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Generalized Lag Output Synchronization of the Multiple Output Coupled Complex Dynamical Networks

Lin-Hao Zhao [1], Shun-Yan Ren [2], Yi-Wei Shi [1], Ye-Peng Liu [1]

1. School of Computer Science and Technology, Tianjin Polytechnic University, Tianjin 300387, China
E-mail: zlh960207@163.com; goglic@163.com; liuyepeng6@163.com

2. School of Mechanical Engineering, Tianjin Polytechnic University, Tianjin 300387, China
E-mail: renshunyan@163.com

Abstract: The generalized lag output synchronization for a complex dynamical network with multiple output couplings is investigated in the present paper, in which the output matrix is positive definite or semi-positive definite. By selecting an appropriate controller and Lyapunov functional and employing inequality techniques, a sufficient condition is obtained to guarantee the generalized lag output synchronization of the complex network with positive definite output matrix. Moreover, we also use the similar method to study the case that output matrix is semi-positive definite. Lastly, two numerical examples are given to demonstrate the effectiveness of the proposed criteria.

Key Words: Complex Networks, Generalized Lag Output Synchronization, Multiple Output Couplings, Output Matrix

1 Introduction

As is known to all, many real systems (e.g. social networks, public traffic roads networks, food webs, and so on [1]) can be characterized by complex network models. Therefore, the dynamical behaviors of various complex networks have been widely studied. Particularly, the synchronization in complex networks has attracted a great deal of attention, and many meaningful results have been reported [2–8]. In [4], the authors discussed a multiple delayed complex network with directed topology, and several exponential synchronization criteria were presented for such network model by selecting appropriate Lyapunov functionals. Based on the Taylor expansion and pinning control method, Yu et al. [5] investigated the synchronization of the delayed complex networks. In [6], Lu et al. studied the pinning synchronization problem for complex networks of networks, and considered the robustness and attack of the pinning control strategy.

It should be noticed that the state synchronization for complex dynamical networks was studied in these existing literatures [2–8]. As a matter of fact, it is difficult to measure and observe the node state in complex networks. Therefore, some researchers have also discussed the output synchronization problem of complex dynamical networks [9–17]. In [10], the authors respectively considered the output synchronization problem for complex dynamical networks with fixed and adaptive coupling strength, and several output synchronization criteria were presented with the help of the output strict passivity. Wang et al. [11] analyzed the output synchronization of directed and undirected coupled neural networks by exploiting the Barbalat's Lemma and matrix theory, and two adaptive control schemes for tuning the coupling weights were developed to guarantee the output synchronization for coupled neural networks. In [12], the output synchronization for complex networks with switching topology and nonlinear nodes was studied under the hypothesis that node dynamics are incrementally dissipative. Unfortunately, the network models in these exist works [9–17] only considered the state coupling. As we all know, besides the state coupling form, the output coupling also has been widely discussed in complex networks, and some significant results about the dynamical behaviors of output coupled complex networks have been reported [18–20]. But, very few authors have studied the output synchronization problem for complex networks with output coupling [21, 22]. Based on the adaptive output feedback control method, Wang et al. [21] studied the output synchronization for output coupled complex networks with positive and semi-positive definite output matrices. In [22], the authors respectively discussed the local and global exponential output synchronization for output coupled complex delayed networks by using the Lyapunov functional method. Therefore, it is very interesting to further investigate the output synchronization for output coupled complex networks.

On the other hand, a great deal of real-world networks, such as public traffic roads networks, social networks, etc., should be modeled by multiple weighted complex network models [23–25]. In [23], the authors proposed two types of multi-weighted coupled neural networks with and without coupling delays, and several finite-time passivity and synchronization criteria were derived for these network models by selecting suitable controllers. Zhang et al. [24] not only analyzed the passivity of multiple weighted complex networks with fixed and switching topologies, but also designed appropriate adaptive state feedback controller for ensuring the passivity. In [25], two exponential stability conditions were given for multi-weighted complex dynamical networks with stochastic disturbances by utilizing graph theory. But, in these existing results [23–25], the complex network models with multiple state couplings were studied. Obviously, it is also very meaningful to further consider the dynamical behaviors of complex networks with multiple output couplings.

