On Stability Condition of Wireless Networked Control Systems under Joint Design of Control Policy and Network Scheduling Policy

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***Abstract*— In this paper, we study a wireless networked control system (WNCS) with $N \geq 2$ sub-systems sharing a common wireless channel. Each sub-system consists of a plant and a controller and the control message must be delivered from the controller to the plant through the shared wireless channel. The wireless channel is unreliable due to interference and fading. As a result, a packet can be successfully delivered in a slot with a certain probability. A network scheduling policy determines how to transmit those control messages generated by such N sub-systems and directly influences the transmission delay of control messages. We characterize the stability condition of such a WNCS under the joint design of the control policy and the network scheduling policy by means of 2^N linear inequalities. We further simplify the stability condition into only one linear inequality for two special cases: the perfect-channel case where the wireless channel can successfully deliver a control message with certainty in each slot, and the symmetric-structure case where all sub-systems have identical system parameters.**

I. Introduction

Networked Control Systems (NCSs) that exchange information between the plant and the controller through a shared communication network have been actively researched for decades in both academia and industry [1]–[3]. Existing communication networks employed in NCS include controller-area network (CAN), Ethernet, and wireless networks (called wireless NCS (WNCS)) [1]. Among them, WNCS is widely used in many applications such as automated highway systems, factories, and unmanned aerial vehicles (UAVs), etc., because wireless communication can be easily deployed in low cost and low complexity [1], [4]–[6]. In this paper, we focus on WNCS.

Many researchers focus on the control-theoretic issues while highly abstracting the network-system performance in terms of transmission delay and packet dropout/loss [7]. For example, with the insertion of a wireless communication network with finite capacity, it is commonly assumed that a packet traveling through the shared communication network could experience a fixed or random delay (with certain distribution) [8] and a fixed dropout rate [9]. Based on such simplification, many researchers focus on how to design the control policy to stabilize the system [10], [11] or optimize the system performance [12]. In [13], Tan and Zhang consider the stabilization problems for NCSs with both packet dropout and transmission delay. By utilizing a delay-dependent algebraic Riccati equation, a necessary and sufficient stabilization condition is derived.

TABLE I

Main results on stability condition

	Symmetric Structure	Asymmetric Structure
Perfect Channel	(23) for $h < N$, one inequality (24) for $h \geq N$, one inequality	(16), one inequality
Imperfect Channel	(21) for $h < N$, one inequality (22) for $h \geq N$, one inequality	(8), 2^N inequalities

However, packet delay and packet dropout incurring in a shared communication network are results of specific network operations that include network protocol and network scheduling policy. To completely understand the behaviour and performance of NCSs, it is important to simultaneously consider both the control policy in the dynamic system and the network scheduling policy in the network system. There are some existing works that consider such joint design for WNCS [7], [14]–[16], (also see a survey [5] and the references therein). Demirel et al. in [7] considers a WNCS with only one plant and one controller, which exchanges information through a multi-hop wireless network. The authors jointly design the control policy and network scheduling policy to minimize the closed-loop loss function and propose a modular co-design framework to solve the problem. Xiao et al. in [16] investigate the effect of channel fading on stabilizability of a WNCS with one plant and one controller. Park et al. in [15] analyze the system performance of WNCS with multiple plants and controllers when the wireless communication network adopts the standard IEEE 802.15.4 protocol. Liu and Goldsmith in [14] also study a WNCS with multiple plants and controllers and they analyze the system performance by jointly considering control policy and cross-layer network design. However, to the best of our knowledge, there does not exist work on characterizing the stability condition of WNCS with multiple plants and controllers sharing a common wireless channel.

In this paper, we study a WNCS with multiple sub-systems sharing a common wireless channel. Each sub-system consists of a plant and a controller and the control message must be delivered from the controller to the plant through the shared wireless channel. We characterize the

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stability condition of such a WNCS under the joint design of the control policy and the network scheduling policy. To the best of our knowledge, we are the first to provide such a stability condition. In particular, we make the following contributions, with the main results on stability condition summarized in Table I:

- For the stated WNCS with general system parameters so that all sub-systems could have different parameters (asymmetric-structure case in Table I) and the wireless channel could be imperfect (imperfect-channel case in in Table I), we characterize the stability condition by means of 2^N linear inequalities where N is the number of sub-systems.
- We simplify the stability condition into one linear inequality for two special cases of the considered WNCS: the perfect-channel case and the symmetric-structure case.
- For perfect-channel case, we show that the system can be stabilized if the sampling period is larger than a certain value.

II. System Model

We consider a wireless networked control system (WNCS) with N sub-systems, indexed from 1 to N. An example of two sub-systems is shown in Fig. 1. Sub-system $i \in [N]$ has a plant (plant i) and a controller (controller i), where $[N] \triangleq \{1, 2, \cdots, N\}$. Each plant has a sensor and an actuator. The sensor can sample and transmit their measurements to the controller over a dedicated channel without incurring packet loss and delay. The controller makes control decision based on sensor's measurements. The control message/packet of the controller is transmitted to the actuator of the plant to influence the dynamics of the plant over a shared wireless channel[1], which could incur packet loss and delay. A typical practical scenario of our model is the remote control of a fleet of unmanned aerial vehicles (UAVs) [17].

