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Lag output synchronization for multiple output coupled complex networks with positive semidefinite or positive definite output matrix

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Abstract

This paper investigates the lag output synchronization of a multiple output coupled complex network, in which the output matrix is positive semidefinite or positive definite. By selecting an appropriate output feedback controller and Lyapunov functional, a criterion is acquired to insure the lag output synchronization for the complex network, in which the output matrix is positive definite. Moreover, an adaptive output feedback control strategy is also developed for ensuring the lag output synchronization in the network with positive definite output matrix. On the other side, we also similarly study the case of positive semidefinite output matrix. Lastly, two numerical examples are given to demonstrate the came up with criteria.

1. Introduction

As is known to all, many real systems (e.g. food webs, social networks, metabolic systems, etc. [1]) can be characterized by complex network models. Therefore, the dynamical behaviors

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of various complex networks have been widely studied. Particularly, the synchronization in complex networks has raised a huge number of attention, and lots of meaningful results have been reported [2–10]. In [4], Wang et al. discussed a directed complex network with multiple delayes, and a number of criteria upon exponential synchronization were presented for such network by selecting Lyapunov functionals. Based on the Taylor expansion and pinning control method, the authors [5] studied the synchronization of the delayed complex networks. Lu et al. [6] investigated the pinning synchronization problem for complex networks of networks, and considered the robustness and attack of the pinning control strategy.

In these existing literatures [2–9], it's worth noting that the state synchronization problem was studied. As the case stands, it is difficult to measure and observe the state of node in complex networks. Therefore, some researchers also have discussed the output synchronization problem of complex dynamical networks [10–22]. In [11], the authors researched the output synchronization for undirected and directed coupled neural networks by exploiting the Barbalat's Lemma and matrix theory, and two kinds of adaptive control schemes for tuning the coupled weights were developed to insure the output synchronization for coupled neural networks. In [12], Wang et al. respectively considered the output synchronization of fixed and adaptive coupled complex networks, and several output synchronization criteria were presented with the help of the output strict passivity. Under the hypothesis that node dynamics are incrementally dissipative, Liu et al. [13], discussed the output synchronization for complex networks with switching topology and nonlinear nodes. In [22], several output synchronization criteria for complex networks with multiple output or output derivative couplings were established, and some adaptive controllers were also devised for insure the output synchronization of these networks. Unfortunately, the network models in these exist works [10–21] only considered the state coupling. As we all know, besides the state coupling form, the output coupling also has been widely discussed in complex networks, and some significant results about the dynamical behaviors for output coupled complex networks have been reported [23–25]. But, there have not been some results about the output synchronization problem for output coupled complex networks [26,27]. Based on the adaptive output feedback control approach, Wang et al. [26] investigated the output synchronization for output coupled complex networks with positive and positive semidefinite output matrices. In [27], the authors discussed the global and local exponential output synchronization of output coupled complex delayed networks by utilizing the Lyapunov functional approach. Thus, it makes the output synchronization for output coupled complex network worthy of being investigated further.

On the other side, a huge number of real-life networks, such as public traffic roads networks, coupled neural networks, etc., may be represented by multi-weighted networks [28–33]. Zhang et al. [28] analyzed the passivity of multiweighted complex networks with fixed and switching topologies, and devised appropriate adaptive state feedback controller for ensuring the passivity. In [29], the authors proposed two types of multi-weighted coupled neural networks with and without coupling delays, and several finite-time passivity and synchronization criteria were derived for these network models by selecting suitable controllers. In [30], two exponential stability conditions were given for multi-weighted complex dynamical networks with stochastic disturbances by utilizing graph theory. But, in these existing results [28–30], the complex network models with multiple state couplings were studied. Obviously, it is also very meaningful to further consider the dynamical behaviors of complex networks with multiple output couplings.

More recently, considering that time delay may occur in complex networks on account of traffic congestion and finite transmission speeds, some authors have studied the lag state synchronization problem for complex dynamical networks [34–41]. Ji et al. [34] studied the lag synchronization between an uncertain complex delayed dynamical network and a nonidentical reference node with uncertain parameter on the basis of the adaptive control approach. In [35], Wang et al. respectively discussed the lag state synchronization for fixed and adaptive coupled multiple weighted complex networks with and without time delay based on the designed state feedback controllers. Li and Cao [38] investigated the influence of parameter mismatch on the lag state synchronization for coupled memristive neural networks by using Halanay inequality and ω-measure method. Nevertheless, there have not been some researches about the lag output synchronization problem for complex networks. Especially, the lag output synchronization of the multiple output coupled complex networks has not been investigated.

This paper discusses a multiple output coupled complex network, in which the output matrix is positive semidefinite or positive definite. The mainly contributions are given as follows: First, a new concept about the lag output synchronization is presented, which generalizes the tradition lag synchronization definition. Second, by employing the output feedback control and adaptive output feedback control approaches, we investigate the lag output synchronization of multiple output coupled complex network, in which the output matrix is positive definite. Third, two criteria of the lag output synchronization are derived for the multiple output coupled complex network with positive semidefinite output matrix on the basic of the designed output feedback controller and adaptive output feedback controller.

Notation. $\mathbb{R} \ni \tau_{ij} > 0$ denotes the time delay between node i and node j. $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ denote the maximum and minimum eigenvalues of a symmetric matrix. In the network, $\mathcal{S} = \{1, 2, \ldots, \mathcal{M}\}$ and $\mathcal{B} \subset \mathcal{S} \times \mathcal{S}$ indicate the node set and the undirected connection set. $\mathcal{D}_i$ represents the neighbors of the node i.

2. Network model and preliminaries

2.1. Lemma

Lemma 2.1 (see [42]). *If the differentiable function $o(t)$ has a finite limit as $t \to +\infty$ and $\dot{o}(t)$ is uniformly continuous, then $\dot{o}(t) \to 0$ as $t \to +\infty$.*

2.2. Network model

The network model with multiple output couplings considered in this paper can be described as follows:

$$\begin{cases} \dot{x}_i(t) = Zx_i(t) + \sum_{q=1}^{m} \sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q y_j(t) + u_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{1}$$

where $i = 1, 2, \ldots, \mathcal{M}$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of the ith node; $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in \mathbb{R}^n$ denotes the output vector of the ith node; $\mathbb{R}^{n\times n} \ni Z = \mathrm{diag}(z_1, z_2, \ldots, z_n)$; $\mathbb{R} \ni c_q (q = 1, 2, \ldots, m)$ represents the coupling strength of the qth coupling form; $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots, u_{in}(t))^T \in \mathbb{R}^n$ is the control input vector of node i; the output matrix $C = \mathrm{diag}(C_1, C_2, \ldots, C_n)$ is positive semidefinite or positive definite; $\mathbb{R}^{\mathcal{M}\times\mathcal{M}} \ni G^q = (G_{ij}^q)_{\mathcal{M}\times\mathcal{M}} (q = 1, 2, \ldots, m)$ represents the outer coupling matrix, where G_{ij}^q

is defined as follows: if there is a connection between nodes j and $i(i \neq j)$, then $G_{ij}^q = G_{ji}^q > 0$; otherwise, $G_{ij}^q = G_{ji}^q = 0$; furthermore, $G_{ii}^q = -\sum_{\substack{j=1 \\ j\neq i}}^{\mathcal{M}} G_{ij}^q$.

In this paper, the complex network (1) needs to be connected. The initial condition of the complex network (1) is given as follows:

$$x_i(0) = x_i^0 \in \mathbb{R}^n, \qquad y_i(0) = Cx_i^0,$$

where $i = 1, 2, \ldots, \mathcal{M}$.

By network model (1), we can get

$$\dot{y}_i(t) = Zy_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q C y_j(t) + Cu_i(t),$$

where $i = 1, 2, \ldots, \mathcal{M}$.

Letting $y^*(t) = \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} y_\rho(t)$, one obtains

$$\begin{aligned}\dot{y}^*(t) &= \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} Zy_\rho(t) + \frac{1}{\mathcal{M}}\sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q\left(\sum_{\rho=1}^{\mathcal{M}} G_{\rho j}^q\right)Cy_j(t) + \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} Cu_\rho(t)\\ &= Zy^*(t) + \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} Cu_\rho(t). \end{aligned} \tag{2}$$

Defining $\varepsilon_i(t) = y_i(t) - y^*(t)$, one has

$$\begin{aligned}\dot{\varepsilon}_i(t) &= \dot{y}_i(t) - \dot{y}^*(t)\\ &= Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q C\varepsilon_j(t) + Cu_i(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} Cu_\rho(t), \end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, \mathcal{M}$.

In what follows, the lag output synchronization for network (1) is defined:

Definition 2.1. The network (1) is lag output synchronized if

$$\lim_{t\to+\infty} \|\varepsilon_i(t) - \varepsilon_j(t-\tau_{ji})\| = 0 \quad \text{for all} \quad i \neq j,$$

where τ_{ji} is time-delay between node i and node j.

