18 Selection Sort and Heapsort

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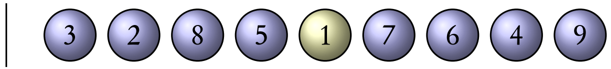
Outline

Selection Sort

Selection sort is to select the minimum element by traversal from the remaining elements and swap it with the element at the least unsorted index.

- The list is again divided into two parts, the front contains the sorted elements, the rear contains the unsorted elements.
- The unsorted part is treated as a priority queue, from which we select the minimum elements one by one.
- We alway append the selected elements to the end of the sorted part, because the selected elements are in ascending order.
- Since the priority queue is implemented as an unsorted list, we can put the original elements, whose places are taken, to the arbitrary places left by the selected elements.



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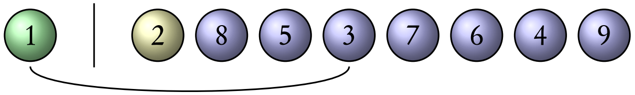

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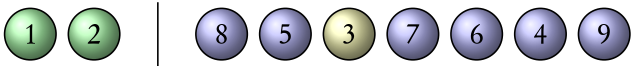

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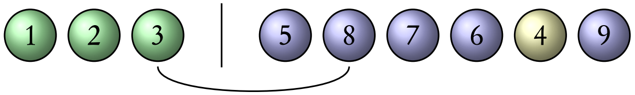

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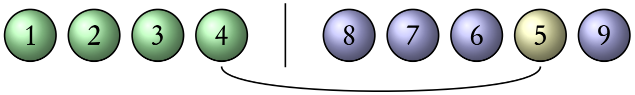

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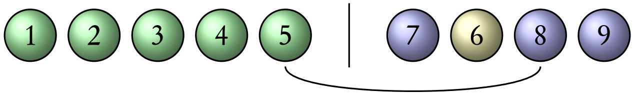

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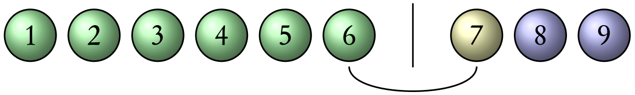

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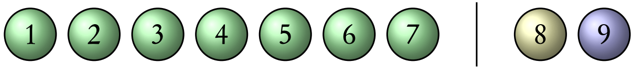

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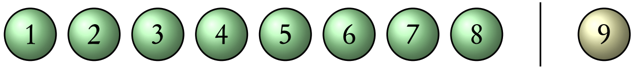

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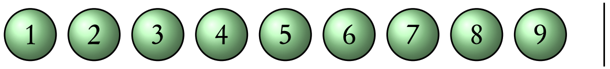


Selection Sort on Array-Based Lists

The straight selection scans all the remaining elements to find the minimum and swaps it with the element just next to the sorted ones. The function *selection_sort_a* sorts the elements of an array-based list *a*.

```
def selection_sort_a(a):
    n = len(a)
    for i in range(n-1):
        m = i
        for j in range(i+1, n):
            if not a[m] <= a[j]:
                m = j
        if i != m:
            a[i], a[m] = a[m], a[i]
```

Selection Sort on Singly Linked Lists

For a singly linked list pointed to by *h*, we initialize an empty list pointed to by *s*, then repeatedly select the *maximum* nodes (why?) from *h* and “push” them to the *front* of *s*. The function *selection_sort_l* sorts a singly linked list *h* and returns the head node of the sorted list.

```
def selection_sort_l(h):
    s = None
    while h is not None:
        m, r = h, None
        p, q = h.nxt, h
        while p is not None:
            if m.elm <= p.elm: m, r = p, q
            p, q = p.nxt, p
        if r is None: h = m.nxt
        else: r.nxt = m.nxt
        m.nxt, s = s, m
    return s
```

Analysis of Straight Selection Sort

For a list of n elements, we only need a fix amount of auxiliary space for selection sort.

- $\mathcal{O}(1)$ auxiliary space.

We count the number of element comparisons.

- In each inner loop, we must compare the first unsorted element with all the rest unsorted elements, thus there are

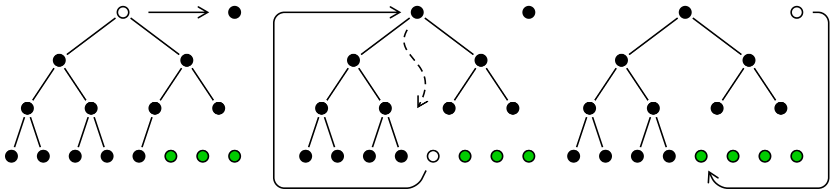
$$(n-1)+(n-2)+\cdots+1=\sum_{i=1}^{n-1} i=\frac{n(n-1)}{2}$$

  comparisons, i.e. $\mathcal{O}(n^2)$.

