Introduction to Ultrafast Laser

Lecture 2: Mathematical Description

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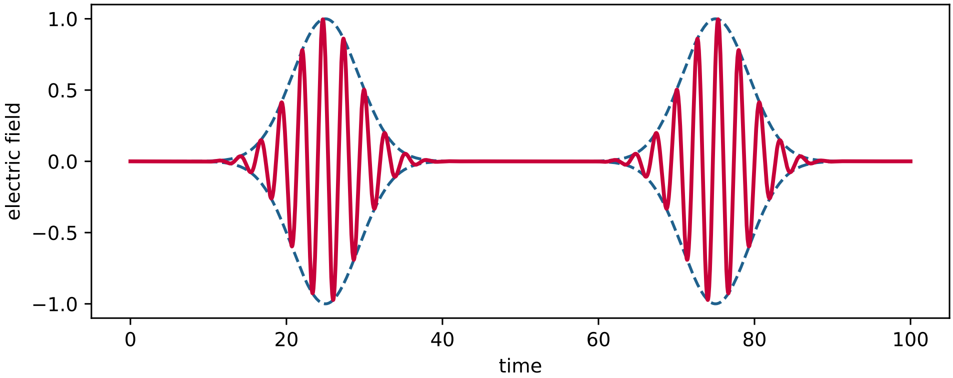
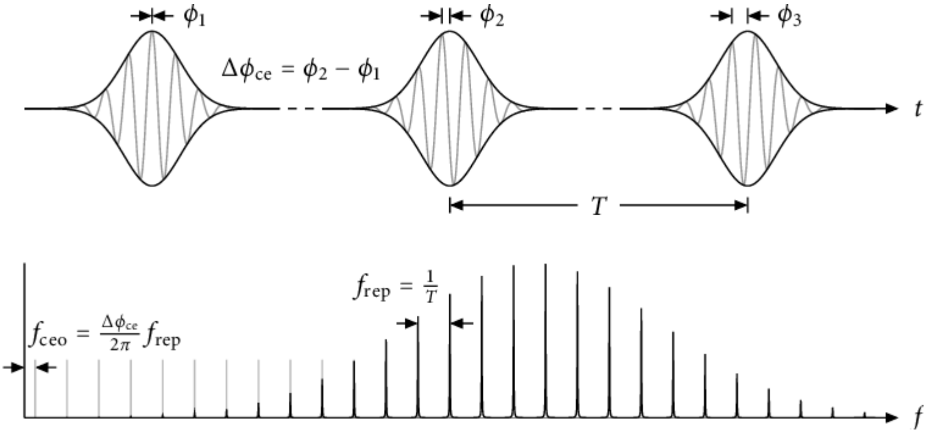
Recap

1. Shape of Ultrafast Laser Pulses: train, envelope and carrier.



Recap

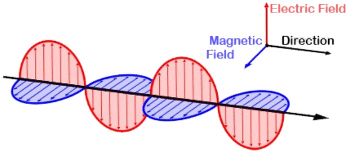
2. Frequency Spectrum: frequency comb.



Overview

Purpose

Find a complete description of E field. Should also be convient in calculation!



- laser profile $\rightarrow$ **E** field (why?)
- Focus on time domain.
- CW laser: $E(t) = E_0 \cos(\omega t - \phi)$

- Laser pulse $=$ Envelope $\times$ Carrier
- First try: $E(t) = E_0(t)\cos(\omega t - \phi)$
- *Not* convinient for practical reasons!
- Complex representation:

$$E(t) = \mathrm{Re}\left\{ E_0(t) e^{-i(\omega t - \phi(t))} \right\}$$

- Negative: propogating.

Fourier Transformation

Convention (note the sign).

Denote the function in time domain as $f(t)$.

- FT (time to frequency):

$$F(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{i\omega t} f(t) \tag{1}$$

- IFT (frequency to time):

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, e^{-i\omega t} F(\omega) \tag{2}$$

Fourier Transformation

Convention (note the sign).

Use frequency instead of angular frequency:

- FT (time to frequency):

$$F(\nu) = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{i2\pi\nu t} f(t) \tag{3}$$

- IFT (frequency to time):

$$f(t) = \int_{-\infty}^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t} F(\nu) \tag{4}$$

Fourier Transformation

Useful properties.

