Political Uncertainty

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Groseclose and Snyder 1996

On Feb. 12, Sebastian and I agreed to focus efforts on finding a base model to facilitate empirical identification. I am pursuing Groseclose & Snyder (1996), "Buying Supermajorities," APSR

- For each legislator i, $v(i) = u_i(x) - u_i(s)$, measured in money; this is the reservation price of i
  - x is an alternative policy proposal; s is the status quo
  - WLOG, label legislators so that $v(i)$ is a non-increasing function
  - Note legislators only have preferences over how they vote, not over which alternative wins
- There are two vote buyers; each prefers to minimize total bribes paid while passing his preferred policy, but each would prefer to concede the issue rather than pay more than his WTP
  - A prefers x; W_A is A's willingness to pay (WTP) for x measured in money
  - B prefers s; W_B is B's WTP for s
- Bribe offer functions: $a(i)$ and $b(i)$ are A and B's offers to i. Legislators take these bribe offers as given and then vote for the alternative that maximizes their payoff
- A moves first; $a(i)$ is perfectly observable to B when he moves
- Goal: characterize SPNE in pure strategies
  - Assume unbribed legislators who are indifferent vote for s; all bribed legislators who are indifferent vote for whoever bribed them last
- Assume continuum of legislators on $\left[-\frac{1}{2}, \frac{1}{2}\right]$
- Assume W_A large enough that x wins in equilibrium (no uncertainty case)
- $m + \frac{1}{2}$ is fraction of legislators who vote for x as opposed to the status quo, s
- Results
  - Prop 1: three types of equilibria in which x wins; depend on size of W_B
  - Prop 2: m^* (the optimal coalition size) is unique, and provides three cases parameterizing its size in terms of W_B, $v(-\frac{1}{2})$ and $v(m^*)$
  - Prop 3/4: special case where $v(z) = \alpha - \beta z$

Our Extension to Uncertainty

General thoughts on extension to uncertainty

- I think, without uncertainty, you would estimate m^* as a function of the parameters of v and WTP
  - It's useful that m^* is unique. Not clear it would extend to case of uncertainty, but I think it's likely so I'm going to assume it for now
- I'm pretty sure this predicts that B should never pay anything when there is no uncertainty, but I don't see where they say it explicitly (I should read more carefully to verify)
  - Uncertainty should reverse this, right?
  - What is uncertainty? Make $v(z)$ stochastic is most natural
    - I'm going to start with linear parameterization of $v(z)$ and add uncertainty. For now, take same form for $v(z)$

      $$v(z) = \alpha - \beta \cdot z$$

      and take z as a random variable distributed normally with mean z and standard deviation σ_z. This is unfortunate nomenclature, since later we'll probably want to take this to a standard normal "Z"
      - This means that the legislators are ordered on the interval according to their ideal points, but there is acknowledged uncertainty surrounding their preferences
      - What is the source of this uncertainty? We take it to vary by legislator and also possibly by issue area when we extend the model to account for that
      - Uncertainty comes from: **log-rolling and cross-pressuring; length of voting record; electoral incentives that we can't control for**
- Note that this model *may* introduce interdependence between the bid functions that does not exist in the continuous model
- What is not clear is whether this happens when the WTP constraint is not binding, which I'm going to assume from the start, and maybe impose later

Solving the Model

Backward induction (legislature moves last; B makes last bribe; A makes first bribe)

1. Legislature

   - Each legislator z will decide whether to vote for x or s given z, $a(z)$ and $b(z)$. Votes for x if
     $$v(z) = \alpha - \beta z + \varepsilon_z + a(z) - b(z)$$
     (whether the inequality is weak or strict depends on tie-breaking rules set out in the paper)
     - Notice that a bribe from B is subtracted because it makes the alternative *less* attractive
     - Note that we will have to worry about separately identifying the parameters. So for now, will assume ε does not vary in z
   - Also going to ignore Vote Buyer A's choice for now, so we have
     $$v(z) = \alpha - \beta z + \varepsilon - b(z)$$
   - Then the probability that legislator z votes against the new proposal is
     $$\Pr\left[v(z) \le 0\right] = \Pr\left[\alpha - \beta z + \varepsilon - b(z) \le 0\right] = \Pr\left[\varepsilon \le \beta z - \alpha + b(z)\right]$$
     Assuming $\varepsilon \sim$ Logistic $(0, 1)$, the probability that legislator z votes "no" is $\frac{1}{1+e^{-(\beta z - \alpha + b(z))}}$.
   - Take a model with three legislators with ideal points at $z = -\frac{1}{2}$, $z = 0$ and $z = \frac{1}{2}$. For ease of notation, let
     (a) $X = -\alpha + b(0)$
     (b) $Y = \frac{\beta}{2} - \alpha + b\left(-\frac{1}{2}\right)$
     (c) $Z = -\frac{\beta}{2} - \alpha + b\left(\frac{1}{2}\right)$
   - Then the three probabilities are
     - $\frac{1}{1+e^{-(-\alpha+b(0))}} = \frac{1}{1+e^{-X}}$, $\frac{1}{1+e^{-\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)}} = \frac{1}{1+e^{-Z}}$, $\frac{1}{1+e^{-\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)}} = \frac{1}{1+e^{-Y}}$
     - Whether a legislator z votes "no" is a random variable, denote it $X(z)$, distributed Bernoulli with this probability:
       $$X(z) \sim \text{Bernoulli}(p(z)) \quad \text{where} \quad p(z) = \frac{1}{1 + e^{-(\beta z - \alpha + b(z))}}$$