Remark 1. Practically, for any connected complex network, the network is synchronization when and only when the state error of any neighbor nodes in this network tends to be zero as time approach infinity. As is well known, the time delay must appears between neighbor nodes in all real-life networks on account of traffic congestion and finite transmission speeds. Therefore, the state error between neighbor nodes should be that the current state of node minus the past state of neighbor node. In fact, this is the main reason why we investigate the lag synchronization in complex networks. Obviously, compared with synchronization, it is more meaningful to investigate the lag synchronization in complex networks.

Remark 2. More recently, the output synchronization in complex dynamical networks has been widely studied, in which the output synchronization is defined as follows:

$$\lim_{t\to+\infty} \|y_i(t) - y_j(t)\| = 0, \;\; i, j = 1, \ldots, \mathcal{M}.$$

Taking $y^*(t) = \frac{1}{\mathcal{M}} \sum_{\rho=1}^{\mathcal{M}} y_\rho(t)$, we can obtain

$$\begin{aligned}\lim_{t\to+\infty} \|y_i(t) - y_j(t)\| = 0 \Longleftrightarrow \lim_{t\to+\infty} &\|\big(y_i(t) - y^*(t)\big) - \big(y_j(t) - y^*(t)\big)\| \\ = &\lim_{t\to+\infty} \|\varepsilon_i(t) - \varepsilon_j(t)\| \\ = & \ 0, \ \ i, j = 1, \ldots, \mathcal{M}. \end{aligned} \tag{4}$$

Based on Eq. (4), the lag output synchronization definition is given [see Definition 2.1], which extends the existing concept for lag synchronization.

Remark 3. In the past few years, many authors have studied the lag synchronization problem for complex dynamical networks, and lots of meaningful results have been obtained [34–41]. But, in these existing work, the lag state synchronization in complex networks was investigated. Considering that node state is difficult to measure and observe in complex networks, thus it is more meaningful to study the lag output synchronization for complex networks. Regretfully, the lag output synchronization for complex networks has not yet been considered.

3. The lag output synchronization of multiple output coupled complex network with positive definite output matrix

This section respectively studies the lag output synchronization problem for the multiple output coupled complex dynamical network (1) with positive definite output matrix, and some output synchronization criteria are derived by selecting appropriate output feedback controller and adaptive output feedback controller.

3.1. Output feedback controller

In order to ensure that the network (1) can realize the lag output synchronization, the following output feedback controller is selected:

$$u_i(t) = -k(y_i(t) - y^*(t)) + k \sum_{j\in\mathcal{D}_i} \Big[(y_j(t-\tau_{ji}) - y^*(t-\tau_{ji})) - (y_i(t) - y^*(t))\Big], \tag{5}$$

where $\mathbb{R} \ni k > 0$.

By Eq. (5), one obtains

$$\begin{aligned}\dot{\varepsilon}_i(t) = Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q C\varepsilon_j(t) - \frac{1}{\mathcal{M}} \sum_{i=1}^{\mathcal{M}} C u_\rho(t) \\ -kC\varepsilon_i(t) + k\sum_{j\in\mathcal{D}_i} C\big(\varepsilon_j(t-\tau_{ji}) - \varepsilon_i(t)\big). \end{aligned} \tag{6}$$

Theorem 3.1. *Based on the output feedback controller (5), the network (1) is lag output synchronized if there is $\mathbb{R}^{n\times n} \ni \varpi = diag(\varpi_1, \varpi_2, \ldots, \varpi_n) > 0$ satisfing*

$$I_{\mathcal{M}} \otimes (\varpi Z) + \left(\sum_{q=1}^{m} c_q G^q - kI_{\mathcal{M}}\right) \otimes (\varpi C) \leqslant 0. \tag{7}$$

Proof. Select a Lyapunov functional for the network Eq. (6) as follows:

$$V_1(t) = \sum_{i=1}^{\mathcal{M}} \varepsilon_i^T(t)\varpi\varepsilon_i(t) + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\int_{t-\tau_{ji}}^{t}\varepsilon_j^T(s)\varpi C\varepsilon_j(s)ds. \tag{8}$$

Consequently, one has

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi\dot{\varepsilon}_i(t) + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\varepsilon_j^T(t)\varpi C\varepsilon_j(t)\\ &\quad -\varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji})\big)\\ &= 2\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi\Big[Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^qC\varepsilon_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}Cu_\rho(t)\\ &\quad -kC\varepsilon_i(t) + k\sum_{j\in\mathcal{D}_i}C\big(\varepsilon_j(t-\tau_{ji}) - \varepsilon_i(t)\big)\Big]\\ &\quad +k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\varepsilon_j^T(t)\varpi C\varepsilon_j(t) - \varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji})\big).\end{aligned} \tag{9}$$

On account of the network (1) is undirected, one obtains

$$\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_i^T(t)\varpi C\varepsilon_i(t) = \sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_j^T(t)\varpi C\varepsilon_j(t). \tag{10}$$

By Eqs. (9) and (10), one gets

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi\left(Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^qC\varepsilon_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}Cu_\rho(t)\right)\\ &\quad -2k\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi C\varepsilon_i(t) + 2k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_i^T(t)\varpi C\big(\varepsilon_j(t-\tau_{ji}) - \varepsilon_i(t)\big)\\ &\quad +k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_j^T(t)\varpi C\varepsilon_j(t) - k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji})\\ &= 2\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi\left(Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^qC\varepsilon_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}Cu_\rho(t)\right)\\ &\quad -2k\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi C\varepsilon_i(t) + 2k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_i^T(t)\varpi C\varepsilon_j(t-\tau_{ji})\\ &\quad -k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_i^T(t)\varpi C\varepsilon_i(t) - k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji}).\end{aligned} \tag{11}$$

Moreover, since

$$\begin{aligned}\sum_{i=1}^{\mathcal{M}}\varepsilon_i(t)&=\sum_{i=1}^{\mathcal{M}}(y_i(t)-y^*(t))\\&=\sum_{i=1}^{\mathcal{M}}\left(y_i(t)-\frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}y_\rho(t)\right)\\&=\sum_{i=1}^{\mathcal{M}}y_i(t)-\sum_{\rho=1}^{\mathcal{M}}y_\rho(t)\\&=0,\end{aligned}$$

one derives

$$\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi\left(\frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}Cu_\rho(t)\right)=0. \tag{12}$$

From Eqs. (9) to (12), one gets

$$\begin{aligned}\dot{V}_1(t)&=2\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi Z\varepsilon_i(t)-2k\sum_{i=1}^{\mathcal{M}}\varepsilon_i^T(t)\varpi C\varepsilon_i(t)\\&\quad+2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^q\varepsilon_i^T(t)\varpi C\varepsilon_j(t)-kH(t)\\&=\varepsilon^T(t)\left[2I_{\mathcal{M}}\otimes(\varpi Z)+2\left(\sum_{q=1}^{m}c_qG^q-kI_{\mathcal{M}}\right)\otimes(\varpi C)\right]\varepsilon(t)-kH(t)\\&\leqslant -kH(t),\end{aligned} \tag{13}$$

where $\varepsilon(t)=(\varepsilon_1^T(t),\varepsilon_2^T(t),\ldots,\varepsilon_{\mathcal{M}}^T(t))^T$ and $H(t)=\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji}))^T\varpi C(\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji}))$.

In light of Eq. (13), we have $\lim_{t\to+\infty}V_1(t)$ exists and

$$H(t)\leqslant-\frac{\dot{V}_1(t)}{k}.$$

Accordingly, one has

$$\begin{aligned}\lim_{t\to+\infty}\int_0^t H(q)dq&\leqslant-\lim_{t\to+\infty}\int_0^t\frac{\dot{V}_1(q)}{k}dq\\&=\frac{V_1(0)}{k}-\frac{V_1(+\infty)}{k}.\end{aligned}$$

Moreover,

$$\dot{H}(t)=2\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji}))^T\varpi C(\dot{\varepsilon}_i(t)-\dot{\varepsilon}_j(t-\tau_{ji})).$$

On account of $\varepsilon_i(t), \dot{\varepsilon}_i(t), \varepsilon_j(t-\tau_{ji}), \dot{\varepsilon}_j(t-\tau_{ji}), i = 1, 2, 3, \ldots, \mathcal{M}$ are bounded for any $t \in [0, +\infty)$, it is easy to infer that $\left|\dot{H}(t)\right|$ is bounded. Consequently, $H(t)$ is uniformly continuous.

