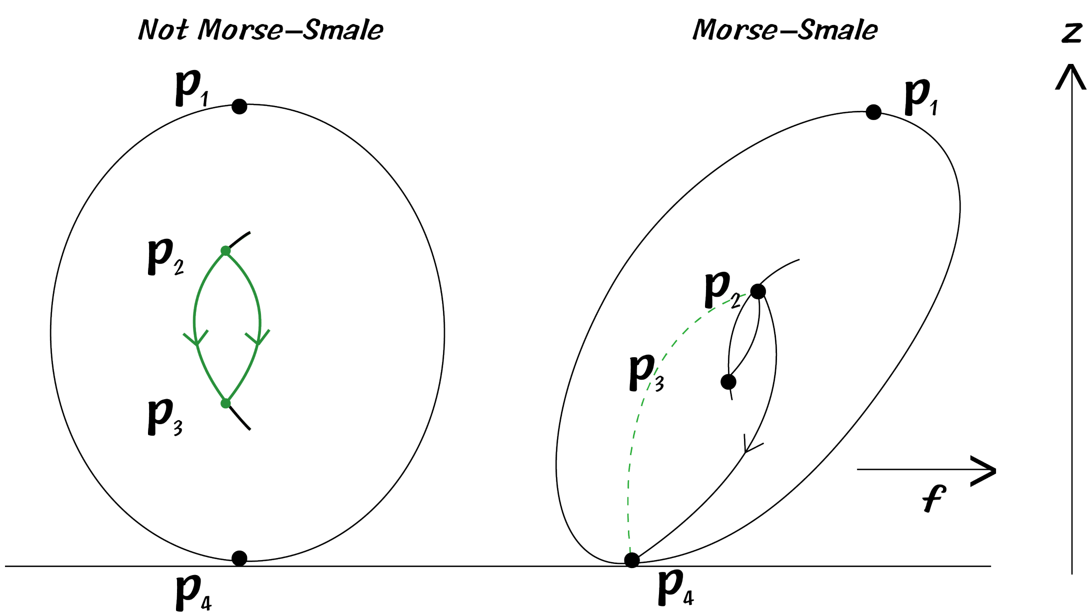
$$W(x,y):=W^u(y)\cap W^s(x).$$

Corollary 3.36. *Let (M,g) be a compact Riemannian manifold of dimension n and let $f\colon M\to\mathbb{R}$ be a Morse-Smale function. Let $x,y\in M$ be critical points for f of index λ_x,λ_y respectively and such that $W(x,y)\neq\emptyset$.*

Then $\lambda_y>\lambda_x$.

Proof. If $W(x,y)\neq\emptyset$ then it contains at least one flow line from y to x. Hence, by Proposition 3.34, we have $\lambda_y-\lambda_x=\dim W(x,y)\geq 1$ and the claim follows. □

Example 3.37. *Let M and f as in Example 3.5. Then there is a flow line connecting two critical points both with index 1. Thus, f is not a Morse-Smale function. On the other hand, this can be fixed by bending the torus a bit and leaving the height function unchanged so that we dot have flow lines beginning and ending at critical points of the same index.*



Corollary 3.38. *Let (M,g) be a compact Riemannian manifold of dimension n and let $f\colon M\to\mathbb{R}$ be a Morse-Smale function. Let $x,y\in M$ be critical points for f of index λ_x,λ_y respectively, such that $\lambda_y-\lambda_x=1$.*

Then $\overline{W(x,y)}=W(x,y)\cup\{x,y\}$ has finitely many components, i.e. the number of gradient flow lines from y to x is finite.

Proof. Since $\lambda_y-\lambda_x=1$, it follows that $W(x,y)\cup\{x,y\}$ is closed because there are no other critical points between y and x as the index is strictly decreasing along flow lines. Since M is compact, it follows that $W(x,y)\cup\{x,y\}$ is also compact.

The gradient flow lines form an open cover of $W(x,y)$ which can be extended to an open cover of $\overline{W(x,y)}$ by taking the union of each gradient flow line with small open subsets in $\overline{W(x,y)}$ around x and y. Thus, by compactness, it follows that the number of flow lines from y to x is finite. □

Let (M,g) be a compact Riemannian manifold of dimension n and let $f\colon M\to\mathbb{R}$ be a Morse-Smale function. Let $x,y\in M$ be critical points for f of index λ_x,λ_y. Then $\mathbb{R}$ acts on $W(x,y)$ by flowing along the flow lines for $t\in\mathbb{R}$. The quotient

$$M(x,y)=W(x,y)/\mathbb{R}$$

is called the **moduli space of flow lines** from y to x. By the previous Corollary, if $\lambda_y-\lambda_x=1$ then $M(x,y)$ is a finite set with each point corresponding to a flow line from y to x. We are now going to assign a sign ±1 to each of these point depending on their orientation.