2. Vote buyer B

   - GS assumption on vote buyers' objective is "each prefers to minimize total bribes paid while passing his preferred policy, but each would prefer to concede the issue rather than pay more than his WTP"; this has to be adapted to our situation with uncertainty
   - With asymmetric $v(z)$ or WTP parameters, it would be easy to get equilibria where only A or B buys votes. But we also have lots of outcomes where both buy votes.

- They can easily both have positive probability of winning. But what do we need for this to be an equilibrium in this three stage game?
- Uncertainty buys us a lot: no longer this knife edge condition of A pushing to the point that B buys no votes
- Note the non-negativity constraint means you can't extract money from a legislator who is past the FOC point in order to reallocate to another legislator
- Some legislators who are past FOC point will not be lobbied
- Need to max probability of at least two voting "no": $\Pr(S \geq 2)$ where $S = X\left(-\frac{1}{2}\right) + X\left(0\right) + X\left(\frac{1}{2}\right)$
- Thus, maximization problem for Vote Buyer B (in absence of Vote Buyer A) is

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Bigg[\Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=1\right)\left(X\left(\frac{1}{2}\right)=0\right)+ \\ \Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=0\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)+ \\ \Pr\left(X\left(-\frac{1}{2}\right)=0\right)\Pr\left(X\left(0\right)=1\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)+ \\ \Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=1\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)\Bigg] - \sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (1)$$

Substituting from the definition of X,

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Bigg[\frac{1}{1+e^{-Z}}\frac{1}{1+e^{-X}}\left(1-\frac{1}{1+e^{-Y}}\right)+ \\ \frac{1}{1+e^{-Z}}\left(1-\frac{1}{1+e^{-X}}\right)\frac{1}{1+e^{-Y}}+ \\ \left(1-\frac{1}{1+e^{-Z}}\right)\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}+ \\ \frac{1}{1+e^{-Z}}\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\Bigg] - \sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (2)$$

This can be simplified as

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Bigg[\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}+\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Z}}+ \\ \frac{1}{1+e^{-Y}}\frac{1}{1+e^{-Z}}-2\frac{1}{1+e^{-Z}}\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\Bigg] - \sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (3)$$

The FOC wrt to $b\left(\frac{1}{2}\right)$ is

$$W_B\left[\left(\frac{1}{1+e^{-X}}+\frac{1}{1+e^{-Y}}-2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\right)\frac{e^{-Z}}{[1+e^{-Z}]^2}\right]=1 \quad (4)$$

Simplifying:

$$W_B\left[\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\frac{e^{-Z}}{\left[1+e^{-Z}\right]^2}\right]=1 \tag{5}$$

Further math related to this equation is on Page 11.

- If we divide through by 2, the left side is the average minus the product
- Remember, I'm ignoring the constraints on WTP and non-negativity of bribes

3. Vote Buyer A

- Vote buyer A's objective function is identical in form to that of Vote buyer B (Expression 3)

$$\max_{a(-\frac{1}{2}),a(0),a(\frac{1}{2})} W_A\left[\frac{1}{1+e^{-X_A}}\frac{1}{1+e^{-Y_A}}+\frac{1}{1+e^{-X_A}}\frac{1}{1+e^{-Z_A}}+\frac{1}{1+e^{-Y_A}}\frac{1}{1+e^{-Z_A}}-2\frac{1}{1+e^{-Z_A}}\frac{1}{1+e^{-X_A}}\frac{1}{1+e^{-Y_A}}\right]-\sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}}a(j) \tag{6}$$

where

(a) $X_A=\alpha-b_0(\boldsymbol{a},W_B,\alpha,\beta)+a_0$

(b) $Y_A=-\frac{\beta}{2}+\alpha-b_{-.5}\left(\boldsymbol{a},W_B,\alpha,\beta\right)+a_{-.5}$

(c) $Z_A=\frac{\beta}{2}+\alpha-b_{.5}\left(\boldsymbol{a},W_B,\alpha,\beta\right)+a_{.5}$

with $\boldsymbol{a}=\left(a_{-.5},a_0,a_{.5}\right)$

- Vote Buyer A's first order condition with respect to a_0 is more complicated because of the dependencies on $\boldsymbol{a}$ (should probably double check this, but it makes sense)

$$\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-Y}}{\left[1+e^{-Y}\right]^2}\frac{\partial b_{-.5}}{\partial a_0}+\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\frac{e^{-Z}}{\left[1+e^{-Z}\right]^2}\frac{\partial b_{.5}}{\partial a_0}+\frac{2+e^{-Y}+e^{-Z}}{\left(1+e^{-Y}\right)\left(1+e^{-Z}\right)}\frac{e^{-X}}{\left[1+e^{-X}\right]^2}\left[1-\frac{\partial b_0}{\partial a_0}\right]=\frac{1}{W_A} \tag{7}$$

