Inner Productt

COMP408 - Linear Algebra
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Dot Product

A ***dot product*** is the numerical product of the lengths of two vectors, multiplied by the cosine of the angle between them, that is $\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|\ cos\ \theta$, where θ represents the angle between the two vectors.

A simply way to calculate a dot product is by multiplying the components of each vector separately and then adding these products together.

Example: $\vec{a} = [4, 3], \vec{b} = [1, 2]$

$\vec{a} \cdot \vec{b} = (4 \text{ x } 3) + (3 \text{ x } 2) = 11$

Inner product

The generalization of the dot product to an arbitrary vector space is called an ***inner product***.

Let V be a vector space. An inner product on V is a rule that assigns to each pair $v, w \in V$ a real number $\langle v, w \rangle$ such that, for all $u, v, w \in V$ and $\alpha \in \mathrm{R}$,

(1) $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$,
(2) $\langle v, w \rangle = \langle w, v \rangle$,
(3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
(4) $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.

Inner product: Matrix space

We get an inner product on $M_{m\times n}$ by defining, for A, $B \in M_{m\times n}$,

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

This inner product is identical to the dot product on R_{mn} if an m × n matrix is viewed as an mn × 1 matrix by stacking its columns.

Example:

$$\left\langle \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -2 \end{bmatrix} \right\rangle = (2)(1) + (-1)(3) + (3)(8) + (5)(0) + (0)(1) + (4)(-2)$$
$$= 15.$$

Inner product: Polynomial space

The idea of inner product is applicable to all vector space, such as polynomial space (What about function space?).

For example, let x_1, x_2, … , x_n be fixed numbers. We get an inner product on P_n by defining, for p, $q \in P_n$,

$$
\begin{aligned}
\langle p, q \rangle &= \sum_{i=1}^{n} p(x_i) q(x_i) \\
&= p(x_1)q(x_1) + p(x_2)q(x_2) + \cdots p(x_n)q(x_n).
\end{aligned}
$$

Example: If $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, then for $p = x^2$ and q = x + 1, we have

$$
\begin{aligned}
<p, q> &= p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\
&= 2
\end{aligned}
$$

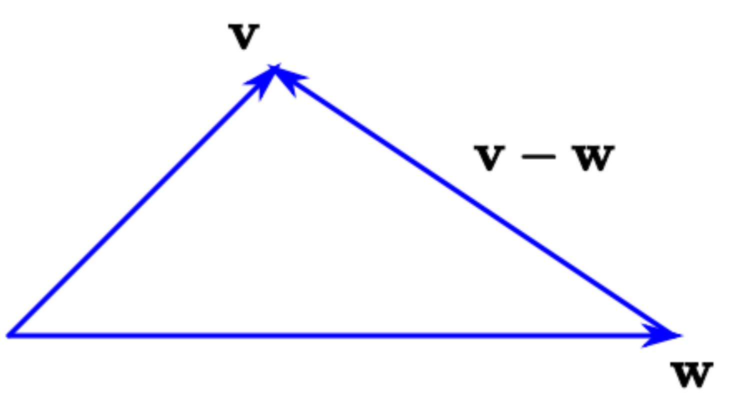
Norm

Let V be an inner product space and let $v \in V$. The ***norm*** (or ***length***) of v is denoted $||v||$ and is defined by

$$||v|| = \sqrt{<v, v>}$$

The ***distance*** between two vectors in V is the norm of their difference

$$\mathrm{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$



Cauchy-Schwarz Theorem

Cauchy-Schwarz Theorem: For all v, w $\in$ V, w have

$$|<v, w>| \leq ||v||\ ||w||.$$

The theorem implies that

$$\theta = \cos^{-1} \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \, \|\mathbf{w}\|}$$

and also $<v, w> = ||v||\ ||w|| \cos \theta$.

We say that *v* is ***orthogonal*** (or ***perpendicular***) to *w* if and only if $<v, w> = 0$:

Cauchy-Schwarz Theorem

The idea of Cauchy-Schwarz theorem can similarly be applied to polynomial space and function space.

Example: If $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, then for $p = x^2$ and q = x + 1. The angle θ between the two functions is as follows.

$\langle p, q\rangle = 2$ (as shown before)
$||p|| = \sqrt{\langle p, p\rangle} = \sqrt{p(-1)^2 + p(0)^2 + p(1)^2} = \sqrt{2}$
$||q|| = \sqrt{\langle q, q\rangle} = \sqrt{q(-1)^2 + q(0)^2 + q(1)^2} = \sqrt{5}$
$\theta = \cos^{-1} (\langle p, q\rangle / (||p||\ ||q||)) = \cos^{-1} (2/\sqrt{10})$

Pythagorean Theorem

Pythagorean theorem: Let v, $w \in V$. If $v \perp w$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.

The idea of Pythagorean theorem can similarly be applied to polynomial space and function space.