Sub-System Dynamics. The underlaying time system is continuous (starting from time 0) but we also create a slotted model (staring from slot 1) for the wireless transmission model where each slot spans $\Delta > 0$ units of time. Details of wireless transmission model will be provided below. Plant i's underlaying state evolves according to the following continuous-time system:

$$\dot{x}^i(t) = A_i x^i(t) + B_i u^i(t), \tag{1}$$

where $x^i(t) \in \mathbb{R}$ is the state and $u^i(t) \in \mathbb{R}$ is the control input at time t. To guarantee that each sub-system is stabilizable, we assume that $A_i \geq 0$, and $B_i \neq 0$.[2] Moreover, we assume that each sensor samples the state every $h \in \mathbb{Z}^+$ slots (i.e., every $h\Delta$ units of time). Namely, we observe plant i every h slots. Staring from slot 1, every T slots forms a frame. The first frame is indexed as 0. For example, frame k is from slot $kh + 1$ to slot $(k + 1)h$. We observe plant i at the beginning of each frame $k = 0, 1, 2, \cdots$ (i.e., at slot $kh + 1$), which is denoted as $x^i_k \in \mathbb{R}$. Controller i can instantaneously obtain plant i's state x^k_i and then makes a control decision $u^i_k \in \mathbb{R}$. The control message/packet u^i_k needs to be transmitted to plant i through a shared wireless channel. If u^i_k cannot be delivered before or at the end of frame k (i.e., slot $(k + 1)h$), then a packet dropout occurs. Denote a random variable γ^i_k which is 1 if message u^i_k is delivered before/at the end of frame k and 0 otherwise. Then based on the analysis in [18], the sampled state of plant i evolves according to the following discrete-time system:

$$x^i_{k+1} = \bar{A}_i x^i_k + \gamma^i_{k-1} \bar{B}_i u^i_{k-1}, \quad k = 0, 1, 2, \cdots \tag{2}$$

where $\bar{A}_i = e^{A_i h\Delta} \geq 1$ and $\bar{B}_i = \int_0^{h\Delta} e^{A_i \tau} B_i d\tau \neq 0$.

Sub-system i is (mean-square) stable if for any initial conditions x^i_0, u^i_{-1} and γ^i_{-1}, the state x^i_k follows

$$\lim_{k\to\infty} \mathbb{E}\left[|x^i_k|^2\right] = 0. \tag{3}$$

Our goal is to design the stabilizing controller $\{u^i_k : i \in [N], k = 0, 1, 2, \cdots\}$ to make all N sub-systems mean-square stable.

Wireless Channel and Scheduler. The wireless channel is shared by all sub-systems and there is a centralized scheduler to collect the control packets and then make scheduling decision to transmit them over the wireless channel. We assume that only one packet can be transmitted over the wireless channel in each slot. As we mentioned before, each slot spans Δ units of time, which is the time length of transmitting a control packet from the scheduler to the plant and getting the acknowledge about whether the packet is delivered or not from the plant to the scheduler.

Wireless channel is usually unreliable because of interference and fading. We model such unreliability by a successful probability p_i. Namely, in a slot, if we transmit the control packet of sub-system i, the message will be delivered successfully with probability $p_i \in (0, 1]$. Different plants could have different channel quality due to the different distances away from the scheduler and different ambient condition. Thus, the successful probability p_i depends on sub-system index i.

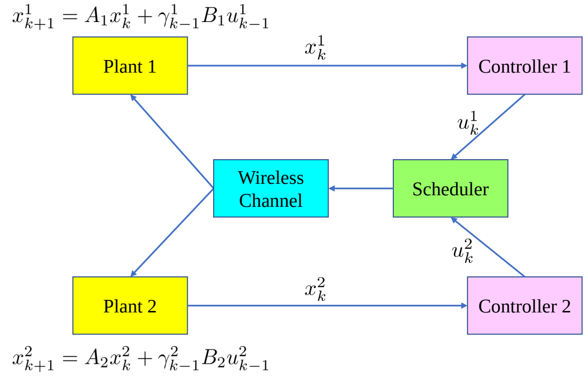

Fig. 1. System model — an example of two sub-systems (i.e., $N = 2$).

[1] Note that it is possible to have multiple channels in practice, which involves another design space of channel allocation. We leave it as an interesting future work.

[2] In this paper, we consider the scalar-state case. It is interesting and important to extend our results to the general vector-state case.

