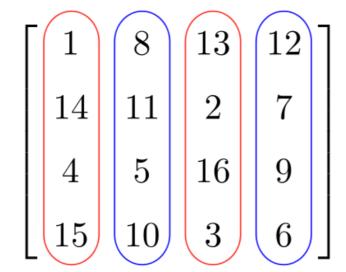
### Column Space and Row Space

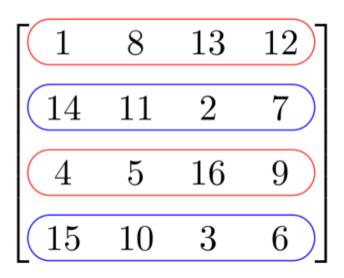
COMP408 - Linear Algebra Dennis Wong

### Column space and row space

The **column space** (range space), col(A), of A is the subspace of  $R^m$  spanned by the columns of A.

The **row space**, row(A), of A is the subspace of  $\mathbb{R}^n$  spanned by the rows of A.





If A and B are matrices with A  $\sim$  B, then row(A) = row(B).

## Bases of column space and row space

If R is a row-echelon matrix, then

- 1. The nonzero rows of R are a basis of row R.
- 2. The columns of *R* containing leading ones are a basis of col *R*.

Example: Consider the following matrix

$$egin{bmatrix} 1 & 3 & 1 & 4 \ 2 & 7 & 3 & 9 \ 1 & 5 & 3 & 1 \ 1 & 2 & 0 & 8 \end{bmatrix} \sim egin{bmatrix} 1 & 3 & 1 & 4 \ 0 & 1 & 1 & 1 \ 0 & 2 & 2 & -3 \ 0 & -1 & -1 & 4 \end{bmatrix} \sim egin{bmatrix} 1 & 0 & -2 & 1 \ 0 & 1 & 1 & 1 \ 0 & 0 & 0 & -5 \ 0 & 0 & 0 & 5 \end{bmatrix} \sim egin{bmatrix} 1 & 0 & -2 & 0 \ 0 & 1 & 1 & 0 \ 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis of the column space are  $(1, 2, 1, 1)^T$ ,  $(3, 7, 5, 2)^T$  and  $(4, 9, 1, 8)^T$ .

#### Rank

Let A denote any  $m \times n$  matrix of **rank** r. Then  $dim(col\ A) = dim(row\ A) = r$ .

Moreover, if A is carried to a row-echelon matrix R by row operations, then

- 1. The *r* nonzero rows of *R* are a basis of row *A*.
- 2. If the leading 1s lie in columns  $j_1, j_2, ..., j_r$  of R, then columns  $j_1, j_2, ..., j_r$  of A are a basis of col A.

If A is any matrix, then rank  $A = \text{rank } (A^T)$ .

#### Rank

Example: Compute the rank and the bases of row A and col A for the following matrix:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank A = 2, and  $\{[1, 2, 2, -1], [0, 0, 1, -3]\}$  is a basis of row A.

Furthermore, columns 1 and 3 of A are a basis {[1, 3, 1]<sup>T</sup>, [2, 5, 1]<sup>T</sup>} of *col* A.

# Null space and image space

The null space of A, denoted null(A), is defined by null(A) =  $\{x \in \mathbb{R}^n \mid Ax = 0\}$ .

The image space of A, denoted im(A), are defined by  $im(A) = \{Ax \mid x \in \mathbb{R}^n \}$ 

In other words, null(A) consists of all solutions x in  $\mathbb{R}^n$  of the homogeneous system Ax = 0, and im(A) is the set of all vectors y in  $\mathbb{R}^m$  such that Ax = y has a solution x.