"Open Source Macroeconomics Laboratory Boot Camp Linearization Methods"

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Outline

Motivation

There are a variety of ways to solve and simulate DSGE models

- Value-function or Policy-function Grid iterations over grids
- Linearization about the steady state: Uhlig (1999) and Christiano (2002).
- Higher-order approximation about the steady state: Collard and Julliard (2001) and Schmitt-Grohé and Uribe (2004).
- Estimation of an appropriate polynomial (GSSA): Judd, Maliar and Maliar (2011) and Hasanhodzic and Kotlikoff (2013).

Illustration of Grid Methods

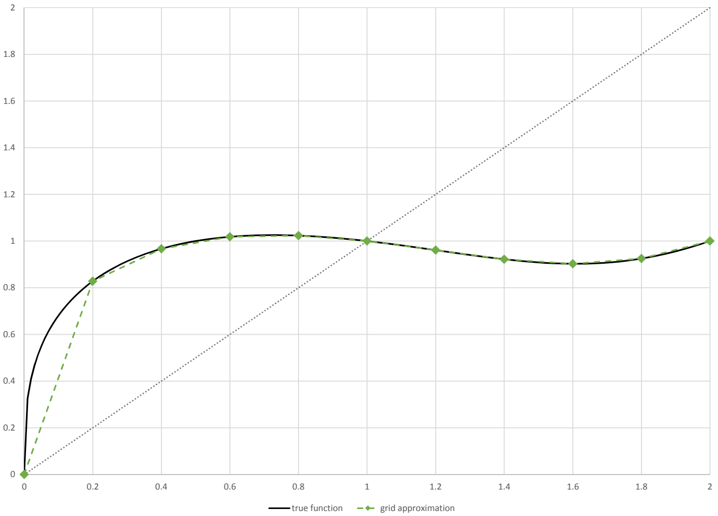
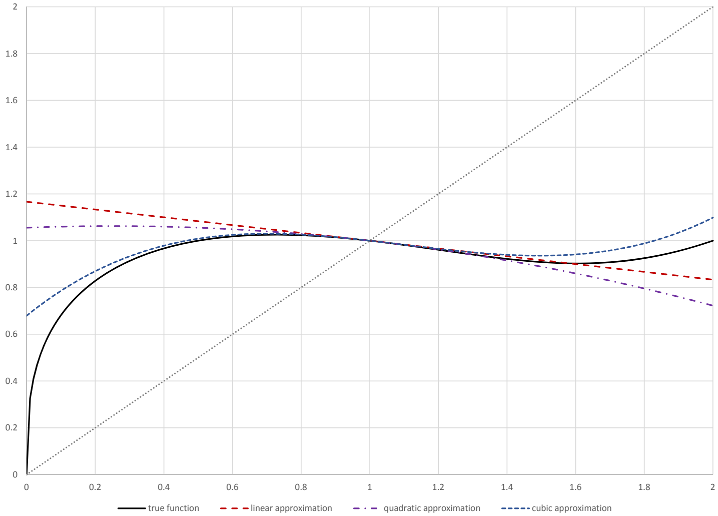


Illustration of Polynomial Approximations



Outline

Types of Variables

- Z_t is a vector of n_Z exogenous state variables.
- X_{t-1} is a vector of n_X endogenous state variables.
- Y_t is a vector of n_Y implicity-defined non-state or "jump" variables.
- D_t is a vector of n_D explicity-defined non-state or "jump" variables.

Note we can lump Y_t and D_t into X_t if we like. This may increase computational cost, but is otherwise sound logically. We are searching for a polucy function $X_t = \Phi(X_{t-1}, Z_t)$, and perhaps a jump function $Y_t = \Psi(X_{t-1}, Z_t)$.

Dynamic Behavior

We assume that Z_t follows first-order vector autoregression for its law of motion.

$$Z_t = (I - N)\bar{Z} + NZ_{t-1} + E_t \tag{1}$$

where N is a $n_Z \times n_Z$ square matrix, and E_t is a vector of stochastic shocks with mean of zero and a variance-covariance matrix of Σ.
Note this law of motion is already linear.

Dynamic Behavior

We will take the characterizing equations for the model and write them as a vector of functions in the following form:

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t)\} = 0 \tag{2}$$

X_{t+1}, X_t and X_{t-1} are $n_X \times 1$ vectors. Z_{t+1} and Z_t are $n_Z \times 1$ vectors and Γ outputs a $n_X \times 1$ vector.