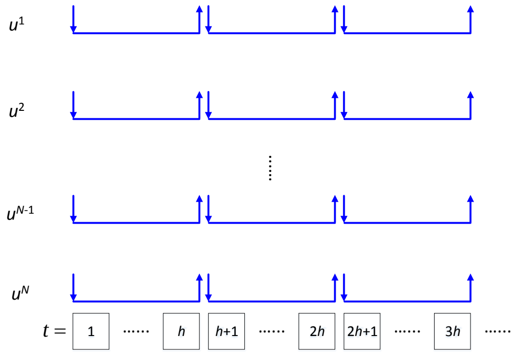

Fig. 2. Control message/packet pattern.

Design Spaces. To make all sub-systems stable, our design spaces include two parts:

- *The control policy*[3] $\{u_k^i : i \in [N], k = 0, 1, 2, \cdots\}$, which determines the control variable u_k^i for each frame k and each sub-system i;
- *The scheduling policy*, which determines the packet to transmit at each slot. Note that the distribution of random variable γ_k^i is completely determined by the scheduling policy.

Both the control policy and the scheduling policy influence the dynamics of the plants according to (2).

The network system and the control system are connected via random variables $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$, whose joint distributions have impact on both systems. In principle, for any sub-system i, the joint distribution of random variables $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ could be completely arbitrary because the scheduling policy is arbitrary. However, it would be difficult to design control policy to stabilize system (2) when the joint distribution of random variables $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ has no pattern. To the best of our knowledge, current literature on NCS only analyzes when $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are identical and independent distributed (i.i.d.) (see [10]). Therefore, to judiciously leverage the existing results on networked control system and delay-constrained wireless communication (see our analysis in the next section), we only consider the case that $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are i.i.d. It turns out that there exists a type of scheduling policies to ensure that $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are i.i.d.

III. Stability Analysis

According to [10], for any sub-system i, if $\{\gamma_k^i : k = 0, 1, 2, \cdots, \}$ are i.i.d. with $\mathbb{P}(\gamma_k^i = 0) = q_i$, where $q_i \geq 0$ is called the packet dropout rate of sub-system i, then sub-system i is (mean-square) stable if and only if[4]

$$q_i < q_{\max}^i (A_i, h) \triangleq \frac{1}{\bar{A}_i^4 - \bar{A}_i^2 + 1} = \frac{1}{e^{4A_i h\Delta} - e^{2A_i h\Delta} + 1}. \tag{4}$$

[3]We consider linear control policy with respect to predictive state in this paper [10].

[4]For the scalar case, the stability of sub-system i does not dependent on parameter B_i. The reason is that the control policy can determine the control variables $\{u_k^i\}$ to compensate the effect of parameter B_i (see system dynamics (2)).

Clearly, random variable γ_k^i depends on the scheduling policy in frame k. To utilize the result in [10], we restrict that the scheduling policy is *frame-periodic*, i.e., it is the same for all frames. Under frame-periodic scheduling policy, $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are i.i.d. but the packet dropout rate $q_i = \mathbb{P}(\gamma_k^i = 0)$ still depends on the detailed scheduling policy.

To make all sub-systems stable, we need to ensure that q_i satisfies (4) for all $i \in [N]$. Given such a dropout rate vector $(q_1, q_2, \cdots, q_N)$, we need to design a frame-periodic scheduling policy such that $\mathbb{P}(\gamma_k^i = 0) = q_i, \forall i \in [N]$. Such a scheduling policy design problem is equivalent to the timely wireless flow problem [19]–[21] with frame-synchronized traffic pattern as shown in Fig. 2. Note that $1 - q_i$ is exactly the per-frame *timely throughput* of sub-system/flow i.[5] Thus, according to [19]–[21], if $(1 - q_i : i \in [N])$ is in the capacity region (which is defined as the set of all feasible timely throughput vectors), there exists a frame-periodic scheduling policy such that $\mathbb{P}(\gamma_k^i = 0) = q_i, \forall i \in [N]$ and then make all sub-systems stable.

Let us denote the capacity region as $\mathcal{R}(\boldsymbol{p}, h)$ where $\boldsymbol{p} = (p_1, p_2, \cdots, p_N)$ is the channel quality vector. Then we have the following result.

***Theorem** 1:* Suppose that we only consider those scheduling policies such that $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are i.i.d. Then given system parameters $\boldsymbol{A} = (A_1, A_2, \cdots, A_N)$, $\boldsymbol{B} = (B_1, B_2, \cdots, B_N)$, $\boldsymbol{p} = (p_1, p_2, \cdots, p_N)$, sampling period h, and slot length Δ, there exists a control policy and a network scheduling policy to make all sub-systems stable if and only if

$$\left(1 - q_{\max}^1 (A_1, h), \cdots, 1 - q_{\max}^N (A_N, h)\right) \in \text{int}\left(\mathcal{R}(\boldsymbol{p}, h)\right), \tag{5}$$

where int$(\mathcal{S})$ denotes the interior of a set $\mathcal{S}$.

Proof: Please see Appendix A. ■

Note that we only show the stability condition in Theorem 1. The corresponding control policy and the network scheduling policy will be described shortly in **Remarks 1 & 2**.