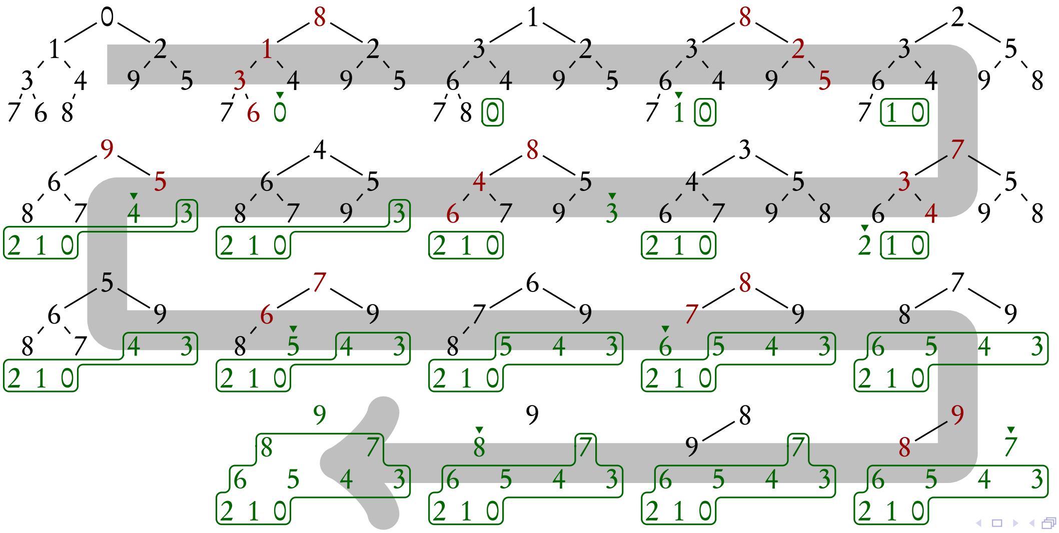
Selection sort is stable on linked lists. However, it is *unstable* on arrays for the algorithm given above, since in the unsorted portion, we swap a front element to the rear.

Selecting by a Heap

- We can also heapify the array-based list, and select the minimum elements by the heap. The array is divided into two parts — a heap and a subarray of sorted elements.
- The heap must occupy the front part of the array, since the tree relations must be computed according to the indices of the elements, and the indices must start from `0`.
- After a *pop_min*, the last element of the heap is moved to the root for being sifted down, leaving the last position of the heap empty.
- We have to put the extracted "min" elements to the empty positions at the end of the heap.
- The *reverse* order of the sorting must be used as the heap order.



Heapsort Illustrated



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Analysis of Heapsort

For an array of n elements, we only need a fix amount of auxiliary space for heapsort.

- $\mathcal{O}(1)$ auxiliary space.

We count the number of element comparisons.

- It takes $\mathcal{O}(n)$ time to heapify an array.
- In the worst case, we sift the elements down the heap $n-1$ times to the bottom, as the heap is shrinking, the maximum depth for all the down-siftings is $\mathcal{O}(\log n)$, but half of the elements are at the bottom, and of depth $\mathcal{O}(\log n)$, so we have the upper bound $n \cdot \mathcal{O}(\log n)$ and the lower bound $\frac{n}{2} \cdot \mathcal{O}(\log n)$ of the time complexity of extracting elements from the heap.
- Heapsort takes overall $\mathcal{O}(n \log n)$ time to sort an array of size n.

Heapsort is *not* stable, since elements of the same key can stay in different sub-trees, be sifted up and down independently.

A General Lower Bound of Sorting

- For a list of n elements, there are $n!$ permutations. We suppose a and b are two elements of the list.
- A general comparison yields a decision out of two possibilities, say, it's $a < b$.
- The number of permutation containing a ahead of b is the same as the number of those containing b ahead of a. The decision eliminates the latter half of the permutations.
- To reach the sorted permutation in all cases, we need at least $\log(n!)$ steps. This is the lower bound for sorting using only the comparisons of two elements.

