Test #2

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6.11

Let H be a subgroup of a group G and suppose that $g_1, g_2 \epsilon G$. Prove that the following conditions are equivalent.

(a)

$g_1H = g_2H$

Proof Let $g_1H = g_2H$. Since H is a subgroup of G, we know $e\epsilon H$. Therefore, $g_1 = g_1e\epsilon g_1H = g_2H$. Similarly, $g_2 = g_2e\epsilon g_2H = g_1H$.

Then since $H \leqq G$ $\exists$ inverses. So $g_1\epsilon g_2H$ $\exists h\epsilon H$ such that $g_1 = g_2h$ and $h^{-1}\epsilon H$.

So $g_1^{-1} = g_2^{-1}h^{-1}$ and by similar argument $g_2^{-1} = g_1^{-1}h^{-1}$. Thus $g_1H = g_2H$.

(b)

$Hg_1^{-1} = Hg_2^{-1}$

Proof

Let $g_1^{-1}H = g_2^{-1}H$ Since H is a subgroup of G, we know $e\epsilon H$. Therefore, $g_1^{-1} = eg_1^{-1}\epsilon Hg_1^{-1} = Hg_2^{-1}$ and similarly $g_2^{-1} = eg_2^{-1}\epsilon Hg_2^{-1} = Hg_1^{-1}$. So, $\exists h^{-1}\epsilon H$ such that $g_1^{-1} = h^{-1}g_2^{-1}$. Then $g_1 = g_2h$. Recall that H is a subgroup so hH will equal H. Thus $g_1H = g_2hH = g_2H$. Therfore $g_1H \subseteq g_2H$

(c)

$g_1H \subset g_2H$

Proof

Let $g_1H \subset g_2H$, By Theorem 6.4 (Judson) As any two left coests of a subgroup H are either disjoint or equal. Thus $g_1H \subset g_2H$ implies that $g_1H = g_2H$. Since $e\epsilon H$, $g_2 = g_2H = g_1H$ Thus $g_2\epsilon g_1H$

(d)

$g_2\epsilon g_1H$

Proof

Let $g_1^{-1}g_2\epsilon H$. So $\exists h\epsilon H$, such that $g_2 = g_1h$. Then $g_2^{-1} = h^{-1}g_1^{-1}$. Then $g_2^{-1}g_2 = h^{-1}g_1^{-1}g_2$. $e = h^{-1}g_1^{-1}g_2$. $he = hh^{-1}g_1^{-1}g_2$. $h = g_1^{-1}g_2$. So, $g_1^{-1}g_2\epsilon H$

(e)

$g_1^{-1}g_2\epsilon H$

Proof Let $g_1^{-1}g_2\epsilon H$. So, $\exists h\epsilon H$ such that $g_1^{-1}g_2\epsilon h$. Thus $g_2 = g_1h$. Thus $g_2H = (g_1h)H = g_1(hH) = g_1H$ $\square$

6.12

If $ghg^{-1}\epsilon H\ \forall\ g\epsilon G$ and $h\epsilon H$, show that right coests are identical to left cosets. That is, show that $gH = Hg \forall g\epsilon G$.

Proof Let $ghg^{-1}\epsilon H\ \forall\ g\epsilon G$ and $h\epsilon H$. Since $h\epsilon H$, then h can be anything in H, so $h = H$. So $ghg^{-1}\epsilon H = gHg^{-1} \subseteq H\ \forall\ g\epsilon G$ and $h\epsilon H$. Consider $g^{-1}\epsilon G$. So, $g^{-1}H(g^{-1})^{-1} \subseteq H = g^{-1}Hg \subseteq H\ \forall g\epsilon G$. Since $g^{-1}Hg \subseteq H$. Then $g(g^{-1}Hg)g^{-1} \subseteq gHg^{-1}$. Thus $H \subseteq gHg^{-1}$.

So, $ghg^{-1} \subseteq H$ and $H \subseteq g^{-1}hg$. Then $gHg^{-1}g \subseteq Hg = gH \subseteq Hg$ and $gH \subseteq gg^{-1}H = gH \subseteq Hg$. Thus $gH = Hg$ and right cosets are equal to left cosets. $\square$

9.5

Show $U(5)$ is isomorphic to $U(10)$, but $U(12)$ is not.

Proof Consider $U(5) = 1, 2, 3, 4$, $U(10) = 1, 3, 7, 9$, $U(12) = 1, 5, 7, 11$.

In order for two groups to be isomorphic the order of each element in a group must correspond to a the order of an element in the other group.

Furthermore by Theorem 9.6 (iv), if an isomorphism exists between a two groups G & H,then if G is cyclic, then H is cyclic.

$U(5)$, $U(10)$, and $U(12)$, all contain the same number of elements which is a necessary condition for isomorphism between groups.

Consider, $U(5)$ is cyclic under the generator $\{2\}$ and thus has order 4. $U(10)$ is also cyclic under the generator $\{3\}$ and also has order 4. $U(12)$ is not cyclic so there exists no element in $U(12)$ with order 4. Therefore $U(12)$ cannot be isomorphic to $U(5)$ or $U(10)$. $\square$ ### 9.25 Prove or disprove: There is a noncyclic abelian group of order 52.

Proof Consider the direct product of two groups G_{26} and G_2.
Then the direct product of G_{26} which is G_{13} and G_2. So, we have G_{13}xG_2xG_2 or G_{13}xG_4 Consider since G_{13} is of prime order then it must be cyclic but G_4 does not necessarily have to be cyclic.
Consider $U(12)$ from the previous question, has order 4 but is non-cyclic.
Since the product of a cyclic group G_{13} and non cyclic group G_4 result in a group of order 52 and they are abelian since both G_{13}xG_4 and G_4xG_{13} result G_{52}, there exists a noncyclic abelian group of order 52. $\square$

9.27

Let $G \cong H$. Show that if G is cyclic, then so is H.

Proof If G is isomorphic to H, then if G is cyclic, H is cyclic. Let G be isomorphic to H and G be cyclic. Then G and H have the same number of elements, say n and each element of the respective groups has the same order. Since G_n is cyclic, $\exists$ a generator $< g >$ that produces every element in G and is of order n by definition of cyclic. Since H is isomorphic then it must have n elements, hence H_n. Then an isomorphism requires each element of G to have a corresponding element in H so by definition of isomorphic there must exist an element that also has order n. Thus $\exists$ a generator of H.

Since there exists an element in H that genereates H, H is cyclic. $\square$

10.11

If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Proof Suppose group G has exactly one subgroup H of order k. Let g_1H and g_2H be two cosets of the H in G.
Since H is the only subgroup we know that $g_1H = g_2H$