Probability and Statistics (IT302) Class No. 30
26th October 2020 Monday 09:45 AM - 10:15 AM

Probability and Statistics (IT302) Class No. 31
27th October 2020 Tuesday 10:30 AM - 11:00 AM

Introduction to Gamma and Exponential Distributions

- Although the normal distribution can be used to solve many problems in engineering and science, there are still numerous situations that require different types of density functions. Two such density functions**, the gamma and exponential distributions.**

- It turns out that the exponential distribution is a special case of the gamma distribution. Both find a large number of applications. The **exponential and gamma distributions play an important role in both queuing theory and reliability problems**.

- Time between arrivals at service facilities and time to failure of component parts and electrical systems often are nicely modeled by the exponential distribution. The relationship between the gamma and the exponential allows the gamma to be used in similar types of problems.

- The gamma distribution derives its name from the well-known **gamma function**, studied in many areas of mathematics.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

The gamma function

Definition: The **gamma function** is defined by $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \qquad \text{for } \alpha > 0.$$

The following are a few simple properties of the gamma function.

(a) $\Gamma(n) = (n-1)(n-2)\cdots(1)\Gamma(1)$, for a positive integer n.

To see the proof, integrating by parts with $u = x^{\alpha-1}$ and $dv = e^{-x}\,dx$, we obtain

$$\Gamma(\alpha) = -e^{-x}\,x^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2}\,dx = (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx,$$

for $\alpha > 1$, which yields the recursion formula

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1).$$

The result follows after repeated application of the recursion formula. Using this result, we can easily show the following two properties.

(b) $\Gamma(n) = (n-1)!$ for a positive integer n.

(c) $\Gamma(1) = 1$.

Source: “Probability & Statistics for Engineers & Scientists”, Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

gamma distribution

The continuous random variable X has a **gamma distribution**, with parameters α and β, if its density function is given by

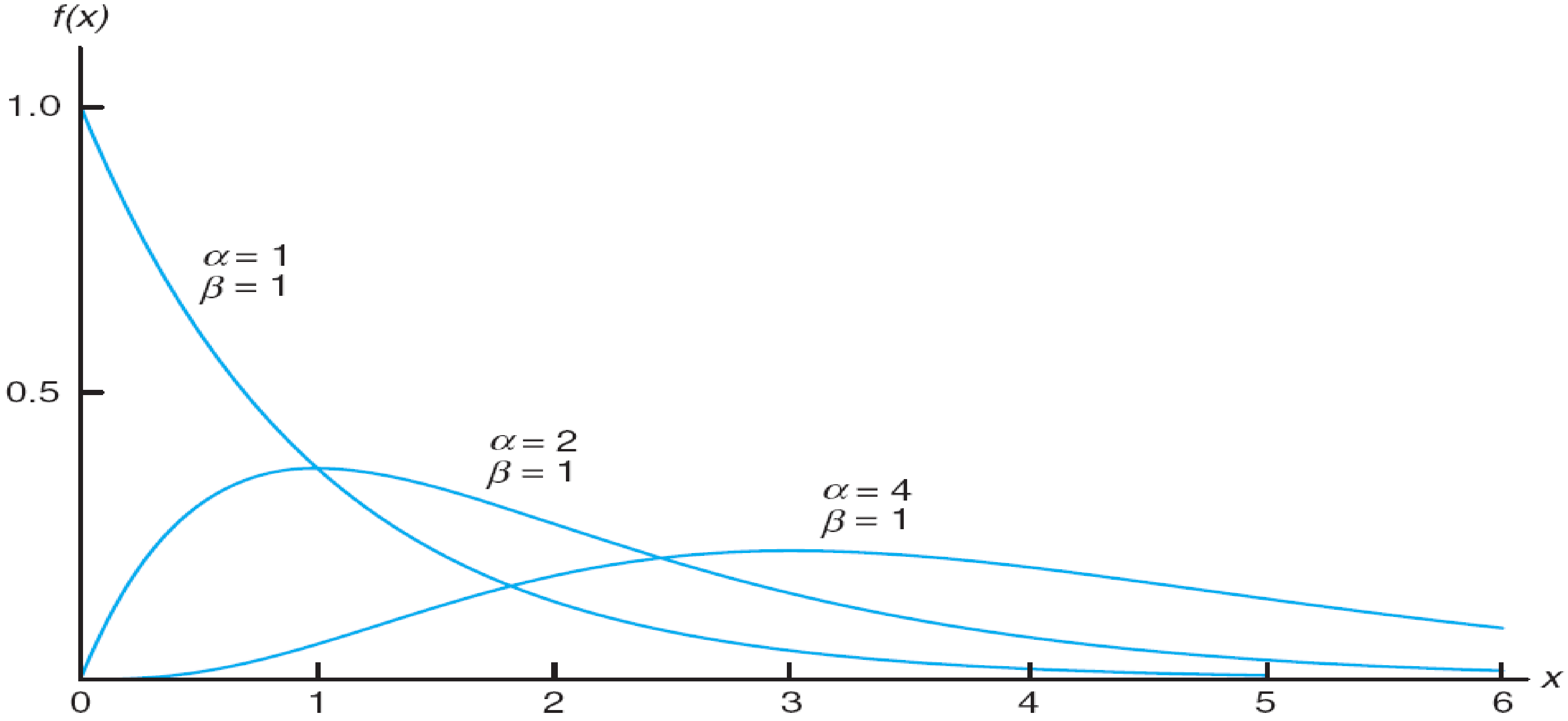
$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9^{th} Edition.