Dynamic Behavior

We can solve for the steady state by noting that Γ is a set of n_X equations in the same number of unknowns when we set $X_t = \bar{X}$ for all t.

$$E_t\{\Gamma(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})\} = 0 \tag{3}$$

We can solve this analytically for simple models, or numerically for more complex ones.

Dynamic Behavior

We can use a first-order Taylor-series approximation of these equations to get a linear approximation of the characterizing equations.

$$E_t \left\{ F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \right\} = 0 \tag{4}$$

where $\tilde{X}_t$ denotes $X_t - \bar{X}$
We can rewrite our law of motion as:

$$\tilde{Z}_t = N\tilde{Z}_{t-1} + \varepsilon_t \tag{5}$$

Dynamic Behavior

The coefficients are matrices of derivatives evaluated at the steady state values.
For example, $F \equiv \Gamma_{X_{t+1}}(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})$,
and $L \equiv \Gamma_{Z_{t+1}}(\bar{X}, \bar{X}, \bar{X}, \bar{Z}, \bar{Z})$.
F, G and H are $n_X \times n_X$ square matrices, while L and M are $n_X \times n_Z$.

Approximate Transition Function

We now hypothesize that the transition function, $X_t = \Phi(X_{t-1}, Z_t)$ can also be log-linearly approximated by

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t \quad (6)$$

where P is a $n_X \times n_X$ square matrix and Q is $n_X \times n_Z$.

Approximate Transition Function

We know the values of F, G, H, L and M since these are functions of the model parameters and steady state values (which are also functions of the parameters). N can be considered as part of the parameter set. We do not know the values of P and Q, however.

We can solve for them by iteratively substituting appropriate versions of the linarized policy function and law of motion into the linearized characterizing equations.

After some tedious matrix algebra this reduces to:

$$[(FP+G)P+H]\tilde{X}_{t-1}+[(FQ+L)N+(FP+G)Q+M]\tilde{Z}_t=0 \quad (7)$$

Approximate Transition Function

For this to be true for all values of X_{t-1} and Z_t, the coefficients on these terms in the equation must be zero. In turn, this gives us two equations that implicitly define P and Q.
Matrix quadratic (related to an Algebraic Riccati Equation):

$$FP^2 + GP + H = 0 \tag{8}$$

Sylvester Equation:

$$FQN + (FP + G)Q + (LN + M) = 0 \tag{9}$$

Obtaining Derivatives

- Analytical or Numerical Derivatives?
  - For simple models analytical derivatives may be tractable when linearizing.
  - Often intractable with more complicated models
- Linearize or Log-Linearize?
  - Analytical derivatives are often easier with log-linearization
  - No such advantage with numerical derivatives

Outline

Brock and Mirman

Let us first solve for the linearized approximation, not the log-linearized one.
Recall the Euler equation from the Brock and Mirman model with stochastic productivity shocks.

$$\frac{1}{e^{z_t} K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} {K_{t+1}}^{\alpha-1}}{e^{z_{t+1}} {K_{t+1}}^{\alpha} - K_{t+2}} \right\}$$

We rewrite this equation in our desired form.

$$E_t \left\{ \beta \frac{\alpha e^{z_{t+1}} {K_{t+1}}^{\alpha-1} (e^{z_t} K_t^{\alpha} - K_{t+1})}{e^{z_{t+1}} {K_{t+1}}^{\alpha} - K_{t+2}} - 1 \right\} = 0 \qquad (10)$$

Here $X_{t-1} = K_t$ and $Z_t = z_t$.

Brock and Mirman

By differentiating with respect to K_{t+2}, K_{t+1}, K_t, z_{t+1} and z_t, and evaluating these at thes steady state values we can recover the Uhlig matrices:

$$
\begin{aligned}
F =& \frac{\alpha \bar{K}^{\alpha-1}}{\bar{K}^{\alpha} - \bar{K}} \\
G =& -\frac{\alpha \bar{K}^{\alpha-1}(\alpha + \bar{K}^{\alpha-1})}{\bar{K}^{\alpha} - \bar{K}} \\
H =& \frac{\alpha^2 \bar{K}^{2(\alpha-1)}}{\bar{K}^{\alpha} - \bar{K}} \\
L =& -\frac{\alpha \bar{K}^{2\alpha-1}}{\bar{K}^{\alpha} - \bar{K}} \\
M =& \frac{\alpha^2 \bar{K}^{2(\alpha-1)}}{\bar{K}^{\alpha} - \bar{K}}
\end{aligned}
$$

Brock and Mirman

To evaluate these, recall $\bar{K} = A^{\frac{1}{1-\alpha}}$. We can then use the Riccatti and Sylveser equations to derive the scalar values P and Q.