Adding Non-negativity Constraints for Bribes

The Lagrangian for the problem is the same as Expression 3, adding on $\lambda_X b(0) + \lambda_Y b\left(-\frac{1}{2}\right) + \lambda_Z b\left(\frac{1}{2}\right)$ The full set of FOCs (in logistic form) is then:

$$W_B \frac{e^{-X}}{\left(1+e^{-X}\right)^2}\left[\frac{e^{-Z}+e^{-Y}}{\left(1+e^{-Z}\right)\left(1+e^{-Y}\right)}\right] - 1 + \lambda_X = 0$$

$$W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\right] - 1 + \lambda_Y = 0$$

$$W_B \frac{e^{-Z}}{\left(1+e^{-Z}\right)^2}\left[\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\right] - 1 + \lambda_Z = 0$$

$$b(0) \geq 0 \quad b\left(\frac{1}{2}\right) \geq 0 \quad b\left(-\frac{1}{2}\right) \geq 0$$

$$\lambda_X \geq 0 \quad \lambda_Y \geq 0 \quad \lambda_Z \geq 0$$

$$\lambda_X \cdot b(0) = 0 \quad \lambda_Y \cdot b\left(-\frac{1}{2}\right) = 0 \quad \lambda_Z \cdot b\left(\frac{1}{2}\right) = 0$$

There are 8 patterns of positive and zero components in the 3-dimensional vector of bribes, $b' = \left(b(0), b\left(\frac{1}{2}\right), b\left(-\frac{1}{2}\right)\right)$.

Note that the complementary slackness condition implies that either a bribe is zero, or its multiplier is zero (or both).

Simulations show that for many sets of parameters, $b(0)$ and $b\left(\frac{1}{2}\right)$ are positive while $b\left(-\frac{1}{2}\right) = 0$.

- Then $Y = \frac{\beta}{2} - \alpha + b\left(-\frac{1}{2}\right) = \frac{\beta}{2} - \alpha$
- λ_Y is likely positive
- $\lambda_X = \lambda_Z = 0$

Parameterizing Uncertainty to Vary by Legislator

I'm going to start by changing the uncertainty for just one legislator when there is just one vote buyer (B). I will do this for the legislator at position -0.5, which corresponds to Y. The FOCs become, for Z (similarly for X, and from Equation 5)

$$W_B\left[\frac{e^{-X}+e^{-\frac{Y}{s_{-.5}}}}{\left(1+e^{-X}\right)\left(1+e^{-\frac{Y}{s_{-.5}}}\right)}\frac{e^{-Z}}{\left[1+e^{-Z}\right]^2}\right]=1 \tag{8}$$

And for Y

$$W_B\left[\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-\frac{Y}{s_{-.5}}}}{s_{-.5}\left[1+e^{-\frac{Y}{s_{-.5}}}\right]^2}\right]=1 \tag{9}$$

The same thing can be done for the legislator at position 0.5. The objective function when both vary is

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\left[\frac{1}{1+e^{-X}}\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}+\frac{1}{1+e^{-X}}\frac{1}{1+e^{-\frac{Z}{s_{.5}}}}+\right.$$
$$\left.\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}-2\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}\frac{1}{1+e^{-X}}\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}\right]-\sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}}b(j) \tag{10}$$

Now what impact does this have on Vote Buyer B's decisions (in the absence of bribes by Vote Buyer A)?

- First, how does the probability that the legislator at 0.5 (most unfriendly to B) votes with B change when $s_{.5}$ goes from 1 to 2 (and there are no bribes)?
  - When $s_{.5}$ is 1, the probability is $\frac{1}{1+e^{-\frac{Z}{s_{.5}}}}=\frac{1}{1+e^{-\frac{-.5}{1}}}=\frac{1}{1+e^{.5}}=0.3775$
  - When $s_{.5}$ is 2, the probability is $\frac{1}{1+e^{-\frac{Z}{s_{.5}}}}=\frac{1}{1+e^{-\frac{-.5}{2}}}=\frac{1}{1+e^{.25}}=0.4378$
  - That is, the probability that the least friendly legislator votes with B *rises*
- Now, how does the probability that the legislator at -0.5 (most friendly to B) votes with B change when $s_{-.5}$ goes from 1 to 2 (and there are no bribes)?
  - When $s_{-.5}$ is 1, the probability is $\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}=\frac{1}{1+e^{-\frac{.5}{1}}}=\frac{1}{1+e^{-.5}}=0.6225$
  - When $s_{-.5}$ is 2, the probability is $\frac{1}{1+e^{-\frac{Y}{s_{-.5}}}}=\frac{1}{1+e^{-\frac{.5}{2}}}=\frac{1}{1+e^{-.25}}=0.5622$
  - That is, the probability that the least most legislator votes with B *falls*
- Numerical example. Take $\alpha=0$ and ignore A. Take $s_{-.5}$ from 1 to 2. B stops bribing legislator -0.5 (most friendly) altogether.

