

# Problem Set: Introduction to Measure Theory

## 1 Measure Spaces

### Exercise 1.

- $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open}\}$  is not closed under complements. For example,  $(0, +\infty)$  is an open set in  $\mathbb{R}$  so it is included in  $\mathcal{G}_1$ , but its complement  $(-\infty, 0]$  is not open. Thus  $\mathcal{G}_1$  is not an algebra.
- If  $\emptyset$  is not included in  $\mathcal{G}_2$ , then it is not an algebra by definition. If  $\emptyset$  is included in  $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$ , then it is an algebra as it is closed under complements (the complement would be a finite intersection of  $(-\infty, a] \cup (b, \infty)$ ,  $(b, \infty)$  and  $(-\infty, a]$ , which is either empty or could be written as a finite union of the above forms) and closed under finite unions (a finite union of the sets that are finite unions of intervals of the form  $(a, b], (-\infty, b]$ , and  $(a, \infty)$  is still a finite union of intervals of these forms). It is not a  $\sigma$ -algebra. For example, the countable union

$$\bigcup_{n=2}^{\infty} \left(0, \frac{n-1}{n}\right] = (0, 1)$$

is not in  $\mathcal{G}_2$ .

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$  is a  $\sigma$ -algebra if we include  $\emptyset$  in it. The proof follows similarly as above but now “finite” applies more generally to “countable” too.

### Exercise 2.

- If  $\mathcal{A}$  is a  $\sigma$ -algebra, then  $\emptyset \in \mathcal{A}$ . The complement of  $\emptyset$ ,  $X$  must also be in  $\mathcal{A}$ . Thus  $\{\emptyset, X\} \subset \mathcal{A}$ .
- Note that if  $S \subset X$ , then  $S^c = X \setminus S \subset X$ . Also note that finite unions of subsets of  $X$  is still a subset of  $X$ . If  $\mathcal{A}$  is a  $\sigma$ -algebra generated from some subsets of  $X$ , then  $\mathcal{A} \subset \mathcal{P}(X)$  as  $\mathcal{P}(X) = \{A : A \subset X\}$  contains all the subsets of  $X$  by definition.

**Exercise 3.**

- Since  $\emptyset \in \mathcal{S}_\alpha, \forall \alpha$ , we thus have  $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$ .
- Pick any  $X \in \cap_\alpha \mathcal{S}_\alpha$  and hence  $X \in \mathcal{S}_\alpha, \forall \alpha$ . Since a  $\sigma$ -algebra is closed under complements, we have  $X^c \in \mathcal{S}_\alpha, \forall \alpha$ . Therefore,  $X^c \in \cap_\alpha \mathcal{S}_\alpha$ .
- Pick countable sets  $X_1, X_2, \dots \in \cap_\alpha \mathcal{S}_\alpha$ , and hence  $X_i \in \mathcal{S}_\alpha, \forall \alpha, \forall i$ . Since a  $\sigma$ -algebra is closed under countable unions, thus  $\cup_i X_i \in \mathcal{S}_\alpha, \forall \alpha$ . Therefore,  $\cup_i X_i \in \cap_\alpha \mathcal{S}_\alpha$ .

The above propositions mean that  $\cap_\alpha \mathcal{S}_\alpha$  is a  $\sigma$ -algebra.

**Exercise 4.**

- Let  $C = B \setminus A = B \cap A^c$ .  $\mu$  is a nonnegative measure and hence  $\mu(C) \geq 0$ . Note that  $B = A \cup C$  and  $A \cap C = \emptyset$ . Therefore

$$\mu(B) = \mu(A) + \mu(C) \geq \mu(A).$$

- Let  $B_n = \cup_{i=1}^n A_i$ . Note that  $B_1 \subset B_2 \subset B_3 \subset \dots$ . By Theorem 1.25 (i), we have  $\lim_{n \rightarrow \infty} \mu(B_n) = \mu(\cup_{i=1}^\infty B_i)$ .
  - Note that  $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$ , hence  $\mu(\cup_{i=1}^\infty B_i) = \mu(\cup_{i=1}^\infty A_i)$ .
  - Note that  $\lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(\cup_{i=1}^n A_i) \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i)$ .
  - Therefore,  $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$ .