$$P = \frac{-G \pm \sqrt{G^2 - 4FH}}{2F} \tag{11}$$

$$Q = -\frac{LN + M}{FN + FP + G} \tag{12}$$

In this case, since we have linearized (not log-linearized) the policy function is:

$$K_{t+1} = \bar{K} + P(K_t - \bar{K}) + Qz_t \tag{13}$$

Outline

Let's use the following functional forms:

$$u(c_t, \ell_t) = \frac{1}{1-\gamma}(c_t^{1-\gamma} - 1) - \chi \frac{1}{1+\theta} \ell_t^{1+\theta} \tag{14}$$

$$f(k_t, \ell_t, z_t) = k_t^{\alpha} (e^{z_t} \ell_t)^{1-\alpha} \tag{15}$$

Baseline Model

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \tag{16}$$

$$r_t = \alpha k_t^{\alpha-1}(e^{z_t}\ell_t)^{1-\alpha} \tag{17}$$

$$w_t = (1-\alpha)k_t^{\alpha} e^{(1-\alpha)z_t}\ell_t^{-\alpha} \tag{18}$$

$$\tau\left[w_t\ell_t + (r_t - \delta)k_t\right] = T_t \tag{19}$$

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)k_t\right] + k_t + T_t - k_{t+1} \tag{20}$$

$$\chi\ell_t^{\theta} = c_t^{-\gamma} w_t(1-\tau) \tag{21}$$

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}[(r_{t+1}-\delta)(1-\tau)+1]\right\} \tag{22}$$

Baseline Model

Define $X_t = \{k_{t-1}, \ell_t\}$, even though ℓ_t is not strictly a state variable.
As before $Z_t = z_t$.
Our characterizing equations could be:

$$\chi \ell_t^{\theta} - c_t^{-\gamma} w_t (1-\tau) = 0 \tag{23}$$

$$E_t \left\{ c_t^{-\gamma} - \beta c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1-\tau) + 1] \right\} = 0 \tag{24}$$

We could also use:

$$\frac{\chi \ell_t^{\theta}}{c_t^{-\gamma} w_t (1-\tau)} - 1 = 0 \tag{25}$$

$$E_t \left\{ \frac{c_t^{-\gamma}}{\beta c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1-\tau) + 1]} - 1 \right\} = 0 \tag{26}$$

Outline

OLG Model

Our dyanmic programming approach to the household's problem cannot be applied to an overlapping generations (OLG) model. This is because the value-function for an n-year-old agent is different from that of an $(n+1)$-year-old agent. However, in our infintely-lived agent models, the problem is the same since the agent still has an infintely number of periods to live every period.

Despite this drawback, we can still express an OLG model in the same notation.

OLG Model

Consider, for example, the Euler equations, for an OLG model with *N*-period-lived agents and fixed labor supplies.

$$\begin{aligned} u_c(c_{1t}) &= \beta E\{u_c(c_{2,t+1})(1 + r_{t+1} - \delta)\} \\ u_c(c_{2t}) &= \beta E\{u_c(c_{3,t+1})(1 + r_{t+1} - \delta)\} \\ &\vdots \\ u_c(c_{N-1,t}) &= \beta E\{u_c(c_{N,t+1})(1 + r_{t+1} - \delta)\} \end{aligned} \tag{27}$$

OLG Model

There are also a set of budget contraints, one for each agent, that define the consumptions, $\{c_{nt}\}_{n=1}^{N}$.

$$\begin{aligned}
c_{1t} &= w_t \ell_{1t} - k_{2,t+1} \\
c_{2t} &= w_t \ell_{2t} + (1 + r_t - \delta) k_{2t} - k_{3,t+1} \\
&\vdots \\
c_{N-1,t} &= w_t \ell_{N-1,t} + (1 + r_t - \delta) k_{N-1,t} - k_{N,t+1} \\
c_{N,t} &= w_t \ell_{N,t} + (1 + r_t - \delta) k_{N,t}
\end{aligned} \tag{28}$$

OLG Model

Aggregate capital (K) and labor (L) will be the sums of the ks and ℓs over all cohorts.

$$\begin{aligned} K_t &= \sum_{n=2}^{N} k_{nt} \\ L_t &= \sum_{n=1}^{N} \ell_{nt} \end{aligned} \tag{29}$$

Wages and interest rates are defined from first-order conditions for firms and will be conditions similar to (3.9) and (3.10).

