Introduction to Ultrafast Laser

Lecture 3: Propagation, Dispersion & Chirp

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- Complex analytical signal (CAS):

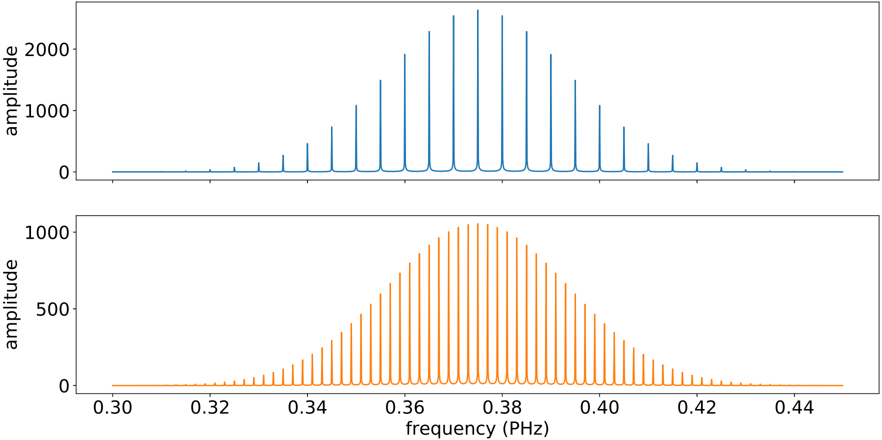
$$x(t) \xrightarrow{\text{FT}} \tilde{x}(\nu) \xrightarrow{\eta(\nu)} \tilde{x}^{+}(\nu) \xrightarrow{\text{IFT}} \tilde{x}^{+}(t)$$

$$\tilde{x}^{+}(t) = \frac{1}{2}[x(t) + iy(t)]$$

- Only considered the time domain, i.e. $E(t) = \mathrm{Re}\left\{E_0(t)e^{-i\omega_0 t}\right\}$.

Recap

2. Code demo: frequency comb.



Overview

Pulse Propagation

In the vacuum.

- Electromagnetic wave equation:

$$\left(\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right) E(z,t) = 0 \tag{1}$$

- Propagation velocity:

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} \quad (= c) \tag{2}$$

$$k(\omega) = \frac{\omega}{c} \tag{3}$$

Pulse Propagation

In the material.

- Electromagnetic wave equation:

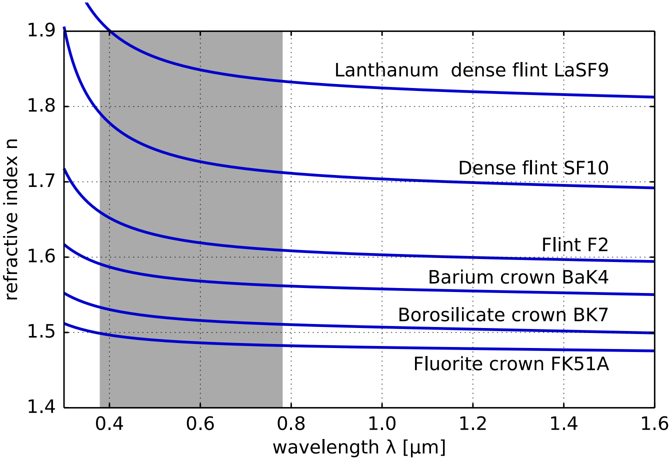
$$\left(\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right) E(z,t) = 0 \tag{4}$$

- Propagation velocity (n depends on λ!):

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \tag{5}$$

Pulse Propagation

In the material: refractive index changes with wavelength (figure from Wikipedia).



Pulse Propagation

In the material.

- FT on the time domain:

$$\left(\nabla^2 + \omega^2 \mu\epsilon\right) \tilde{E}(z, \omega) = 0 \tag{6}$$

- Solution:

$$\tilde{E}(z, \omega) = \tilde{E}(0, \omega) e^{ik(\omega) z}, \tag{7}$$

where

$$k(\omega) = \sqrt{\omega^2 \mu\epsilon} = \omega \frac{n(\omega)}{c}. \tag{8}$$

- General picture:

$$\tilde{E}(0, \omega) \xrightarrow[\text{transfer function}]{H(z,\omega)} \tilde{E}(z, \omega)$$

CAS Envelope

Pulse = Envelope × Carrier.