**Exercise 5.**

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ .
- For any  $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$  s.t.  $A_i \cap A_j = \emptyset \forall i \neq j$ , we have

$$\begin{aligned} \lambda(\cup_{i=1}^\infty A_i) &= \mu((\cup_{i=1}^\infty A_i) \cap B) = \mu(\cup_{i=1}^\infty (A_i \cap B)) \\ &= \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i). \end{aligned}$$

**Exercise 6.** Let  $B_n = A_1 \cap A_n^c$ . Since

$$\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty (A_1 \cap A_i^c) = A_1 \cap (\cup_{i=1}^\infty A_i^c) = A_1 \setminus (\cap_{i=1}^\infty A_i),$$

we have  $\mu(\cup_{i=1}^{\infty} B_i) = \mu(A_1) - \mu(\cap_{i=1}^{\infty} A_i)$ . In addition,

$$\lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(A_1 \cap A_n^c) = \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n).$$

Note that  $B_1 \subset B_2 \subset B_3 \subset \dots$ . By Theorem 1.25 (i) we have

$$\lim_{n \rightarrow \infty} \mu(B_n) = \mu(\cup_{i=1}^{\infty} B_i).$$

Therefore,

$$\mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n) = \mu(A_1) - \mu(\cap_{i=1}^{\infty} A_i) \implies \lim_{n \rightarrow \infty} \mu(A_n) = \mu(\cap_{i=1}^{\infty} A_i).$$

## 2 Construction of Lebesgue Measure

**Exercise 7.** Since  $\mu^*$  is an outer measure, it is countably subadditive. Note that  $(B \cap E) \cup (B \cap E^c) = B$ , thus

$$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c).$$

If in addition  $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ , it must be that  $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$ .

**Exercise 8.** Denote  $\mathcal{O}$  the collection of open sets of  $\mathbb{R}$ . By definition, the Borel  $\sigma$ -algebra of  $\mathbb{R}$  is the  $\sigma$ -algebra generated by  $\mathcal{O}$ , i.e.,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$ . Let  $\nu$  be a premeasure on  $\mathbb{R}$  and  $\mu^*$  the outer measure generated by  $\nu$ , and  $\mathcal{M}$  the  $\sigma$ -algebra from the Caratheodory construction. By Theorem 2.12, we have  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) \subset \mathcal{M}$ .

## 3 Measurable Functions

**Exercise 9.** Consider a countable set  $\{x_n\}_{n=1}^{\infty}$ . Note that  $\{x_n\} \subset (x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}], \forall \varepsilon > 0$ . Note that

$$\sum_{n=1}^{\infty} \left[ \left( x_n + \frac{\varepsilon}{2^{n+1}} \right) - \left( x_n - \frac{\varepsilon}{2^{n+1}} \right) \right] = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

By definition of Lebesgue measure,

$$\begin{aligned}\lambda(\{x_n\}_{n=1}^\infty) &= \inf \left\{ \sum_{n=1}^\infty (b_n - a_n) : \{x_n\}_{n=1}^\infty \subset \bigcup_{i=1}^\infty (a_i, b_i] \right\} \\ &\leq \inf \left\{ \sum_{n=1}^\infty \left[ \left(x_n + \frac{\varepsilon}{2^{n+1}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+1}}\right) \right] : \{x_n\}_{n=1}^\infty \subset \bigcup_{i=1}^\infty \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right], \varepsilon > 0 \right\} \\ &= \inf \{ \varepsilon : \varepsilon > 0 \} = 0.\end{aligned}$$

Therefore,  $\lambda(\{x_n\}_{n=1}^\infty) = 0$ .

**Exercise 10.** Since  $\mathcal{M}$  is a  $\sigma$ -algebra, it is closed under complements and countable unions.

1. If  $\forall a, \{x \in X : f(x) \geq a\} \in \mathcal{M}$ , then its complement  $\{x \in X : f(x) < a\} \in \mathcal{M}$  too.  
So (\*) can be replaced by  $\{x \in X : f(x) \geq a\}$ .
2. If  $\forall a, \{x \in X : f(x) > a\} \in \mathcal{M}$ , then a countable union

$$\bigcup_{n=1}^\infty \{x \in X : f(x) > a + \frac{1}{n}\} = \{x \in X : f(x) \geq a\}$$

is also in  $\mathcal{M}$ . By (1), (\*) can be replaced by  $\{x \in X : f(x) > a\}$ .

3. If  $\forall a, \{x \in X : f(x) \leq a\} \in \mathcal{M}$ , then its complement  $\{x \in X : f(x) > a\} \in \mathcal{M}$  too.  
By (2), (\*) can be replaced by  $\{x \in X : f(x) \leq a\}$ .