Lemma 3.39. *Let (M,g) be a compact Riemannian manifold of dimension n and let $f\colon M\to\mathbb{R}$ be a Morse-Smale function. Let $x,y\in M$ be critical points for f of index λ_x,λ_y. Let $z\in W(x,y)$ and let γ be the orbit of z under the $\mathbb{R}$-action.*

Then there are canonical isomorphisms:

1. $T_zW^u(y)\simeq T_zW(x,y)\oplus T_zM/T_zW^s(x)$;
2. $T_zW(x,y)\simeq T_zM(x,y)\oplus T_z\gamma$;
3. $T_zW^u(y)\simeq T_zM(x,y)\oplus T_z\gamma\oplus T_zM/T_zW^s(x)$.

Proof. We first prove (1). By Proposition 3.34, the Riemannian metric induces the decomposition

$$T_zW^u(y)\simeq T_zW(x,y)\oplus T_zW^u(y)/T_zW(x,y).$$

Consider the map

$$\phi\colon T_zW^u(x)\to T_zM/T_zW^s(x)$$

obtained as the composition of the inclusion with the quotient map. Then,

$$\mathrm{Ker}\phi=T_zW(x,y).$$

By definition of Morse-Smale function, $W^u(x)$ and $W^s(y)$ are transverse. Thus,

$$T_zW^u(y)+T_zW^s(x)=T_zM.$$

It follows that ϕ is surjective and

$$T_zW^u(y)/T_zW(x,y) \simeq T_zM/T_zW^s(x).$$

Thus, (1) holds.

We now prove (2). Let $\pi\colon W(x,y) \to W(x,y)/\mathbb{R}$ be the quotient map. Then

$$D\pi_z\colon T_zW(x,y) \to T_{\pi(z)}W(x,y)/\mathbb{R}$$

is a surjective linear map and the kernel of $D\pi_z$ is exactly $T_z\gamma$. Thus, by the nullity theorem, we get a canonical isomorphism

$$T_zW(x,y)/T_z\gamma \simeq T_{\pi(z)}W(x,y)/\mathbb{R}.$$

Using the metric, we obtain (2).

(3) follows immediately from (1) and (2). □

Assume now that (M,g) is a compact oriented Riemannian manifold. In the set-up of the Lemma above, we now want to choose an orientation which is compatible with the flow. To this end, we first choose an orientation on M and an orientation on $T_xW^u(x)$ for any critical point x. By the Stable Manifold Theorem (cf. Theorem 3.29), $W^u(x)$ is an oriented manifold. Thus, the orientation on $T_xW^u(x)$ induces an orientation on $W^u(x)$ for all $z \in W^u(x)$. Since, by transversality, we have $T_xM = T_xW^u(x) \oplus T_xW^s(x)$, the orientation on T_zM and $T_zW^u(x)$ induces an orientation on $T_xW^s(x)$. By the Stable Manifold Theorem, $W^s(x)$ is also oriented and so this induces an orientation on $W^s(x)$ for all $z \in W^s(x)$.

Assume now that $x, y \in M$ are critical points for f of index λ_x, λ_y respectively, such that $\lambda_y - \lambda_x = 1$. By Corollary 3.38, it follows that $M(x,y)$ is a finite set. Let $z \in W(x,y)$. Then the gradient flow with respect of f induces an orientation on $T_z\gamma$, given by $-\nabla f(z)$. The above orientation on $W^u(y)$, provides an ordered basis

$$(-\nabla f(z), B^u(z))$$

of $T_zW^u(y)$. Similarly, the orientation on $W^u(x)$ induces an orientation on $T_xW^s(x)$ and thus an ordered basis $B^s(z)$ of $T_zW^s(x)$. Since $-\nabla f(z)$ spans $T_zW^u(y) \cap T_zW^s(x)$, it follows that

$$(B^u(x), B^s(x))$$

is an ordered basis of T_zM. If this coincides with the orientation on T_zM then we assign $+1$ as the orientation of the corresponding element in $M(x,y)$, otherwise, we assign -1. Note that the choice of these orientations is not unique, as the depends on the choice of the manifold $W^u(x)$.

Thus, given a smooth compact oriented Riemannian manifold (M,g) with a Morse-Smale function $f\colon M \to \mathbb{R}$, after we assign the orientation of the unstable manifolds of f, we may define the number $n(x,y) \in \mathbb{Z}$ as the algebraic sum of signed flow lines from $y \in Cr_k(f)$ to $x \in Cr_{k-1}(f)$, where, for any $k \geq 0$,

$$Cr_k(f) := \{\text{critical point for } f \text{ of index } k\}.$$

Example 3.40. *Let $M = S^1 \subset \mathbb{R}^2$ be the circe, let*

$$f\colon S^1 \to \mathbb{R} \qquad (x, y) \mid y$$

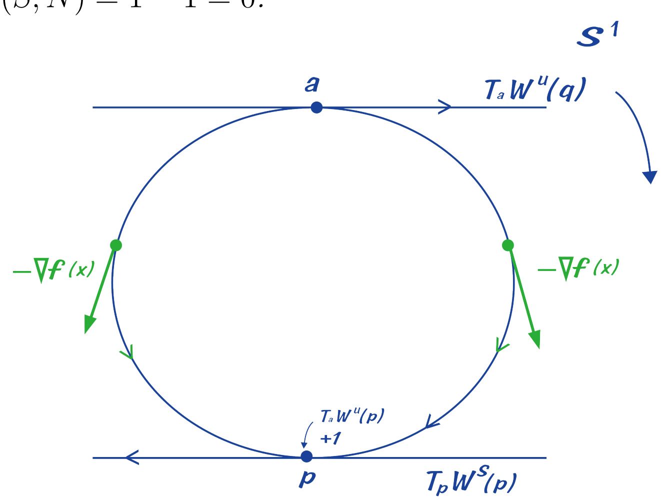
be the height function and let g be the metric on M induced by the Euclidean metric on $\mathbb{R}^2$. As in Example 3.30, f admits two critical points S and N and we have

