If a particle (with mass m) has potential $V(\boldsymbol{r},t)$, then, it's wave function $\psi(\boldsymbol{r},t)$ satisfies the Schrödinger equation

$$\boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r},t) + V(\boldsymbol{r},t)\psi(\boldsymbol{r},t).} \tag{1}$$

Case: zero potential If $V(\boldsymbol{r},t)=0$, Eq. (1) becomes

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r},t). \tag{2}$$

Lemma 1. *The* ***general solution*** *to the free-particle Schrödinger equation (2) is:*

$$\boxed{\psi(\mathbf{r},t) = \int A(\mathbf{k})\, e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \hbar k^2 t/2m)}\, d^3 k,}$$

representing a ***wave packet*** *(superposition of plane waves). For a single plane wave, the solution is:*

$$\boxed{\psi_{\mathbf{k}}(\mathbf{r},t) = e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \hbar k^2 t/2m)}.}$$

This describes a particle with definite momentum $\mathbf{p}=\hbar\mathbf{k}$ *and energy* $E=\hbar\omega=\hbar^2 k^2/2m$.

Proof. Assume the solution can be written as a product of a spatial part and a temporal part:

$$\psi(\mathbf{r},t) = \phi(\mathbf{r})\, T(t).$$

Substitute into the Schrödinger equation:

$$\mathrm{i}\hbar\,\phi(\mathbf{r})\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\hbar^2}{2\,m}T(t)\,\Delta\phi(\mathbf{r}).$$

Divide both sides by $\phi(\mathbf{r})\,T(t)$:

$$\frac{\mathrm{i}\hbar}{T(t)}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\hbar^2}{2\,m}\frac{\Delta\phi(\mathbf{r})}{\phi(\mathbf{r})}.$$

The left side depends only on t, and the right side only on $\mathbf{r}$. Since they are equal for all $\mathbf{r}$ and t, both sides must equal a constant, say E (the energy):

$$\frac{\mathrm{i}\hbar}{T(t)}\frac{\mathrm{d}T}{\mathrm{d}t} = E, \qquad -\frac{\hbar^2}{2\,m}\frac{\Delta\phi(\mathbf{r})}{\phi(\mathbf{r})} = E.$$

From the time equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\mathrm{i}E}{\hbar}T(t).$$

This is a first-order ODE with solution:

$$T(t) = e^{-\mathrm{i}Et/\hbar}.$$

From the spatial equation:

$$-\frac{\hbar^2}{2\,m}\,\Delta\phi(\mathbf{r}) = E\,\phi(\mathbf{r}),$$

or equivalently:

$$\Delta\phi(\mathbf{r}) + k^2\,\phi(\mathbf{r}) = 0, \quad \text{where} \quad k^2 = \frac{2\,m\,E}{\hbar^2}.$$

This is the **Helmholtz equation**, whose solutions are **plane waves**:

$$\phi(\mathbf{r}) = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}},$$

where $\mathbf{k}$ is the wavevector, and $|\mathbf{k}| = k = \sqrt{2\,m\,E}\,/\,\hbar$.

Combining the time and spatial parts, the general solution is a **superposition of plane waves**:

$$\psi(\mathbf{r},t) = \int A(\mathbf{k})\,e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,d^3\,k,$$

where:

- $A(\mathbf{k})$ is an amplitude determined by initial conditions,

- The **dispersion relation** is $\omega = \frac{E}{\hbar} = \frac{\hbar\,k^2}{2\,m}$.

Check that this satisfies the original equation:

$$\frac{\partial\,\psi}{\partial\,t} = -\mathrm{i}\omega\,\psi, \quad \Delta\psi = -k^2\,\psi.$$

Substitute:

$$\mathrm{i}\hbar\,(-\mathrm{i}\omega\psi) = -\frac{\hbar^2}{2\,m}\,(-k^2\,\psi),$$

which simplifies to:

$$\hbar\,\omega\,\psi = \frac{\hbar^2 k^2}{2\,m}\,\psi,$$

true by the dispersion relation $\omega = \hbar\,k^2\,/\,2\,m$.

The **general solution** to the free-particle Schrödinger equation is:

$$\boxed{\psi(\mathbf{r},t) = \int A(\mathbf{k})\,e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\hbar k^2 t/2m)}\,d^3\,k,}$$

representing a **wave packet** (superposition of plane waves). For a single plane wave, the solution is:

$$\boxed{\psi_{\mathbf{k}}(\mathbf{r},t) = e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\hbar k^2 t/2m)}.}$$

This describes a particle with definite momentum $\mathbf{p}=\hbar\mathbf{k}$ and energy $E = \hbar\,\omega = \hbar^2\,k^2\,/\,2\,m$. □