Political Uncertainty

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Groseclose and Snyder 1996

On Feb. 12, Sebastian and I agreed to focus efforts on finding a base model to facilitate empirical identification. I am pursuing Groseclose & Snyder (1996), "Buying Supermajorities," APSR

- For each legislator i, $v(i) = u_i(x) - u_i(s)$, measured in money; this is the reservation price of i
  - x is an alternative policy proposal; s is the status quo
  - WLOG, label legislators so that $v(i)$ is a non-increasing function
  - Note legislators only have preferences over how they vote, not over which alternative wins
- There are two vote buyers; each prefers to minimize total bribes paid while passing his preferred policy, but each would prefer to concede the issue rather than pay more than his WTP
  - A prefers x; W_A is A's willingness to pay (WTP) for x measured in money
  - B prefers s; W_B is B's WTP for s
- Bribe offer functions: $a(i)$ and $b(i)$ are A and B's offers to i. Legislators take these bribe offers as given and then vote for the alternative that maximizes their payoff
- A moves first; $a(i)$ is perfectly observable to B when he moves
- Goal: characterize SPNE in pure strategies
  - Assume unbribed legislators who are indifferent vote for s; all bribed legislators who are indifferent vote for whoever bribed them last
- Assume continuum of legislators on $\left[-\frac{1}{2}, \frac{1}{2}\right]$
- Assume W_A large enough that x wins in equilibrium (no uncertainty case)
- $m + \frac{1}{2}$ is fraction of legislators who vote for x as opposed to the status quo, s
- Results
  - Prop 1: three types of equilibria in which x wins; depend on size of W_B
  - Prop 2: m^* (the optimal coalition size) is unique, and provides three cases parameterizing its size in terms of W_B, $v(-\frac{1}{2})$ and $v(m^*)$
  - Prop 3/4: special case where $v(z) = \alpha - \beta z$

Our Extension to Uncertainty

General thoughts on extension to uncertainty

- I think, without uncertainty, you would estimate m^* as a function of the parameters of v and WTP
  - It's useful that m^* is unique. Not clear it would extend to case of uncertainty, but I think it's likely so I'm going to assume it for now
- I'm pretty sure this predicts that B should never pay anything when there is no uncertainty, but I don't see where they say it explicitly (I should read more carefully to verify)
  - Uncertainty should reverse this, right?
  - What is uncertainty? Make $v(z)$ stochastic is most natural
    - I'm going to start with linear parameterization of $v(z)$ and add uncertainty. For now, take same form for $v(z)$

      $$v(z) = \alpha - \beta \cdot z$$

      and take z as a random variable distributed normally with mean z and standard deviation σ_z. This is unfortunate nomenclature, since later we'll probably want to take this to a standard normal "Z"
      - This means that the legislators are ordered on the interval according to their ideal points, but there is acknowledged uncertainty surrounding their preferences
      - What is the source of this uncertainty? We take it to vary by legislator and also possibly by issue area when we extend the model to account for that
      - Uncertainty comes from: **log-rolling and cross-pressuring; length of voting record; electoral incentives that we can't control for**

Solving the Model

Backward induction (legislature moves last; B makes last bribe; A makes first bribe)

1. Legislature
   - Each legislator z will decide whether to vote for x or s given z, $a(z)$ and $b(z)$. Votes for x if

     $$v(z) = \alpha - \beta \cdot z + a(z) - b(z) > 0$$

     (whether the inequality is weak or strict depends on tie-breaking rules set out in the paper)
     - Let's start out by thinking of z as being a random variable distributed $N(z, \sigma_z)$

- Notice that a bribe from B is subtracted because it makes the alternative *less* attractive

- Payoff for vote buyer A if x wins is $U_A(x) - \int_{-\frac{1}{2}}^{\frac{1}{2}} a(j)dj$
- Payoff for vote buyer A if s wins is $U_A(s) - \int_{-\frac{1}{2}}^{\frac{1}{2}} a(j)dj$
- I think the cleanest way to write the condition for whether x wins is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{1}\left[v(j) \geq 0\right] dj \geq \frac{1}{2}$$

2. Vote buyer B

- GS assumption on vote buyers' objective is "each prefers to minimize total bribes paid while passing his preferred policy, but each would prefer to concede the issue rather than pay more than his WTP"
- This has to be adapted to our situation with uncertainty
  - Groseclose and Snyder formulation would suggest something like

$$\min_{b(z)} \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj \quad \text{subject to} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj \leq U_B(s) - U_B(x)$$

and however we write the "while passing his preferred policy" constraint

  - With asymmetric $v(z)$ or WTP parameters, it would be easy to get equilibria where only A or B buys votes. But we also have lots of outcomes where both buy votes.
  - They can easily both have positive probability of winning. But what do we need for this to be an equilibrium in this three stage game?
  - Uncertainty buys us a lot: no longer this knife edge condition of A pushing to the point that B buys no votes
- Let's start with the simple maximize expected value of winning (WTP times probability of winning) net of bribes:

$$\max_{b(j)} W_B \left\{ \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{1}\left[\alpha - \beta \cdot j + a(j) - b(j) \leq 0\right] dj \right] \geq \frac{1}{2} \right\} - \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj$$
$$\text{subject to} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj \leq W_B \quad \text{and} \quad b(j) \geq 0 \;\forall j \quad (1)$$

  - Problem is that the indicator function will never equal 1 when $J \sim N(j, \sigma_j)$
  - So what's a reasonable objective function?
    * Maximizing the total probability mass where $v(j) \leq 0$?

$$\max_{b(j)} W_B \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \Pr\left[\alpha - \beta \cdot j + a(j) - b(j) \leq 0\right] dj \right] - \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj$$
$$\text{subject to} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j)dj \leq W_B \quad \text{and} \quad b(j) \geq 0 \;\forall j \quad (2)$$

· Note the non-negativity constraint means you can't extract money from a legislator who is past the FOC point in order to reallocate to another legislator

· Some legislators who are past FOC point will not be lobbied

∗ Suppressing the constraint for now and using the fact that $J \sim N(j, \sigma_j)$, this can be re-written as

$$\max_{b(j)} W_B \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \Pr\left[\frac{\alpha + a(j) - b(j)}{\beta} \leq j \right] dj \right] - \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j) dj \tag{3}$$

or

$$\max_{b(j)} W_B \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \Pr\left[\frac{\frac{\alpha + a(j) - b(j)}{\beta} - j}{\sigma_j} \leq z \right] dj \right] - \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j) dj \tag{4}$$

where $Z \sim N(0,1)$.

