An example of uncountable set

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1 Countable set

Are there differences in size between different infinities? Before we answer this question, what is the "size" of an infinity?

Definition 1. A set is **countable** if it is either:

1. **Finite** (has a specific number of elements), or
2. **Countably infinite** (has the same "size" as the set of natural numbers $\mathbb{N}$, meaning its elements can be put into a one-to-one correspondence with $\mathbb{N}$).

The answer is Yes if we can find a set that is uncountable.

2 An example of uncountable set

The following proof is insipired by Rudin et al. 1953 [1]

Theorem 2. *The set of real number* $\mathbb{R}$ *is uncountable*

Proof. To prove that the set of real numbers is uncountable, we use Cantor's diagonal argument. Here are the key steps:

1. **Assume the contrary**: Suppose the interval $(0,1)$ is countable. Then, there exists a bijection $f\colon \mathbb{N} \rightarrow (0,1)$. This means we can list all real numbers in $(0,1)$ as a sequence r_1, $r_2, r_3, \ldots$.
2. **Decimal expansions**: Each real number r_i in the list can be written in decimal form as:

$$r_1 = 0.d_{11}d_{12}d_{13}\ldots$$

$$r_2 = 0.d_{21}d_{22}d_{23}\ldots$$

$$r_3 = 0.d_{31}d_{32}d_{33}\ldots$$

   and so on, where d_{ij} is the j-th digit after the decimal point of r_i.

3. **Construct a new number**: Create a new number $x = 0.x_1x_2x_3\ldots$ where each digit x_i is chosen such that $x_i \neq d_{\mathrm{ii}}$. To avoid issues with dual decimal representations (e.g., 0.999... = 1.000...), we can choose x_i to be 1 if d_{ii} is not 1, and 2 if d_{ii} is 1. This ensures x has a unique decimal expansion.
4. **Contradiction**: The number x differs from each r_i in the list at the i-th digit. Therefore, x is not in the list, contradicting the assumption that the list contains all real numbers in $(0,1)$.
5. **Conclusion**: Since our assumption leads to a contradiction, the interval $(0,1)$ must be uncountable. As $(0,1)$ is a subset of $\mathbb{R}$, the set of all real numbers $\mathbb{R}$ is also uncountable.

$$\mathbb{R} \text{ is uncountable}$$

□

Bibliography

[1] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1953.