Graphs of several gamma distributions

Graphs of several gamma distributions are shown in below Figure for certain specified values of the parameters α and β. The special gamma distribution for which α = 1 is called the exponential distribution.



Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Exponential Distribution

The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\beta > 0$.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Theorem

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

Corollary

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Relationship to the Poisson Process

- The most important applications of the exponential distribution are situations where the Poisson process applies.

- Recall that the Poisson distribution is used to compute the probability of specific numbers of “events” during a particular period of time or span of space.

- In many applications, the time period or span of space is the random variable. For example, an industrial engineer may be interested in modeling the time T between arrivals at a congested intersection during rush hour in a large city. An arrival represents the Poisson event.

Source: “Probability & Statistics for Engineers & Scientists”, Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Poisson Distribution The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is the average number of outcomes per unit time, distance, area, or volume and $e = 2.71828\ldots$.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Relationship to the Poisson Process Contd

- The relationship between the exponential distribution (often called the negative exponential) and the Poisson process is quite simple.

- Poisson distribution was developed as a single-parameter distribution with parameter λ, where λ may be interpreted as the mean number of events per unit "time."

- Consider now the random variable described by the time required for the first event to occur. Using the Poisson distribution, we find that the probability of no events occurring in the span up to time t is given by

$$p(0; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Relationship to the Poisson Process Contd

We can now make use of the above and let X be the time to the first Poisson event. The probability that the length of time until the first event will exceed x is the same as the probability that no Poisson events will occur in x. The latter, of course, is given by $e^{-\lambda x}$. As a result,

$$P(X > x) = e^{-\lambda x}.$$

Thus, the cumulative distribution function for *X* is given by $$P(0 \leq X \leq x) = 1 - e^{-\lambda x}.$$

Now, in order that we may recognize the presence of the exponential distribution, we differentiate the cumulative distribution function above to obtain the density

function

$$f(x) = \lambda e^{-\lambda x},$$

which is the density function of the exponential distribution with $\lambda = 1/\beta$.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Applications of the Exponential and Gamma Distributions

- Notice that the mean of the exponential distribution is the parameter β, the reciprocal of the parameter in the Poisson distribution.

- The reader should recall that it is often said that the Poisson distribution has no memory, implying that occurrences in successive time periods are independent.

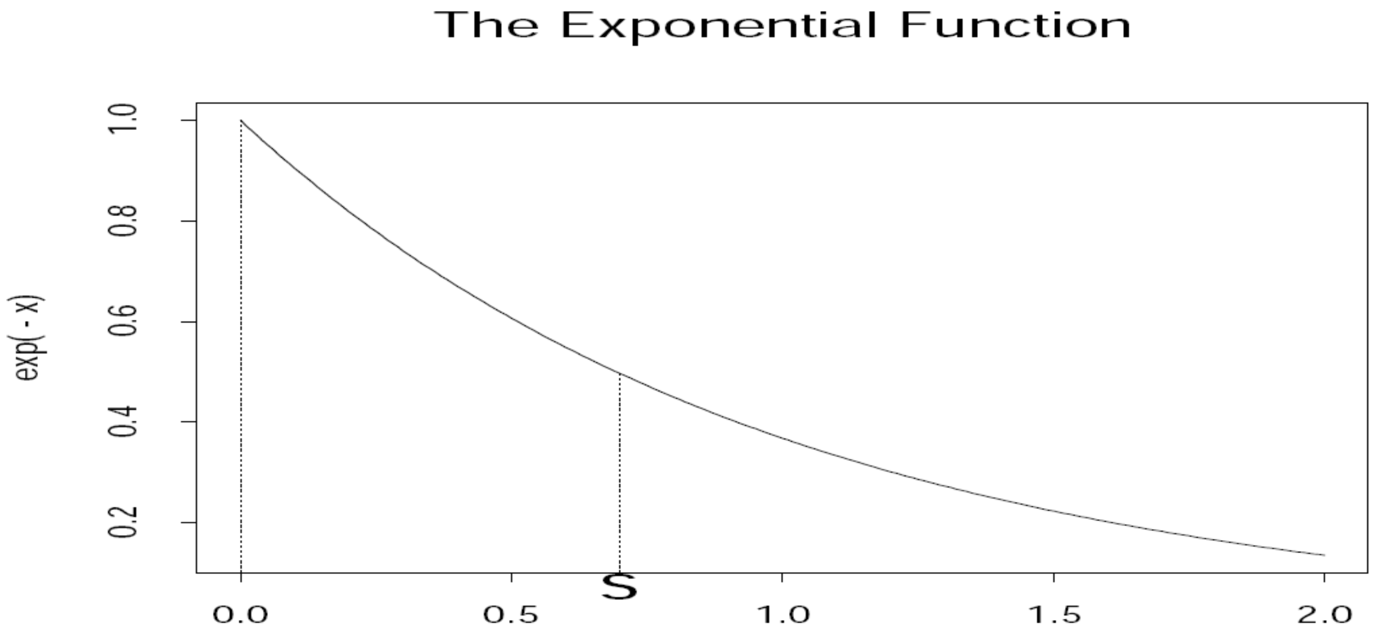
- The important parameter β is the mean time between events. In reliability theory, where equipment failure often conforms to this Poisson process, β is called mean time between failures. Many equipment breakdowns do follow the Poisson process, and thus the exponential distribution does apply.