∗ So, this can be written as

$$\max_{b(j)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[W_B \left(1 - \Phi\left(\frac{\frac{\alpha + a(j) - b(j)}{\beta} - j}{\sigma_j} \right) \right) - b(j) \right] dj$$

$$\text{subject to} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} b(j) dj \leq W_B \quad \text{and} \quad b(j) \geq 0 \; \forall j \tag{5}$$

∗ If the realizations of Z are i.i.d. and the bribes are independent across legislators, (I believe) this can be maximized pointwise–except for the constraint

∗ The FOC for each j is then

$$-W_B \cdot \frac{\partial \Phi\left(\frac{\frac{\alpha + a(j) - b(j)}{\beta} - j}{\sigma_j} \right)}{\partial b(j)} = 1 \tag{6}$$

as long as the constraint is satisfied.

· I don't know how decisions are made if the constraint binds—how the rationing happens

· If the constraint is not binding, vote buyer B will pay all legislators (we could put in some kind of fixed cost of dealing with each legislator to short circuit this outcome) except those who are already so favorable to him that they are past the FOC point

· Perhaps there's something in equilibrium I'm not seeing yet that will reduce further the number of legislators that are paid

- Let's examine the FOC, assuming pointwise maximization is okay as in Equation 6
  - It's easier for me to think of the CDF in terms of the non-standardized Normal distribution before taking the derivative. Then, if my math is right, the FOC becomes

$$\frac{1}{\beta} \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{\left(\frac{\alpha + a(j) - b(j)}{\beta} - j \right)^2}{2\sigma_j^2}} = \frac{1}{W_B} \tag{7}$$

* This just says the pdf has to equal β divided by the willingness to pay
* Rearranging to solve for $b(j)$:

$$e^{-\frac{\left(\frac{\alpha+a(j)-b(j)}{\beta}-j\right)^2}{2\sigma_j^2}} = \frac{\beta\sigma_j\sqrt{2\pi}}{W_B} \tag{8}$$

$$-\left(\frac{\alpha+a(j)-b(j)}{\beta}-j\right)^2 = 2\sigma_j^2 \ln\left(\frac{\beta\sigma_j\sqrt{2\pi}}{W_B}\right) \tag{9}$$

Since the log statement on the RHS will almost certainly be negative, I can take the negative sign to the right and take the square root without worrying about imaginary numbers

$$\frac{\alpha+a(j)-b(j)}{\beta} - j = \pm\sqrt{-2\sigma_j^2 \ln\left(\frac{\beta\sigma_j\sqrt{2\pi}}{W_B}\right)} \tag{10}$$

· Notice there are two roots: could be to the left of the mode (less than 0.5 cumulative probability), or to the right (more than 0.5 cumulative probability)

$$\alpha + a(j) - b(j) = \pm\beta\sigma_j\sqrt{-2\ln\left(\frac{\beta\sigma_j\sqrt{2\pi}}{W_B}\right)} + j\beta \tag{11}$$

$$b(j) = \alpha + a(j) - j\beta \pm \beta\sigma_j\sqrt{-2\ln\left(\frac{\beta\sigma_j\sqrt{2\pi}}{W_B}\right)} \tag{12}$$

$$b(j) = \alpha + a(j) - \beta j \pm \beta\sigma_j\sqrt{2\left(\ln W_B - \ln\beta\sigma_j - \ln\sqrt{2\pi}\right)} \tag{13}$$

* So, the bribe for legislator j takes legislator j's preferences back to zero in expectation (the first three terms) and then relative to that, either adds or subtracts a term that involves WTP, variance and β
  · Do we have a way of distinguishing between the two? They have very different implications for the CDF, even though the same implications for the pdf
  · It makes more intuitive sense that the bribe would be increasing in W_B (WTP)
  · Perhaps we derive the different implications and let the data speak
  · All the comparative statics will come out with same magnitude, both signs
  · SOC is obviously okay (essentially $-xe^{\frac{x^2}{2}}$)

– How do we deal with this r.v. with continuous support when the range of ideal points is posited to be finite ($[-\frac{1}{2}, \frac{1}{2}]$)?

3. Vote buyer A

• Whatever we decide for vote buyer B will be the same for vote buyer A

- Adapting Expression 5 for Vote Buyer A

$$\max_{a(j)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[W_A \cdot \Phi\left(\frac{\frac{\alpha+a(j)-b(j)}{\beta} - j}{\sigma_j} \right) - a(j) \right] dj$$

$$\text{subject to} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} a(j)dj \leq W_A \quad \text{and} \quad a(j) \geq 0 \ \forall j \tag{14}$$

  – Major problem if we solve this game sequentially: when substituting the FOC for Vote Buyer B (Equation 13) into Vote Buyer A's objective function, the $a(j)$'s cancel out ($b(j)$ is linear in $a(j)$) so the benefit term is constant in $a(j)$
  – Static best response

$$W_A \cdot \frac{\partial \Phi\left(\frac{\frac{\alpha+a(j)-b(j)}{\beta} - j}{\sigma_j} \right)}{\partial a(j)} = 1 \tag{15}$$