On the basis of Lemma 2.1, one concludes that

$$\lim_{t\to+\infty} H(t) = 0.$$

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$$\lim_{t\to+\infty} \|\varepsilon_i(t) - \varepsilon_j(t-\tau_{ji})\| = 0, \;\; i = 1, 2, \ldots, \mathcal{M}, \;\; j \in \mathcal{D}_i.$$

Therefore, the network (1) is lag output synchronized. □

Remark 4. For any node $i \in \{1, 2, \ldots, \mathcal{M}\}$ and node $j \in \mathcal{D}_i$, we always hypothesize that there exists the same time delay τ in this paper. On the other hand, there may be lots of paths between nodes φ and $\iota(\varphi \neq \iota)$. Under the circumstance, we select

$$\tau_{\varphi\iota} = \tau * \eta,$$

where η represents that the number of connections in the shortest path between node φ and node ι.

3.2. Adaptive output feedback controller

In this subsection, $\kappa^q(t) = (\kappa^q_{ij}(t))_{\mathcal{M}\times\mathcal{M}}$ denotes a matrix varied with time, in which $\kappa^q_{ij}(t)$ has the following definition: if node j and node $i(i\neq j)$ are connected, then $\kappa^q_{ij}(t) = \kappa^q_{ji}(t) > 0$; elsewise, $\kappa^q_{ij}(t) = \kappa^q_{ji}(t) = 0$; moreover,

$$\kappa^q_{ii}(t) = -\sum_{\substack{j=1\\ j\neq i}}^{\mathcal{M}} \kappa^q_{ij}(t), \quad i = 1, 2, \ldots, \mathcal{M}.$$

Design an appropriate adaptive output feedback controller for network (3):

$$u_i(t) = \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q \kappa^q_{ij}(t)\varepsilon_j(t) + \sum_{j\in\mathcal{D}_i} k\big(\varepsilon_j(t-\tau_{ji}) - \varepsilon_i(t)\big), \;\; i = 1, 2, \ldots, \mathcal{M},$$
$$\dot{\kappa}^q_{ij}(t) = \begin{cases} a_{ij}(y_i(t) - y_j(t))^T \varpi C(y_i(t) - y_j(t)), & \text{if } (i, j) \in \mathcal{B}, \\ -\sum_{\substack{s=1\\ s\neq i}}^{\mathcal{M}} \dot{\kappa}^q_{is}(t), & \text{if } \;\; i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

where $\mathbb{R} \ni k > 0$, $\mathbb{R} \ni a_{ij} = a_{ji} > 0$ and $\mathbb{R}^{n\times n} \ni \varpi = \mathrm{diag}(\varpi_1, \varpi_2, \ldots, \varpi_n) > 0$.

Apparently,

$$\begin{aligned} \dot{\varepsilon}_i(t) = {} & Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G^q_{ij} C\varepsilon_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} Cu_\rho(t) \\ & + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q \kappa^q_{ij}(t) C\varepsilon_j(t) + \sum_{j\in\mathcal{D}_i} kC\big(\varepsilon_j(t-\tau_{ji}) - \varepsilon_i(t)\big), \end{aligned} \tag{15}$$

where $i = 1, 2, \ldots, \mathcal{M}$.

Theorem 3.2. *Based on the adaptive output feedback controller (14), the network (1) achieves the lag output synchronization.*

Proof. Choose an appropriate Lyapunov functional for network (15) as follows:

$$V_2(t) = \sum_{i=1}^{\mathcal{M}} \varepsilon_i^T(t)\varpi\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\frac{c_q(\kappa_{ij}^q(t)-B_{ij}^q)^2}{2a_{ij}} + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\int_{t-\tau_{ji}}^{t}\varepsilon_j^T(s)\varpi C\varepsilon_j(s)ds, \tag{16}$$

where $\mathbb{R}^{\mathcal{M}\times\mathcal{M}} \ni B_{ji}^q = B_{ij}^q\ (i\neq j) \geqslant 0, B_{ij}^q = 0$ when and only when $\kappa_{ij}^q(t) = 0$.

Consequently, one gets

$$\begin{aligned}
\dot{V}_2(t) &= 2\sum_{i=1}^{\mathcal{M}} \varepsilon_i^T(t)\varpi\dot{\varepsilon}_i(t) + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\varepsilon_j^T(t)\varpi C\varepsilon_j(t) \\
&\quad -\varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji})\big) \\
&\quad + \sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\frac{c_q(\kappa_{ij}^q(t)-B_{ij}^q)}{a_{ij}}\dot{\kappa}_{ij}^q(t) \\
&= 2\sum_{i=1}^{\mathcal{M}} \varepsilon_i^T(t)\varpi\Big[Z\varepsilon_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q C\varepsilon_j(t) \\
&\quad + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q\kappa_{ij}^q(t)C\varepsilon_j(t) + \sum_{j\in\mathcal{D}_i} kC\big(\varepsilon_j(t-\tau_{ji})-\varepsilon_i(t)\big)\Big] \\
&\quad + \sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i} c_q(\kappa_{ij}^q(t)-B_{ij}^q)(\varepsilon_i(t)-\varepsilon_j(t))^T\varpi C(\varepsilon_i(t)-\varepsilon_j(t)) \\
&\quad + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\varepsilon_j^T(t)\varpi C\varepsilon_j(t) - \varepsilon_j^T(t-\tau_{ji})\varpi C\varepsilon_j(t-\tau_{ji})\big).
\end{aligned} \tag{17}$$

Moreover, one derives

$$\begin{aligned}
&\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\kappa_{ij}^q(t)-B_{ij}^q)(\varepsilon_i(t)-\varepsilon_j(t))^T\varpi C(\varepsilon_i(t)-\varepsilon_j(t)) \\
&= -2\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}}(\kappa_{ij}^q(t)-B_{ij}^q)\varepsilon_i^T(t)\varpi C\varepsilon_j(t),
\end{aligned} \tag{18}$$

where $B_{ii}^q = -\sum_{\substack{j=1\\ j\neq i}}^{\mathcal{M}} B_{ij}^q$.

By Eqs. (17) and (18), one obtains

$$\dot{V}_2(t) = 2\sum_{i=1}^{\mathcal{M}} \varepsilon_i^T(t)\varpi Z\varepsilon_i(t) + 2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q\varepsilon_i^T(t)\varpi C\varepsilon_j(t)$$

$$+2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}} c_q B_{ij}^q \varepsilon_i^T(t)\varpi C\varepsilon_j(t) - kH(t)$$

$$\leqslant 2\varepsilon^T(t)\left[I_{\mathcal{M}}\otimes(\varpi Z) + \sum_{\substack{q=1\\ q\neq r}}^{m} c_q B^q\otimes(\varpi C) + c_r B^r\otimes(\varpi C)\right]\varepsilon(t) - kH(t)$$

$$\leqslant 2\varepsilon^T(t)\big[I_{\mathcal{M}}\otimes(\varpi Z) + c_r B^r\otimes(\varpi C)\big]\varepsilon(t) - kH(t), \tag{19}$$

where $H(t) = \sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji}))^T\varpi C(\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji}))$ and $B^r = (B_{ij}^r)_{\mathcal{M}\times\mathcal{M}} \in \mathbb{R}^{\mathcal{M}\times\mathcal{M}}$.

Apparently, we can acquire an orthogonal matrix $\Phi = (\phi_1, \phi_2, \ldots, \phi_{\mathcal{M}}) \in \mathbb{R}^{\mathcal{M}\times\mathcal{M}}$ such that

$$\Phi^T B^r \Phi = \beta = \text{diag}(\beta_1, \beta_2, \ldots, \beta_{\mathcal{M}}) \in \mathbb{R}^{\mathcal{M}\times\mathcal{M}},$$

where $0 = \beta_1 > \beta_2 \geqslant \beta_3 \geqslant \ldots \geqslant \beta_{\mathcal{M}}$. Define $l(t) = (l_1^T(t), l_2^T(t), \ldots, l_{\mathcal{M}}^T(t))^T = (\Phi^T\otimes I_n)\varepsilon(t)$. On account of $\phi_1 = \frac{1}{\sqrt{\mathcal{M}}}(1, 1, \ldots, 1)^T$, it is easy to see that $l_1(t) = (\phi_1^T\otimes I_n)\varepsilon(t) = 0$. Consequently, one can obtain

$$\dot{V}_2(t) \leqslant 2\varepsilon^T(t)\Big\{I_{\mathcal{M}}\otimes(\varpi Z) + c_r(\Phi\otimes I_n)[\beta\otimes(\varpi C)](\Phi^T\otimes I_n)\Big\}\varepsilon(t) - kH(t)$$

$$= 2\varepsilon^T(t)[I_{\mathcal{M}}\otimes(\varpi Z)]\varepsilon(t) + 2c_r l^T(t)[\beta\otimes(\varpi C)]l(t) - kH(t)$$

$$\leqslant 2\varepsilon^T(t)\Big\{I_{\mathcal{M}}\otimes(\varpi Z) + c_r\beta_2[I_{\mathcal{M}}\otimes(\varpi C)]\Big\}\varepsilon(t) - kH(t). \tag{20}$$

By choosing B_{ij}^r sufficiently large such that

$$\lambda_M(\varpi Z) + c_r\beta_2\lambda_m(\varpi C) \leqslant 0. \tag{21}$$

By Eqs. (20) and (21), one has

$$\dot{V}_2(t) \leqslant -kH(t). \tag{22}$$

Similarly to the Theorem 3.1 proving, it can be verified that, with the help of the adaptive output feedback controller (14), the network (1) is the lag output synchronized. □

4. The lag output synchronization of multiple output coupled complex network with positive semidefinite output matrix

This section mainly investigates the lag output synchronization of multiple output coupled complex network with positive semidefinite output matrix, and some criteria of the output synchronization are derived by choosing appropriate output feedback controller and adaptive output feedback controller.

