Sequences

COMP406 - Calculus
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Sequences

A ***sequence***, denoted by $\{a_n\}$, is a function from a subset of the set of integers to a set S.

We use the notation a_n to denote the image of the integer n. We also call a_n as a ***term*** of the sequence.

Example 1: $a_n = (-1)^n$, where $n \in \{0, 1, 2, 3, 4,\ldots\}$.
The elements of the sequence are: 1, -1, 1, -1, 1,…

Example 2: $a_n = 2^n$, where $n \in \{0, 1, 2, 3, 4,\ldots\}$.
The elements of the sequence are: 1, 2, 4, 8, 16,…

Arithmetic Progression

An ***arithmetic progression*** is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the ***initial term*** a and the ***common difference*** d are real numbers.

Example: $a_n = -1 + 4n$, where $n \in \{0, 1, 2, 3, 4, \ldots\}$.
The elements of the sequence are: -1, 3, 7, 11,…, where -1 is the initial term, and 4 is the common difference.

Geometric Progression

A ***geometric progression*** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the ***initial term*** a and the ***common ratio*** r are real numbers.

Example: $a_n = (1/2)^n$, where $n \in \{0, 1, 2, 3, 4, \ldots\}$.
The elements of the sequence are: 1, 1/2, 1/4, 1/8,…, where 1 is the initial term, and 1/2 is the common ratio.

Recurrence Relations

A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation (a.k.a: ***recurrence equation***) that expresses the term a_n in terms of some previous terms, namely a_0, a_1, a_2, …, a_n, of the sequence for some positive integer n.

The ***initial terms*** of a recurrence relation specifies the terms precedes the first term where the recurrence relation take effect.

Example: The famous ***Fibonacci sequence***, f_0, f_1, f_2, …, is a recurrence relation with the initial terms $f_0 = 0$ and $f_1 = 1$ with the following recurrence equation: $f_n = f_{n-1} + f_{n-2}$, where $n \in \{2, 3, \ldots\}$.