- The complex analytical signal is

$$
\begin{aligned}
\tilde{E}^{+}(z,t) &= \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(z,\omega)e^{-i\omega t} &(9)\\
&= \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega)e^{ik(\omega)z-i\omega t}\\
&= \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega)e^{i[k(\omega)-k(\omega_0)]z-i(\omega-\omega_0)t}\cdot e^{ik(\omega_0)z-i\omega_0 t}\\
&= \frac{1}{2}\mathcal{E}(z,t)e^{ik(\omega_0)z-i\omega_0 t} &(10)
\end{aligned}
$$

CAS Envelope

Pulse = Envelope × Carrier.

- The CAS envelope is defined as

$$\frac{1}{2}\mathcal{E}(z,t) = \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega) e^{i[k(\omega)-k(\omega_0)]z - i(\omega-\omega_0)t} \tag{11}$$

which is *slowly* varying both temporally and spatially.

Taylor Expansion of Dispersion Relation

- Taylor expansion of $k(\omega)$ to the 2nd order:

$$k(\omega) = k(\omega_0) + k'(\omega_0)(\omega - \omega_0) + \frac{1}{2}k''(\omega_0)(\omega - \omega_0)^2 + O(\Delta\omega^3) \tag{12}$$

$$\Rightarrow k(\omega) - k(\omega_0) = k'(\omega_0)(\omega - \omega_0) + \frac{1}{2}k''(\omega_0)(\omega - \omega_0)^2 \tag{13}$$

Envelope Equation

- Start from the CAS envelope:

$$\frac{1}{2}\mathcal{E}(z,t) = \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega) e^{i[k(\omega)-k(\omega_0)]z - i(\omega-\omega_0)t}$$

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega) e^{i[k(\omega)-k(\omega_0)]z - i(\omega-\omega_0)t} \cdot \{2i[k(\omega) - k(\omega_0)]\} \tag{14}$$

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega) e^{i[k(\omega)-k(\omega_0)]z - i(\omega-\omega_0)t} \cdot \{-2i(\omega - \omega_0)\} \tag{15}$$

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = \frac{1}{2\pi}\int_0^{+\infty} \mathrm{d}\omega\, \tilde{E}(0,\omega) e^{i[k(\omega)-k(\omega_0)]z - i(\omega-\omega_0)t} \cdot \{-2(\omega - \omega_0)^2\} \tag{16}$$

Envelope Equation

- Integral on both sides of Eq.(13):

$$\int_0^{+\infty} \mathrm{d}\omega \{\cdots\} [k(\omega) - k(\omega_0)] = \int_0^{+\infty} \mathrm{d}\omega \{\cdots\} \left[k'(\omega_0)(\omega - \omega_0) + \frac{1}{2} k''(\omega_0)(\omega - \omega_0)^2 \right]$$

where $\{\cdots\} = \frac{1}{2\pi} \tilde{E}(0, \omega) e^{i[k(\omega) - k(\omega_0)]z - i(\omega - \omega_0)t}$.

- Substitute the former relations into this equation, we have

$$\frac{1}{2i} \frac{\partial \mathcal{E}}{\partial z} = \frac{1}{-2i} \frac{\partial \mathcal{E}}{\partial t} k'(\omega_0) + \frac{1}{-2} \frac{\partial^2 \mathcal{E}}{\partial t^2} \frac{1}{2} k''(\omega_0)$$

$$\Rightarrow \frac{\partial \mathcal{E}}{\partial z} + k'(\omega_0) \frac{\partial \mathcal{E}}{\partial t} + i \frac{1}{2} k''(\omega_0) \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0 \tag{17}$$

which is the equation governing the complex envelope in propagation.

Remark

Envelope Equation

- The 1st-order dispersion is related to the group velocity:

$$k'(\omega_0) = \frac{1}{v_g}. \tag{18}$$

- The 2nd-order dispersion is the dispersion of group velocity, thus is given the name "group velocity dispersion" (GVD).
- Keep to the 2nd order in the dispersion relation should be safe for us (i.e. 100 fs laser pulse). For shorter pulse, higher order should be included (since the frequency range is much wider).
- The nonlinear effect (e.g. the Kerr effect) should be also considered, since it is comparable to the 2nd-order dispersion in most cases.