4.1. Output feedback controller

Reorganized the order of output variables and state variables such that

$$C = \text{diag}(C_1, C_2, \ldots, C_\xi, 0, \ldots, 0),$$

where $\xi \in [1, n)$.

For the purpose of getting main results, one defines

$$\begin{aligned}
&\bar{C} = \mathrm{diag}(C_1, C_2, \ldots, C_\xi), \qquad \tilde{C} = (\bar{C}, 0) \in \mathbb{R}^{\xi \times n}\\
&\tilde{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{i\xi}(t))^T \in \mathbb{R}^\xi,\\
&\tilde{y}_i(t) = \tilde{C}\tilde{x}_i(t), y_i(t) = \big((\tilde{y}_i(t))^T, 0, 0, \ldots, 0\big)^T,\\
&\tilde{u}_i(t) = \big(u_{i1}(t), u_{i2}(t), \ldots, u_{i\xi}(t)\big)^T \in \mathbb{R}^\xi,\\
&\bar{Z} = \mathrm{diag}(z_1, z_2, \ldots, z_\xi).
\end{aligned}$$

Thus, one obtains

$$\dot{\tilde{y}}_i(t) = \bar{Z}\tilde{y}_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \bar{C}\tilde{y}_j(t) + \bar{C}\tilde{u}_i(t), \tag{23}$$

where $i = 1, 2, \ldots, \mathcal{M}$.

Defining $\tilde{y}^*(t) = \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} \tilde{y}_\rho(t)$ and $\check{\varepsilon}_i(t) = \tilde{y}_i(t) - \tilde{y}^*(t)$, one gets

$$\dot{\check{\varepsilon}}_i(t) = \bar{Z}\check{\varepsilon}_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \bar{C}\check{\varepsilon}_j(t) + \bar{C}\tilde{u}_i(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} \bar{C}\tilde{u}_\rho(t), \tag{24}$$

where $i = 1, 2, \ldots, \mathcal{M}$.

In order to ensure that the network (1) can realize the lag output synchronization, the following output feedback controller is selected:

$$\tilde{u}_i(t) = -k\big(\tilde{y}_i(t) - \tilde{y}^*(t)\big) + k\sum_{j\in\mathcal{D}_i}\Big[\big(\tilde{y}_j(t-\tau_{ji}) - \tilde{y}^*(t-\tau_{ji})\big) - \big(\tilde{y}_i(t) - \tilde{y}^*(t)\big)\Big], \tag{25}$$

where $k > 0$.

From Eqs. (24) and (25), one has

$$\begin{aligned}
\dot{\check{\varepsilon}}_i(t) = {}& \bar{Z}\check{\varepsilon}_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \bar{C}\check{\varepsilon}_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} \bar{C}\tilde{u}_\rho(t)\\
& -k\bar{C}\check{\varepsilon}_i(t) + k\bar{C}\sum_{j\in\mathcal{D}_i}\big((\check{\varepsilon}_j(t-\tau_{ji}) - \check{\varepsilon}_i(t)\big).
\end{aligned} \tag{26}$$

Theorem 4.1. *Based on the output feedback controller (25), the network (1) is the lag output synchronized if there is a matrix* $\mathbb{R}^{\xi\times\xi} \ni \Upsilon = \mathrm{diag}(\upsilon_1, \upsilon_2, \ldots, \upsilon_\xi) > 0$ *satisfing*

$$I_{\mathcal{M}} \otimes (\Upsilon\bar{Z}) + \left(\sum_{q=1}^{m} c_q G^q - kI_{\mathcal{M}}\right) \otimes (\Upsilon\bar{C}) \leqslant 0.$$

Proof. Select a Lyapunov functional for network model (26) as follows:

$$V_3(t) = \sum_{i=1}^{\mathcal{M}} \check{\varepsilon}_i^T(t)\Upsilon\check{\varepsilon}_i(t) + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\int_{t-\tau_{ji}}^{t} \check{\varepsilon}_j^T(s)\Upsilon\bar{C}\check{\varepsilon}_j(s)ds. \tag{27}$$

Then, one has

$$\dot{V}_3(t) = 2\sum_{i=1}^{\mathcal{M}} \check{\varepsilon}_i^T(t)\Upsilon\dot{\check{\varepsilon}}_i(t) + k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\check{\varepsilon}_j^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t)$$

$$
\begin{aligned}
&-\check{\varepsilon}_j^T(t-\tau_{ji})\Upsilon\bar{C}\check{\varepsilon}_j(t-\tau_{ji})\big)\\
&=2\sum_{i=1}^{\mathcal{M}}\check{\varepsilon}_i^T(t)\Upsilon\Big[\bar{Z}\check{\varepsilon}_i(t)+\sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^q\bar{C}\check{\varepsilon}_j(t)-\frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}}\bar{C}\tilde{u}_\rho(t)\\
&-k\bar{C}\check{\varepsilon}_i(t)+k\bar{C}\sum_{j\in\mathcal{D}_i}\big((\check{\varepsilon}_j(t-\tau_{ji})-\check{\varepsilon}_i(t)\big)\Big]\\
&+k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\check{\varepsilon}_j^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t)-\check{\varepsilon}_j^T(t-\tau_{ji})\Upsilon\bar{C}\check{\varepsilon}_j(t-\tau_{ji})\big)\\
&=2\sum_{i=1}^{\mathcal{M}}\check{\varepsilon}_i^T(t)\Upsilon\bar{Z}\check{\varepsilon}_i(t)+2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}}c_qG_{ij}^q\check{\varepsilon}_i^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t)\\
&-2k\sum_{i=1}^{\mathcal{M}}\check{\varepsilon}_i^T(t)\Upsilon\bar{C}\check{\varepsilon}_i(t)-k\bar{H}(t)\\
&=\check{\varepsilon}^T(t)\Big[2I_{\mathcal{M}}\otimes(\Upsilon\bar{Z})+2\Big(\sum_{q=1}^{m}c_qG^q-kI_{\mathcal{M}}\Big)\otimes(\Upsilon\bar{C})\Big]\check{\varepsilon}(t)-k\bar{H}(t)\\
&\leqslant -k\bar{H}(t),
\end{aligned}
\tag{28}
$$

where $\check{\varepsilon}(t)=(\check{\varepsilon}_1^T(t),\check{\varepsilon}_2^T(t),\dots,\check{\varepsilon}_{\mathcal{M}}^T(t))^T$, $\bar{H}(t)=\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\check{\varepsilon}_i(t)-\check{\varepsilon}_j(t-\tau_{ji}))^T\Upsilon\bar{C}(\check{\varepsilon}_i(t)-\check{\varepsilon}_j(t-\tau_{ji}))$.

Similarly to the Theorem 3.1 proving, it can be verified that, with the help of the output feedback controller (25), the network (1) is the lag output synchronized. □

4.2. Adaptive output feedback controller

In this subsection, $\check{\kappa}^q(t)=(\check{\kappa}_{ij}^q(t))_{\mathcal{M}\times\mathcal{M}}$ denotes a matrix varied with time, where $\check{\kappa}_{ij}^q(t)$ has the following definition: if node j and node $i(i\neq j)$ are connected, then $\check{\kappa}_{ij}^q(t)=\check{\kappa}_{ji}^q(t)>0$; elsewise, $\check{\kappa}_{ij}^q(t)=\check{\kappa}_{ji}^q(t)=0$; moreover,

$$
\check{\kappa}_{ii}^q(t)=-\sum_{\substack{j=1\\ j\neq i}}^{\mathcal{M}}\check{\kappa}_{ij}^q(t),\quad i=1,2,\dots,\mathcal{M}.
$$

Design an appropriate adaptive output feedback controller for network (24):

$$
\tilde{u}_i(t)=\sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}}c_q\check{\kappa}_{ij}^q(t)\check{\varepsilon}_j(t)+k\sum_{j\in\mathcal{D}_i}\big(\check{\varepsilon}_j(t-\tau_{ji})-\check{\varepsilon}_i(t)\big),\ i=1,2,\dots,\mathcal{M}.
$$

$$
\dot{\check{\kappa}}_{ij}^q(t)=\begin{cases}\bar{a}_{ij}(\tilde{y}_i(t)-\tilde{y}_j(t))^T\Upsilon\bar{C}(\tilde{y}_i(t)-\tilde{y}_j(t)), & \text{if } (i,j)\in\mathcal{B},\\ -\sum_{\substack{s=1\\ s\neq i}}^{\mathcal{M}}\dot{\check{\kappa}}_{is}^q(t), & \text{if } i=j,\\ 0, & \text{otherwise},\end{cases}
\tag{29}
$$

where $\mathbb{R}^{\xi\times\xi}\ni\Upsilon=\mathrm{diag}(\upsilon_1,\upsilon_2,\dots,\upsilon_\xi)>0$, $\mathbb{R}\ni\bar{a}_{ij}=\bar{a}_{ji}>0,\mathbb{R}\ni k>0$.