- This is true even with $s_{-.5}$ as low as 1.2 (taking $s_{.5} = s_0 = 1$)
- For $1 < s_{-.5} < 1.2$, no bribe for -0.5 guy until W_B gets significantly higher, and then his net position is lower than $X = Z$
- Reverse if $s_{-.5} < 1$: move -0.5 guy further left than $X = Z$ (but not further than X when just these two); in some cases leave 0.5 guy at zero

• Numerical example. Take $\alpha = 0$ and ignore A. Take $s_{.5}$ from 1 to 1.5. B stops bribing legislator 0.5 (least friendly) altogether. Does better in welfare terms because the probability this guy votes in his favor shifts up

- For $s_{.5} = 1.1$, no bribe for 0.5 guy until W_B gets significantly higher, and then his net position is lower than $X = Y$
- At $s_{.5} = .75$: ONLY Z for $W_B \in [6, 8]$ (8 is usually where it's X alone). Then Z and Y together but Z further on net than Y. Eventually all three, with $X = Y$, Z further

Intuition

- A legislator votes for x (A's preferred policy, against the status quo) if

$$v(z) = \alpha - \beta z + a(z) - b(z) > 0$$

- B needs 2 "no" votes, or two legislators with $v(z) < 0$. When $a(z) = 0$, this is

$$\alpha - \beta z - b(z) < 0$$

or

$$\beta z - \alpha + b(z) > 0$$

- If $\alpha > 0$, legislators on average are biased against B.
  - If $\alpha < 0$, legislators on average are biased in favor of B.
  - If α is negative enough, bribe only legislator furthest to the left (most against B)
- If X, Y, or Z is positive, it means the probability of legislator $0, -\frac{1}{2}$, or $\frac{1}{2}$ voting for the status quo / against the new proposal / with interest group B is greater than 0.5.

Recall:

1. $X = -\alpha + b(0)$
2. $Y = \frac{\beta}{2} - \alpha + b\left(-\frac{1}{2}\right)$
3. $Z = -\frac{\beta}{2} - \alpha + b\left(\frac{1}{2}\right)$

Pattern more or less is to get the guy who is WHAT?

- Frank: Imagine a see-saw; these guys are weights on the see-saw.
- From looking at overlaided graphs of the pdfs (pdf-graph.R), it appears that the legislator who is chosen first is the one who has the most probability mass just to the left of 0 (it's probability mass to the right of zero that counts for winning the vote).
  - But this is not quite right: sometimes the non-negativity constraint binds on the PDF of that variables (as in the case of $\alpha = -0.3$ and $W_B = 8$).

The only intuition I see for why the next legislator is added in is that it's when the additional expense is justified by the increase in winning probability, conditional on the rule that 2+ legislators have to have the same gross ideal point

- I think it's actually that the next legislator is added when the non-negativity constraint stops binding; that is, when W_B becomes high enough that the additional probability of winning is worth the additional cost
- What's the intuition for why they should be equal? As far as I can tell, it's that the PDF is unimodal. If the ideal points are very close, you don't want to continue moving one and leave the other behind because the marginal increase will be greater for whichever is to the left.
- **Next question: How general is this result? That is, how much asymmetry can I have in the logistic distribution, βs, and for how broad a class of distributions would it hold? My conjecture is that it leans on having an interior single mode**

Math for Discrete Model

The math that follows from Equation 5:

- Then Equation 5 is equivalent to:

$$\frac{1}{1+e^{-X}} + \frac{1}{1+e^{-Y}} - 2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}} \tag{11}$$

$$\frac{2+e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)} - 2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}}$$

$$\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}}$$

$$\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\frac{e^{-Z}}{\left(1+e^{-Z}\right)^2} = \frac{1}{W_B} \tag{12}$$

Likewise, we have

$$\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-Y}}{\left(1+e^{-Y}\right)^2} = \frac{1}{W_B} \tag{13}$$

and

$$\frac{e^{-Y}+e^{-Z}}{\left(1+e^{-Y}\right)\left(1+e^{-Z}\right)}\frac{e^{-X}}{\left(1+e^{-X}\right)^2} = \frac{1}{W_B} \tag{14}$$