$$W^u(N) = S^n \setminus \{S\} \qquad W^s(N) = \{N\}$$

$$W^u(S) = S \qquad W^s(N) = S^n \setminus \{S\}.$$

Thus, the function f is Morse-Smale.

We now want to compute $n(N, P)$. We pick the "clockwise" orientation on S^1. We now have to pick an orientation for the unstable manifolds at the two critical points. We pick the basis on $T_N W^u(N)$ to be the tangent vector "from left to right". This determines the orientation of $T_x W^u(N, P)$ for all $x \in W(S, N)$. Since $W^u(S)$ is just a point, we assign the orientation $+1$. This yields an orientation on all stable manifolds such that at the critical points, the orientations of the unstable and stable manifolds are compatible with the one of S^1, i.e. for example the basis of $T_S W^s(S)$ has to be the tangent vector "from right to left". Consider now the negative gradient $-\nabla f(x)$. It agrees with the orientation of $T_x W^u(N)$ on the right side and has the opposite orientation on the left side. Since $T_x W^u(N)$ is one dimensional, $B^u(x)$ is just a sign which turns the basis $-\nabla f(x)$ into a positive basis of $T_x W^u(N)$. Hence $B^u(x) = +1$ on the right side and $B^u(x) = -1$ on the left side. Since $B^s(x)$ is a positive basis of $T_x W^s(S)$ by our choice, and $W^u(S)$ and $W^s(S)$ are oriented compatibly with M at S, we see that the orientation $+1$ of $T_S W^u(S)$ together with a positive basis of $T_S W^s(S)$ gives a positive basis of $T_S M$. There also at x, $+1$ together with a positive basis of $T_x W^s(p)$ yields a positive basis of $T_x M$. We conclude finally that $(B^u(x), B^s(x))$ is positive on the right flow line and negative on the left. Thus, $n(S, N) = 1 - 1 = 0$.



(the flow lines are in gree, the orientation in blue)

Let $C_k(f) := \mathbb{Z}[Cr_k(f)]$ be the free abelian group generated by all the critical points of index k and define the linear map

$$\partial_k : C_k(f) \to C_{k-1}(f) \qquad \partial_k(y) := \sum_{x \in Cr_{k-1}(f)} n(x,y)x.$$

Theorem 3.41. [6] *Under the set-up above, the pair $(C_\bullet(f), \partial_\bullet)$ is a chain complex. If we denote*

$$\mathcal{Z}_k(C_\bullet(f), \partial_\bullet) := \mathrm{Ker}(\partial_k \colon C_k \to C_{k-1})$$

and

$$\mathcal{B}_k(C_\bullet(f), \partial_\bullet) := \mathrm{Im}(\partial_{k+1} \colon C_{k+1} \to C_k).$$

The quotient

$$H_k(M,f) := \frac{\mathcal{Z}_k(C_\bullet(f), \partial_\bullet)}{\mathcal{B}_k(C_\bullet(f), \partial_\bullet)}$$

does not depend on f and it coincides with the singular homology group $H_k(M)$.

The pair $(C_\bullet(f), \partial_\bullet)$ is called the **Morse-Smale-Witten** *chain complex. The group $H_k(M,f)$ is called the k-th* **Morse Homology group** *of M with respect to f.*

Example 3.42. *Consider again the example $M = S^1$ with f and g as in Example 3.40. Since $n(S,N) = 0$, we have $\partial_1 = 0$ and the Morse-Smale-Witten chain complex is*

$$C_1(f) \xrightarrow{\partial_1} C_0(f) \to 0,$$

where $C_1(f) = C_0(f) = \mathbb{Z}$, since there is only one critical point of index 0 and one critical point of index 1. Thus,

$$H_p(M,F) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

4. Singular homology

Definition 4.1. *A* **standard n-simplex** *is*

$$\Delta_n := \left\{ (t_0, \dots t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

*A k-***face** *of Δ_n is defined as $[e_{i_0}, \dots, e_{i_k}]$ with $0 \leq i_0 \leq \dots i_k \leq n$, where $e_0, \dots, e_n$ denotes the standard basis of $\mathbb{R}^{n+1}$.*

The i-th **face map** *of Δ_n is defined to be the map*

$$F_i^n := [e_0, \dots, \hat{e}_i, \dots e_n] : \Delta_{n-1} \to \Delta_n.$$

[6] See "Lectures on Morse homology", A. Banyaga and D. Hurtubise - Theorem 7.4.