Truncated Exponential Distribution

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Abstract

This document specifies how to generate random numbers from a truncated exponential distribution.

Probability Density Function

Consider the exponential distribution in the range $[a, b]$ with the rate parameter λ and a normalizing constant C:

$$P(x|\lambda, a, b) = Ce^{-\lambda x} \tag{1}$$

For it to be a valid distribution, the integral over the specified range should be one:

$$\int_a^b P(x|\lambda, a, b)dx = \int_a^b Ce^{-\lambda x}dx = 1 \tag{2}$$

$$\begin{aligned}\int_a^b Ce^{-\lambda x}dx &= C\left[-\frac{e^{-\lambda x}}{\lambda} + K\right]_a^b \\ &= \frac{C}{\lambda}(e^{-\lambda a} - e^{-\lambda b})\end{aligned}$$

Equating the integral to 1:

$$\begin{aligned}\frac{C}{\lambda}(e^{-\lambda a} - e^{-\lambda b}) &= 1 \\ C &= \frac{\lambda}{e^{-\lambda a} - e^{-\lambda b}} \qquad (3)\end{aligned}$$

Cumulative Distribution Function

$$\begin{aligned} F(x) &= \int Ce^{-\lambda x} dx \\ &= C\left[-\frac{e^{-\lambda x}}{\lambda} + K\right] \\ &= \frac{\lambda}{e^{-\lambda a} - e^{-\lambda b}}\left[-\frac{e^{-\lambda x}}{\lambda} + K\right] \\ &= \frac{-e^{-\lambda x}}{e^{-\lambda a} - e^{-\lambda b}} + K \end{aligned}$$

CDF at lower limit is 0:

$$F(a) = \frac{-e^{-\lambda a}}{e^{-\lambda a} - e^{-\lambda b}} + K = 0$$

$$K = \frac{e^{-\lambda a}}{e^{-\lambda a} - e^{-\lambda b}}$$

Replacing K in CDF:

$$F(x) = \frac{e^{-\lambda a} - e^{-\lambda x}}{e^{-\lambda a} - e^{-\lambda b}} \tag{4}$$

Inverse CDF

Inverting equation 4, we get an Inverse CDF (Percent Point Function) of the distribution:

$$F^{-1}(x) = -\frac{ln}{\lambda}(e^{-\lambda a} - x(e^{-\lambda a} - e^{-\lambda b})) \tag{5}$$

Inverse Transform Sampling

We want to generate a random variable X with CDF as F(x). Steps for inverse transform sampling algorithm include:

1. Generate $U \sim Uniform(0, 1)$
2. Apply Inverse CDF: $X = F^{-1}(U)$

Expected value

The expected value of a random variable X that follows a truncated exponential distribution is given by:

$$
\begin{aligned}
E(X) &= \int_a^b xP(x|\lambda, a, b)dx \\
&= \frac{\lambda}{e^{-\lambda a} - e^{-\lambda b}} \int_a^b xe^{-\lambda x} dx \\
&= \frac{\lambda}{e^{-\lambda a} - e^{-\lambda b}} \frac{1}{\lambda} \left[e^{-\lambda x} - xe^{-\lambda x} \right]_a^b \\
&= \frac{(1-b)e^{-\lambda b} - (1-a)e^{-\lambda a}}{e^{-\lambda a} - e^{-\lambda b}}
\end{aligned}
$$