  – Math looks exactly the same as Eqn. 7 to 13 except with W_A replacing W_B and all the opposite signs, so

$$a(j) = -\alpha + b(j) + \beta j \pm \beta\sigma_j \sqrt{2\left(\ln W_A - \ln \beta\sigma_j - \ln \sqrt{2\pi}\right)} \tag{16}$$

- Plugging either one of the vote buyers' FOCs into the other just implies that $W_A = W_B$, but doesn't say anything about the level of bribes. Use some kind of pareto ranking to choose one?
  – Note that to get a solution, I've taken both FOCs to have the same root (either both positive or both negative; Otherwise, we need a very special combination of parameters to get an equilibrium)
  – Intuition for this equilibrium: each vote buyer exactly brings legislator j back to neutral (compensating both for inherent preference $\alpha - \beta \cdot j$) and the other vote buyer's bribe; they also each tack on a term related to WTP, β, and σ_j, but these terms cancel out as well as long as $W_A = W_B$. If $W_A \neq W_B$, then there is not such an interior equilibrium as far as I can tell.
    * Need to look for more equilibria, perhaps in mixed strategies.
    * Especially a way to get asymmetric pure strategies (I'm happy to have asymmetry between W_A and W_B if necessary)

Discrete Model

Discrete version (Conversation between Kristy and Frank, 3/27/15)

- Purpose: try to understand what the right objective function is. Does maximizing total probability also maximize the probability of winning?
  - Note that this model *may* introduce interdependence between the bid functions that does not exist in the continuous model
  - What is not clear is whether this happens when the WTP constraint is not binding, which I'm going to assume from the start, and maybe impose later
- Frank prefers to write
$$v(z) = \alpha - \beta z + \varepsilon_z + a(z) - b(z)$$
  - Note that, in a probit we will have to worry about separately identifying the parameters. So for now, will assume ε does not vary in z
  - Also going to ignore Vote Buyer A's choice for now, so we have
$$v(z) = \alpha - \beta z + \varepsilon - b(z)$$
- Then the probability that legislator z votes against the new proposal is
$$\Pr\left[v(z) \leq 0\right] = \Pr\left[\alpha - \beta z + \varepsilon - b(z) \leq 0\right] = \Pr\left[\varepsilon \leq \beta z - \alpha + b(z)\right]$$
Assuming $\varepsilon \sim N(0,1)$, the probability that legislator z votes "no" is $\Phi\left(\beta z - \alpha + b(z)\right)$
- Take a model with three legislators with ideal points at $z = -\frac{1}{2}$, $z = 0$ and $z = \frac{1}{2}$. Then the three probabilities are
  - $\Phi\left(-\frac{\beta}{2} - \alpha + b(-\frac{1}{2})\right)$, $\Phi\left(-\alpha + b(0)\right)$, and $\Phi\left(\frac{\beta}{2} - \alpha + b(\frac{1}{2})\right)$
  - Whether a legislator z votes "no" is a random variable, denote it $X(z)$, distributed Bernoulli with this probability:
$$X(z) \sim \text{Bernoulli}(p(z)) \quad \text{where} \quad p(z) = \Phi\left(\beta z - \alpha + b(z)\right)$$
  - Need to max probability of at least two voting "no": $\Pr(S \geq 2)$ where $S = X\left(-\frac{1}{2}\right) + X\left(0\right) + X\left(\frac{1}{2}\right)$

– Thus, maximization problem for Vote Buyer B (in absence of Vote Buyer A) is

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Big[\Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=1\right)\left(X\left(\frac{1}{2}\right)=0\right)+$$
$$\Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=0\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)+$$
$$\Pr\left(X\left(-\frac{1}{2}\right)=0\right)\Pr\left(X\left(0\right)=1\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)+$$
$$\Pr\left(X\left(-\frac{1}{2}\right)=1\right)\Pr\left(X\left(0\right)=1\right)\Pr\left(X\left(\frac{1}{2}\right)=1\right)\Big]-\sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (17)$$

Substituting from the definition of X,

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Big[\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)\Phi\left(-\alpha+b(0)\right)\left(1-\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\right)+$$
$$\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)\left(1-\Phi\left(-\alpha+b(0)\right)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)+$$
$$\left(1-\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)\right)\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)+$$
$$\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\Big]-\sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (18)$$

Substituting from the definition of X but substituting Logit for Probit and using the shorthand notation on the top of page 12,

$$\max_{b(-\frac{1}{2}),b(0),b(\frac{1}{2})} W_B\Big[\frac{1}{1+e^{-Z}}\frac{1}{1+e^{-X}}\left(1-\frac{1}{1+e^{-Y}}\right)+$$
$$\frac{1}{1+e^{-Z}}\left(1-\frac{1}{1+e^{-X}}\right)\frac{1}{1+e^{-Y}}+$$
$$\left(1-\frac{1}{1+e^{-Z}}\right)\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}+$$
$$\frac{1}{1+e^{-Z}}\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\Big]-\sum_{j\in\{-\frac{1}{2},0,\frac{1}{2}\}} b(j) \quad (19)$$

The FOC wrt to $b\left(-\frac{1}{2}\right)$ is (in terms of probit)

$$W_B\Bigg[\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)^2}{2}}\Phi\left(-\alpha+b(0)\right)\left(1-\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\right)+$$
$$\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)^2}{2}}\left(1-\Phi\left(-\alpha+b(0)\right)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)-$$
$$\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)^2}{2}}\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)+$$
$$\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)^2}{2}}\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\Bigg]=1 \quad (20)$$

