Augmented Matrix

COMP408 - Linear Algebra
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Augmented matrix

Let's simplify the system of linear equations.

$$x_1 - 2x_2 + 3x_3 = 1$$
$$2x_1 - 3x_2 + 5x_3 = 0$$
$$-x_1 + 4x_2 - x_3 = -1$$

We use a rectangular array of numbers to represent the coefficients of each variable, and the constant term of each linear equation is represented by the numbers on the right.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 5 & 0 \\ -1 & 4 & -1 & -1 \end{array}\right]$$

Row operations

The operations we have been using to reduce a system of equations are called ***row operations***.

We summarize the three types of operation below
1. Interchange two rows
2. Multiply a row by a nonzero number
3. Add a multiple of one row to another row
(4). Add a multiple of one row to a nonzero multiple of another row

The row operations of type 1, 2, and 3 are the ***elementary row operations***. Type 4 is a combination of type 2 and 3.

Two matrices A and B are ***row equivalent*** (written A $\sim$ B) if B is obtained from A by applying one or more elementary row operations.

Row operations

Applying row operations on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 5 & 0 \\ -1 & 4 & -1 & -1 \end{array}\right]\begin{array}{l} -2 \quad 1 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 2 & 0 \end{array}\right]\begin{array}{l} \\ -2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 4 & 4 \end{array}\right]\begin{array}{l} \\ \\ 1/4 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We can then solve the system by using back substitution as before, which we have $x_1 = -4$, $x_2 = -1$ and $x_3 = 1$.

Row echelon form

A matrix is in ***row echelon form*** if (a) its nonzero rows come before its zero rows, (b) each of its pivot entries is to the right of the pivot entry in the row above (if any).

The first nonzero entry in each nonzero row is called that row's ***pivot*** entry.

$$\begin{bmatrix} 2 & -3 & 0 & 4 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(REF)} \qquad \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & -1 \\ 0 & -8 & 6 \end{bmatrix} \quad \text{(not REF)}$$

Reduced row echelon form

A matrix is in ***reduced row echelon form*** if (a) it is in row echelon form, (b) each entry above (and below) a pivot entry is 0, (c) each pivot entry is 1.

$$
\begin{bmatrix} 0 & 0 & 6 & 10 & -1 \\ 3 & 1 & -2 & -5 & -3 \\ 6 & 2 & 0 & -9 & -1 \\ -3 & -1 & 4 & 3 & 8 \end{bmatrix}
\overset{\text{Ex. 1.5.1}}{\sim}
\begin{bmatrix} 3 & 1 & -2 & -5 & -3 \\ 0 & 0 & 6 & 10 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} \\ \\ -10 \quad 5 \\ \\ \end{array}
$$

$$
\sim \begin{bmatrix} 3 & 1 & -2 & 0 & -8 \\ 0 & 0 & 6 & 0 & 9 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} 3 \\ 1 \\ \\ \\ \end{array}
$$

$$
\sim \begin{bmatrix} 9 & 3 & 0 & 0 & -15 \\ 0 & 0 & 6 & 0 & 9 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} \frac{1}{9} \\ \frac{1}{6} \\ \\ \\ \end{array}
$$

$$
\sim \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\qquad \text{(RREF)}
$$

Reduced row echelon form

Once we have the matrix in reduced row echelon form, it is easy to write down the solution.

There are again three possibilities:
1. Unique solution
2. Infinitely many solutions
3. No solution

Example (unique solution):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2$, $x_2 = -3$, and $x_3 = 5$.

Reduced row echelon form

Example (Infinitely many solutions):

$$\left[\begin{array}{cccccc|c} 1 & -5 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & -9 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \end{array}\right]$$

$x_1 = 4 + 5t - 3s$, $x_3 = 2 + 9s$, $x_5 = 8$, and $x_6 = -6$ for ***every possible choice of the numbers*** s and t.

Example (No solution):

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Not possible to find a solution such that $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$.