Setting the left-hand size of these last two equations (the FOCs for the bribes for 0 and $\frac{1}{2}$ equal to each other,

$$\frac{e^{-Y}+e^{-Z}}{\left(1+e^{-Y}\right)\left(1+e^{-Z}\right)}\frac{e^{-X}}{\left(1+e^{-X}\right)^2} = \frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}$$

$$\frac{e^{-X-Y}+e^{-X-Z}}{1+e^{-X}} = \frac{e^{-X-Y}+e^{-Y-Z}}{1+e^{-Y}}$$

$$e^{-X-Y}+e^{-X-Z}+e^{-X-2Y}+e^{-X-Y-Z} = e^{-X-Y}+e^{-Y-Z}+e^{-2X-Y}+e^{-X-Y-Z}$$

$$e^{-X-Z}+e^{-X-2Y} = e^{-Y-Z}+e^{-2X-Y}$$

$$e^{-Z}+e^{-2Y} = e^{X-Y-Z}+e^{-X-Y}$$

$$e^{Y-Z}+e^{-Y} = e^{X-Z}+e^{-X} \tag{15}$$

Equation 15 can be rearranged into two similar forms. First:

$$e^{Y-Z}-e^{X-Z} = e^{-X}-e^{-Y}$$

$$e^{Y}-e^{X} = e^{Z}\left(e^{-X}-e^{-Y}\right) \tag{16}$$

Second, multiplying through by zero:

$$e^{X}-e^{Y} = e^{Z}\left(e^{-Y}-e^{-X}\right) \tag{17}$$

In this case where the non-negativity constraint binds for the variable associated with Z, that is $b\left(\frac{1}{2}\right)$, there are three possibilities, $Z = 0$, $Z > 0$ and $Z < 0$. I will take each in turn.

1. When $Z = 0$, $e^Z = 1$. Then the only values of X and Y with which both Equation 16 and 17 are consistent are $X = Y$.

   - Note that this is the only case in which $e^{-2Z} = 1$, i.e. $\frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} = 0$. In the other two cases, the right hand side of 22 is positive so that the two second derivatives on the left hand side must be either both positive or both negative. This means that X and Y must be either both positive or both negative.

2. When $Z > 0$, $e^Z > 1$. In this case, X and Y must be positive since they are greater than Z.

   - Equation 16 is consistent with $Y \geq X \geq 0$.
   - Equation 17 is consistent with $X \geq Y \geq 0$.
   - Since both equations must hold, we have $X = Y \geq 0$

3. When $Z < 0$, $e^Z < 1$. In this case,

   - Equation 16 is consistent with $X \geq Y \geq 0$ or $0 \geq Y \geq X$.
   - Equation 17 is consistent with $Y \geq X \geq 0$ or $0 \geq X \geq Y$.
   - Since both equations must hold, we have either $X = Y \geq 0$ or $X = Y \leq 0$

Once we know that both $X = Y$ (or $X = Z$), go back to one of the equations like 14 and substitute for one of the variables. Given that the third equals zero, for a particular value of (α, W_B), we can solve for the two relevant bribes.

Second Order Conditions

Perhaps more importantly, let's look at the second order condition (using the same substitutions at the top of the previous page). The first derivative of the objective function w.r.t. $b(\frac{1}{2})$ is

$$W_B \frac{e^{-Z}}{(1+e^{-Z})^2}\left[\frac{1}{1+e^{-X}} + \frac{1}{1+e^{-Y}} - 2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\right] - 1 \tag{18}$$

$$W_B \frac{e^{-Z}}{(1+e^{-Z})^2}\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right] - 1 \tag{19}$$

Now, the second derivative:

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}2(1+e^{-Z})e^{-Z} - (1+e^{-Z})^2 e^{-Z}}{(1+e^{-Z})^4}\right]$$

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}2e^{-Z} - (1+e^{-Z}) e^{-Z}}{(1+e^{-Z})^3}\right]$$

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{2e^{-Z} - (1+e^{-Z})\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{2e^{-Z} - 1 - e^{-Z}\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{e^{-Z} - 1\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B \left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}}{(1+e^{-Z})^3}\right]\left\{e^{-Z} - 1\right\}$$

Everything except for the expression in the curly braces is positive. The expression in the curly braces is positive when $Z < 0$ and negative when $Z > 0$. So this objective function appears to have an inflection point at $Z = 0$.

We also need cross partials. Let's start with the first derivative for $b\left(-\frac{1}{2}\right)$, analogous to Expression 19 for $b\left(\frac{1}{2}\right)$:

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{e^{-X}+e^{-Z}}{(1+e^{-X})(1+e^{-Z})}\right] - 1 \tag{20}$$

Taking the derivative of this expression with respect to $b(0)$:

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{(1+e^{-X})(1+e^{-Z})(-1)e^{-X} - (e^{-X}+e^{-Z})(-e^{-X} - e^{-X-Z})}{(1+e^{-X})^2(1+e^{-Z})^2}\right]$$

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{(e^{-X}+e^{-Z})(e^{-X} + e^{-X-Z}) - (1+e^{-X})(1+e^{-Z})e^{-X}}{(1+e^{-X})^2(1+e^{-Z})^2}\right]$$

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{\left(e^{-2X}+e^{-2X-Z}+e^{-X-Z}+e^{-X-2Z}\right)-\left(1+e^{-X}+e^{-Z}+e^{-X-Z}\right)e^{-X}}{(1+e^{-X})^2(1+e^{-Z})^2}\right]$$

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{e^{-2X}+e^{-2X-Z}+e^{-X-Z}+e^{-X-2Z}-e^{-X}-e^{-2X}-e^{-X-Z}-e^{-2X-Z}}{(1+e^{-X})^2(1+e^{-Z})^2}\right]$$

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{e^{-X-2Z}-e^{-X}}{(1+e^{-X})^2(1+e^{-Z})^2}\right]$$

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{e^{-X}\left(e^{-2Z}-1\right)}{(1+e^{-X})^2(1+e^{-Z})^2}\right] \tag{21}$$

The sign of this expression is the same as the sign of $e^{-2Z}-1$. It's 0 when $Z=0$, positive when $Z<0$ and negative when $Z>0$.