Simplifying:

$$W_B\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)^2}{2}}\Bigg[\Phi\left(-\alpha+b(0)\right)\left(1-\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\right)+$$
$$\left(1-\Phi\left(-\alpha+b(0)\right)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)-\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)+$$
$$\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)\Bigg]=1 \quad (21)$$

Simplifying further:

$$\Phi\left(-\alpha+b(0)\right)+\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)-2\Phi\left(-\alpha+b(0)\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)$$
$$=\frac{\sqrt{2\pi}}{W_B\,e^{-\frac{\left(-\frac{\beta}{2}-\alpha+b\left(-\frac{1}{2}\right)\right)^2}{2}}} \quad (22)$$

The two symmetric conditions would be:

$$\Phi\left(-\alpha+b(0)\right)+\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)-2\Phi\left(-\alpha+b(0)\right)\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)$$
$$=\frac{\sqrt{2\pi}}{W_B\,e^{-\frac{\left(\frac{\beta}{2}-\alpha+b\left(\frac{1}{2}\right)\right)^2}{2}}} \quad (23)$$

$$\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)+\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)-2\Phi\left(-\frac{\beta}{2}-\alpha+b(-\frac{1}{2})\right)\Phi\left(\frac{\beta}{2}-\alpha+b(\frac{1}{2})\right)$$
$$=\frac{\sqrt{2\pi}}{W_B\,e^{-\frac{(-\alpha+b(0))^2}{2}}} \quad (24)$$

- Some notes

- If we divide through by 2, the left side is the average minus the product
- If that is set equal to zero, the only solutions (for just one equation) are at (0,0) and (1,1). The right hand side modifies that.
- Need to check SOC at some point
- Remember, I'm ignoring the constraints on WTP and non-negativity of bribes
- First I'm going to try to get closed form answer / intuition by substituting logistic for normal CDF/PDF

• Substituting logistic for normal in Equation 22

$$\frac{1}{1+e^{\alpha-b(0)}}+\frac{1}{1+e^{-\frac{\beta}{2}+\alpha-b\left(\frac{1}{2}\right)}}-2\frac{1}{1+e^{\alpha-b(0)}}\frac{1}{1+e^{-\frac{\beta}{2}+\alpha-b\left(\frac{1}{2}\right)}}=\frac{\left(1+e^{\frac{\beta}{2}+\alpha-b\left(-\frac{1}{2}\right)}\right)^2}{W_B\, e^{\frac{\beta}{2}+\alpha-b\left(-\frac{1}{2}\right)}} \tag{25}$$

- A LOT of math (I will put it somewhere else) leads to three equations:

$$[-\alpha+b(0)]=\left[\frac{\beta}{2}-\alpha+b\left(\frac{1}{2}\right)\right]+\left[-\frac{\beta}{2}-\alpha+b\left(-\frac{1}{2}\right)\right]$$

$$\left[\frac{\beta}{2}-\alpha+b\left(\frac{1}{2}\right)\right]=[-\alpha+b(0)]+\left[-\frac{\beta}{2}-\alpha+b\left(-\frac{1}{2}\right)\right]$$

$$\left[-\frac{\beta}{2}-\alpha+b\left(-\frac{1}{2}\right)\right]=\left[\frac{\beta}{2}-\alpha+b\left(\frac{1}{2}\right)\right]+[-\alpha+b(0)]$$

- A little more math implies that all three quantities in brackets must equal zero
  * That is, Vote Buyer B's bribe should make each legislator just indifferent between the new proposal and the status quo
  * First question: why doesn't this depend on willingness-to-pay?
    · FOC says each bribe should be calibrated so that W_B·marginal contribution to probability = 1, or $\frac{\partial\text{probability}}{\partial\text{bribe}}=\frac{1}{W_B}$
    · Something happens when all three conditions come together...what's the intuition?
  * To do next: see what non-negativity of bribes implies
    · Also, insert Vote Buyer A's behavior
    · And give the various legislators different variances

Adding Non-negativity Constraints for Bribes

The Lagrangian for the problem is the same as Expression 19, adding on $\lambda_X b(0)+\lambda_Y b\left(\frac{1}{2}\right)+\lambda_Z b\left(-\frac{1}{2}\right)$ The full set of FOCs (in logistic form) is then:

$$W_B\frac{e^{-X}}{(1+e^{-X})^2}\left[\frac{e^{-Z}+e^{-Y}}{(1+e^{-Z})(1+e^{-Y})}\right]-1+\lambda_X=0$$

$$W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\right]-1+\lambda_Y=0$$

$$W_B \frac{e^{-Z}}{\left(1+e^{-Z}\right)^2}\left[\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\right]-1+\lambda_Z=0$$

$$b(0) \geq 0 \quad b\left(\frac{1}{2}\right) \geq 0 \quad b\left(-\frac{1}{2}\right) \geq 0$$

$$\lambda_X \geq 0 \quad \lambda_Y \geq 0 \quad \lambda_Z \geq 0$$

$$\lambda_X \cdot b(0)=0 \quad \lambda_Y \cdot b\left(\frac{1}{2}\right)=0 \quad \lambda_Z \cdot b\left(-\frac{1}{2}\right)=0$$

There are 8 patterns of positive and zero components in the 3-dimensional vector of bribes, $b' = \left(b(0), b\left(\frac{1}{2}\right), b\left(-\frac{1}{2}\right)\right)$.

Note that the complementary slackness condition implies that either a bribe is zero, or its multiplier is zero (or both).

Simulations show that for many sets of parameters, $b(0)$ and $b\left(\frac{1}{2}\right)$ are positive while $b\left(-\frac{1}{2}\right)=0$.

