1. Translate the following problem into a system of equations and then solve. A river cruise ship sailed 60 kilometers downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution. Let x be the rate of the ship in still water and y be the rate of the current. Since rate times time is distance, we can write the system of equations:

$$4(x+y) = 60$$
$$5(x-y) = 60$$

Multiply the top equation by 5 and the bottom equation by 4:

$$20(x+y) = 300$$
$$20(x-y) = 240$$

Add equations and solve for x:

$$\begin{aligned}(20x+20y)+(20x-20y) &= 300+240\\ 40x &= 540\\ x &= 13.5\end{aligned}$$

Substitute $x = 13.5$ into one of the original equation:

$$\begin{aligned}20(x-y) &= 240\\ 20(13.5-y) &= 240\\ 13.5-y &= 12\\ y &= 1.5\end{aligned}$$

Therefore, the rate of the ship is $x = 13.5$ km/h and the rate of the current is $y = 1.5$ km/h.

2. One can often describe a given set in different ways. For instance, the sets $\{x^2 : x \in \mathbb{R}\}$ and $\{x \in \mathbb{R} : x \geq 0\}$ are the same set. Each of the six sets given below is equal to one of the others. Match up each set with the other set on the list it is equal to.

(a) $\left\{\frac{x}{2} : x \in \mathbb{Z}\right\}$

(b) $\left\{\cos^2(x) + 1 : x \in \mathbb{R}\right\}$

(c) $\mathbb{Z} \cap \left\{x \in \mathbb{R} : -\frac{1}{2} \leq x \leq \frac{9}{2}\right\}$

(d) $\left\{x - \frac{1}{2} : x \in \mathbb{Z}\right\} \cup \mathbb{Z}$

(e) $\{|x| : x \in \mathbb{Z}, \ -4 \leq x \leq 4\}$

(f) $\left\{2(x-1) : \frac{3}{2} \leq x \leq 2\right\}$

Solution. To match up the sets, it is easiest to first work out what each set is, and then it will be clear which ones are equal.

(a) $\left\{\frac{x}{2} : x \in \mathbb{Z}\right\} = \left\{..., \frac{-3}{2}, \frac{-2}{2}, \frac{-1}{2}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, ...\right\} = \left\{..., -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, ...\right\}$
The set in (a) is the set of all integers, divided by 2.

(b) $\left\{\cos^2(x) + 1 : x \in \mathbb{R}\right\} = \{y \in \mathbb{R} : 1 \le y \le 2\} = [1, 2]$
As x runs through the real numbers, the output of $\cos(x)$ is all the real numbers between -1 and 1 (including the endpoints). Taking the square of all these numbers gives all real numbers between 0 and 1 (including the endpoints). Adding 1, this is all the real numbers between 1 and 2 (again including the endpoints), that is, the interval $[1, 2]$.

(c) $\mathbb{Z} \cap \left\{x \in \mathbb{R} : -\frac{1}{2} \le x \le \frac{9}{2}\right\} = \{0, 1, 2, 3, 4\}$
The intersection of $\mathbb{Z}$ and $\left\{x \in \mathbb{R} : -\frac{1}{2} \le x \le \frac{9}{2}\right\}$ is the intersection of $\mathbb{Z}$ and the interval $[-\frac{1}{2}, \frac{9}{2}]$, and is therefore the integers between $-\frac{1}{2}$ and $\frac{9}{2}$, that is, the set $\{0, 1, 2, 3, 4\}$.

(d) $\left\{x - \frac{1}{2} : x \in \mathbb{Z}\right\} \cup \mathbb{Z}$

$$\begin{aligned}
&= \left\{..., -2 - \frac{1}{2}, -1 - \frac{1}{2}, 0 - \frac{1}{2}, 1 - \frac{1}{2}, 2 - \frac{1}{2}, ...\right\} \cup \mathbb{Z} \\
&= \left\{..., -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ...\right\} \cup \{..., -2, -1, 0, 1, 2, ...\} \\
&= \left\{..., -\frac{5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}
\end{aligned}$$

(e) $\{|x| : x \in \mathbb{Z}, \ -4 \le x \le 4\} = \{0, 1, 2, 3, 4\}$
From the description, the set consists of magnitudes (absolute values) of integers between -4 and 4 (endpoints included), i.e., the magnitudes of -4, -3, -2, -1, 0, 1, 2, 3, and 4. The set of these magnitudes is

$$\{|-4|, |-3|, |-2|, |-1|, |0|, |1|, |2|, |3|, |4|\} = \{4, 3, 2, 1, 0, 1, 2, 3, 4\} = \{0, 1, 2, 3, 4\}.$$

(f) $\left\{2(x-1) : \frac{3}{2} \le x \le 2\right\} = [1, 2]$
As x runs through the real numbers between $\frac{3}{2}$ and 2 (endpoints included), it actually runs through the interval $[\frac{3}{2}, 2]$, and so $x - 1$ runs through the real numbers in the interval $[\frac{1}{2}, 1]$. Then $2(x-1)$ runs through the real numbers in the interval $[1, 2]$.