The SOCs when one non-negativity constraint binds; bordered Hessian is 4x4.

$$\begin{matrix} 0 & \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial g_3}{\partial b(-0)} & \frac{\partial g_3}{\partial b(\frac{1}{2})} \\ \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\ \frac{\partial g_3}{\partial b(0)} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\ \frac{\partial g_3}{\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2} \end{matrix}$$

Evaluating the constraint terms:

$$\begin{matrix} 0 & 0 & 0 & -1 \\ 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\ 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\ -1 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2} \end{matrix}$$

The second order condition is on the last two principal minors. The largest must be negative. The next to last should then be positive.

- First we need the determinant of the next-to-last principal minor, the 3x3 matrix that is the upper-lefthand corner of this matrix. This determinant should be non-negative. It's actually equal to zero because of the row / column of zeros, so gives no further conditions (note that this is NOT the case when either of the other two bribes are the single non-negative bribe).

- Second, we need the determinant of the last principal minor, which is the whole 4x4 matrix. This determinant can be simplified as follows:

$$0\cdot|\text{something}| - 0\cdot|\text{something}| + 0\cdot|\text{something}| - (-1)\cdot\begin{vmatrix} 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} \\ 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} \\ -1 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2} & \frac{\partial^2 L}{\partial b(0)\partial b\left(-\frac{1}{2}\right)} \\ 0 & \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} \\ -1 & \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)\partial b\left(\frac{1}{2}\right)} & \frac{\partial^2 L}{\partial b(0)\partial b\left(\frac{1}{2}\right)} \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2} & \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)\partial b(0)} \\ \frac{\partial^2 L}{\partial b(0)\partial b\left(-\frac{1}{2}\right)} & \frac{\partial^2 L}{\partial b(0)^2} \end{vmatrix}$$

$$= -1 \cdot \left(\frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2} - \left[\frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right) \partial b\left(0\right)} \right]^2 \right)$$

$$= \left[\frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right) \partial b\left(0\right)} \right]^2 - \frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2}$$

We need this determinant to be non-positive, that is

$$\frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2} \geq \left[\frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right) \partial b\left(0\right)} \right]^2 \tag{22}$$

The right hand side is non-negative because it's a square. The left hand side must then be non-negative.

Note that nothing useful seems to come out of trying to expand and then simplify this expression. I get:

$$2e^{-X}e^{-Y} + 2 > e^{-2Z} + e^{-X}e^{-2Z} + e^{-Y}e^{-2Z} + e^{-X}e^{-Y}e^{-2Z}$$

In general

- n is the number of choice variables ($n = 3$ in my case for the three bribes)
- If there are k inequality constraints ($k = 3$ in my case, for 3 non-negativity constraints)
- And g_1 through g_b are binding (i.e. b is the number of bribes that are zero)
- m is the number of equality constraints (0 in my case)
- Then there are restrictions on the last $(n - b + m)$ minors
  - If one bribe is zero, then need the last two $3 - 1 = 2$ minors to alternate in sign with the last being $(-1)^n = (-1)^3 = -1 < 0$, i.e. negative

  - If no bribes are zero, there are restrictions on the last three minors, but the matrix is 3x3 (no constraint rows)
  - If two bribes are binding, there is only a restriction on the last minor, but it's a 5x5 matrix.
    * But can't wipe out $\frac{1}{W_B}$ using two FOCs because only one holds with equality
- If all three non-negativity constraints bind, we know all bribes are zero. No restriction on SOC?

One Non-negative Bribe

When $\alpha = 0$ and $WB = 8$, we find that $b\left(-\frac{1}{2}\right)$ and $b\left(\frac{1}{2}\right)$ are zero. Only $b(0)$ is positive.

- Only the FOC for X holds with equality (i.e. $\lambda_X = 0$)

$$\frac{e^{-Y} + e^{-Z}}{\left(1 + e^{-Y}\right)\left(1 + e^{-Z}\right)} \frac{e^{-X}}{\left(1 + e^{-X}\right)^2} = \frac{1}{W_B}$$

- We know exactly what Y and Z will be since $b\left(-\frac{1}{2}\right)$ and $b\left(\frac{1}{2}\right)$ are zero. Given WB, we can solve for X and then $b(0)$.