More recently, considering that time delay may occur in complex networks on account of traffic congestion and finite transmission speeds, some authors have studied the lag state synchronization problem for complex dynamical networks [26–30]. In [26], Wang et al. respectively discussed the lag state synchronization for fixed and adaptive coupled multiple weighted complex networks with and without time delay based on the designed state feedback controllers. Li and Cao [29] investigated the influence of parameter mismatch on the lag state synchronization for coupled memristive neural networks by using Halanay inequality and ω-measure method. Nevertheless, very few authors have studied the lag output synchronization problem of complex networks. Especially, the lag output synchronization for complex networks with output coupling has not yet been investigated.

In this paper, we discuss the lag output synchronization problem for a complex dynamical network with multiple output couplings. First, a new concept about the lag output synchronization is presented, which generalizes the tradition lag state synchronization definition. Second, we study the lag output synchronization of multiple output coupled complex network with positive definite output matrix by employing the output feedback control method. Third, a lag output synchronization criterion is derived for the complex network with multiple output couplings and semi-positive definite output matrix based on the designed output feedback controller and Barbalat's Lemma.

2 Network model and Preliminaries

Lemma 2.1. (see [31]) If the differentiable function $o(t)$ has a finite limit as $t \to +\infty$ and if $\dot{o}(t)$ is uniformly continuous, then $\dot{o}(t) \to 0$ as $t \to +\infty$.

The model of the network can be described as follows:

$$\begin{cases} \dot{x}_i(t) = Fx_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q y_j(t) \\ \qquad\quad +u_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{1}$$

where $i = 1, 2, \cdots, N$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in \mathbb{R}^n$, is the state vector of the i th node; $y_i(t) = (y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t))^T \in \mathbb{R}^n$ denotes the output vector of the i th node; $\mathbb{R}^{n\times n} \ni F = \text{diag}(F_1, F_2, \cdots, F_n); \mathbb{R} \ni c_q(q = 1, 2, \cdots, m)$ denotes coupling strength of the qth coupling form; $u_i(t) = (u_{i1}(t), u_{i2}(t), \cdots, u_{in}(t))^T \in \mathbb{R}^n$ is the control input vector of node i; the output matrix C represents a positive definite or semi-positive definite matrix; $G^q = (G_{ij}^q)_{N\times N}(q = 1, 2, \cdots m)$ means the outer coupling matrix which represents the weight between nodes; where G_{ij}^q is defined as follows: if there exists a connection between nodes j and $i(i \neq j)$, then $G_{ij}^q = G_{ji}^q > 0(i \neq j)$; elsewise, $G_{ij}^q = G_{ji}^q = 0$; moreover $G_{ii}^q = -\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}^q$.

In this paper, the complex network (1) needs to be connected. The initial condition of the complex network (1) is given as follows:

$$x_i(0) = x_i^0 \in \mathbb{R}^n, \quad y_i(0) = Cx_i^0,$$

where $i = 1, 2, \cdots, N$.

By network model (1), one gets

$$\dot{y}_i(t) = Fy_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q Cy_j(t) + Cu_i(t), \tag{2}$$

where $i = 1, 2, \cdots, N$.