From Eqs. (24) and (29), one gets

$$\dot{\check{\varepsilon}}_i(t) = \bar{Z}\check{\varepsilon}_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \bar{C}\check{\varepsilon}_j(t) - \frac{1}{\mathcal{M}}\sum_{\rho=1}^{\mathcal{M}} \bar{C}\tilde{u}_\rho(t)$$
$$+ \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q \check{\kappa}_{ij}^q(t)\bar{C}\check{\varepsilon}_j(t) + k\bar{C}\sum_{j\in\mathcal{D}_i}\big(\check{\varepsilon}_j(t-\tau_{ji}) - \check{\varepsilon}_i(t)\big), \tag{30}$$

where $i = 1, 2, \ldots, \mathcal{M}$.

Theorem 4.2. *Based on the adaptive output feedback controller (29), the network (1) achieves the lag output synchronization.*

Proof. Choose a Lyapunov functional for network model (30) as follows:

$$V_4(t) = \sum_{i=1}^{\mathcal{M}} \check{\varepsilon}_i^T(t)\Upsilon\check{\varepsilon}_i(t) + \sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i} \frac{c_q(\check{\kappa}_{ij}^q(t) - B_{ij}^q)^2}{2\bar{a}_{ij}}$$
$$+ k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\int_{t-\tau_{ji}}^{t} \check{\varepsilon}_j^T(s)\Upsilon\bar{C}\check{\varepsilon}_j(s)ds,$$

where $\mathbb{R}^{\mathcal{M}\times\mathcal{M}} \ni B_{ji}^q = B_{ij}^q \ (i \neq j) \geqslant 0, B_{ij}^q = 0$ when and only when $\check{\kappa}_{ij}^q(t) = 0$.

Then, one has

$$\dot{V}_4(t) = 2\sum_{i=1}^{\mathcal{M}} \check{\varepsilon}_i^T(t)\Upsilon\Bigg[\bar{Z}\check{\varepsilon}_i(t) + \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \bar{C}\check{\varepsilon}_j(t)$$
$$+ \sum_{q=1}^{m}\sum_{j=1}^{\mathcal{M}} c_q\check{\kappa}_{ij}^q(t)\bar{C}\check{\varepsilon}_j(t) + \sum_{j\in\mathcal{D}_i} k\bar{C}\big(\check{\varepsilon}_j(t-\tau_{ji}) - \check{\varepsilon}_i(t)\big)\Bigg]$$
$$+ \sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i} c_q(\check{\kappa}_{ij}^q(t) - B_{ij}^q)(\check{\varepsilon}_i(t) - \check{\varepsilon}_j(t))^T\Upsilon\bar{C}(\check{\varepsilon}_i(t) - \check{\varepsilon}_j(t))$$
$$+ k\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}\big(\check{\varepsilon}_j^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t) - \check{\varepsilon}_j^T(t-\tau_{ji})\Upsilon\bar{C}\check{\varepsilon}_j(t-\tau_{ji})\big)$$
$$= 2\sum_{i=1}^{\mathcal{M}} \check{\varepsilon}_i^T(t)\Upsilon\bar{Z}\check{\varepsilon}_i(t) + 2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}} c_q B_{ij}^q \check{\varepsilon}_i^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t)$$
$$+ 2\sum_{q=1}^{m}\sum_{i=1}^{\mathcal{M}}\sum_{j=1}^{\mathcal{M}} c_q G_{ij}^q \check{\varepsilon}_i^T(t)\Upsilon\bar{C}\check{\varepsilon}_j(t) - k\bar{H}(t)$$
$$\leqslant 2\check{\varepsilon}^T(t)\left[I_{\mathcal{M}}\otimes(\Upsilon\bar{Z}) + \sum_{q=1}^{m} c_q B^q \otimes (\Upsilon\bar{C})\right]\check{\varepsilon}(t) - k\bar{H}(t)$$
$$= 2\check{\varepsilon}^T(t)\left[I_{\mathcal{M}}\otimes(\Upsilon\bar{Z}) + \sum_{\substack{q=1\\ q\neq r}}^{m} c_q B^q \otimes (\Upsilon\bar{C}) + c_r B^r \otimes (\Upsilon\bar{C})\right]\check{\varepsilon}(t)$$

$$-k\bar{H}(t)$$
$$\leqslant 2\check{\varepsilon}^T(t)\Big[I_{\mathcal{M}}\otimes(\Upsilon\bar{Z})+c_rB^r\otimes(\Upsilon\bar{C})\Big]\check{\varepsilon}(t)-k\bar{H}(t), \tag{31}$$

where $\bar{H}(t)=\sum_{i=1}^{\mathcal{M}}\sum_{j\in\mathcal{D}_i}(\check{\varepsilon}_i(t)-\check{\varepsilon}_j(t-\tau_{ji}))^T\Upsilon\bar{C}(\check{\varepsilon}_i(t)-\check{\varepsilon}_j(t-\tau_{ji}))$.

Apparently, we can acquire an orthogonal matrix $\Phi=(\phi_1,\phi_2,\ldots,\phi_{\mathcal{M}})\in\mathbb{R}^{\mathcal{M}\times\mathcal{M}}$ such that

$$\Phi^TB^r\Phi=\beta=\mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_{\mathcal{M}})\in\mathbb{R}^{\mathcal{M}\times\mathcal{M}},$$

where $0=\beta_1>\beta_2\geqslant\beta_3\geqslant\ldots\geqslant\beta_{\mathcal{M}}$. Define $\bar{l}(t)=(\bar{l}_1^T(t),\bar{l}_2^T(t),\ldots,\bar{l}_{\mathcal{M}}^T(t))^T=(\Phi^T\otimes I_\xi)\check{\varepsilon}(t)$. On account of $\phi_1=\frac{1}{\sqrt{\mathcal{M}}}(1,1,\ldots,1)^T$, it is easy to see that $\bar{l}_1(t)=(\phi_1^T\otimes I_\xi)\check{\varepsilon}(t)=0$. Consequently, one gets

$$\begin{aligned}\dot{V}_4(t)&\leqslant 2\check{\varepsilon}^T(t)\Big[I_{\mathcal{M}}\otimes(\Upsilon\bar{Z})+c_r(\Phi\otimes I_\xi)(\beta\otimes(\Upsilon\bar{C}))(\Phi^T\otimes I_\xi)\Big]\check{\varepsilon}(t)-k\bar{H}(t)\\&=2\check{\varepsilon}^T(t)\big[I_{\mathcal{M}}\otimes(\Upsilon\bar{Z})\big]\check{\varepsilon}(t)+2c_r\bar{l}^T(t)[\beta\otimes(\Upsilon\bar{C})]\bar{l}(t)-k\bar{H}(t)\\&\leqslant 2\check{\varepsilon}^T(t)\Big\{I_{\mathcal{M}}\otimes(\Upsilon\bar{Z})+c_r\beta_2[I_{\mathcal{M}}\otimes(\Upsilon\bar{C})]\Big\}\check{\varepsilon}(t)-k\bar{H}(t).\end{aligned}\tag{32}$$

By choosing B_{ij}^r sufficiently large such that

$$\lambda_M(\Upsilon\bar{Z})+c_r\beta_2\lambda_m(\Upsilon\bar{C})\leqslant 0. \tag{33}$$

By Eqs. (32) and (33), one obtains

$$\dot{V}_4(t)\leqslant -k\bar{H}(t).$$

Similarly to the Theorem 3.2 proving, it can be verified that, with the help of the adaptive output feedback controller (29), the network (1) is the lag output synchronized. □