In the literature on delay-constrained wireless communication, there are two equivalent characterizations for the capacity region $\mathcal{R}(\boldsymbol{p}, h)$: one idle-time-based in [19], [20] and one MDP-based in [21], both of which have a complexity of $O(2^N)$. The idle-time-based one is easy to analyze and we show it as follows: the capacity region $\mathcal{R}(\boldsymbol{p}, h)$ is the set of all timely throughput vectors $(R_1, R_2, \cdots, R_N)$ satisfying

$$R_i \in [0, 1], \quad \forall i \in [N] \tag{6a}$$

$$\sum_{i \in \mathcal{S}} \frac{R_i}{p_i} + \mathbb{E}[I_{\mathcal{S}}] \leq h, \quad \forall \mathcal{S} \subset [N], \tag{6b}$$

where random variable $I_{\mathcal{S}}$ is the number of idle slots in a frame when we only schedule the control packets in sub-system set $\mathcal{S}$ in any work-conserving manner.

[5]The timely throughput of sub-system i is the long-term-wise per-frame average number of packets that are successfully delivered before their deadlines [19]–[21]. Under the frame-periodic scheduling policy, the average number of packets that are successfully delivered before their deadlines is identical for all frames, meaning that the timely throughput of sub-system i is equal to $P(\gamma_k^i = 1) = 1 - q_i$.

Denote δ_i as the number of transmissions to get a successful delivery for sub-system i's control message, which is a geometric random variable with mean $1/p_i$. Thus, the number of idle slots is

$$I_{\mathcal{S}} = \max\{h - \sum_{i\in\mathcal{S}} \delta_i, 0\}, \tag{7}$$

and the expected number of idle slots is $\mathbb{E}[I_{\mathcal{S}}]$.

Thus, the stability condition (5) is equivalent to the following inequalities

$$\sum_{i\in\mathcal{S}} \frac{1-q^i_{\max}(A_i,h)}{p_i} + \mathbb{E}[I_{\mathcal{S}}] < h, \quad \forall \mathcal{S} \subset [N]. \tag{8}$$

Note that in (8) we have in total 2^N linear inequalities.

Remark 1. Though we characterize the stability condition (see (8)) using the idle-time-based approach in [19], [20], we should note that we need to use the MDP-based characterization in [21, Equ. (11)] to design a frame-periodic network scheduling policy (called RAC policy [21, Theorem 2-(iii)]) to simultaneously achieve any feasible timely throughput vector and ensure that $\{\gamma^i_k : k = 0, 1, 2, \cdots\}$ are i.i.d.

Remark 2. For given dropout rate $q_i = P(\gamma^i_k = 0)$, we only mentioned the stability condition (4) of sub-system i but ignore the design of control policy. In fact, if q_i satisfies (4), we can solve the following delay dependent algebraic Riccati equation (DARE) whose variable is P_i,

$$P_i = \bar{A}_i' P_i \bar{A}_i + I - M_i' \Upsilon_i^{-1} M_i, \tag{9}$$

with

$$\begin{aligned}\Upsilon_i =& (1-q_i)^2 \bar{B}_i' P_i \bar{B}_i + q_i(1-q_i)\bar{B}_i'\bar{A}_i' P_i \bar{A}_i \bar{B}_i \\ &+ q_i(1-q_i)\bar{B}_i'\bar{B}_i + I, \\ M_i =& (1-q_i)\bar{B}_i' P_i \bar{A}_i. \end{aligned}\tag{10}$$

Note that (9) and (10) is for general vector-state case. For our scalar-state case, we can simplify them by applying $\bar{A}_i' = \bar{A}_i$, $\bar{B}_i' = \bar{B}_i$, $M_i' = M_i$, and $I = 1$. In this case, the stabilizing control policy is given as

$$\begin{aligned} u^i_k &= -\Upsilon_i^{-1} M_i \hat{x}^i_{k|k-1} \\ &= -\Upsilon_i^{-1} M_i \left[\bar{A}_i x^i_k + (1-q_i)\bar{B}_i u^i_{k-1}\right]. \end{aligned}\tag{11}$$

Please refer to [10] for the details and proofs.

Next we simplify the stability condition (8) for two special cases: the perfect-channel case (Sec. IV) and the symmetric-structure case (Sec. V), both of which characterize the stability condition by means of only one linear inequality.

IV. The Perfect-Channel Case

When $p_i = 1$ for all $i \in [N]$, i.e., the wireless channel is perfect in the sense that it can successfully deliver a packet in each slot with certainty, then we can get a simple capacity region characterization according to (6). Note that $I_{\mathcal{S}} = \max\{h-|\mathcal{S}|, 0\}$ for perfect channel. Thus, (6) becomes

$$R_i \in [0,1], \quad \forall i \in [N] \tag{12a}$$

$$\sum_{i\in\mathcal{S}} R_i + \max\{h - |\mathcal{S}|, 0\} \le h, \quad \forall \mathcal{S} \subset [N], \tag{12b}$$

If $|\mathcal{S}| \le h$, then $\max\{h-|\mathcal{S}|, 0\} = h-|\mathcal{S}|$ and thus (12b) becomes

$$\sum_{i\in\mathcal{S}} R_i \le |\mathcal{S}|, \tag{13}$$

which is implied by (12a).