Defining $y^*(t) = \frac{1}{N}\sum_{\rho=1}^{N} y_\rho(t)$, one has

$$\begin{aligned} \dot{y}^*(t) &= \frac{1}{N}\sum_{\rho=1}^{N}\Big(Fy_i(t) + \sum_{j=1}^{N} c_1 G_{\rho j}^1 Cy_j(t) \\ &\quad + \sum_{j=1}^{N} c_2 G_{\rho j}^2 Cy_j(t) + \cdots \\ &\quad + \sum_{j=1}^{N} c_q G_{\rho j}^q Cy_j(t) + Cu_\rho(t)\Big) \\ &= \frac{1}{N}\sum_{\rho=1}^{N} Fy_\rho(t) + \frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t). \end{aligned} \tag{3}$$

Letting $\eta_i(t) = y_i(t) - y^*(t)$, one obtains

$$\begin{aligned} \dot{\eta}_i(t) &= \dot{y}_i(t) - \dot{y}^*(t) \\ &= F\eta_i(t) - \frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t) \\ &\quad + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q C\eta_j(t) + Cu_i(t), \end{aligned} \tag{4}$$

where $i = 1, 2, \cdots, N$.

Definition 2.1. The network (1) is generalized lag output synchronization if

$$\lim_{t\to+\infty} \|\eta_i(t) - \eta_j(t - \tau_{ji})\| = 0 \ \text{ for all } i \neq j.$$

Remark 1. Recently, the output synchronization of complex networks has been researched, and the output synchronization is defined as follows:

$$\lim_{t\to+\infty} \|y_i(t) - y_j(t)\| = 0, \quad i, j = 1, \cdots, N.$$

Then, one has

$$\begin{aligned} & \lim_{t\to+\infty} \|y_i(t) - y_j(t)\| \\ =\ & \lim_{t\to+\infty} \|y_i(t) - y^*(t) + y^*(t) - y_j(t)\| \\ =\ & \lim_{t\to+\infty} \|\eta_i(t) - \eta_j(t)\| \\ =\ & 0. \end{aligned} \tag{5}$$

Based on (5), the lag output synchronization definition is given [see Definition 2.1], which extends the existing concept for lag state synchronization.

Remark 2. In this paper, there may be lots of paths between any nodes i and nodes j $(i \neq j)$. We always suppose that there exists the same delay τ for nodes $i \in \{1, 2, \ldots, N\}$ and $j \in \mathcal{N}_i$. In the case, we choose

$$\tau_{ji} = \alpha * \tau,$$

where α represents the least number of the connections in the paths between node i and node j.

3 The generalized lag output synchronization of multiple output coupled complex dynamical network

This section mainly discusses the multiple output coupled generalized lag output synchronization for a complex dynamical network, in which the output matrix is positive definite or semi-positive definite. In order to ensure that the network (1) can achieve the generalized lag output synchronization, we design the controller as follows:

$$u_i(t) = -k(y_i(t) - y^*(t)) + k \sum_{j\in\mathcal{N}_i} \Big[(y_j(t-\tau_{ji}) - y^*(t-\tau_{ji})) - (y_i(t) - y^*(t))\Big], \qquad (6)$$

where $\mathbb{R} \ni k > 0$.

3.1 Positive definite output matrix

In this subsection, we mainly investigated the network (1) with positive definite output matrix.

Obviously, we can obtain

$$\dot{\eta}_i(t) = F\eta_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q C\eta_j(t) - \frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t) - kC\eta_i(t) + kC\sum_{j\in\mathcal{N}_i}\big(\eta_j(t-\tau_{ji}) - \eta_i(t)\big). \qquad (7)$$

Theorem 3.1. The network (1) is generalized lag output synchronized under the controller (6) if there is a matrix $\mathbb{R}^{n\times n} \ni P = \mathrm{diag}(P_1, P_2, \cdots, P_n) > 0$ such that

$$I_N \otimes (PF) + \left(\sum_{q=1}^{m} c_q G^q - kI_N\right) \otimes (PC) \leqslant 0.$$

Proof: Choose a Lyapunov functional for the network (6) as follows:

$$V_1(t) = \sum_{i=1}^{N} \eta_i^T(t) P\eta_i(t) + k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_{t-\tau_{ji}}^{t} \eta_j^T(s) PC\eta_j(s)ds. \qquad (8)$$