- Looking at the SOC, when there are three inequality constraints and 2 bind, we only need a condition on the last principal minor of a suitably constructed bordered Hessian.
  - It should be negative because $n = 3$
  - It's a 5x5 matrix because there are two binding inequality constraints.

The SOCs when one non-negativity constraint binds; bordered Hessian is 5x5.

$$\begin{array}{ccccc}
0 & 0 & \frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial g_1}{\partial b(-0)} & \frac{\partial g_1}{\partial b(\frac{1}{2})} \\
0 & 0 & \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial g_3}{\partial b(-0)} & \frac{\partial g_3}{\partial b(\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(0)} & \frac{\partial g_3}{\partial b(-0)} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
\frac{\partial g_1}{\partial b(\frac{1}{2})} & \frac{\partial g_3}{\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}$$

Evaluating the constraint terms:

$$\begin{array}{ccccc}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
-1 & 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
0 & -1 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}$$

Solving for the determinant of this matrix:

$$(-1) \cdot \begin{vmatrix}
0 & 0 & 0 & -1 \\
-1 & 0 & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
0 & -1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{vmatrix} = (-1) \cdot (-1) \cdot (-1) \begin{vmatrix}
-1 & 0 & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(0)^2} \\
0 & -1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})}
\end{vmatrix} =$$

$$(-1) \cdot (-1) \begin{vmatrix}
0 & \frac{\partial^2 L}{\partial b(0)^2} \\
-1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})}
\end{vmatrix} = 0 - (-1) \cdot \frac{\partial^2 L}{\partial b(0)^2} = \frac{\partial^2 L}{\partial b(0)^2}$$

- Thus we need $\frac{\partial^2 L}{\partial b(0)^2} < 0$
  - This boils down to $\left(e^{-X} - 1\right) < 0$
  - Implies $X > 0$
  - i.e. $-\alpha + b(0) > 0$ or $b(0) > \alpha$
- It is NOT the case that $b(0)$ has to be such that $X = Y$ even though $b\left(\frac{1}{2}\right) = 0$. In fact, it seems in numerical examples that this does *not* happen.
  - However, I do see that this existence of the one non-negative bribe (NNB) is not common. It exists for $\alpha = 0$ and WB = 8, but goes away as soon as $\alpha = 0.1$ or W_B = 9.
  - In both cases, I can see that changing the parameter would push the value of $b(0)$ over 0.5, making $X > Y$. This is when the solution switches to 2 NNB's.
  - CONJECTURE: Group B chooses 1 NNB until it would be large enough to make $X > Y$ (when $\alpha \approx 0$) or $Z > Y$ (when α large and negative), then switch to 2 NBB
    - This is not strictly true around -0.7: but I can see for $\alpha = .75$, when $W_B = 10$ it's only X, then for $W_B = 11$ it's only Z. This is because the non-negativity constraint for Z binds at $W_B = 10$, as it does for Y. X is the only variable there for which it does not bind, and it is worthwhile shifting the middle legislator just a bit.
    - When α is large and positive, go straight from 0 NNB to 2 NNB
- Can write out SOC and FOC, sub together to get

$$\frac{1}{1+e^{-X}}\left(e^{-X} - 1\right) < 0$$

Not sure what this means

All Three Bribes are Non-negative

In many cases all three bribes are non-negative. Here are very loose results

- I always observe that $X = Y = Z$. I think it's easy to show this analytically.
- From the FOCs, when X and Y are non-negative, it can be shown that a set of inequalities just below Equations 16 and 17 must hold.
- The SOCs when 0 non-negativity constraints bind are on the 3x3 matrix of second derivatives. There are three conditions:

$$\frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2}\left\{\frac{\partial^2 L}{\partial b(0)^2}\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} - \left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)\partial b\left(0\right)}\right]^2\right\} \leq 0$$

$$\frac{\partial^2 L}{\partial b(0)^2}\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} - \left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)\partial b\left(0\right)}\right]^2 \geq 0$$

$$\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} \leq 0$$

  The last one says that Y must be non-negative. The first and second together say that Z must be non-negative.

- Y non-negative combined with the inequalities below Equations 16 and 17 tells us that $X = Y > 0$.
- Similar logic that leads to those inequalities can be used to derive analagous inequalities for X and Z (or Y and Z). Then Z non-negative tells us $Z = Y > 0$ so that $X = Y = Z > 0$.

Now as to when this case occurs

- When $X = Z$ (the two left-most legislators), bribe to $\frac{1}{2}$ is added in before X would move to the right of Y, i.e. when $X \to 0.5$.
- This is NOT the case when X and Y are the two non-negative bribes. They can get very large before the left-most legislator is bribed.

Simultaneous Model

For most of the parameter space, there is no PSNE. What must a MSNE look like?

- In a fully mixed strategy equilibrium, a probability distribution F_b over strategies

  $$\boldsymbol{b_1} = \left(b_{-.5,1}, b_{0,1}, b_{.5,1}\right), \boldsymbol{b_2}, \ldots$$

  must make Vote Buyer A indifferent between at least two vectors of strategies

  $$\boldsymbol{a_1} = \left(a_{-.5,1}, a_{0,1}, a_{.5,1}\right), \boldsymbol{a_2}$$

Numerical Results

For all of these, $\beta = 1$ and there are three legislators with $b \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$.