Retarded Time Frame

Move with the pulse!

- Define new parameters:

$$\begin{cases} \tau = t - \dfrac{z}{v_g} \\ z' = z \end{cases} \Rightarrow \begin{cases} \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z'} - \dfrac{1}{v_g}\dfrac{\partial}{\partial \tau} \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \tau} \end{cases}$$

- Substitute into the envelope equation:

$$\frac{\partial \mathcal{E}}{\partial z'} + \frac{1}{2} i k'' \frac{\partial^2 \mathcal{E}}{\partial \tau^2} = 0 \tag{19}$$

where $k'' = k''(\omega_0)$.

Retarded Time Frame

Move with the pulse!

- FT on τ:

$$\tilde{\mathcal{E}}(z,\Omega) = \int \mathrm{d}\tau\, e^{i\Omega\tau} \mathcal{E}(z,\tau) \tag{20}$$

$$\Rightarrow \frac{\partial \tilde{\mathcal{E}}}{\partial z} - i\frac{k''}{2}\Omega^2 \tilde{\mathcal{E}} = 0 \tag{21}$$

- Solving this equation, we have

$$\tilde{\mathcal{E}}(z,\Omega) = \tilde{\mathcal{E}}(0,\Omega) e^{i\frac{k''}{2}\Omega^2 z}. \tag{22}$$

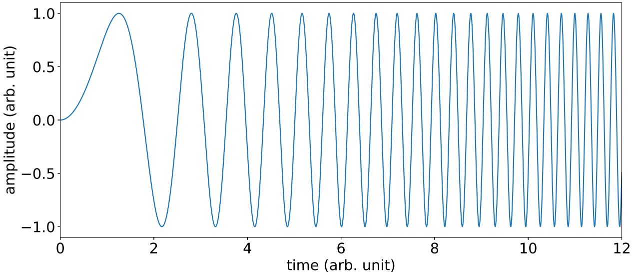
- The transfer function is

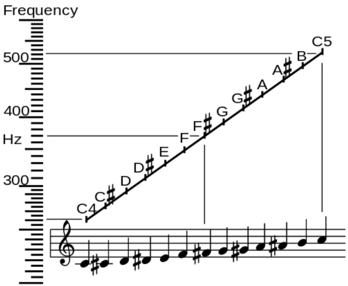
$$H(z,\Omega) = \exp\left(i\frac{k''}{2}\Omega^2 z\right). \tag{23}$$

Chirp

What is chirp?

- Chirp: the frequency increases (up-chirp) or decreases (down-chirp) with time.





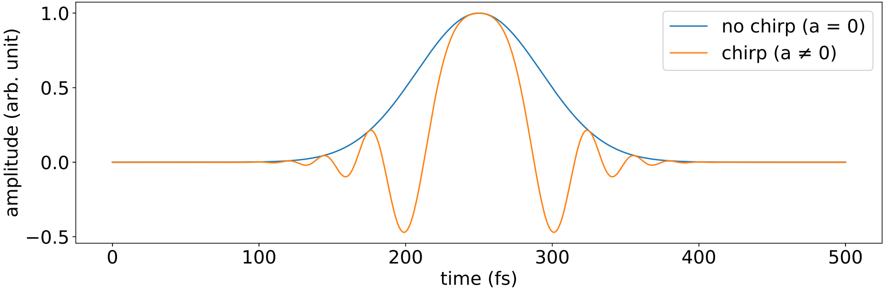
Example: Chirp from Dispersion

Gaussian pulse propagation.

- The initial pulse envelope is

$$\mathcal{E}(0,t) = \exp\left[-(1+ia)\frac{t^2}{t_0^2}\right] \tag{24}$$

where a is the chirp term.



Example: Chirp from Dispersion

Gaussian pulse propagation.

- FT on t (at $z = 0$, we have $t = \tau$):

$$\tilde{\mathcal{E}}(0, \Omega) = \sqrt{\frac{\pi t_0^2}{1 + ia}} \exp\left[-\frac{t_0^2}{4(1 + ia)}\Omega^2\right] \tag{25}$$

- Use Eq.(23), we have

$$\begin{aligned}\tilde{\mathcal{E}}(z, \Omega) &= \tilde{\mathcal{E}}(0, \Omega)e^{i\frac{k''}{2}\Omega^2 z} \\ &= \sqrt{\frac{\pi t_0^2}{1 + ia}} \exp\left[-\frac{t_0^2}{4(1 + ia)}\Omega^2\right] \cdot \exp\left(i\frac{k''}{2}\Omega^2 z\right).\end{aligned} \tag{26}$$

Example: Chirp from Dispersion

Gaussian pulse propagation.