Then, one has

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{N}\eta_i^T(t)P\dot{\eta}_i(t) + k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big(\eta_j^T(t)PC\eta_j(t) - \eta_j^T(t-\tau_{ji})PC\eta_j(t-\tau_{ji})\big)\\ &= 2\sum_{i=1}^{N}\eta_i^T(t)P\Bigg[F\eta_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q C\eta_j(t) - \frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t) - kC\eta_i(t)\\ &\quad + kC\sum_{j\in\mathcal{N}_i}\big(\eta_j(t-\tau_{ji}) - \eta_i(t)\big)\Bigg]\\ &\quad + k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big(\eta_j^T(t)PC\eta_j(t) - \eta_j^T(t-\tau_{ji})PC\eta_j(t-\tau_{ji})\big). \end{aligned} \qquad (9)$$

The network (1) is undirected, one gets

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\eta_i^T(t)P\eta_i(t) = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\eta_j^T(t)P\eta_j(t). \qquad (10)$$

Consequently, one obtains

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{N}\eta_i^T(t)P\Bigg(F\eta_i(t) - \frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t) + \sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q C\eta_j(t)\Bigg)\\ &\quad + 2k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\eta_i^T(t)PC\eta_j(t-\tau_{ji})\\ &\quad - k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\eta_j^T(t-\tau_{ji})PC\eta_j(t-\tau_{ji})\\ &\quad - k\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\eta_i^T(t)PC\eta_i(t)\\ &\quad - 2k\sum_{i=1}^{N}\eta_i^T(t)PC\eta_i(t). \end{aligned} \qquad (11)$$

Then, one gets

$$\begin{aligned}\sum_{i=1}^{N}\eta_i(t) &= \sum_{i=1}^{N}(y_i(t) - y^*(t))\\ &= \sum_{i=1}^{N}\left(y_i(t) - \frac{1}{N}\sum_{\rho=1}^{N} y_\rho(t)\right)\\ &= \sum_{i=1}^{N} y_i(t) - \sum_{\rho=1}^{N} y_\rho(t)\\ &= 0.\end{aligned}$$

Obviously, one has

$$\sum_{i=1}^{N}\eta_i^T(t)P\left(\frac{1}{N}\sum_{\rho=1}^{N} Cu_\rho(t)\right) = 0. \qquad (12)$$

From (9) to (12), one gets

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{N}\eta_i^T(t)PF\eta_i(t) - 2k\sum_{i=1}^{N}\eta_i^T(t)PC\eta_i(t)\\ &\quad + 2\sum_{i=1}^{N}\sum_{q=1}^{m}\sum_{j=1}^{N} c_q G_{ij}^q \eta_i^T(t)PC\eta_j(t) - kH(t)\\ &= 2\eta^T(t)\Bigg[\left(c_q\sum_{q=1}^{m} G^q - kI_N\right)\otimes(PC)\\ &\quad + I_N\otimes(PF)\Bigg]\eta(t) - kH(t)\\ &\leqslant -kH(t), \end{aligned} \qquad (13)$$

where $\eta(t) = (\eta_1^T(t), \eta_2^T(t), \cdots, \eta_N^T(t))^T$, the function $H(t) = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}(\eta_i(t) - \eta_j(t-\tau_{ji}))^T PC(\eta_i(t) - \eta_j(t-\tau_{ji}))$.