- Then $Z=-\frac{\beta}{2}-\alpha+b\left(-\frac{1}{2}\right)=-\frac{\beta}{2}-\alpha$
- λ_Z is likely positive
- $\lambda_X=\lambda_Y=0$

Math for Discrete Model

The math that follows from Equation 25:

- Let

  1. $X = -\alpha + b(0)$
  2. $Y = \frac{\beta}{2} - \alpha + b\left(\frac{1}{2}\right)$
  3. $Z = -\frac{\beta}{2} - \alpha + b\left(-\frac{1}{2}\right)$

- Then Equation 25 is equivalent to:

$$\frac{1}{1+e^{-X}} + \frac{1}{1+e^{-Y}} - 2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}} \tag{26}$$

$$\frac{2+e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)} - 2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}}$$

$$\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)} = \frac{\left(1+e^{-Z}\right)^2}{W_B\, e^{-Z}}$$

$$\frac{e^{-X}+e^{-Y}}{\left(1+e^{-X}\right)\left(1+e^{-Y}\right)}\frac{e^{-Z}}{\left(1+e^{-Z}\right)^2} = \frac{1}{W_B} \tag{27}$$

Likewise, we have

$$\frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-Y}}{\left(1+e^{-Y}\right)^2} = \frac{1}{W_B} \tag{28}$$

and

$$\frac{e^{-Y}+e^{-Z}}{\left(1+e^{-Y}\right)\left(1+e^{-Z}\right)}\frac{e^{-X}}{\left(1+e^{-X}\right)^2} = \frac{1}{W_B} \tag{29}$$

Setting the left-hand size of these last two equations (the FOCs for the bribes for 0 and $\frac{1}{2}$ equal to each other,

$$\frac{e^{-Y}+e^{-Z}}{\left(1+e^{-Y}\right)\left(1+e^{-Z}\right)}\frac{e^{-X}}{\left(1+e^{-X}\right)^2} = \frac{e^{-X}+e^{-Z}}{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)}\frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}$$

$$\frac{e^{-X-Y}+e^{-X-Z}}{1+e^{-X}} = \frac{e^{-X-Y}+e^{-Y-Z}}{1+e^{-Y}}$$

$$e^{-X-Y}+e^{-X-Z}+e^{-X-2Y}+e^{-X-Y-Z} = e^{-X-Y}+e^{-Y-Z}+e^{-2X-Y}+e^{-X-Y-Z}$$

$$e^{-X-Z}+e^{-X-2Y} = e^{-Y-Z}+e^{-2X-Y}$$

$$e^{-Z}+e^{-2Y} = e^{X-Y-Z}+e^{-X-Y}$$

$$e^{Y-Z}+e^{-Y} = e^{X-Z}+e^{-X} \tag{30}$$

Equation 30 can be rearranged into two similar forms. First:

$$e^{Y-Z} - e^{X-Z} = e^{-X} - e^{-Y}$$

$$e^{Y} - e^{X} = e^{Z}\left(e^{-X} - e^{-Y}\right) \tag{31}$$

Second, multiplying through by zero:

$$e^{X} - e^{Y} = e^{Z}\left(e^{-Y} - e^{-X}\right) \tag{32}$$

In this case where the non-negativity constraint binds for the variable associated with Z, that is $b\left(-\frac{1}{2}\right)$, we know that $Z < 0$. Note that this is due to the construction of the problem. This means $e^{Z} < 1$. In this case,

- Equation 31 is consistent with $X > Y > 0$, $X > 0 > Y$, or $0 > X > Y$.
- Equation 32 is consistent with $Y > X > 0$, $Y > 0 > X$, or $0 > Y > X$.

The second order conditions will pin down the solution in this case of two non-negative bribes.

- SOCs imply that both X and Y are positive.

Once we know that both $X = Y > 0$ (or $X = Z > 0$), go back to one of the equations like 29 and substitute for one of the variables. Given that the third equals zero, for a particular value of (α, W_B), we can solve for the two relevant bribes.

Second Order Conditions

Perhaps more importantly, let's look at the second order condition (using the same substitutions at the top of the previous page). The first derivative of the objective function w.r.t. $b(-\frac{1}{2})$ is (in logistic form)

$$W_B \frac{e^{-Z}}{(1+e^{-Z})^2}\left[\frac{1}{1+e^{-X}}+\frac{1}{1+e^{-Y}}-2\frac{1}{1+e^{-X}}\frac{1}{1+e^{-Y}}\right]-1 \tag{33}$$

$$W_B \frac{e^{-Z}}{(1+e^{-Z})^2}\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]-1 \tag{34}$$

Now, the second derivative:

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}2(1+e^{-Z})e^{-Z}-(1+e^{-Z})^2e^{-Z}}{(1+e^{-Z})^4}\right]$$

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}2e^{-Z}-(1+e^{-Z})e^{-Z}}{(1+e^{-Z})^3}\right]$$

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{2e^{-Z}-(1+e^{-Z})\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{2e^{-Z}-1-e^{-Z}\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}\left\{e^{-Z}-1\right\}}{(1+e^{-Z})^3}\right]$$

$$W_B\left[\frac{e^{-X}+e^{-Y}}{(1+e^{-X})(1+e^{-Y})}\right]\left[\frac{e^{-Z}}{(1+e^{-Z})^3}\right]\left\{e^{-Z}-1\right\}$$

Everything except for the expression in the curly braces is positive. The expression in the curly braces is positive when $z \leq 0$ and negative when $z \geq 0$. So this objective function appears to have an inflection point at $z = 0$.