$a < b$	d,b,c,a	d,b,a,c	d,c,b,a	d,c,a,b	d,a,b,c	d,a,c,b
$b < c$	c,b,a,d	c,b,d,a	c,a,b,d	c,a,d,b	c,d,b,a	c,d,a,b
$c < d$	b,a,c,d	b,a,d,c	b,c,a,d	b,c,d,a	b,d,a,c	b,d,c,a
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$c<d$	~~b,a,c,d~~	~~b,a,d,c~~	~~b,c,a,d~~	~~b,c,d,a~~	~~b,d,a,c~~	~~b,d,c,a~~
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$\cancel{b < c}$	~~c,b,a,d~~	~~c,b,d,a~~	$\cancel{c,a,b,d}$	$\cancel{c,a,d,b}$	~~c,d,b,a~~	$\cancel{c,d,a,b}$
$c < d$	~~b,a,c,d~~	~~b,a,d,c~~	~~b,c,a,d~~	~~b,c,d,a~~	~~b,d,a,c~~	~~b,d,c,a~~
	a,b,c,d	a,b,d,c	$\cancel{a,c,b,d}$	$\cancel{a,c,d,b}$	a,d,b,c	$\cancel{a,d,c,b}$

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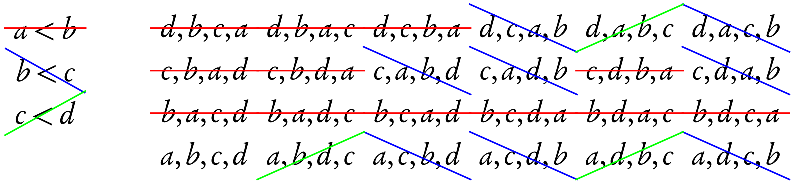
$a < b$
$b < c$
$c < d$
d,b,c,a d,b,a,c d,c,b,a d,c,a,b d,a,b,c d,a,c,b
c,b,a,d c,b,d,a c,a,b,d c,a,d,b c,d,b,a c,d,a,b
b,a,c,d b,a,d,c b,c,a,d b,c,d,a b,d,a,c b,d,c,a
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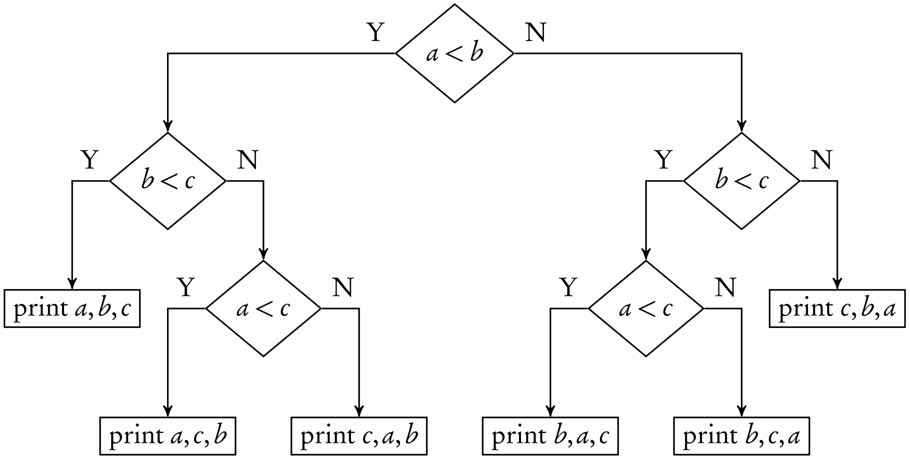
A Decision Tree of Three Elements



Approximating $\log(n!)$

$$
\begin{aligned}
\log(n!) &= \log(n(n-1)(n-2)\cdots 2\cdot 1) \\
&= \log n + \log(n-1) + \cdots + \log 2 + \log 1 \\
&\geqslant \underbrace{\log n + \log(n-1) + \cdots + \log\frac{n}{2}}_{\frac{n}{2}\text{ items}} && \text{omit the latter half items} \\
&\geqslant \underbrace{\log\frac{n}{2} + \log\frac{n}{2} + \cdots + \log\frac{n}{2}}_{\frac{n}{2}\text{ items}} && \text{decrement each item to } \frac{n}{2} \\
&= \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}(\log n - 1).
\end{aligned}
$$

Therefore, $\log(n!) = \Omega(n\log n)$ is the lower bound of the worst-case time complexity of any sorting algorithm based on comparisons.