By (13), one obtains

$$H(t) \leqslant -\frac{\dot{V}_1(t)}{k}.$$

Then, one has

$$\begin{aligned}\lim_{q\to+\infty}\int_0^q H(t)dt &\leqslant \lim_{q\to+\infty} -\int_0^q \frac{\dot V_1(t)}{k}dt\\ &= -\frac{V_1(+\infty)}{k}+\frac{V_1(0)}{k}.\end{aligned}$$

Moreover,

$$\begin{aligned}\dot H(t) &= 2\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}(\eta_i(t)-\eta_j(t-\tau_{ji}))^T PC(\dot\eta_i(t)\\ &\quad -\dot\eta_j(t-\tau_{ji})).\end{aligned}$$

On account of $\eta_i(t),\dot\eta_i(t), i=1,2,3,\cdots,N$ are bounded for any $t\in[0,+\infty)$, it is easy to derive that $\left|\dot H(t)\right|$ is bounded. Accordingly, $H(t)$ is uniformly continuous.

On the basis of Lemma 2.1, we can conclude that

$$\lim_{q\to+\infty} H(t)=0.$$

Namely,

$$\lim_{t\to+\infty}\|\eta_i(t)-\eta_j(t-\tau_{ji})\|=0 \ \ i=1,2,\cdots,N,\ j\in\mathcal{N}_i.$$

Therefore, the network (1) is generalized lag output synchronized.□

3.2 Semi-positive definite output matrix

Rearrange the order of output variables and state variables such that

$$C=\mathrm{diag}(c_1,c_2,\cdots,c_\xi,0,\cdots,0),$$

where $\xi\in[1,n)$.

For the purpose of getting main results, one defines

$$\begin{aligned}&\bar C=\mathrm{diag}(c_1,c_2,\cdots,c_\xi),\quad \tilde C=(\bar C,0)\in\mathbb{R}^{\xi\times n},\\ &\tilde x_i(t)=(x_{i1}(t),x_{i2}(t),\cdots,x_{i\xi}(t))^T\in\mathbb{R}^\xi,\\ &\tilde y_i(t)=\bar C\tilde x_i(t), y_i(t)=\left((\tilde y_i(t))^T,0,0,\cdots,0\right)^T,\\ &\tilde u_i(t)=\left(u_{i1}(t),u_{i2}(t),\cdots,u_{i\xi}(t)\right)^T\in\mathbb{R}^\xi,\\ &\bar F=\mathrm{diag}(F_1,F_2,\cdots,F_\xi).\end{aligned}$$

Then, we can get from (1) that

$$\dot{\tilde y}_i(t)=\bar F\tilde y_i(t)+\sum_{q=1}^m\sum_{j=1}^N c_qG_{ij}^q\tilde C\tilde y_j(t)+\tilde C\tilde u_i(t), \tag{14}$$

where $i=1,2,\cdots,N$.

Theorem 3.2. The network (1) is generalized lag output synchronized under the controller (6) if there is a matrix $\mathbb{R}^{\xi\times\xi}\ni\Upsilon=\mathrm{diag}(\upsilon_1,\upsilon_2,\cdots,\upsilon_\xi)>0$ such that

$$I_N\otimes(\Upsilon\bar F)+\left(\sum_{q=1}^m c_qG^q-kI_N\right)\otimes(\Upsilon\bar C)\leqslant 0.$$

Proof. Define $\tilde y^*(t)=\frac{1}{N}\sum_{\rho=1}^N\tilde y_i(t)$, and $\tilde\eta_i(t)=\tilde y_i(t)-\tilde y^*(t)$. Then, one obtains

$$\begin{aligned}\tilde\eta_i(t) &= \bar F\tilde\eta_i(t)+\sum_{q=1}^m\sum_{j=1}^N c_qG_{ij}^q\bar C\tilde\eta_j(t)-\frac{1}{N}\sum_{\rho=1}^N\bar C\tilde u_\rho(t)\\ &\quad -k\bar C\tilde\eta_i(t)+k\bar C\sum_{j\in\mathcal{N}_i}\left((\tilde\eta_j(t-\tau_{ji})-\tilde\eta_i(t)\right),\end{aligned} \tag{15}$$

where $i=1,2,\cdots,N$.