If $|\mathcal{S}| > h$, then $\max\{h - |\mathcal{S}|, 0\} = 0$ and thus (12b) becomes

$$\sum_{i\in\mathcal{S}} R_i \le h. \tag{14}$$

When $|\mathcal{S}| = [N] = \{1, 2, \cdots, N\}$, (14) becomes

$$\sum_{i=1}^{N} R_i \le h, \tag{15}$$

which implies (14) for any $\mathcal{S} \subset [N]$. Therefore, the capacity region for perfect channel becomes

$$\mathcal{R}(\mathbf{1}, h) = \left\{(R_1, R_2, \cdots, R_N) : \sum_{i=1}^{N} R_i \le h, R_i \in [0,1] (\forall i \in [N])\right\}$$

Thus, in the perfect-channel case, the stability condition (5) becomes

$$\begin{aligned}&\sum_{i=1}^{N} \left[1 - q^i_{\max}(A_i, h)\right] \\ &= \sum_{i=1}^{N} \left[1 - \frac{1}{e^{4A_i h\Delta} - e^{2A_i h\Delta} + 1}\right] < h. \end{aligned}\tag{16}$$

Hence, we characterize the stability condition of the WNCS in the perfect-channel case by means of one linear inequality.

We further show a property for the perfect-channel case.

Theorem 2: There exists an $h_{\min} \in [N]$ such that (16) holds if $h \ge h_{\min}$.

Proof: Please see Appendix B. ■

Theorem 2 shows that $h \ge h_{\min}$ is a sufficient condition for stability. Readers may wonder whether it is also necessary. However, it may not be true, as shown later in Fig. 4 in the simulation section.

V. The Symmetric-Structure Case

The capacity region $\mathcal{R}(\boldsymbol{p}, h)$ becomes much more complicated in the imperfect-channel case, i.e., when some $p_i < 1$. For general channels (which could be perfect or imperfect), in this section, we analyze a special case of our system, called symmetric-structure case in the sense that $A_i = A$, $p_i = p$ for all $i \in [N]$. Thus,

$$\begin{aligned} q^i_{\max}(A_i, h) &= \frac{1}{e^{4Ah\Delta} - e^{2Ah\Delta} + 1} \\ &= \frac{1}{\bar{A}^4 - \bar{A}^2 + 1} \triangleq q_{\max}(A, h), \quad \forall i \in [N] \end{aligned}\tag{17}$$

where we define $\bar{A} \triangleq e^{Ah\Delta}$. Due to the symmetric structure, if $|\mathcal{S}_1| = |\mathcal{S}_2|$, we have $\mathbb{E}[I_{\mathcal{S}_1}] = \mathbb{E}[I_{\mathcal{S}_2}]$. Then the stability condition (8) becomes

$$\frac{1-q_{\max}(A,h)}{p}+\mathbb{E}[I_{\{1\}}]<h, \tag{18a}$$

$$\frac{2\left(1-q_{\max}(A,h)\right)}{p}+\mathbb{E}[I_{\{1,2\}}]<h, \tag{18b}$$

$$\cdots \tag{18c}$$

$$\frac{N\left(1-q_{\max}(A,h)\right)}{p}+\mathbb{E}[I_{\{1,2,\cdots,N\}}]<h, \tag{18d}$$

which is equivalent to

$$\frac{1-q_{\max}(A,h)}{p}<h-\mathbb{E}[I_{\{1\}}], \tag{19a}$$

$$\frac{1-q_{\max}(A,h)}{p}<\frac{h-\mathbb{E}[I_{\{1,2\}}]}{2}, \tag{19b}$$

$$\cdots \tag{19c}$$

$$\frac{1-q_{\max}(A,h)}{p}<\frac{h-\mathbb{E}[I_{\{1,2,\cdots,N\}}]}{N}, \tag{19d}$$

We will further simplify (19) based on the following result.

***Theorem** 3:* In the symmetric-structure case, we have

$$h-\mathbb{E}[I_{\{1\}}]\geq\frac{h-\mathbb{E}[I_{\{1,2\}}]}{2}\geq\cdots\geq\frac{h-\mathbb{E}[I_{\{1,2,\cdots,N\}}]}{N}.$$

Proof: See Appendix C. ■

Theorem 3 shows that the stability condition (19) can be further simplified into one linear inequality,

$$\frac{1-q_{\max}(A,h)}{p}<\frac{h-\mathbb{E}[I_{\{1,2,\cdots,N\}}]}{N}. \tag{20}$$