$$\begin{aligned} r_t &= f_K(K_t, L_t, z_t) \\ w_t &= f_L(K_t, L_t, z_t) \end{aligned} \tag{30}$$

OLG Model

Finally, we have a law of motion for the exogenous productivity shock.

$$z_t = \rho z_{t-1} + \varepsilon_t \tag{31}$$

This is a dynamic system of $2N + 4$ equations and variables. We can categorize our variables as before.

$$\begin{aligned} X_t &= (\{k_{n,t+1}\}_{n=2}^{N}) \\ Y_t &= (\{c_{n,t}\}_{n=1}^{N}, K_t, L_t, r_t, w_t) \\ Z_t &= z_t \end{aligned} \tag{32}$$

We can solve and simulate this system in exactly the same way we do an infinitely-lived agent model

Outline

Exercise 1

For the Brock and Mirman model in use Uhlig's notation to analytically find the values of the following matrices: F, G, H, L, M & N as functions of the parameters. Given these find the values of P & Q, also as functions of the parameters. Imposing our calibrated parameter values, plot the three-dimensional surface plot for the policy function $K' = H(K, z)$. Compare this with the closed form solution and the solution you found using the grid search method from the DSGE chapter.

Exercise 2

Repeat the above exercise using $k \equiv \ln K$ in place of K as the endogenous state variable.

Exercise 3

Do the necessary tedious matrix algebra necessary to transform

$$E_t \left\{ F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \right\} = 0$$

into

$$[(FP + G)P + H]\tilde{X}_{t-1} + [(FQ + L)N + (FP + G)Q + M]\tilde{Z}_t = 0$$

Exercise 4

For the baseline tax model, find the steady state values of k, c, r, w, ℓ, T, y and i, numerically. Assuming $u(c_t, \ell_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi}-1}{1-\xi}$ and $F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$. Use the following parameter values: $\gamma = 2.5$, $\xi = 1.5$, $\beta = .98$, $\alpha = .40$, $a = .5$, $\delta = .10$, $\bar{z} = 0$, $\rho_z = .9$ and $\tau = .05$.

Exercise 5

For the same model as above, find $\frac{\partial y}{\partial x}$ for $y \in \{\bar{k}, \bar{c}, \bar{r}, \bar{w}, \bar{\ell}, \bar{T}, \bar{y}, \bar{i}\}$ and $x \in \{\delta, \tau, \bar{z}, \alpha, \gamma, \xi, \beta, a\}$ using numerical techniques.

Exercise 6

For the same model as above, let $X_t = \{k_{t-1}, \ell_{t-1}\}$. Find the values of F, G, H, L, M, N, P and Q.

Exercise 7

For the same model as above, generate 10,000 artificial time series for an economy where each time series is 250 periods long. Start each simulation off with a starting value for k equal to the steady state value, and a value of $z = 0$.
Use $\sigma_z^2 = .0004$.
For each simulation save the time-series for GDP, consumption, investment, and the labor input. When all 10,000 simulations have finished generate a graph for each of these time-series showing the average value over the simulations for each period, and also showing the five and ninety-five percent confidence bands for each series each period.

Exercise 8

For the same model as above, calculate: the mean, volatility (standard deviation), coefficient of variation (mean divided by standard deviation), relative volatility (standard deviation divided bu the standard deviation of output), persistence (autocorrelation), and cyclicality (correlation with output); for each series over each simulation and report the average values and standard errors for these moments over the 10,000 simulations.

Exercise 9

For the same model as above, generate impulse response functions for: GDP, consumption, investment and total labor input; with lags from zero to forty periods.

Exercise 10

Recall the TPI exercises from the OLG chapter. Use linear approximation methods to solve for the non-steady state equilibrium transition path of the economy from $(k_{2,1}, k_{3,1}) = (0.8\bar{k}_2, 1.1\bar{k}_3)$ to the steady-state $(\bar{k}_2, \bar{k}_3)$. Use the same value for T as you used when you solved the problem using time path iteration (TPI). Plot the time paths for k_2, k_3 and K. Compare the resulting time paths for the two methods. Comment on the tradeoff between accuracy and compute time based upon this comparison.

Exercise 11

Repeat the exercise above only this time assume there is a stochastic shock to the model, so that A is replaced by e^{z_t} with $z_t = \rho_z z_{t-1} + \epsilon_t$; $\epsilon_t \sim$ i.i.d$(0, \sigma_z^2)$. Let $\sigma_z = .02$ and $\rho_z = .9^{20}$. As with exercise 8 above, run 10,000 simulations and report the average and confidence bands for GDP, total consumption and investment.