Select a Lyapunov function for network model (15)

$$\begin{aligned}V_2(t) &= \sum_{i=1}^N\tilde\eta_i^T(t)\Upsilon\tilde\eta_i(t)\\ &\quad +k\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\int_{t-\tau_{ji}}^t\tilde\eta_j^T(s)\Upsilon\bar C\tilde\eta_j(s)ds.\end{aligned} \tag{16}$$

Obviously, one gets

$$\begin{aligned}\dot V_2(t) &= 2\sum_{i=1}^N\tilde\eta_i^T(t)\Upsilon\dot{\tilde\eta}_i(t)+k\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\big(\tilde\eta_j^T(t)\Upsilon\bar C\tilde\eta_j(t)\\ &\quad -\tilde\eta_j^T(t-\tau_{ji})\Upsilon\bar C\tilde\eta_j(t-\tau_{ji})\big)\\ &= 2\sum_{i=1}^N\tilde\eta_i^T(t)\Upsilon\Bigg[\bar F\tilde\eta_i(t)+\sum_{q=1}^m\sum_{j=1}^N c_qG_{ij}^q\bar C\tilde\eta_j(t)\\ &\quad -\frac{1}{N}\sum_{\rho=1}^N\bar Cu_\rho(t)-k\bar C\tilde\eta_i(t)\\ &\quad +k\bar C\sum_{j\in\mathcal{N}_i}\left((\tilde\eta_j(t-\tau_{ji})-\tilde\eta_i(t)\right)\Bigg]\\ &\quad +k\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\big(\eta_j^T(t)\Upsilon\bar C\eta_j(t)\\ &\quad -\eta_j^T(t-\tau_{ji})\Upsilon\bar C\eta_j(t-\tau_{ji})\big)\\ &= 2\sum_{i=1}^N\tilde\eta_i^T(t)\Upsilon\bar F\tilde\eta_i(t)-2k\sum_{i=1}^N\tilde\eta_i^T(t)\Upsilon\bar C\tilde\eta_i(t)\\ &\quad +2\sum_{i=1}^N\sum_{q=1}^m\sum_{j=1}^N c_qG_{ij}^q\tilde\eta_i^T(t)\Upsilon\bar C\tilde\eta_j(t)-k\bar H(t)\\ &= 2\tilde\eta^T(t)\Bigg[\left(\sum_{q=1}^m c_qG^q-kI_N\right)\otimes(\Upsilon\bar C)\\ &\quad +I_N\otimes(\Upsilon\bar F)\Bigg]\tilde\eta(t)-k\bar H(t)\\ &\leqslant -k\bar H(t),\end{aligned} \tag{17}$$

where $\tilde\eta(t)=(\tilde\eta_1^T(t),\tilde\eta_2^T(t),\cdots,\tilde\eta_n^T(t))^T$, the function $\bar H(t)=\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}(\tilde\eta_i(t)-\tilde\eta_j(t-\tau_{ji}))^T\Upsilon\bar C(\tilde\eta_i(t)-\tilde\eta_j(t-\tau_{ji}))$.

Finally, similarly to the proof of Theorem 3.1, we can easily get the conclusion.□

4 Numerical example

Example 4.1. Consider the following multiweighted complex dynamical network with output couplings:

$$\begin{cases}\dot x_i(t) &= Fx_i(t)+0.3\sum_{j=1}^6 G_{ij}^1y_j(t)\\ &\quad +0.4\sum_{j=1}^6 G_{ij}^2y_j(t)\\ &\quad +0.5\sum_{j=1}^6 G_{ij}^3y_j(t)+u_i(t),\\ y_i(t) &= Cx_i(t),\ i=1,2,\cdots,6,\end{cases}$$

where $F=\mathrm{diag}(1,2,3), C=\mathrm{diag}(0.5,0.6,0.7), x_i(t)=(x_{i1}(t),x_{i2}(t),x_{i3}(t))^T\in\mathbb{R}^3, y_i(t)=(y_{i1}(t),y_{i2}(t),$