We next show how to calculate $h-\mathbb{E}[I_{\{1,2,\cdots,N\}}]$. Note that $h-\mathbb{E}[I_{\{1,2,\cdots,N\}}]=\mathbb{E}[h-I_{\{1,2,\cdots,N\}}]$ is the expected number of transmissions in a frame of h slots when scheduling all sub-systems' control packets. Denote random variable $X\triangleq h-I_{\{1,2,\cdots,N\}}$. Then when $h\leq N$, we have that $P(X=h)=1$. When $h>N$, we have that

$$P(X=N)=p^N,$$
$$P(X=N+k)=\binom{N+k-1}{k}(1-p)^k p^N,\forall k\in[h-N-1]$$
$$P(X=h)=\sum_{i=0}^{N-1}\binom{h}{i}(1-p)^{h-i}p^i+\binom{h-1}{h-N}(1-p)^{h-N}p^N.$$

Then when $h\leq N$, stability condition (20) becomes,

$$\frac{1-q_{\max}(A,h)}{p}<\frac{h}{N}. \tag{21}$$

Otherwise, when $h>N$, stability condition (20) becomes,

$$\begin{aligned}&\frac{1-q_{\max}(A,h)}{p}<\frac{\mathbb{E}[X]}{N}=\\&\frac{NP(X=N)+\sum_{k=1}^{h-N-1}(N+k)P(X=N+k)+hP(X=h)}{N}\\&=\frac{Np^N+\sum_{k=1}^{h-N-1}(N+k)\binom{N+k-1}{k}(1-p)^kp^N}{N}\\&+\frac{h\left(\sum_{i=0}^{N-1}\binom{h}{i}(1-p)^{h-i}p^i+\binom{h-1}{h-N}(1-p)^{h-N}p^N\right)}{N}.\end{aligned} \tag{22}$$

Note that when $q=1$, i.e., in the perfect-channel case, the stability condition (21) for $h\leq N$ becomes

$$N(1-q_{\max}(A,h))<h, \tag{23}$$

which coincides with (16) under the symmetric structure; the stability condition (22) for $h>N$ becomes

$$1-q_{\max}(A,h)<1, \tag{24}$$

which always holds. This result again coincides with the analysis in Sec. IV that the system can be stabilized when $h>N$.

Readers may wonder whether we have a similar result like Theorem 2 for the non-perfect-channel case. However, it turns out that this may not be true, as shown later in Fig. 5 in the simulation section.

VI. Simulation

In this section, we use simulation to confirm our theoretic analysis. First, we consider a general (imperfect-channel asymmetric-structure) system with system parameters,

$$\begin{aligned}&N=3,h=5,\Delta=0.01,\\&A_1=6.5137,A_2=5.8265,A_3=8.8964,\\&B_1=1,B_2=1,B_3=1,\\&p_1=0.7690,p_2=0.7277,p_3=0.2846.\end{aligned} \tag{25}$$

According to our stability condition (8), we can check that the system can be stabilized. We then construct the network scheduling policy and the control policy to get the per-slot state of each sub-system, which is shown in Fig. 3. As we can see, indeed, the states of all three sub-systems converge to 0 and thus all three sub-systems are stabilized. This confirms our stability condition (8) for general system.

Second, we consider a perfect-channel case with system parameters,

$$\begin{aligned}&N=6,\Delta=0.0114,\\&\boldsymbol{A}=(A_1,A_2,A_3,A_4,A_5,A_6)\\&\quad=(3.7482,8.7512,7.7711,8.5482,6.8823,5.6830),\\&\boldsymbol{p}=(p_1,p_2,p_3,p_4,p_5,p_6)=(1,1,1,1,1,1).\end{aligned} \tag{26}$$

We change the sampling period h from 1 slot to 10 slots. The stability result is shown in Fig. 4. We can see that there exists an $h_{\min}=3\in\{1,2,3,4,5,6\}$ such that the system can be stabilized if $h\geq h_{\min}$. This is consistent with Theorem 2. From this figure, we can also see that $h\geq h_{\min}$ is not necessary for stability in (16), because the system can be stabilized when $h=1<h_{\min}=3$.

Finally, we consider a symmetric-structure (imperfect-channel) case with parameters,

$$N=2,\Delta=0.1,A_1=A_2=A=1. \tag{27}$$

We change the sampling period h from 1 slot to 10 slots and consider three different levels of channel quality $p_1=p_2=p\in\{0.300,0.425,0.500\}$. The stability result is shown in Fig. 5. We can see that the system is unstable for all sampling periods when the channel quality is bad,

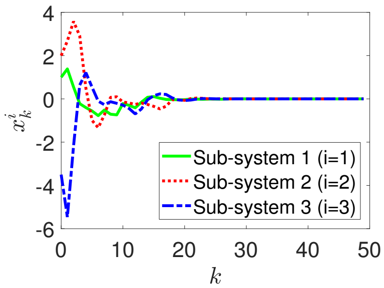

Fig. 3. State evolution of three sub-systems with system parameters in (25).