**Exercise 11.**

- Let  $F(f, g) = f + g$  which is continuous. By (4),  $f + g$  is measurable.
- Let  $F(f, g) = f \cdot g$  which is continuous. By (4),  $f \cdot g$  is measurable.
- Let  $f_n = f$  for  $n$  odd and  $f_n = g$  for  $n$  even. Thus  $\sup_{n \in \mathbb{N}} f_n(x) = \max(f, g)$ . By (2),  $\max(f, g)$  is measurable.
- Let  $f_n = f$  for  $n$  odd and  $f_n = g$  for  $n$  even. Thus  $\inf_{n \in \mathbb{N}} f_n(x) = \min(f, g)$ . By (2),  $\min(f, g)$  is measurable.
- First,  $g(x) = -1$  is measurable. Thus  $-f = f \cdot g$  is measurable. Next note that  $|f| = \max(f, -f)$ , so  $|f|$  is also measurable.

**Exercise 12.** We construct a partition as in the proof. If  $f$  is bounded,  $\exists M$  s.t.  $f(x) < M, \forall x$ . For  $n > M$ ,  $\forall x \in X$ , there exists some  $i$  s.t.  $x \in E_i^n$ . Thus  $s_n(x) = \frac{i-1}{2^n}$  for

this  $i$  and  $|f(x) - s_n(x)| < \frac{1}{2^n}$ .  $\forall \varepsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $\frac{1}{2^N} < \varepsilon$ . Therefore,  $\forall n > \max(N, M)$ , we have

$$|f(x) - s_n(x)| < \frac{1}{2^n} < \varepsilon,$$

hence the convergence in is uniform.

## 4 Lebesgue Integration

**Exercise 13.** Since  $f^+ = \max\{f(x), 0\} \in [0, M)$ , and  $\mu(E) < \infty$ , then

$$0 \leq \int_E f^+ d\mu \leq M\mu(E) < \infty.$$

Similarly,  $f^- = \max\{-f(x), 0\} \in [0, M)$ , then

$$0 \leq \int_E f^- d\mu \leq M\mu(E) < \infty.$$

Both  $\int_E f^+ d\mu$  and  $\int_E f^- d\mu$  are finite, so  $f \in \mathcal{L}^1(\mu, E)$ .

**Exercise 14.** Suppose there exists  $X \subset E$  with  $\mu(X) > 0$ , and  $f(x) = \infty, \forall x \in X$ . Then

$$\int_E |f| d\mu \geq \int_A |f| d\mu \geq \int_A f d\mu = \infty,$$

contradictory to  $f$  being integrable.

**Exercise 15.**  $f \leq g$  implies that  $\{s : 0 \leq s \leq f, s \text{ simple, measurable}\} \subset \{s : 0 \leq s \leq g, s \text{ simple, measurable}\}$ , and thus  $\sup\{\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\} \leq \sup\{\int_E s d\mu : 0 \leq s \leq g, s \text{ simple, measurable}\}$ . By definition this is  $\int_E f d\mu \leq \int_E g d\mu$ .

**Exercise 16.** Consider any  $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$  simple, measurable.  $A \subset E$  implies that  $A \cap E_i \subset E \cap E_i, \forall i$ . Hence  $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ . Thus

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu.$$

Therefore,

$$\begin{aligned}\int_A f^+ d\mu &= \sup \left\{ \int_A s d\mu : 0 \leq s \leq f^+, \text{ s simple, measurable} \right\} \\ &\leq \sup \left\{ \int_E s d\mu : 0 \leq s \leq f^+, \text{ s simple, measurable} \right\} \\ &= \int_E f^+ d\mu < \infty,\end{aligned}$$

and similarly  $\int_A f^- d\mu < \infty$ . So  $f \in \mathcal{L}^1(\mu, A)$ .

**Exercise 17.** Let  $X_1 = A \cap B$  and  $X_2 = A - B$ . Note that  $X_1 \cap X_2 = \emptyset$  and  $A = X_1 \cup X_2$ . Since  $B \subset A$ , we have  $A \cap B = B$ . Thus

$$\int_A f d\mu = \int_B f d\mu + \int_{A-B} f d\mu.$$

Since  $\mu(A - B) = 0$ , by Proposition 4.6 we have  $\int_{A-B} f d\mu = 0$ . Therefore  $\int_A f d\mu = \int_B f d\mu$ .