- Reality condition:

$$f(t) \text{ is real} \Rightarrow F(-\omega) = F^*(\omega) \tag{5}$$

- Scaling formula:

$$h(t) = f(at) \Rightarrow H(\omega) = \frac{1}{a}F(\frac{\omega}{a}) \tag{6}$$

- Time-delay formula:

$$h(t) = f(t-\tau) \Rightarrow H(\omega) = F(\omega)e^{i\omega\tau} \tag{7}$$

- Frequency-offset formula:

$$h(t) = f(t)e^{i\omega_0 t} \Rightarrow H(\omega) = F(\omega + \omega_0) \tag{8}$$

Fourier Transformation

Useful properties.

- Convolution formula:

$$h(t) = f(t) * g(t) = \int \mathrm{d}t'\, f(t')g(t-t') \tag{9}$$

$$\Rightarrow H(\omega) = F(\omega)G(\omega) \tag{10}$$

- Parseval's theorem:

$$\int \mathrm{d}t\, f(t)f^*(t) = \frac{1}{2\pi}\int \mathrm{d}\omega\, F(\omega)F^*(\omega) \tag{11}$$

where $|F(\omega)|^2 = F(\omega)F^*(\omega)$ is called the power spectral density.

Fourier Transformation

Special cases.

- Delta function:

$$f(t) = \delta(t) \Rightarrow F(\omega) = 1$$

$$f(t) = e^{i\omega_0 t} \Rightarrow F(\omega) = 2\pi\delta(\omega - \omega_0)$$

- Gaussian function:

$$f(t) = e^{-t^2/t_p^2} \Rightarrow F(\omega) = t_p\sqrt{\pi}e^{-\omega^2 t_p^2/4} \tag{12}$$

Complex Analytical Signal

Step by step.

- Start with real field $x(t)$:

$$\tilde{x}(\nu) = \mathrm{F.T.}\{x(t)\}, \ x(t) = \mathrm{I.F.T.}\{\tilde{x}(\nu)\}$$

- By reality condition (5):

$$\tilde{x}(-\nu) = \tilde{x}^*(\nu)$$

Negative frequency doesn't contain *any more* information!

Complex Analytical Signal

Step by step.

- Define new complex field in frequency domain $\tilde{x}^+(\nu)$:

$$\tilde{x}^+(\nu) = \begin{cases} \tilde{x}(\nu) & \text{if } \nu \geq 0 \\ 0 & \text{if } \nu < 0 \end{cases}. \tag{13}$$

- I.F.T. to time domain:

$$\tilde{x}^+(t) = \int_{-\infty}^{+\infty} \mathrm{d}\nu \, e^{-i2\pi\nu t} \tilde{x}^+(\nu) = \int_{0}^{+\infty} \mathrm{d}\nu \, e^{-i2\pi\nu t} \tilde{x}(\nu) \tag{14}$$

which is the **complex analytical signal (CAS)** of $x(t)$, a complete discription of E-field (both amplitude and phase).

Properties of Complex Analytical Signal

Real part.

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t}\tilde{x}(\nu) \\
&= \int_{0}^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t}\tilde{x}(\nu) + \int_{-\infty}^{0} \mathrm{d}\nu\, e^{-i2\pi\nu t}\tilde{x}(\nu) \\
&= \int_{0}^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t}\tilde{x}(\nu) + \int_{0}^{+\infty} \mathrm{d}\nu\, e^{i2\pi\nu t}\tilde{x}(-\nu) \\
&= \int_{0}^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t}\tilde{x}(\nu) + \int_{0}^{+\infty} \mathrm{d}\nu\, e^{i2\pi\nu t}\tilde{x}^{*}(\nu) \\
&= \tilde{x}^{+}(t) + [\tilde{x}^{+}(t)]^{*} \\
&= 2\,\mathrm{Re}\left\{\tilde{x}^{+}(t)\right\}
\end{aligned}
$$

Properties of Complex Analytical Signal

Imaginary part.