Thus, the sets in the list match up in the following way :

$$(a) \Leftrightarrow (d), \ (b) \Leftrightarrow (f), \text{ and } (c) \Leftrightarrow (e)$$

3. Let A and B be sets. Prove that if $A \subset B$ then $A \cap B = A$.

Hints: To show that $X = Y$, first show that the conditions imply that $X \subset Y$, then show that the conditions also imply $Y \subset X$.

Proof. By the definition of intersection, all elements of $A \cap B$ are also elements of A, that is, $A \cap B \subset A$. We therefore need to show the opposite inclusion: $A \subset A \cap B$.

Suppose that $x \in A$, i.e., that x is an element of A. Then, since $A \subset B$ means that every element of A is also an element of B, we know that x is also an element of B. Thus $x \in A$ and $x \in B$. From the definition of intersection, x is an element of $A \cap B$. Therefore every element of A is also an element of $A \cap B$, i.e., $A \subset A \cap B$.

4. For each of the cases below, determine whether the given rule defines a valid function between the domain and target set given. If it does, also determine whether the function is injective, surjective, bijective, or none of these.

(a) $f : \mathbb{N} \mapsto \mathbb{N}$, $f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$

(b) $f : \mathbb{Z} \mapsto \mathbb{Z}$, $f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}$

(c) $g : \mathbb{Z} \mapsto \mathbb{Z}$, $g(x) = \sin\left(\frac{\pi x}{2}\right)$

(d) $g : \mathbb{Z} \mapsto \mathbb{Z}$, $g(x) = \frac{x}{2}\cos\left(\frac{\pi x}{2}\right)$

(e) $g : \mathbb{R} \mapsto \mathbb{R}^2$, $g(t) = \left(t^2, t^3\right)$

(f) $h : \mathbb{Q}_+ \mapsto \mathbb{N}$, $h\left(\frac{m}{n}\right) = 2^m 3^n$, where $\mathbb{Q}_+ = \left\{\frac{m}{n} : m, n \in \mathbb{N}, m > 0, n > 0\right\}$ denote the set of positive rational numbers (positive fractions).

Solution.

(a) $f : \mathbb{N} \mapsto \mathbb{N}$, $f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$

This formula **does define** a function from $\mathbb{N}$ to $\mathbb{N}$. If $x \in \mathbb{N}$ is even, then $f(x) = x + 1$ is again a non-negative integer, i.e., an element of $\mathbb{N}$. If $x \in \mathbb{N}$ is odd, then $x \geq 1$, and so $x - 1$ is a nonnegative even integer. Thus $f(x) = \frac{x-1}{2}$ is an integer (since dividing an even integer by two gives an integer) and non-negative (since $\frac{x-1}{2} \geq 0$ if $x - 1 \geq 0$).

The function is **not injective**; for instance $f(0) = 1 = f(3)$, and $0 \neq 3$.

However the function is **surjective**. For any $y \in \mathbb{N}$ if we set $x = 2y + 1$ then x is odd and so $f(x) = \frac{x-1}{2} = \frac{2y}{2} = y$.

(b) $f : \mathbb{Z} \mapsto \mathbb{Z}$, $f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}$

The formula **does define** a function from $\mathbb{Z}$ to $\mathbb{Z}$. For any $x \in \mathbb{Z}$ (i.e., x an integer), both $x+1$ and $x - 1$ are integers.

This function is both injective and surjective. Here is how we could write out an argument.

First, to see that f is **injective**, suppose that $f(x_1) = f(x_2)$. We want to show that this implies that $x_1 = x_2$. The function f exchanges odd and even numbers.

- If $f(x_1)$ is odd then $f(x_2)$ is also odd (since we are assuming that $f(x_1) = f(x_2)$), so both x_1 and x_2 must be even. Then $f(x_1) = x_1 + 1$ and $f(x_2) = x_2 + 1$, and from

$$x_1 + 1 = f(x_1) = f(x_2) = x_2 + 1$$

  we conclude that $x_1 = x_2$.
- On the other hand, if $f(x_1)$ is even we conclude that both x_1 and x_2 are odd, and therefore that $f(x_1) = x_1 - 1$ and $f(x_2) = x_2 - 1$. From

$$x_1 - 1 = f(x_1) = f(x_2) = x_2 - 1$$

  we again conclude that $x_1 = x_2$.

