Proof of sub-modularity of a class of separable concave utility maximization problem

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***Abstract*— We establish the sub-modularity of a class of separable concave utility maximization problem.**

I. INTRODUCTION

We consider the following optimization problem (P):

$$\begin{aligned} \underset{x_t}{\text{maximize}} \quad & \sum_{t=1}^{T} U_t(x_t) \\ \text{subject to} \quad & \sum_{t=1}^{T} x_t \leq \Delta \\ & x_t \geq 0 \\ & x_t \leq z_t \Delta \end{aligned} \tag{1}$$

, where $z_t \in \{0,1\}$ is the parameter of the maximization problem. Here we make the following assumptions about the utility function U_t:

Assumption 1. *U_t is continuous on $[0,+\infty]$, non-decreasing, concave, continuously differentiable on $(0,+\infty)$. Further, $U_t^{'}$ is strictly decreasing in its positive part. Also, $U_t(0) = 0$.*

We furthur define the following function:

Definition 1. *f: $2^T \mapsto \mathbf{R}, f(\emptyset) = 0$ and $f(A) =$ optimal value of (P) with $z_t = 1$ if and only if $t \in A$.*

II. PROOF OF SUB-MODULARITY

By definition, we only need to show that for any $A \subset B \subset [T]$ and $i \in [T] \setminus \{i\}$, we have

$$f(A+i) - f(A) \geq f(B+i) - f(B) \tag{2}$$

Here, we only need to consider the non-trival case, that is to say, $A \neq \emptyset$ and $A \neq B$. To characterize the maximizer of (P), we have the following lemma.

Lemma 1. *The maximizer x^* of (P) satisfies:*
(1) $U_t^{'}(x_t^*) = G \geq 0$ if $x_t^* > 0$.
(2) $G > \sup_{x>0} U_t^{'}(x)$ or $z_t = 0$ if $x_t^* = 0$

where G is a constant dependent on $\{t : z_t = 1\}$ and Δ. We then have lemma2.

Lemma 2. *$G(A,\Delta)$ is increasing in A and decreasing in Δ.*

Based on the above two lemmas, we can establish the following theorem:

Theorem 1. *The function f defined in definition 1 is sub-modular.*

Proof: Let the maximizer be $x_t^*(A,\Delta)$,then we have

$$\begin{aligned} & f(A+i) - f(A) \geq \\ & \textstyle\sum_{t\in A} \int_{U_t' \geq G(A,\Delta - x_i^*(B+i,\Delta))} U_t^{'} dx + \int_{U_i' \geq G(B+i,\Delta)} U_i^{'} dx \\ & - \textstyle\sum_{t\in A} \int_{U_t' \geq G(A,\Delta)} U_t^{'} dx \\ & = - \textstyle\sum_{t\in A} \int_{G(A,\Delta - x_i^*(B+i,\Delta)) \geq U_t' \geq G(A,\Delta)} U_t^{'} dx + \\ & \textstyle\int_{x \leq x_i^*(B+i,\Delta)} U_i^{'} dx \end{aligned}$$

We then have:

$$\begin{aligned} & - \textstyle\sum_{t\in A} \int_{G(A,\Delta - x_i^*(B+i,\Delta)) \geq U_t' \geq G(A,\Delta)} U_t^{'} dx + \\ & \textstyle\int_{x \leq x_i^*(B+i,\Delta)} U_i^{'} dx - (f(B+i) - f(B)) = \\ & \textstyle\sum_{t\in B\setminus A} \int_{G(B+i,\Delta) \geq U_t' \geq G(B,\Delta)} U_t^{'} dx + \\ & \textstyle\sum_{t\in A} \int_{G(B+i,\Delta) \geq U_t' \geq G(A,\Delta - x_i^*(B+i,\Delta))} U_t^{'} dx - \\ & \textstyle\sum_{t\in A} \int_{G(B,\Delta) \geq U_t' \geq G(A,\Delta)} U_t^{'} dx \\ & \geq G(B,\Delta)(\textstyle\sum_{t\in B\setminus A} \int_{G(B+i,\Delta) \geq U_t' \geq G(B,\Delta)} 1 dx + \\ & \textstyle\sum_{t\in A} \int_{G(B+i,\Delta) \geq U_t' \geq G(A,\Delta - x_i^*(B+i,\Delta))} 1 dx - \\ & \textstyle\sum_{t\in A} \int_{G(B,\Delta) \geq U_t' \geq G(A,\Delta)} 1 dx) \\ & \geq 0 \end{aligned}$$

Finally, we have $f(A+i) - f(A) \geq f(B+i) - f(B)$.