- When $z < 0$ (as when this legislator is naturally against you and you don't bribe her), this expression is negative

We also need cross partials. Let's start with the first derivative for $b\left(\frac{1}{2}\right)$, analogous to Expression 34 for $b\left(-\frac{1}{2}\right)$:

$$W_B \frac{e^{-Y}}{(1+e^{-Y})^2}\left[\frac{e^{-X}+e^{-Z}}{(1+e^{-X})(1+e^{-Z})}\right]-1 \tag{35}$$

Taking the derivative of this expression with respect to $b\,(0)$:

$$
\begin{gathered}
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{\left(1+e^{-X}\right)\left(1+e^{-Z}\right)(-1)e^{-X}-\left(e^{-X}+e^{-Z}\right)\left(-e^{-X}-e^{-X-Z}\right)}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right] \\
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{\left(e^{-X}+e^{-Z}\right)\left(e^{-X}+e^{-X-Z}\right)-\left(1+e^{-X}\right)\left(1+e^{-Z}\right)e^{-X}}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right] \\
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{\left(e^{-2X}+e^{-2X-Z}+e^{-X-Z}+e^{-X-2Z}\right)-\left(1+e^{-X}+e^{-Z}+e^{-X-Z}\right)e^{-X}}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right] \\
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{e^{-2X}+e^{-2X-Z}+e^{-X-Z}+e^{-X-2Z}-e^{-X}-e^{-2X}-e^{-X-Z}-e^{-2X-Z}}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right] \\
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{e^{-X-2Z}-e^{-X}}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right] \\
W_B \frac{e^{-Y}}{\left(1+e^{-Y}\right)^2}\left[\frac{e^{-X}\left(e^{-2Z}-1\right)}{\left(1+e^{-X}\right)^2\left(1+e^{-Z}\right)^2}\right]
\end{gathered}
\tag{36}
$$

The SOCs when one non-negativity constraint binds; bordered Hessian is 4x4.

$$
\begin{array}{cccc}
0 & \frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial g_1}{\partial b(-0)} & \frac{\partial g_1}{\partial b(\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(0)} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
\frac{\partial g_1}{\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}
$$

Evaluating the constraint terms:

$$
\begin{array}{cccc}
0 & -1 & 0 & 0 \\
-1 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}
$$

The second order condition is on the last two principal minors. The largest must be negative. The next to last should then be positive.

- First we need the determinant of the next-to-last principal minor, the 3x3 matrix that is the upper-lefthand corner of this matrix. This determinant should be positive. We can use the trick by which one copies the first two rows to the right of the matrix and then multiplies down the columns to the right and up the columns to the right (these terms get a negative sign). Then the determinant is

$$0+0+0-\left[0+0+(-1)(-1)\frac{\partial^2 L}{\partial b(0)^2}\right]=-\frac{\partial^2 L}{\partial b(0)^2}$$

Because we need this to be positive, we need $\frac{\partial^2 L}{\partial b(0)^2} < 0$. This corresponds to the X compositive variable, and happens only when X is non-negative

- Second, we need the determinant of the last principal minor, which is the whole 4x4 matrix. This determinant can be simplified as follows:

$$0 \cdot |\text{something}| - (-1) \cdot \begin{vmatrix} -1 & \frac{\partial^2 L}{\partial b(0) \partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(-\frac{1}{2})} \\ 0 & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(0)} \\ 0 & \frac{\partial^2 L}{\partial b(0) \partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2} \end{vmatrix} + 0 \cdot |\text{something}| - 0 \cdot |\text{something}|$$

$$= \begin{vmatrix} -1 & \frac{\partial^2 L}{\partial b(0) \partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(-\frac{1}{2})} \\ 0 & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(0)} \\ 0 & \frac{\partial^2 L}{\partial b(0) \partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2} \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(0)} \\ \frac{\partial^2 L}{\partial b(0) \partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2} \end{vmatrix}$$

$$= -1 \cdot \left(\frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} - \left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right) \partial b\left(0\right)} \right]^2 \right)$$

$$= \left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right) \partial b\left(0\right)} \right]^2 - \frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2}$$

We need this determinant to be non-positive, that is

$$\frac{\partial^2 L}{\partial b(0)^2} \frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} \geq \left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right) \partial b\left(0\right)} \right]^2$$

The right hand side is positive because it's a square. Moreover, it is strictly positive because $\frac{\partial^2 L}{\partial b(\frac{1}{2}) \partial b(0)}$ is strictly positive: examine Equation 36. Since $Z < 0$ in the case under examination, Equation 36 must be strictly positive. (This strengthens us from the negative semi-definite condition condition that supplies only a weak inequality).

The left hand side must then be positive, and since we know that one term is positive from the previous principal minor result, the other must be positive as well. Thus both X and Y must be positive.

Note that nothing useful seems to come out of trying to expand and then simplify this expression. I get:

$$2e^{-X}e^{-Y} + 2 > e^{-2Z} + e^{-X}e^{-2Z} + e^{-Y}e^{-2Z} + e^{-X}e^{-Y}e^{-2Z}$$

In general

- n is the number of choice variables ($n = 3$ in my case for the three bribes)
- If there are k inequality constraints ($k = 3$ in my case, for 3 non-negativity constraints)
- And g_1 through g_b are binding (i.e. b is the number of bribes that are zero)
- m is the number of equality constraints (0 in my case)
- Then there are restrictions on the last $(n - b + m)$ minors
  - If one bribe is zero, then need the last two $3 - 1 = 2$ minors to alternate in sign with the last being $(-1)^n = (-1)^3 = -1 < 0$, i.e. negative
  - If no bribes are zero, there are restrictions on the last three minors, but the matrix is 3x3 (no constraint rows)
  - If two bribes are binding, there is only a restriction on the last minor, but it's a 5x5 matrix.
    - But can't wipe out $\frac{1}{W_B}$ using two FOCs because only one holds with equality
- If all three non-negativity constraints bind, we know all bribes are zero. No restriction on SOC?