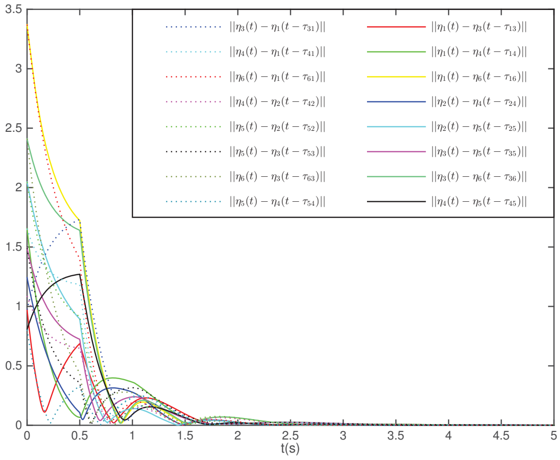

Fig. 1: The norms of the vectors $\eta_i(t) - \eta_j(t - \tau_{ji})$, $i = 1, 2, \cdots, 6$, $j \in \mathcal{N}_i$.

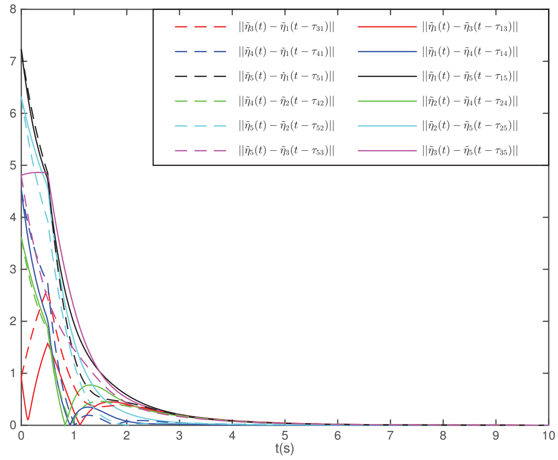

Fig. 2: The norms of the vectors $\tilde{\eta}_i(t) - \tilde{\eta}_j(t - \tau_{ji})$, $i = 1, 2, \cdots, 5. j \in \mathcal{N}_i$.

$y_{i3}(t))^T \in \mathbb{R}^3, u_i(t) = (u_{i1}(t), u_{i2}(t), u_{i3}(t))^T \in \mathbb{R}^3$, and the matrices G^1, G^2, G^3 are respectively chosen as follows:

$$G^1 = \begin{pmatrix} -0.8 & 0 & 0.3 & 0.4 & 0 & 0.1 \\ 0 & -0.4 & 0 & 0.2 & 0.2 & 0 \\ 0.3 & 0 & -0.9 & 0 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0 & -0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.4 & 0.2 & -0.8 & 0 \\ 0.1 & 0 & 0.2 & 0 & 0 & -0.3 \end{pmatrix},$$

$$G^2 = \begin{pmatrix} -1.1 & 0 & 0.4 & 0.5 & 0 & 0.2 \\ 0 & -0.4 & 0 & 0.2 & 0.2 & 0 \\ 0.4 & 0 & -0.8 & 0 & 0.1 & 0.3 \\ 0.5 & 0.2 & 0 & -0.8 & 0.1 & 0 \\ 0 & 0.2 & 0.1 & 0.1 & -0.4 & 0 \\ 0.2 & 0 & 0.3 & 0 & 0 & -0.5 \end{pmatrix},$$

$$G^3 = \begin{pmatrix} -0.7 & 0 & 0.2 & 0.3 & 0 & 0.2 \\ 0 & -0.5 & 0 & 0.3 & 0.2 & 0 \\ 0.2 & 0 & -0.9 & 0 & 0.5 & 0.2 \\ 0.3 & 0.3 & 0 & -0.7 & 0.1 & 0 \\ 0 & 0.2 & 0.5 & 0.1 & -0.8 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0 & -0.4 \end{pmatrix}.$$

Taking $k = 5$ and $\varpi = I_3$, it is easily to demonstrate that the requirement of Theorem 3.1 is met. Therefore, the network (1) based on the controller (6) is lag output synchronized. The results of simulation are shown in Fig. 1.