Therefore in all cases $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, and so f is injective.

To see that f is **surjective**, let y be any element of $\mathbb{Z}$. If y is even then set $x = y + 1$, which is odd. Since x is odd, $f(x) = x - 1 = (y + 1) - 1 = y$. On the other hand, if y is odd, then set $x = y - 1$, which is even. Since x is even, $f(x) = x + 1 = (y - 1) + 1 = y$. Therefore, for each $y \in \mathbb{Z}$ there is some $x \in \mathbb{Z}$ so that $f(x) = y$. Therefore f is surjective.

Since f is both **injective and surjective**, f is **bijective**.

(c) $g : \mathbb{Z} \mapsto \mathbb{Z}$, $g(x) = \sin\left(\frac{\pi x}{2}\right)$

The function $\sin(x)$ takes values in the real numbers, so perhaps it seems that the output does not lie in $\mathbb{Z}$. However, when x is an integer, $\sin(\frac{\pi x}{2})$ is either -1, 0, or 1, so this is a **function** from $\mathbb{Z}$ to the target set $\mathbb{Z}$.

The function g is **not injective**: $g(0) = \sin(0) = 0$ while $g(2) = \sin(\pi) = 0$.

The function is **not surjective**, since, for example, $2 \in \mathbb{Z}$ (the target set) and there is no $x \in \mathbb{Z}$ for which $g(x) = 2$.

(d) $g : \mathbb{Z} \mapsto \mathbb{Z}$, $g(x) = \frac{x}{2}\cos\left(\frac{\pi x}{2}\right)$

Note that if x is an integer, then $\cos(\frac{\pi x}{2})$ is either -1, 0, or 1. When x is an odd integer, then $\cos(\frac{\pi x}{2}) = 0$, so $f(x) = \frac{x}{2}\cos(\frac{\pi x}{2}) = 0$; when x is even integer, then $\frac{x}{2}$ is an integer, so $\frac{x}{2}\cos(\frac{\pi x}{2})$ is an integer. Thus $f(x) = \frac{x}{2}\cos(\frac{\pi x}{2})$ is a well-defined **function** from $\mathbb{Z}$ to target set $\mathbb{Z}$.

The function f is **not injective** since, for example, $f(-1) = 0 = f(1)$.

The function is **surjective** since if y is a positive even integer, then $x = 2y$ gives $y = f(x)$ and if y is a positive odd integer, then $x = -2y$ gives $y = f(x)$. Similarly, if y is a negative even integer, then $y = 2x$ gives $y = f(x)$ and if y is a negative odd integer, then $x = -2y$ gives $y = f(x)$. Since $f(0) = 0$, these imply that for any $y \in \mathbb{Z}$ we can find an $x \in \mathbb{Z}$ such that $f(x) = y$, so f is surjective.

(e) $g : \mathbb{R} \mapsto \mathbb{R}^2$, $g(t) = \left(t^2, t^3\right)$

The rule does give a well-defined **function** from $\mathbb{R}$ to $\mathbb{R}^2$. The function defined is injective but not surjective.

One way to see that g is **injective** is to see that knowing $g(t)$ allows us to recover t. Given $g(t) = \left(t^2, t^3\right) = (x, y)$, we recover t as $t = y^{\frac{1}{3}}$. Since it is possible to deduce t from $g(t)$, the function g is injective.

The function g is **not surjective**. For instance, there is no t so that $g(t) = \left(t^2, t^3\right) = (4, 4)$. The solutions to $t^2 = 4$ are $t = \pm 2$, and then $t^3 = \pm 8$, not equal to 4.

(f) $h : \mathbb{Q}_+ \mapsto \mathbb{N}$, $h\left(\frac{m}{n}\right) = 2^m 3^n$, where $\mathbb{Q}_+ = \left\{\frac{m}{n} : m, n \in \mathbb{N}, m > 0, n > 0\right\}$ denote the set of positive rational numbers (positive fractions).

The rule does **not** define a function since it violates the condition that for each input there should be a well-defined (unique) output. The reason is that a rational number cannot be expressed as a fraction of integers in a unique way.

For example, $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = ...$ Letting $x = \frac{1}{2}$ gives $h(x) = 2^1 \cdot 3^2 = 18$ while $\bar{x} = \frac{2}{4}$ gives $h(\bar{x}) = 2^2 \cdot 3^4 = 324 \neq 18$. Since $\frac{1}{2} = \frac{2}{4}$ but $h(\frac{1}{2}) \neq h(\frac{2}{4})$, this demonstrates that the rule given does not define a function.