5. Numerical examples

Example 5.1. Consider the following multiweighted complex dynamical network with output couplings:

$$\begin{cases}\dot{x}_i(t)=Zx_i(t)+0.3\sum_{j=1}^{6}G_{ij}^1y_j(t)+0.4\sum_{j=1}^{6}G_{ij}^2y_j(t)+0.5\sum_{j=1}^{6}G_{ij}^3y_j(t)+u_i(t),\\ y_i(t)=Cx_i(t),\ \ i=1,2,\ldots,6,\end{cases}$$

where $x_i(t)=(x_{i1}(t),x_{i2}(t),x_{i3}(t))^T\in\mathbb{R}^3$, $y_i(t)=(y_{i1}(t),y_{i2}(t),y_{i3}(t))^T\in\mathbb{R}^3$, $u_i(t)=(u_{i1}(t),u_{i2}(t),u_{i3}(t))^T\in\mathbb{R}^3$, $C=\mathrm{diag}(0.5,0.6,0.7)$, $Z=\mathrm{diag}(1,2,3)$, and the matrices G^1, G^2, G^3 are respectively chosen as follows:

$$G^1=\begin{pmatrix}-0.8&0&0.3&0.4&0&0.1\\0&-0.4&0&0.2&0.2&0\\0.3&0&-0.9&0&0.4&0.2\\0.4&0.2&0&-0.8&0.2&0\\0&0.2&0.4&0.2&-0.8&0\\0.1&0&0.2&0&0&-0.3\end{pmatrix},$$

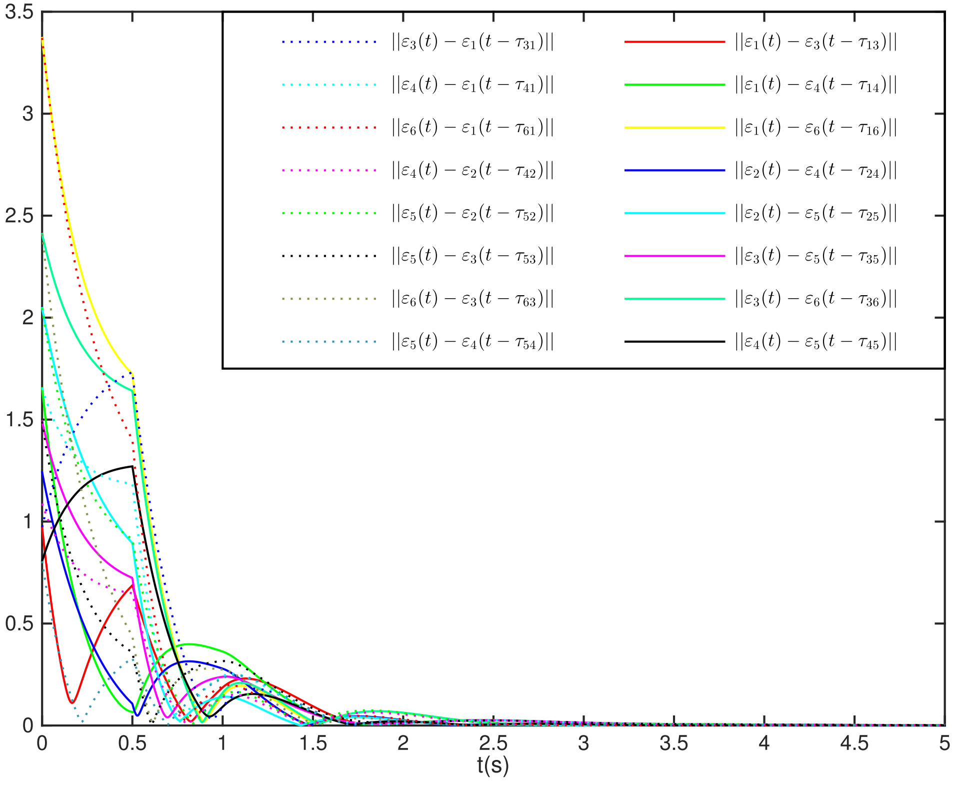

Fig. 1. $\|\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji})\|,\ i=1,2,\ldots,6,\ j\in\mathcal{D}_i$.

$$G^2=\begin{pmatrix}-1.1 & 0 & 0.4 & 0.5 & 0 & 0.2\\ 0 & -0.4 & 0 & 0.2 & 0.2 & 0\\ 0.4 & 0 & -0.8 & 0 & 0.1 & 0.3\\ 0.5 & 0.2 & 0 & -0.8 & 0.1 & 0\\ 0 & 0.2 & 0.1 & 0.1 & -0.4 & 0\\ 0.2 & 0 & 0.3 & 0 & 0 & -0.5\end{pmatrix},$$

$$G^3=\begin{pmatrix}-0.7 & 0 & 0.2 & 0.3 & 0 & 0.2\\ 0 & -0.5 & 0 & 0.3 & 0.2 & 0\\ 0.2 & 0 & -0.9 & 0 & 0.5 & 0.2\\ 0.3 & 0.3 & 0 & -0.7 & 0.1 & 0\\ 0 & 0.2 & 0.5 & 0.1 & -0.8 & 0\\ 0.2 & 0 & 0.2 & 0 & 0 & -0.4\end{pmatrix}.$$

Case 1: Taking $k=5$ and $\varpi=I_3$, it is easily to demonstrate that the Theorem 3.1 is met. Therefore, the network (1) based on the controller (5) is lag output synchronized. The results of simulation are shown in Fig. 1.

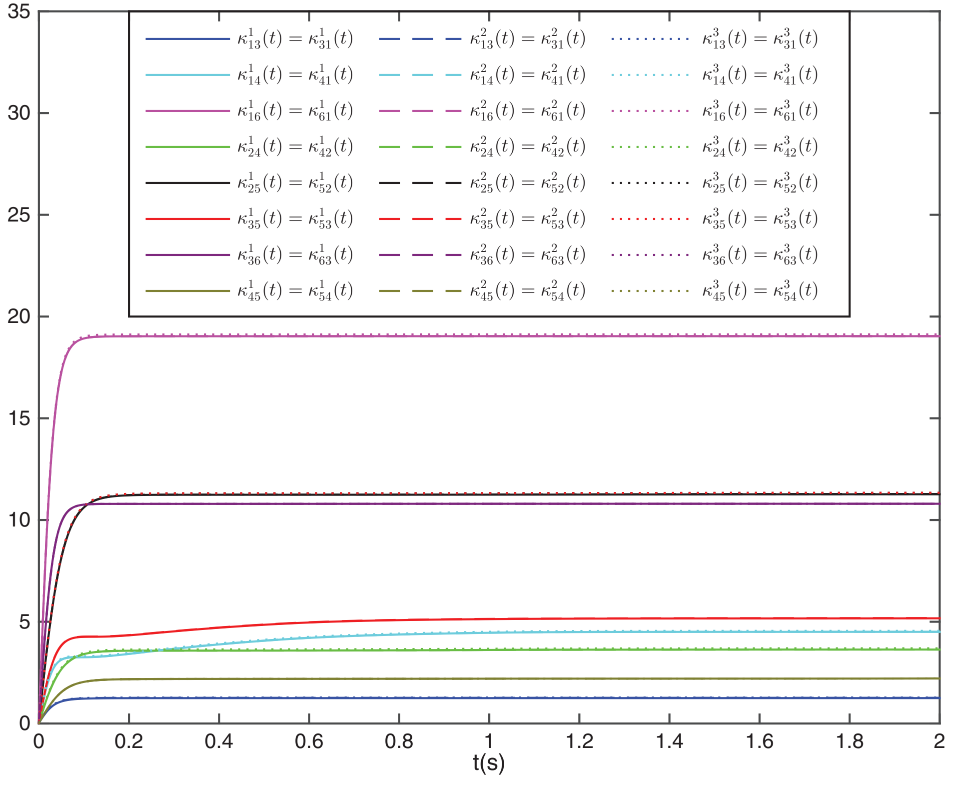

Fig. 2. κ^s_{ij}, $s=1,2,3,\ i=1,2,\dots,6,\ j\in\mathcal{D}_i$.