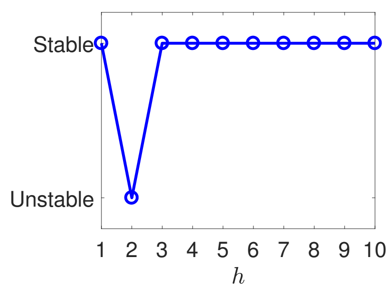

Fig. 4. Stability condition for the perfect-channel case with system parameters in (26).

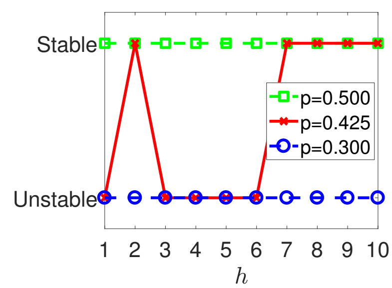

Fig. 5. Stability condition for the symmetric-structure case with system parameters in (27).

i.e., $p = 0.30$. However, the system can be stabilized for all sampling periods when the channel quality is good, i.e., $p = 0.5$. When the channel quality is medium, i.e., $p = 0.425$, the system can be stabilized when the sampling period $h \in \{2, 7, 8, 9, 10\}$ and unstable otherwise. Thus, for such imperfect-channel case, we do not have a similar result like Theorem 2 for perfect-channel case. Instead, the stability result becomes more complicated depending on the channel quality.

VII. Conclusion

In this paper, we characterize the stability condition of a WNCS with multiple plants and controllers sharing a common wireless channel under the joint design of control policy and network scheduling policy. To solve our WNCS problem, we have leveraged the recent results in the research area of delay-constrained wireless communication [19]–[21]. In the future, it is interesting and important to generalize our system in several aspects. First, we have only considered the scalar-state case and the general vector-state case is worth studying. Second, we assume that the hard deadline of each control message is equal to the sampling period corresponding to $d = 1$ in [18]. It is also worth studying the case of $d > 1$. Finally, our results are based on the assumption that the dropout random variables $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ are i.i.d., which can simplify the design of the control policy but also shrinks the design space. It would be challenging to consider the full design space where $\{\gamma_k^i : k = 0, 1, 2, \cdots\}$ may not be i.i.d.

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Appendix

A. Proof of Theorem 1

We first prove the "if" part and then prove the "only if" part.

1) "If" Part.: If (5) holds, then there exists a timely throughput vector $\boldsymbol{R} = (R_1, R_2, \cdots R_N) \in \mathcal{R}(\boldsymbol{p}, h)$ such that

$$1 - q^i_{\max}(A_i, h) < R_i. \tag{28}$$

According to Theorem 2 in [21], $\boldsymbol{R} = (R_1, R_2, \cdots R_N)$ can be achieved by a randomized almost cyclostationary (RAC) policy. Since our traffic pattern for the control messages (see Fig. 2) is frame-synchronized, the RAC policy becomes cyclostationary (frame-periodic) in the sense that the policy (in terms of the action distribution conditioning on any state, see Definition 1 in [21]) is the same for all frames. Therefore, the probability of delivering sub-system i's control message in any frame k is equal to the probability of delivering sub-system i's control message in any frame $k' \neq k$, which must be equal to the achieved time throughput R_i, i.e.,

$$P(\gamma^i_k = 1) = R_i, \forall k, k' = 0, 1, 2, \cdots. \tag{29}$$

Combining (28) and (29), we have

$$1 - q^i_{\max}(\bar{A}_i, h) < R_i = P(\gamma^i_k = 1) = 1 - P(\gamma^i_k = 0) = 1 - q_i,$$

implying that

$$q_i < q^i_{\max}(A_i, h).$$

Thus, any sub-system i can be stabilized according to Theorem 3 in [10].

2) "Only If" Part.: If the system can be stabilized under the condition that $\{\gamma^i_k : k = 0, 1, 2, \cdots\}$ are i.i.d., then according to Theorem 3 in [10], we have

$$P(\gamma^i_k = 0) = q_i < q^i_{\max}(A_i, h). \tag{30}$$

The scheduling policy such that $\{\gamma^i_k : k = 0, 1, 2, \cdots\}$ are i.i.d. with $P(\gamma^i_k = 0) = q_i$ achieves the timely throughput $R_i = P(\gamma^i_k = 1) = 1 - q_i$ for any sub-system i. Thus,

$$\begin{aligned}\boldsymbol{R} &= (R_1, R_2, \cdots, R_N) \\ &= (1 - q_1, 1 - q_2, \cdots, 1 - q_N) \in \mathcal{R}(\boldsymbol{p}, h).\end{aligned} \tag{31}$$

Combining (30) and (31), we prove that (5) holds.