- What about the imaginary part?
- Recall how we get the CAS:

$$\tilde{x}^{+}(t) = \int_{0}^{+\infty} \mathrm{d}\nu \, e^{-i2\pi\nu t} \tilde{x}(\nu) = \int_{-\infty}^{+\infty} \mathrm{d}\nu \, e^{-i2\pi\nu t} \tilde{x}(\nu)\eta(\nu) \tag{15}$$

  where $\eta(\nu)$ is the step function.
- Using the convolution formula:

$$\tilde{x}^{+}(t) = \text{I.F.T.}\{\tilde{x}(\nu)\eta(\nu)\} = x(t) * \text{I.F.T.}\{\eta(\nu)\} \tag{16}$$

Properties of Complex Analytical Signal

Imaginary part.

$$
\begin{aligned}
\text{I.F.T.}\{\eta(\nu)\} &= \int_0^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu t} \\
&= \lim_{\epsilon\to 0} \int_0^{+\infty} \mathrm{d}\nu\, e^{-i2\pi\nu(t-i\epsilon)} \\
&= \lim_{\epsilon\to 0} \frac{1}{-i2\pi(t-i\epsilon)}\, e^{-i2\pi\nu(t-i\epsilon)}\Big|_{\nu=0}^{+\infty} \\
&= \lim_{\epsilon\to 0} \frac{1}{2\pi i}\frac{1}{t-i\epsilon} \\
&= \frac{1}{2\pi i}\left[\text{p.v.}\left(\frac{1}{t}\right) + i\pi\delta(t)\right] \\
&= \frac{1}{2}\delta(t) - \text{p.v.}\left(\frac{1}{2\pi t}\right) i
\end{aligned}
$$

A useful relation

$$
\frac{1}{x \pm i\delta} = \text{p.v.}\left(\frac{1}{x}\right) \mp i\pi\delta(x)
$$

Properties of Complex Analytical Signal

Imaginary part.

- Insert into the convolution:

$$
\begin{aligned}
\tilde{x}^{+}(t) &= x(t) * \text{I.F.T.}\{\eta(\nu)\} \\
&= x(t) * \left\{ \frac{1}{2}\delta(t) - \text{p.v.} \left(\frac{1}{2\pi t} \right) i \right\} \\
&= \frac{1}{2}x(t) - i\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \frac{x(\tau)}{t-\tau} \\
&= \frac{1}{2}[x(t) + iy(t)]
\end{aligned}
$$

where

$$
y(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \frac{x(\tau)}{t-\tau}. \tag{17}
$$

Properties of Complex Analytical Signal

Imaginary part.

Hilbert transform

The Hilbert transform of $u(t)$ is defined as

$$H(u)(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} \, \mathrm{d}\tau \, ,$$

which is the convolution of $u(t)$ and $1/\pi t$:

$$H(u)(t) = u(t) * \frac{1}{\pi t}.$$

Notice that

$$H[H(u)](t) = -u(t).$$

Properties of Complex Analytical Signal

Analytic continuation.

- The complex analytical signal is

$$\tilde{x}^{+}(t) = \frac{1}{2}[x(t) + iy(t)] \tag{18}$$

where

$$y(t) = -H(x)(t),\ x(t) = H(y)(t). \tag{19}$$

- $\tilde{x}^{+}(t)$ is the complex analytical correspondence of the real function $x(t)$!

Application of CAS

Express the E-field.

- Amplitude and phase representation with carrier:

$$\tilde{x}^{+}(t) = \frac{1}{2}A(t)e^{i\phi(t)}e^{-i\omega_0 t}, \tag{20}$$

where $A(t) = \sqrt{x^2(t) + y^2(t)}$, $\phi(t) = \tan^{-1}[y(t)/x(t)] + \omega_0 t$.

- Real E-field:

$$x(t) = 2\,\mathrm{Re}\left\{\tilde{x}^{+}(t)\right\} = \mathrm{Re}\left\{A(t)e^{-i[\omega_0 t - \phi(t)]}\right\}. \tag{21}$$

- $\Rightarrow$ Mathematical description of propagation, interaction with material, etc.

Practical Data Analysis

Demonstration with Python.

- Continous Fourier transform $\rightarrow$ discrete Fourier transform;
- Demonstrate the appearance of “frequency comb”;
- Abstract several key properties of ultrafast laser pulses from data file:
  - repetition rate;
  - center wavelength.
- ...

Summary

- Fourier transform;
- Complex analytical signal (CAS):

$$x(t) \xrightarrow{\text{FT}} \tilde{x}(\nu) \xrightarrow{\eta(\nu)} \tilde{x}^{+}(\nu) \xrightarrow{\text{IFT}} \tilde{x}^{+}(t)$$

$$\tilde{x}^{+}(t) = \frac{1}{2}[x(t) + iy(t)]$$

- Next time: pulse propagation and dispersion.

The End