Intuition

- A legislator votes for x (A's preferred policy, against the status quo) if

$$v(z) = \alpha - \beta z + a(z) - b(z) > 0$$

- B needs 2 "no" votes, or two legislators with $v(z) < 0$. When $a(z) = 0$, this is

$$\alpha - \beta z - b(z) < 0$$

or

$$\beta z - \alpha + b(z) > 0$$

- If $\alpha > 0$, legislators on average are biased against B.
  - If $\alpha < 0$, legislators on average are biased in favor of B.
  - If α is negative enough, bribe only legislator furthest to the left (most against B)
- If X, Y, or Z is positive, it means the probability of legislator $0, \frac{1}{2}$, or $-\frac{1}{2}$ voting for the status quo / against the new proposal / with interest group B is greater than 0.5.

Pattern more or less is to get the guy who is WHAT?

- Imagine a see-saw; these guys are weights on the see-saw.

One Non-negative Bribe

When $\alpha = 0$ and $WB = 8$, we find that $b\left(-\frac{1}{2}\right)$ and $b\left(\frac{1}{2}\right)$ are zero. Only $b(0)$ is positive.

- Only the FOC for X holds with equality (i.e. $\lambda_X = 0$)

$$\frac{e^{-Y} + e^{-Z}}{\left(1 + e^{-Y}\right)\left(1 + e^{-Z}\right)} \frac{e^{-X}}{\left(1 + e^{-X}\right)^2} = \frac{1}{W_B}$$

- We know exactly what Y and Z will be since $b\left(-\frac{1}{2}\right)$ and $b\left(\frac{1}{2}\right)$ are zero. Given WB, we can solve for X and then $b(0)$.

- Looking at the SOC, when there are three inequality constraints and 2 bind, we only need a condition on the last principal minor of a suitably constructed bordered Hessian.
  - It should be negative because $n = 3$
  - It's a 5x5 matrix because there are two binding inequality constraints.

The SOCs when one non-negativity constraint binds; bordered Hessian is 5x5.

$$\begin{array}{ccccc}
0 & 0 & \frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial g_1}{\partial b(-0)} & \frac{\partial g_1}{\partial b(\frac{1}{2})} \\
0 & 0 & \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial g_3}{\partial b(-0)} & \frac{\partial g_3}{\partial b(\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(-\frac{1}{2})} & \frac{\partial g_3}{\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
\frac{\partial g_1}{\partial b(0)} & \frac{\partial g_3}{\partial b(-0)} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
\frac{\partial g_1}{\partial b(\frac{1}{2})} & \frac{\partial g_3}{\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}$$

Evaluating the constraint terms:

$$\begin{array}{ccccc}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
-1 & 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})^2} & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(0)} & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
0 & -1 & \frac{\partial^2 L}{\partial b(-\frac{1}{2})\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{array}$$

Solving for the determinant of this matrix:

$$(-1) \cdot \begin{vmatrix}
0 & 0 & 0 & -1 \\
-1 & 0 & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(0)^2} & \frac{\partial^2 L}{\partial b(\frac{1}{2})\partial b(0)} \\
0 & -1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})} & \frac{\partial^2 L}{\partial b(\frac{1}{2})^2}
\end{vmatrix} = (-1) \cdot (-1) \cdot (-1) \begin{vmatrix}
-1 & 0 & \frac{\partial^2 L}{\partial b(0)\partial b(-\frac{1}{2})} \\
0 & 0 & \frac{\partial^2 L}{\partial b(0)^2} \\
0 & -1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})}
\end{vmatrix} =$$

$$(-1) \cdot (-1) \begin{vmatrix}
0 & \frac{\partial^2 L}{\partial b(0)^2} \\
-1 & \frac{\partial^2 L}{\partial b(0)\partial b(\frac{1}{2})}
\end{vmatrix} = 0 - (-1) \cdot \frac{\partial^2 L}{\partial b(0)^2} = \frac{\partial^2 L}{\partial b(0)^2}$$

- Thus we need $\frac{\partial^2 L}{\partial b(0)^2} < 0$
  - This boils down to $\left(e^{-X} - 1\right) < 0$
  - Implies $X > 0$
  - i.e. $-\alpha + b(0) > 0$ or $b(0) > \alpha$
- It is NOT the case that $b(0)$ has to be such that $X = Y$ even though $b\left(\frac{1}{2}\right) = 0$. In fact, it seems in numerical examples that this does *not* happen.
  - However, I do see that this existence of the one non-negative bribe (NNB) is not common. It exists for $\alpha = 0$ and WB = 8, but goes away as soon as $\alpha = 0.1$ or $W_B = 9$.
  - In both cases, I can see that changing the parameter would push the value of $b(0)$ over 0.5, making $X > Y$. This is when the solution switches to 2 NNB's.
  - CONJECTURE: Group B chooses 1 NNB until it would be large enough to make $X > Y$ (when $\alpha \approx 0$) or $Z > Y$ (when α large and negative), then switch to 2 NBB
    - This is not strictly true around -0.7
    - When α is large and positive, go straight from 0 NNB to 2 NNB
- Can write out SOC (just after equation 33) and FOC, sub together to get

$$\frac{1}{1 + e^{-X}}\left(e^{-Z} - 1\right) < 0$$

Not sure what this means

All Three Bribes are Non-negative

In many cases all three bribes are non-negative. Here are very loose results

- I always observe that $X = Y = Z$. I think it's easy to show this analytically.
- From the FOCs, when X and Y are non-negative, it can be shown that a set of inequalities just below Equations 31 and 32 must hold.
- The SOCs when 0 non-negativity constraints bind are on the 3x3 matrix of second derivatives. There are three conditions:

$$\frac{\partial^2 L}{\partial b\left(-\frac{1}{2}\right)^2}\left\{\frac{\partial^2 L}{\partial b(0)^2}\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2}-\left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)\partial b\left(0\right)}\right]^2\right\} \leq 0$$

$$\frac{\partial^2 L}{\partial b(0)^2}\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2}-\left[\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)\partial b\left(0\right)}\right]^2 \geq 0$$

$$\frac{\partial^2 L}{\partial b\left(\frac{1}{2}\right)^2} \leq 0$$

  The last one says that Y must be non-negative. The first and second together say that Z must be non-negative.