- Other applications include survival times in biomedical experiments and computer response time.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

The Memoryless Property

The following plot illustrates a key property of the exponential distribution. The graph after the point **s** is an exact copy of the original function. The important consequence of this is that the distribution of X conditioned on {X > s} is again exponential.



To see how this works, imagine that at time 0 we start an alarm clock which will ring after a time X that is exponentially distributed with rate . Let us call X the lifetime of the clock. For any t > 0, we have that

$$P(X > t) = \int_t^\infty \lambda e^{-\lambda x} dx = \lambda \left. \frac{-e^{-\lambda x}}{\lambda} \right|_t^\infty = e^{-\lambda t}.$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

The Memoryless Property Contd.

Now we go away and come back at time s to discover that the alarm has not yet gone off. That is, we have observed the event {X > s}. If we let Y denote the remaining lifetime of the clock given that {X > s}, then

$$\begin{aligned} P(Y > t \mid X > s) &= P(X > s+t \mid X > s) \\ &= \frac{P(X > s+t, X > s)}{P(X > s)} \\ &= \frac{P(X > s+t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t}. \end{aligned}$$

But this implies that the remaining lifetime after we observe the alarm has not yet gone off at time s has the same distribution as the original lifetime X. **The really important thing to note, though, is that this implies that the distribution of the remaining lifetime does not depend on s. In fact, if you try setting X to have any other continuous distribution, then ask what would be the distribution of the remaining lifetime after you observe {X > s}, the distribution will depend on s.**

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

The Memoryless Property Contd.

- **This property is called the memoryless property of the exponential distribution** because I don't need to remember when I started the clock.

- If the distribution of the lifetime X is Exponential(), then if I come back to the clock at any time and observe that the clock has not yet gone off, regardless of when the clock started I can assert that the distribution of the time till it goes off, starting at the time I start observing it again, is Exponential().

- Put another way, given that the clock has currently not yet gone off, I can forget the past and still know the distribution of the time from my current time to the time the alarm will go off.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Example 6.17

Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure β = 5. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Solution:

The probability that a given component is still functioning after 8 years is given by

$$P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}\, dt = e^{-8/5} \approx 0.2.$$

Let X represent the number of components functioning after 8 years. Then using the binomial distribution, we have

$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2) = 1 - \sum_{x=0}^{1} b(x; 5, 0.2) = 1 - 0.7373 = 0.2627.$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Chi-Squared Distribution

Another very important special case of the gamma distribution is obtained by letting **α = v/2 and β = 2**, where v is a positive integer. The result is called the chi-squared distribution. The distribution has a single parameter, **v, called the degrees of freedom**.

The continuous random variable X has a **chi-squared distribution**, with v **degrees of freedom**, if its density function is given by

$$f(x; v) = \begin{cases} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where v is a positive integer.

The mean and variance of the chi-squared distribution are

$$\mu = v \text{ and } \sigma^2 = 2v.$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Beta Distribution

An extension to the uniform distribution is a beta distribution. Let us start by defining a beta function.

A **beta function** is defined by

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ for } \alpha, \beta > 0,$$

where $\Gamma(\alpha)$ is the gamma function.

Beta Distribution

The continuous random variable X has a **beta distribution** with parameters $\alpha > 0$ and $\beta > 0$ if its density function is given by

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that the uniform distribution on $(0, 1)$ is a beta distribution with parameters $\alpha = 1$ and $\beta = 1$.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Theorem 6.6

The mean and variance of a beta distribution with parameters α and β are

$$\mu = \frac{\alpha}{\alpha + \beta} \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)},$$

respectively.

For the uniform distribution on $(0, 1)$, the mean and variance are

$$\mu = \frac{1}{1+1} = \frac{1}{2} \text{ and } \sigma^2 = \frac{(1)(1)}{(1+1)^2(1+1+1)} = \frac{1}{12},$$

respectively.

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.

Lognormal Distribution

The lognormal distribution is used for a wide variety of applications. The distribution applies in cases where a **natural log transformation results in a normal distribution**.

Lognormal Distribution

The continuous random variable X has a **lognormal distribution** if the random variable $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ. The resulting density function of X is

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu]^2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The mean and variance of the lognormal distribution are

$$\mu = e^{\mu+\sigma^2/2} \text{ and } \sigma^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

Source: "Probability & Statistics for Engineers & Scientists", Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition.