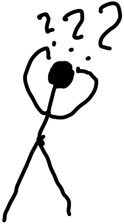
4-valued coalgebraic modal logic

Tomáš Jakl

28 January 2015

$$\underbrace{\text{4-valued}}_{?}\ \underbrace{\text{coalgebraic}}_{???}\ \underbrace{\text{modal logic}}_{?}$$



4-valued logics

1. true
2. false
3. ?
4.

4-valued logics

1. true
2. false
3. $\perp$
4.

$\perp =$ Not enough information

- At the beginning of a computation.
- Program hangs (example: trying to evaluate halting problem).

4-valued logics

1. true
2. false
3. $\perp$
4. $\top$

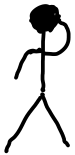
$\top =$ Inconsisten information

- Obtained information from database don't make any sense.
- Different threads returning contradicting results.

4-valued logics

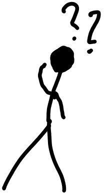
$$\begin{tikzcd} & \top \arrow[dl, dash] \arrow[dr, dash] & \\ \text{false} & & \text{true} \\ & \bot \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

1. true
2. false
3. $\bot$
4. $\top$



$$\text{true} \sqcup \text{false} = \top$$
$$\text{true} \sqcap \text{false} = \bot$$

$$\underbrace{\text{4-valued}}_{\checkmark}\ \underbrace{\text{coalgebraic}}_{???}\ \underbrace{\text{modal logic}}_{?}$$



Modal logics

$$\boldsymbol{s} \models \Box\varphi$$

Starting in the state s, the proposition φ is going to hold *after one step of a computation*.

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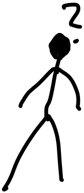
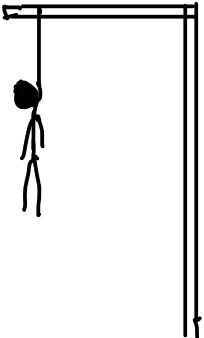


What are coalgebras?

Simple answer:

They are just algebras in the opposite category.



What are coalgebras?

Simple answer:

They are just algebras in the opposite category.

More useful answer:

Any state based system is a coalgebra. For example

- streams of bytes, single thread computation,
- concurrent computation,
- complex inteligent networks (brain, internet, finantial systems),
- ...

Coalgebras

states

shape/type

$$\sigma : S \to T(S)$$

transition function

Coalgebras

$$\sigma\colon S \to T(S)$$

- Printing coalgebra ($T(S) = A$):

$$\sigma\colon S \longrightarrow A$$
$$\sigma\colon s \in S \longmapsto \sigma(s) \in A$$

Coalgebras

$$\sigma\colon S \to T(S)$$

- Printing coalgebra ($T(S) = A$):

$$\sigma\colon S \longrightarrow A$$
$$\sigma\colon s \in S \longmapsto \sigma(s) \in A$$

- Changing-states coalgebra ($T(S) = S$):

$$\sigma\colon S \longrightarrow S$$

Coalgebras of combined shapes

- Choice:

$$\sigma\colon S \to T_1(S) \uplus T_2(S)$$

Coalgebras of combined shapes

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$$\sigma \colon S \to T_1(S) \times T_2(S)$$

Coalgebras of combined shapes

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- Parallel composition:

$$\sigma \colon S \to T_1(S) \times T_2(S)$$

- Reading input:

$$\sigma \colon S \to T(S)^B$$

where

$$T(S)^B = \{f \mid f \colon B \to T(S)\}$$

Coalgebras of combined shapes

- Choice:

$$\sigma\colon S \to T_1(S) \mathbin{\dot\cup} T_2(S)$$

- Parallel composition:

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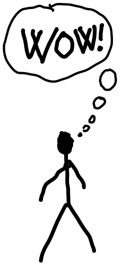
- Nondeterminism:

$$\sigma\colon S \to \mathcal{P}(T(S)) = \{X \mid X \subseteq T(S)\}$$



Example

An automaton is: $(S,\ \delta\colon S \times \Sigma \to S,\ F \subseteq S)$.

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- $\widetilde{\delta}\colon S \to S^{\Sigma}$

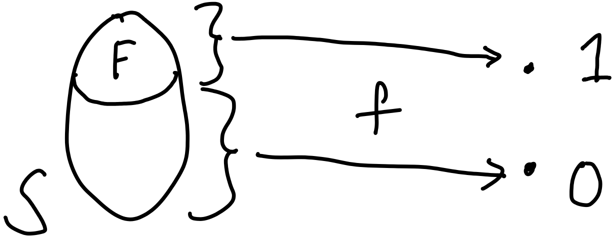
$$\widetilde{\delta}\colon s \in S \mapsto \lambda x.\ \delta(s, x)$$

Example

An automaton is: $(S,\ \delta\colon S \times \Sigma \to S,\ F \subseteq S)$.

- $\widetilde{\delta}\colon S \to S^{\Sigma}$
- $f\colon S \to \{0, 1\}$

$$f^{-1}(1) = F$$



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Similarly, nondeterministic automata are coalgebras

$$S \to \{0,1\} \times (\mathcal{P}(S))^{\Sigma}.$$

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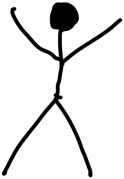

4-valued coalgebraic modal logic

= a suitable logic for systems!

Thank you!