The elements of the sequence are: 1, 2, 3, 5, 8,…

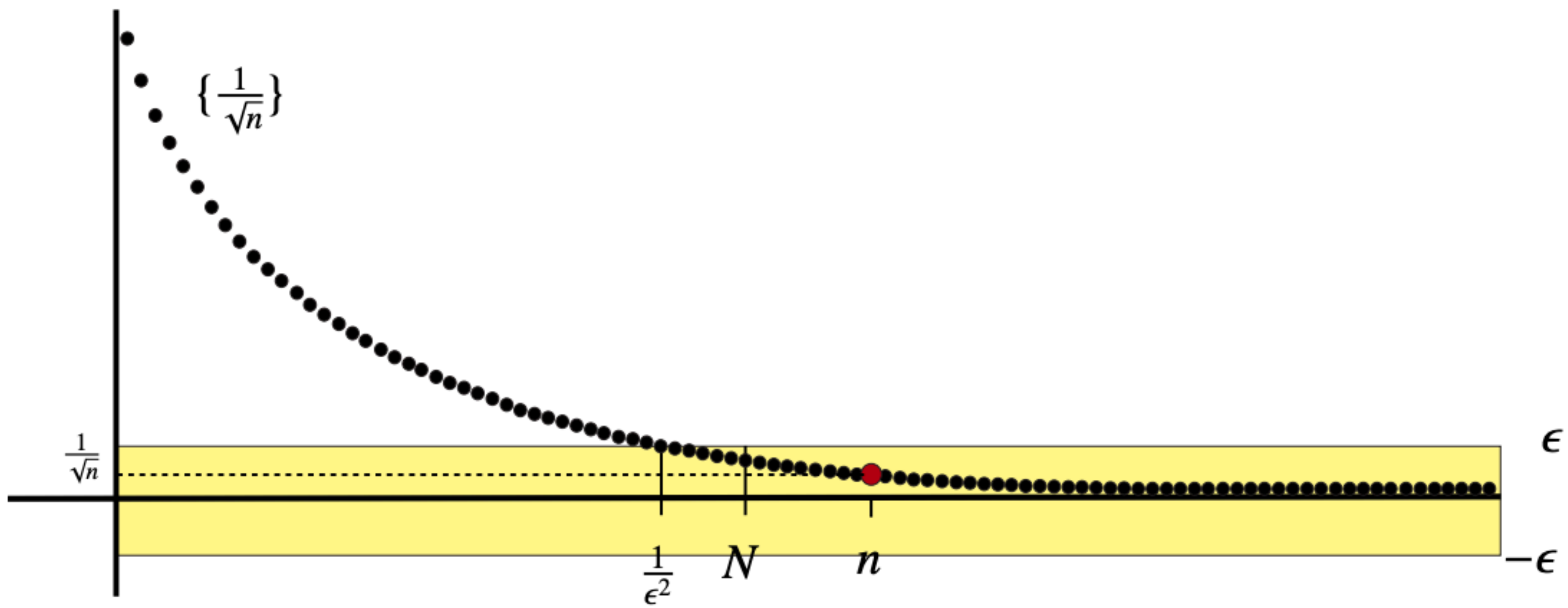
Limits of Sequences

We say that L is the ***limit*** of the sequence $\{a_n\}$ as n goes to infinity if for every $\varepsilon > 0$ there exists an natural number N such that if $n \geq N$, then $|a_n - L| < \varepsilon$.

If such an L exists, we say that the sequence is ***convergent*** and write $\lim\limits_{n \to \infty} a_n = L$.

If no such L exists, then we say that the sequence ***diverges***.

Limits of Sequences



Note that it is usually not easy to show directly that a particular sequence has a limit.

One of the purposes of this course is to learn a few tools to find the limits for some sequences.

Heron's algorithm

Consider the following recursive defined sequence:

$a_1 = 4$ and $a_{n+1} = 1/2\,(a_n + 17/a_n)$

The first 10 terms of the sequence are as below:

n	a_n
1	4
2	4.125
3	4.1231060606
4	4.1231056256
5	4.1231056256
6	4.1231056256
7	4.1231056256
8	4.1231056256
9	4.1231056256
10	4.1231056256

The terms of this sequence actually approach the value $\sqrt{17}$, and thus we say $\lim_{n \to \infty} a_n = \sqrt{17}$.

Summations

Summations of the terms of a sequence:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

where the variable j, m and n are referred as the ***index***, ***lower limit*** and ***upper limit*** of the summation respectively.

Example: Sum of the first 4 terms of $\{n^2\}$ with $n = 1, 2, 3, \ldots$

$$\sum_{j=1}^{4} a_j = \sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

Summation of Arithmetic Progression

The sum of the first n terms of a arithmetic sequence a, $a + d$, $a + 2d$, …, $a + (n - 1)d$ is

$$S = \sum_{j=0}^{n-1} (a + jd) = n\,(a + a + (n - 1)d) / 2$$
$$= na + n\,(n - 1)d / 2$$

Example: Sum of the first 5 terms of $\{2 + 3n\}$ with $n = 0, 1, 2, \ldots$

$$S = \sum_{j=0}^{4} (2 + 3j) = (5 \times 2 + 5 \times (2 + 4 \times 3)) / 2$$
$$= (10 + 70) / 2 = 40$$

Summation of Geometric Progression

The sum of the first n terms of a geometric sequence a, ar, ar^2, …, ar^{n-1} is

$$S = \sum_{j=0}^{n-1} ar^j = a \sum_{j=0}^{n-1} r^j$$

$$= a\,(r^n - 1) / (r - 1)$$

Example 1: Sum of the first 3 terms of $\{2(5)^n\}$ with $n = 0, 1, 2,\ldots$

$$S = \sum_{j=0}^{2} 2\,(5)^j = 2\,(5^3 - 1) / (5 - 1)$$

$$= 2 \times (125 - 1) / 4 = 62$$

Example 2: Sum of all terms of $\{(1/2)^n\}$ with $n = 0, 1, 2,\ldots$

$$S = \sum_{j=0}^{\infty} (1/2)^j = ((1/2)^\infty - 1) / (1/2 - 1)$$

$$\simeq -1 / (-1/2) = 2$$