- Y non-negative combined with the inequalities below Equations 31 and 32 tells us that $X = Y > 0$.
- Similar logic that leads to those inequalities can be used to derive analagous inequalities for X and Z (or Y and Z). Then Z non-negative tells us $Z = Y > 0$ so that $X = Y = Z > 0$.

Now as to when this case occurs

- When $X = Z$ (the two left-most legislators), bribe to $\frac{1}{2}$ is added in before X would move to the right of Y, i.e. when $X \rightarrow 0.5$.
- This is NOT the case when X and Y are the two non-negative bribes. They can get very large before the left-most legislator is bribed.

Numerical Results

For all of these, $\beta = 1$ and there are three legislators with $b \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$.

- Below, the column headings are $v(z)$ before the bribe.
- That is, from left to right, $-Z, -Y, -X$.

		-2	-1.5	-1
For $\alpha = -1.5$	up to WB=19	0	0	0
	only WB = 20	0	0	just a little

		-1.8	-1.3	-0.8
For $\alpha = -1.3$	up to WB=15	0	0	0
	WB = 16 to 20	0	0	just a little

		-1.6	-1.1	-0.6
For $\alpha = -1.1$	up to WB=13	0	0	0
	WB = 14 to 15	0	0	just a little
	WB = 16 to 20	0	some, always increasing	some + .5

		-1.4	-0.9	-0.4
For $\alpha = -0.9$	up to WB=11	0	0	0
	WB = 12 to 13	0	0	just a little
	WB = 14 to 19	0	some, always increasing	some + .5
	WB = 20	some	some + .5	some + 1

		-1.1	-0.6	-0.1
For $\alpha = -0.6$	up to WB=8	0	0	0
	WB = 9 to 10	0	some	0
	WB = 11 to 14	0	some, always increasing	some + .5
	WB = 15 to 20	some	some + .5	some + 1

$\alpha = -0.6$ pattern holds to $\alpha = -0.2$

		-0.6	-0.1	0.4
For $\alpha = -0.1$	up to WB=7	0	0	0
	WB = 8	0	some	0
	WB = 9	some	some + .5	0
	WB = 15 to 20	some	some + .5	some + 1

$\alpha = -0.1$ pattern holds to $\alpha = 0.5$ at least

		0.0	0.5	1.0
For $\alpha = 0.5$	up to WB=8	0	0	0
	WB = 9 to 11	> 1	$> 1 + .5$	0
	WB = 12 to 20	significant	significant + .5	significant + 1

		0.6	1.1	1.6
For $\alpha = 1.1$	up to WB=13	0	0	0
	WB = 9 to 20	> 2	$> 2 + .5$	0

Going to explore in-depth case of $\alpha = 0.0$

	-0.5	0.0	0.5
up to WB=7	0	0	0
WB = 8	0	some	0
WB = 9	some	some + .5	0
WB = 10 to 20	some	some + .5	some + 1

Some Comparative Statics

Comparative statics

- Vote buyer B (3/25/15) — note these are not super useful yet — they're comparative statics of the best response function, but not of the equilibrium
  - Those for j, $a(j)$ and α don't involve the root term, so don't differ depending on whether we take the plus or minus version

$$\frac{\partial b(j)}{\partial j} = -\beta \qquad \frac{\partial b(j)}{\partial a(j)} = 1 \qquad \frac{\partial b(j)}{\partial \alpha} = 1$$

  - Those for β and σ_j *do* involve the root term. I will only display the version for the positive root; there is also a version with all the signs reversed

$$\frac{\partial b(j)}{\partial \beta} = -j + \sigma_j \sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)} - \frac{\sigma_j}{\sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)}}$$

$$\frac{\partial b(j)}{\partial \sigma_j} = \beta \sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)} - \frac{\beta}{\sqrt{2\left(\ln W_B - \ln \beta\sigma_j - \ln\sqrt{2\pi}\right)}}$$

Notes from March

Notes from 3/16 Skype chat (Kristy and Sebastian)

- Seems like FOC for lobby will boil down to each one buying votes until marginal benefit = marginal cost
- May want to use some simplified rule / heuristic for lobby's decision: perhaps they reorder the legislators by their ± 2 std. dev. and make some decision based on that ordering who to lobby
- Will some kind of rule that looks like RMSE come out of the math? Is it possible to get anything closed-form at all?

Notes from 3/19 Skype chat (Kristy and Sebastian)

- We can use the data we have to horse-race this model against other hypotheses about how lobbyists distribute bribes
  - Allocate equally among legislators
  - Groseclose Snyder with full information: only one side pays
  - Our model with one dimension
    - Could we use the econometric model to benchmark to one where uncertainty disappears? Or, as it does theoretically, does the model have to change completely?
  - Our model with multiple dimensions
- We can think of this as looking for the effect of uncertainty on prices—we'd be pricing uncertainty relative to a model with uncertainty
  - "a metric in dollars of uncertainty"
  - this is a model of vote buying under uncertainty
- Given how different the environment with uncertainty is, Kristy should explore both the sequential model that parrots GS96 and a simultaneous model that is more like GH94 (menu auction)
- When mapping to the data, we're going to want to know from the model whether/when total contributions represent WTP.
  - It clearly doesn't in the case of certainty. One side pays nothing; the other pays either as much as is necessary to shut down its opponent, or nothing at all the necessary amount exceeds WTP
  - May be able to show it doesn't matter, that the proportion of total expenditures is a sufficient statistic

- Finish writing the model, and then see if we can use the estimates we already have to write a first, very rough draft