B. Proof of Theorem 2

The stability condition (16) is equivalent to

$$f(h) \triangleq \sum_{i=1}^{N} \left[1 - \frac{1}{e^{4A_i h\Delta} - e^{2A_i h\Delta} + 1}\right] - h < 0. \tag{32}$$

Instead of considering integer h, we consider $h \in \mathbb{R}$ for function $f(h)$. First, it is easy to see that $f(0) = 0$ and $f(h) < 0$ for all $h \geq N$. We trace back $f(h)$ from $h = N$ to $h = 0$ and find the first h such that $f(h) = 0$ and we denote it as h^*, i.e.,

$$h^* \triangleq \max\{h \in \mathbb{R} : 0 \leq h \leq N, f(h) = 0\}. \tag{33}$$

Note that h^* is well-defined because $f(0) = 0$ and $h^* \in [0, N)$ because $f(N) < 0$. Since $f(h)$ is continuous and $f(N) < 0$, we have that

$$f(h) < 0, \quad \forall h \in (h^*, N]. \tag{34}$$

In addition, since $f(h) < 0$ for all $h \geq N$, we have that

$$f(h) < 0, \quad \forall h \in (h^*, \infty) \tag{35}$$

Then we denote

$$h_{\min} = \begin{cases} h^* + 1, & \text{if } h^* \text{ is an integer;} \\ \lceil h^* \rceil, & \text{otherwise.} \end{cases} \tag{36}$$

Since $h^* \in [0, N)$, we obtain that $h_{\min} \in [N]$. Clearly, $f(h) < 0$ when $h \geq h_{\min}$. The proof is completed.

C. Proof of Theorem 3

Define function

$$f(\mathcal{S}) \triangleq h - \mathbb{E}[I_{\{\mathcal{S}\}}], \quad \forall \mathcal{S} \subset [N]. \tag{37}$$

Then to prove Theorem 3, we need to prove the following inequality,

$$f(\{1\}) \geq \frac{f(\{1, 2\})}{2} \geq \cdots \geq \frac{f(\{1, 2, \cdots, N\})}{N} \tag{38}$$

Clearly $f(\emptyset) = 0$ and

$$f(\mathcal{S}_1) = f(\mathcal{S}_2), \text{ if } |\mathcal{S}_1| = |\mathcal{S}_2| \tag{39}$$

due to the symmetry. The authors in [22] show that $f(\mathcal{S})$ is a submodular function. Therefore, for any $\mathcal{S}_1 \subset [N], \mathcal{S}_2 \subset [N]$, we have

$$f(\mathcal{S}_1) + f(\mathcal{S}_2) \geq f(\mathcal{S}_1 \cup \mathcal{S}_2) + f(\mathcal{S}_1 \cap \mathcal{S}_2), \tag{40}$$

Now we prove (38) by induction.

First, when setting $\mathcal{S}_1 = \{1\}, \mathcal{S}_2 = \{2\}$ in (40), we have

$$\begin{aligned}f(\{1\}) + f(\{2\}) &\geq f(\{1\} \cup \{2\}) + f(\{1\} \cup \{2\}) \\ &= f(\{1, 2\}) + f(\emptyset) = f(\{1, 2\}).\end{aligned} \tag{41}$$

In addition, due to (39), we have $f(\{2\}) = f(\{1\})$. This implies that

$$f(\{1\}) \geq \frac{f(\{1, 2\})}{2}. \tag{42}$$

Second, suppose that

$$\frac{f(\{1, 2, \cdots, k-1\})}{k-1} \geq \frac{f(\{1, 2, \cdots, k\})}{k} \tag{43}$$

holds for $k \geq 2$. Now in (40), we consider two sets $\mathcal{S}_1 = \{1, 2, \cdots, k\}$ and $\mathcal{S}_2 = \{2, 3, \cdots, k, k+1\}$. Clearly both $\mathcal{S}_1$ and $\mathcal{S}_2$ are of size k and thus $f(\mathcal{S}_2) = f(\mathcal{S}_1) = f(\{1, 2, \cdots, k\})$. In addition, we have $\mathcal{S}_1 \cup \mathcal{S}_2 = \{1, 2, \cdots, k+1\}$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{2, 3, \cdots, k\} \triangleq \mathcal{S}_3$. Since $\mathcal{S}_3$ is of size $k-1$, we have $f(\mathcal{S}_3) = f(\{1, 2, \cdots, k-1\})$. Thus, we have

$$\begin{aligned}&2f(\{1, 2, \cdots, k\}) = f(\mathcal{S}_1) + f(\mathcal{S}_2) \geq f(\mathcal{S}_1 \cup \mathcal{S}_2) + f(\mathcal{S}_3) \\ &= f(\{1, 2, \cdots, k+1\}) + f(\{1, 2, \cdots, k-1\}) \\ &\geq f(\{1, 2, \cdots, k+1\}) + \frac{k-1}{k} f(\{1, 2, \cdots, k\}),\end{aligned} \tag{44}$$

where the last inequality follows from hypothesis (43). Rearranging (44), we have

$$\frac{f(\{1, 2, \cdots, k\})}{k} \geq \frac{f(\{1, 2, \cdots, k+1\})}{k+1}, \tag{45}$$

which shows that (43) also holds for $k+1$. Thus we complete the proof for (38).