Example 4.2. Consider a complex dynamical network consisting of 5 identical nodes, in which each node is a 4-dimensional linear system described by

$$\begin{cases} \dot{x}_i(t) &= Fx_i(t) + 0.2\sum_{j=1}^{5} G^1_{ij} y_j(t) \\ & \quad +0.3\sum_{j=1}^{5} G^2_{ij} y_j(t) \\ & \quad +0.4\sum_{j=1}^{5} G^2_{ij} y_j(t) \\ & \quad +0.5\sum_{j=1}^{5} G^3_{ij} y_j(t) + u_i(t), \\ y_i(t) &= Cx_i(t),\ i = 1, 2, \cdots, 5, \end{cases}$$

where $F = \mathrm{diag}(1, 2, 3), C = \mathrm{diag}(0.3, 0.4, 0), x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \in \mathbb{R}^3, y_i(t) = (y_{i1}(t), y_{i2}(t), y_{i3}(t))^T \in \mathbb{R}^3, u_i(t) = (u_{i1}(t), u_{i2}(t), u_{i3}(t))^T \in \mathbb{R}^3$, and the matrices G^1, G^2, G^3, G^4 are chosen as follows:

$$G^1 = \begin{pmatrix} -0.9 & 0 & 0.2 & 0.4 & 0.3 \\ 0 & -0.5 & 0 & 0.3 & 0.2 \\ 0.2 & 0 & -0.3 & 0 & 0.1 \\ 0.4 & 0.3 & 0 & -0.7 & 0 \\ 0.3 & 0.2 & 0.1 & 0 & -0.6 \end{pmatrix},$$

$$G^2 = \begin{pmatrix} -1 & 0 & 0.3 & 0.5 & 0.2 \\ 0 & -0.4 & 0 & 0.2 & 0.2 \\ 0.3 & 0 & -0.5 & 0 & 0.2 \\ 0.5 & 0.2 & 0 & -0.7 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & -0.6 \end{pmatrix},$$

$$G^3 = \begin{pmatrix} -0.8 & 0 & 0.3 & 0.3 & 0.2 \\ 0 & -0.5 & 0 & 0.2 & 0.3 \\ 0.3 & 0 & -0.6 & 0 & 0.3 \\ 0.3 & 0.2 & 0 & -0.5 & 0 \\ 0.2 & 0.3 & 0.3 & 0 & -0.8 \end{pmatrix},$$

$$G^4 = \begin{pmatrix} -1.3 & 0 & 0.5 & 0.7 & 0.1 \\ 0 & -0.5 & 0 & 0.2 & 0.3 \\ 0.5 & 0 & -0.6 & 0 & 0.1 \\ 0.7 & 0.2 & 0 & -0.9 & 0 \\ 0.1 & 0.3 & 0.1 & 0 & -0.5 \end{pmatrix}.$$

Taking $k = 5$ and $\Upsilon = I_2$, it is easily to demonstrate that the requirement of Theorem 3.2 is met. Therefore, the network (1) based on the controller (6) is lag output synchronized. The results of simulation are shown in Fig. 2.

5 Conclusion

In this paper, we have considered the generalized lag output synchronization problem for complex dynamical network with multiple output couplings. And then we have discussed two kinds of output matrices which are positive definite and semi-positive definite. On the basic of a suitable controller and Barlalat's lemma, a sufficient condition has been obtained to guarantee generalized lag output synchronization of the network model. Moreover, a generalized lag output synchronized criterion is also established for multi-weighted complex dynamical network by employing the designed controller. Finally, two numerical examples have been provided to verify the correctness and effectiveness of the acquired results.

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