Invertibility

COMP408 - Linear Algebra
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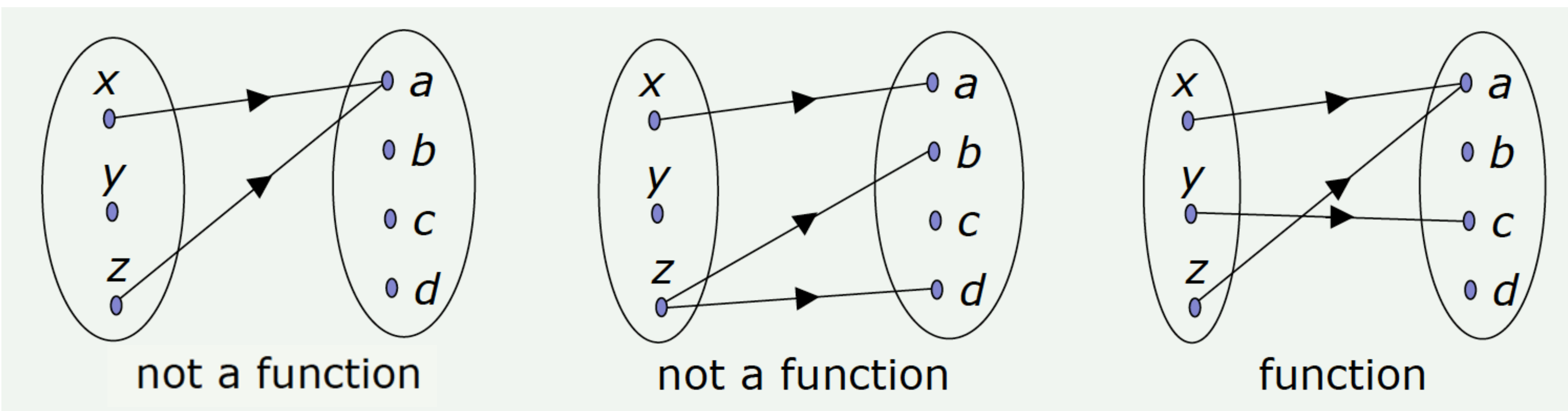
Functions

A ***function*** *f* is a mapping between 2 sets A and B, denoted by *f: A -> B*, such that each $a \in A$ maps to exactly one element in B.

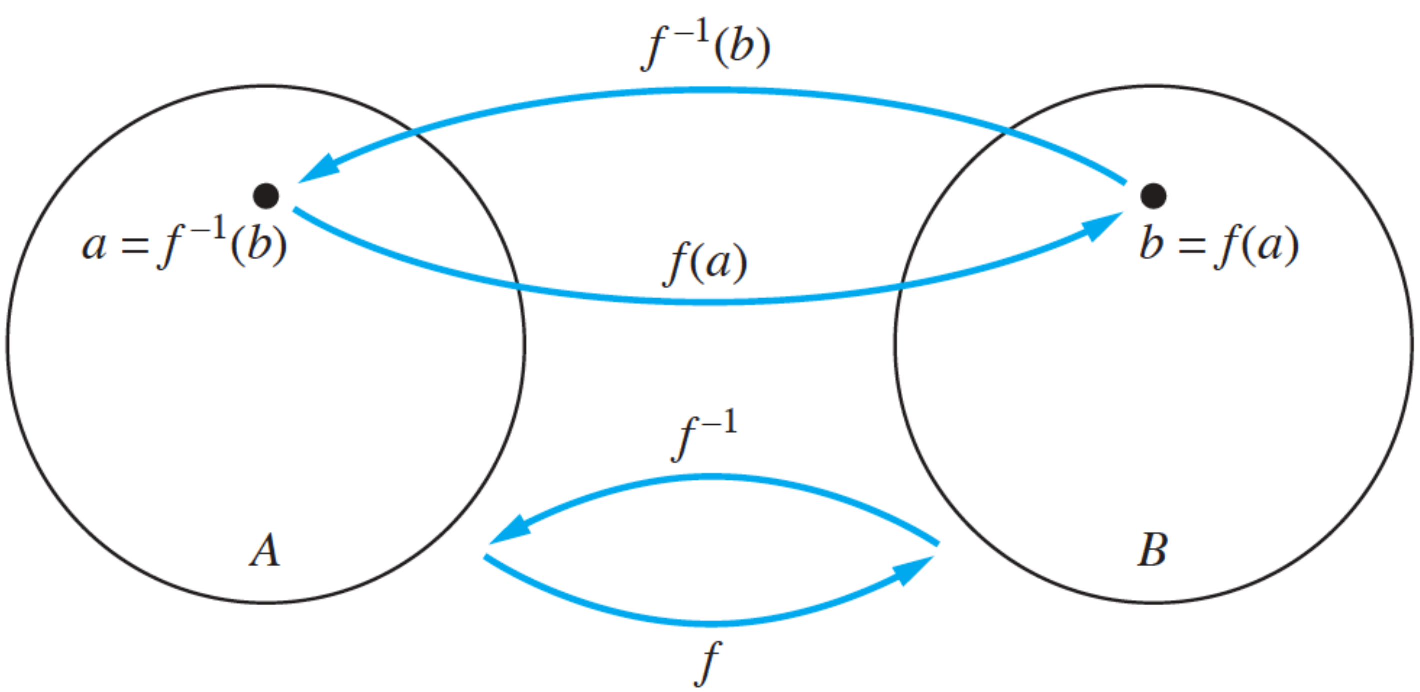


We write $f(a) = b$ if the function f maps the element $a \in A$ to the element $b \in B$.

Inverse function

An ***inverse function*** f^{-1} is a mapping between elements in codomain to the domain of the function *f*.

Theorem: The inverse function is a function if and only if *f* is bijective.



Example: Let *f*: **Z** -> **Z** be such that $f(x) = x + 1$, then $f^{-1}(x) = x - 1$.

Inverse matrix

Let A be an $n \times n$ matrix. An inverse of A is an $n \times n$ matrix A^{-1} such that

$$A^{-1} A = I \text{ and } A A^{-1} = I,$$

where I is the $n \times n$ identity matrix. The matrix A is invertible (or nonsingular) if it has an inverse.

Example: Given $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$ we have $\mathbf{A}^{-1} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ since

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

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Finding inverse matrix

Let A be an $n \times n$ matrix. The inverse of A (if it exists) can be found by applying row operations to the augmented matrix $[A \mid I]$:

$$[A \mid I] \sim \cdots \sim [I \mid A^{-1}].$$

If A is not row equivalent to I, then A^{-1} does not exist.

In general, when A is a 2 x 2 matrix with $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = ad - bc$, then

$$\mathbf{A}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Finding inverse matrix

Example: Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]\begin{matrix} -3 \\ \end{matrix} \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]\begin{matrix} \\ 1 \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]\begin{matrix} \\ -\frac{1}{2} \end{matrix}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right].$$

Therefore, the inverse of the matrix is $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.