Case 2: Take $k=0.5$ and $\varpi=I_3$. On the basic of the Theorem 3.2, the network (1) with the help of the controller (14) is lag output synchronized. Selecting the matrices $\kappa^1(0)$, $\kappa^2(0)$, $\kappa^3(0)$ as follows:

$$\kappa^1(0)=\begin{pmatrix} -0.1 & 0 & 0.03 & 0.04 & 0 & 0.03\\ 0 & -0.05 & 0 & 0.03 & 0.02 & 0\\ 0.03 & 0 & -0.09 & 0 & 0.02 & 0.04\\ 0.04 & 0.03 & 0 & -0.1 & 0.03 & 0\\ 0 & 0.02 & 0.02 & 0.03 & -0.07 & 0\\ 0.03 & 0 & 0.04 & 0 & 0 & -0.07 \end{pmatrix},$$

$$\kappa^2(0)=\begin{pmatrix} -0.12 & 0 & 0.045 & 0.045 & 0 & 0.03\\ 0 & -0.06 & 0 & 0.03 & 0.03 & 0\\ 0.045 & 0 & -0.135 & 0 & 0.03 & 0.06\\ 0.045 & 0.03 & 0 & -0.12 & 0.045 & 0\\ 0 & 0.03 & 0.03 & 0.045 & -0.105 & 0\\ 0.03 & 0 & 0.06 & 0 & 0 & -0.09 \end{pmatrix},$$

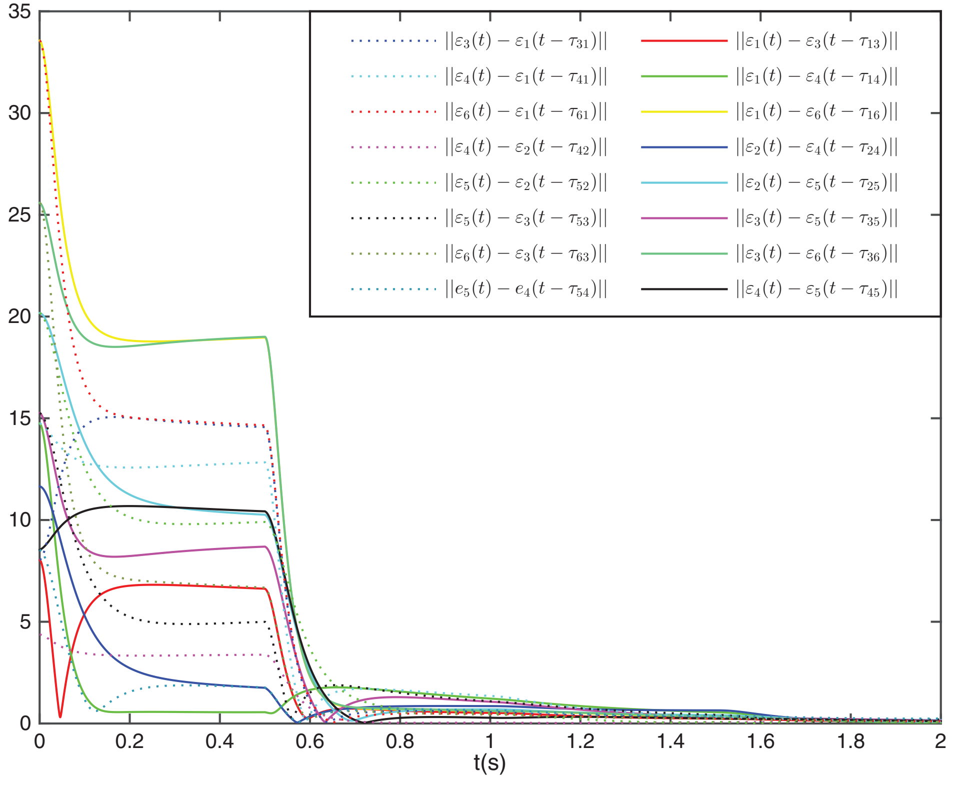

Fig. 3. $\|\varepsilon_i(t)-\varepsilon_j(t-\tau_{ji})\|,\ i=1,2,\ldots,6,\ j\in\mathcal{D}_i$.

$$\kappa^3(0)=\begin{pmatrix} -0.27 & 0 & 0.06 & 0.09 & 0 & 0.12 \\ 0 & -0.18 & 0 & 0.09 & 0.09 & 0 \\ 0.06 & 0 & -0.27 & 0 & 0.06 & 0.15 \\ 0.09 & 0.09 & 0 & -0.21 & 0.03 & 0 \\ 0 & 0.09 & 0.06 & 0.03 & -0.18 & 0 \\ 0.12 & 0 & 0.15 & 0 & 0 & -0.27 \end{pmatrix},$$

the results of emulation are shown in Figs. 2 and 3.

Remark 5. From (22), we can derive that $V_2(t)$ is nonincreasing and bounded. Obviously, the parameters $\kappa_{ij}^q(t)$, $(i,j)\in\mathcal{B}$, are also bounded. Moreover, the parameters $\kappa_{ij}^q(t)$, $(i,j)\in\mathcal{B}$, are monotonically increasing [see (14)]. Therefore, we can easily obtain that the parameters $\kappa_{ij}^q(t)$, $(i,j)\in\mathcal{B}$, converge to some positive finite values. Based on Fig. 2, we can clearly observe that the parameters $\kappa_{ij}^q(t)$, $(i,j)\in\mathcal{B}$, converge to constants and increase progressively.

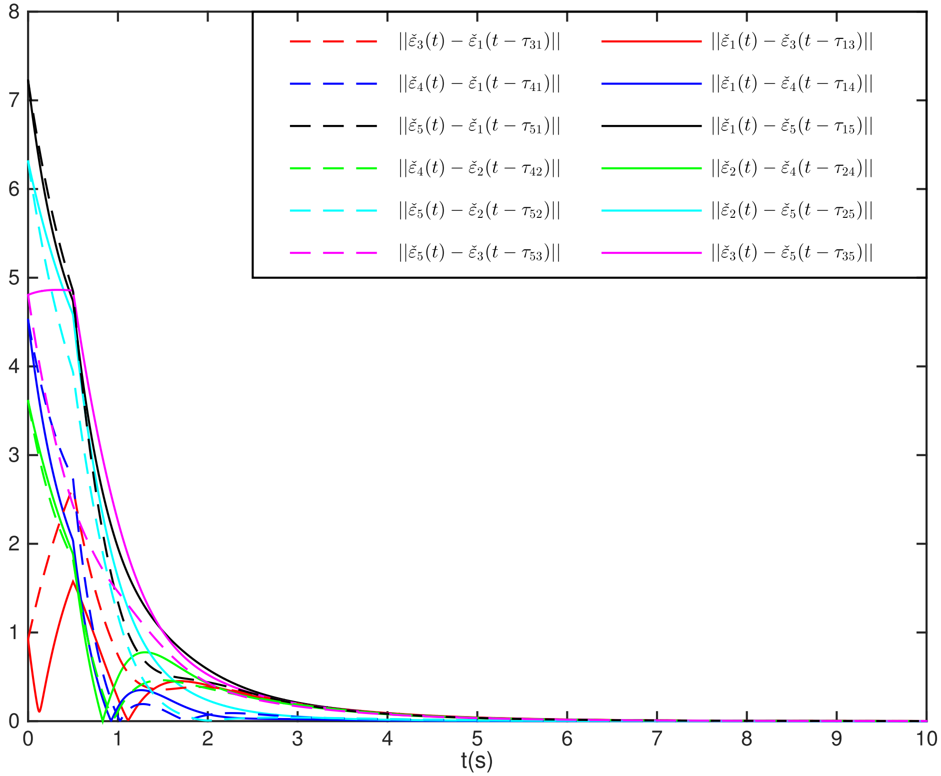

Fig. 4. $\|\check{\varepsilon}_i(t)-\check{\varepsilon}_j(t-\tau_{ji})\|,\ i=1,2,\ldots,5,\ j\in\mathcal{D}_i$.

Example 5.2. Consider the following complex dynamical network with multiple output couplings:

$$\begin{cases} \dot{x}_i(t)=Zx_i(t)+0.2\sum\limits_{j=1}^{5}G^1_{ij}y_j(t)+0.3\sum\limits_{j=1}^{5}G^2_{ij}y_j(t) \\ \qquad\qquad +0.4\sum\limits_{j=1}^{5}G^3_{ij}y_j(t)+0.5\sum\limits_{j=1}^{5}G^4_{ij}y_j(t)+u_i(t), \\ y_i(t)=Cx_i(t),\quad i=1,2,\ldots,5, \end{cases}$$

where $x_i(t)=(x_{i1}(t),x_{i2}(t),x_{i3}(t))^T\in\mathbb{R}^3$, $y_i(t)=(y_{i1}(t),y_{i2}(t),y_{i3}(t))^T\in\mathbb{R}^3$, $u_i(t)=(u_{i1}(t),u_{i2}(t),u_{i3}(t))^T\in\mathbb{R}^3$, $Z=\mathrm{diag}(1,2,3)$, $C=\mathrm{diag}(0.3,0.4,0)$, and the matrices G^1, G^2, G^3, G^4 are chosen as follows:

$$G^1=\begin{pmatrix} -0.9 & 0 & 0.2 & 0.4 & 0.3 \\ 0 & -0.5 & 0 & 0.3 & 0.2 \\ 0.2 & 0 & -0.3 & 0 & 0.1 \\ 0.4 & 0.3 & 0 & -0.7 & 0 \\ 0.3 & 0.2 & 0.1 & 0 & -0.6 \end{pmatrix},$$