- Below, the column headings are $v(z)$ before the bribe.
- That is, from left to right, $-Z, -Y, -X$.

		-2	-1.5	-1
For $\alpha = -1.5$	up to WB=19	0	0	0
	only WB = 20	0	0	just a little

		-1.8	-1.3	-0.8
For $\alpha = -1.3$	up to WB=15	0	0	0
	WB = 16 to 20	0	0	just a little

		-1.6	-1.1	-0.6
For $\alpha = -1.1$	up to WB=13	0	0	0
	WB = 14 to 15	0	0	just a little
	WB = 16 to 20	0	some, always increasing	some + .5

		-1.4	-0.9	-0.4
For $\alpha = -0.9$	up to WB=11	0	0	0
	WB = 12 to 13	0	0	just a little
	WB = 14 to 19	0	some, always increasing	some + .5
	WB = 20	some	some + .5	some + 1

		-1.1	-0.6	-0.1
For $\alpha = -0.6$	up to WB=8	0	0	0
	WB = 9 to 10	0	some	0
	WB = 11 to 14	0	some, always increasing	some + .5
	WB = 15 to 20	some	some + .5	some + 1

$\alpha = -0.6$ pattern holds to $\alpha = -0.2$

		-0.6	-0.1	0.4
For $\alpha = -0.1$	up to WB=7	0	0	0
	WB = 8	0	some	0
	WB = 9	some	some + .5	0
	WB = 15 to 20	some	some + .5	some + 1

$\alpha = -0.1$ pattern holds to $\alpha = 0.5$ at least

		0.0	0.5	1.0
For $\alpha = 0.5$	up to WB=8	0	0	0
	WB = 9 to 11	> 1	$> 1 + .5$	0
	WB = 12 to 20	significant	significant + .5	significant + 1

		0.6	1.1	1.6
For $\alpha = 1.1$	up to WB=13	0	0	0
	WB = 9 to 20	> 2	$> 2 + .5$	0

Going to explore in-depth case of $\alpha = 0.0$

	-0.5	0.0	0.5
up to WB=7	0	0	0
WB = 8	0	some	0
WB = 9	some	some + .5	0
WB = 10 to 20	some	some + .5	some + 1

Some Comparative Statics

Comparative statics

- Vote buyer B (3/25/15) — note these are not super useful yet — they're comparative statics of the best response function, but not of the equilibrium
  - Those for j, $a(j)$ and α don't involve the root term, so don't differ depending on whether we take the plus or minus version

$$\frac{\partial b(j)}{\partial j} = -\beta \qquad \frac{\partial b(j)}{\partial a(j)} = 1 \qquad \frac{\partial b(j)}{\partial \alpha} = 1$$

  - Those for β and σ_j *do* involve the root term. I will only display the version for the positive root; there is also a version with all the signs reversed

$$\frac{\partial b(j)}{\partial \beta} = -j + \sigma_j \sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)} - \frac{\sigma_j}{\sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)}}$$

$$\frac{\partial b(j)}{\partial \sigma_j} = \beta \sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)} - \frac{\beta}{\sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)}}$$

Notes from March

Notes from 3/16 Skype chat (Kristy and Sebastian)

- Seems like FOC for lobby will boil down to each one buying votes until marginal benefit = marginal cost
- May want to use some simplified rule / heuristic for lobby's decision: perhaps they reorder the legislators by their ± 2 std. dev. and make some decision based on that ordering who to lobby
- Will some kind of rule that looks like RMSE come out of the math? Is it possible to get anything closed-form at all?

Notes from 3/19 Skype chat (Kristy and Sebastian)

- We can use the data we have to horse-race this model against other hypotheses about how lobbyists distribute bribes
  - Allocate equally among legislators
  - Groseclose Snyder with full information: only one side pays
  - Our model with one dimension
    - Could we use the econometric model to benchmark to one where uncertainty disappears? Or, as it does theoretically, does the model have to change completely?
  - Our model with multiple dimensions
- We can think of this as looking for the effect of uncertainty on prices—we'd be pricing uncertainty relative to a model with uncertainty
  - "a metric in dollars of uncertainty"
  - this is a model of vote buying under uncertainty
- Given how different the environment with uncertainty is, Kristy should explore both the sequential model that parrots GS96 and a simultaneous model that is more like GH94 (menu auction)
- When mapping to the data, we're going to want to know from the model whether/when total contributions represent WTP.
  - It clearly doesn't in the case of certainty. One side pays nothing; the other pays either as much as is necessary to shut down its opponent, or nothing at all the necessary amount exceeds WTP
  - May be able to show it doesn't matter, that the proportion of total expenditures is a sufficient statistic

- Finish writing the model, and then see if we can use the estimates we already have to write a first, very rough draft