- IFT:

$$\mathcal{E}(z,t) = \exp\left[-(1+ib)\frac{t^2}{t_0'^2}\right], \tag{27}$$

where

$$\begin{cases} t_0' = \sqrt{\dfrac{t_0^2}{1+a^2} + \left(\dfrac{t_0^2 a}{1+a^2} + 2k''z\right)^2 \cdot \dfrac{1+a^2}{t_0^2}} \\ b = a + \dfrac{2k''z}{t_0^2/(1+a^2)} \end{cases}.$$

Example: Chirp from Dispersion

Gaussian pulse propagation.

- If $a = 0$ (no chirp at the beginning):

$$\begin{cases} t_0' = \sqrt{t_0^2 + \dfrac{(2k''z)^2}{t_0^2}} & \Rightarrow \text{pulse broadened} \\ b = \dfrac{2k''z}{t_0^2} & \Rightarrow \text{pulse chirped} \end{cases}.$$

- Normal dispersion ($k'' > 0$, *higher frequencies travel slower*): up-chirp.
- Abnormal dispersion ($k'' < 0$, *higher frequencies travel faster*): down-chirp.

Example: Chirp from Dispersion

Gaussian pulse propagation.

- A complete description of pulse:

$$
\begin{aligned}
\tilde{E}^{+} &= \frac{1}{2} \exp\left[-(1+ia)\frac{t^2}{t_0^2}\right] \exp\left(-i\omega_0 t\right) \\
&= \frac{1}{2} \exp\left(-t^2/t_0^2\right) \exp\left[-i\phi(t)\right]
\end{aligned} \tag{28}
$$

where

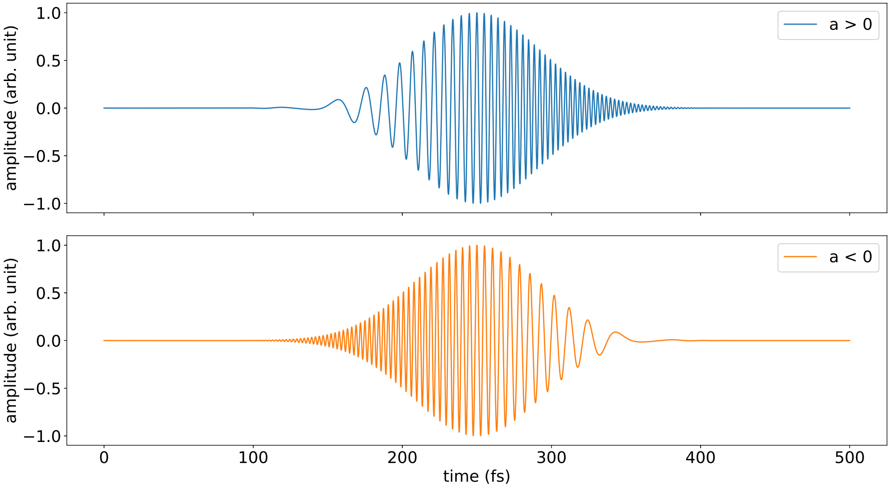
$$
\phi(t) = \omega_0 t + a\frac{t^2}{t_0^2}. \tag{29}
$$

- The instant frequency:

$$
\omega(t) = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 + \frac{2a}{t_0^2} t. \tag{30}
$$

Example: Chirp from Dispersion

Gaussian pulse propagation.



Summary

- Transfer function:

$$\tilde{E}(0,\omega) \xrightarrow[\text{transfer function}]{H(z,\omega)} \tilde{E}(z,\omega)$$

- Envelope equation:

$$\frac{\partial \mathcal{E}}{\partial z} + k' \frac{\partial \mathcal{E}}{\partial t} + \frac{1}{2} i k'' \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

- Dispersion $\rightarrow$ chirp.
- Next time: nonlinear effect.

The End