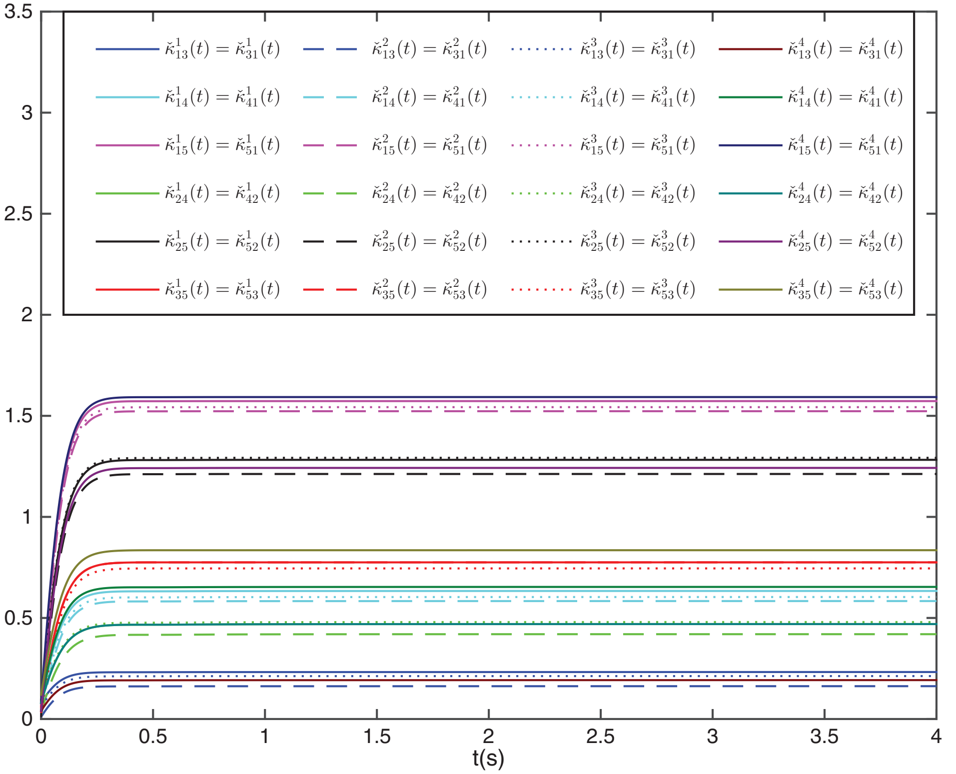

Fig. 5. $\breve{\kappa}^s_{ij},\ s=1,2,3,\ i=1,2,\ldots,5,\ j\in\mathcal{D}_i$.

$$G^2=\begin{pmatrix}-1 & 0 & 0.3 & 0.5 & 0.2\\ 0 & -0.4 & 0 & 0.2 & 0.2\\ 0.3 & 0 & -0.5 & 0 & 0.2\\ 0.5 & 0.2 & 0 & -0.7 & 0\\ 0.2 & 0.2 & 0.2 & 0 & -0.6\end{pmatrix},$$

$$G^3=\begin{pmatrix}-0.8 & 0 & 0.3 & 0.3 & 0.2\\ 0 & -0.5 & 0 & 0.2 & 0.3\\ 0.3 & 0 & -0.6 & 0 & 0.3\\ 0.3 & 0.2 & 0 & -0.5 & 0\\ 0.2 & 0.3 & 0.3 & 0 & -0.8\end{pmatrix},$$

$$G^4=\begin{pmatrix}-1.3 & 0 & 0.5 & 0.7 & 0.1\\ 0 & -0.5 & 0 & 0.2 & 0.3\\ 0.5 & 0 & -0.6 & 0 & 0.1\\ 0.7 & 0.2 & 0 & -0.9 & 0\\ 0.1 & 0.3 & 0.1 & 0 & -0.5\end{pmatrix}.$$

Case 1: Taking $k=5$ and $\Upsilon=I_2$, it is easily to demonstrate that the Theorem 4.1 is met. Therefore, the network (1) based on the controller (25) is lag output synchronized. The results of simulation are shown in Fig. 4.

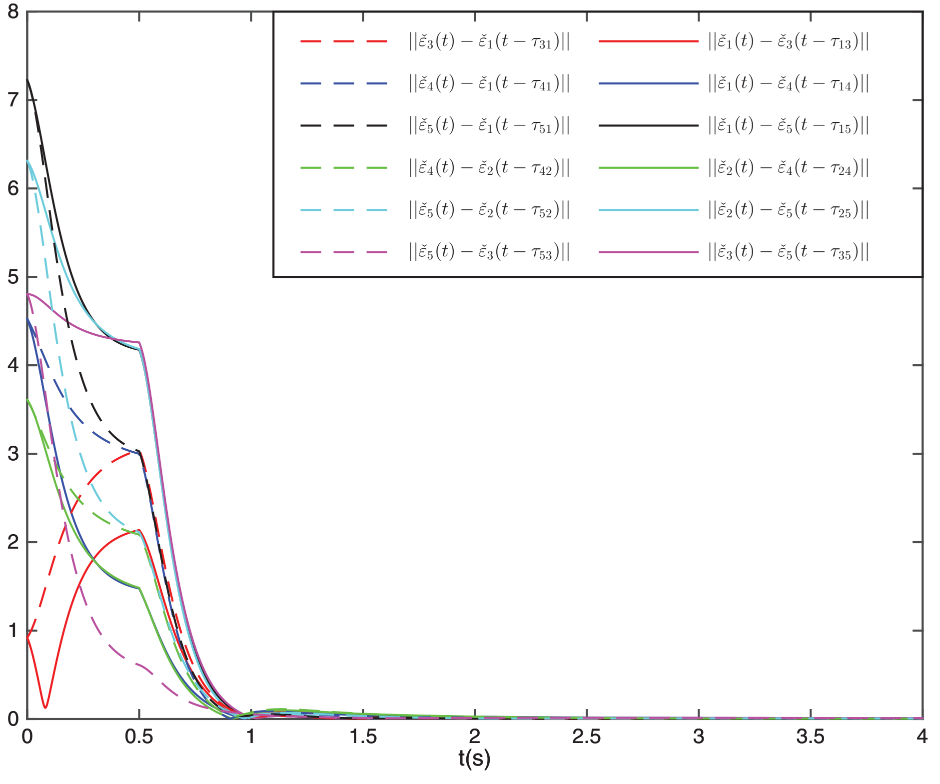

Fig. 6. $\|\check{\varepsilon}_i(t) - \check{\varepsilon}_j(t-\tau_{ji})\|,\ i = 1, 2, \ldots, 5,\ j \in \mathcal{D}_i$.

Case 2: Take $k = 0.5$ and $\Upsilon = I_2$. On the basic of the Theorem 4.2, the network (1) with the help of the controller (29) is lag output synchronized. Choosing the matrices $\check{\kappa}^1(0), \check{\kappa}^2(0), \check{\kappa}^3(0), \check{\kappa}^4(0)$ as follows:

$$\check{\kappa}^1(0) = \begin{pmatrix} -0.2 & 0 & 0.08 & 0.06 & 0.06 \\ 0 & -0.16 & 0 & 0.08 & 0.08 \\ 0.08 & 0 & -0.14 & 0 & 0.06 \\ 0.06 & 0.08 & 0 & -0.14 & 0 \\ 0.06 & 0.08 & 0.06 & 0 & -0.2 \end{pmatrix},$$

$$\check{\kappa}^2(0) = \begin{pmatrix} -0.03 & 0 & 0.01 & 0.01 & 0.01 \\ 0 & -0.04 & 0 & 0.03 & 0.01 \\ 0.01 & 0 & -0.05 & 0 & 0.04 \\ 0.01 & 0.03 & 0 & -0.04 & 0 \\ 0.01 & 0.01 & 0.04 & 0 & -0.06 \end{pmatrix},$$

$$\check{\kappa}^3(0) = \begin{pmatrix} -0.12 & 0 & 0.06 & 0.03 & 0.03 \\ 0 & -0.18 & 0 & 0.09 & 0.09 \\ 0.06 & 0 & -0.09 & 0 & 0.03 \\ 0.03 & 0.09 & 0 & -0.12 & 0 \\ 0.03 & 0.09 & 0.03 & 0 & -0.15 \end{pmatrix},$$

$$\check{\kappa}^4(0) = \begin{pmatrix} -0.2 & 0 & 0.04 & 0.08 & 0.08 \\ 0 & -0.12 & 0 & 0.08 & 0.04 \\ 0.04 & 0 & -0.16 & 0 & 0.12 \\ 0.08 & 0.08 & 0 & -0.16 & 0 \\ 0.08 & 0.04 & 0.12 & 0 & -0.24 \end{pmatrix},$$

the results of emulation are shown in Figs. 5 and 6.

6. Conclusion

In the present paper, we have discussed the lag output synchronization for two kinds of complex networks with multiple output couplings and different output matrices, in which the output matrix is positive definite or positive semidefinite. On one side, based on the devised output feedback controllers, two criteria have been obtained to insure lag output synchronization for the network models. On the other side, some criteria of the lag output synchronization also have been established for these networks by utilizing the adaptive output feedback controllers. At last, two examples have been given to demonstrate the effectiveness and correctness of the acquired consequences.

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