Basis and Dimension

COMP408 - Linear Algebra
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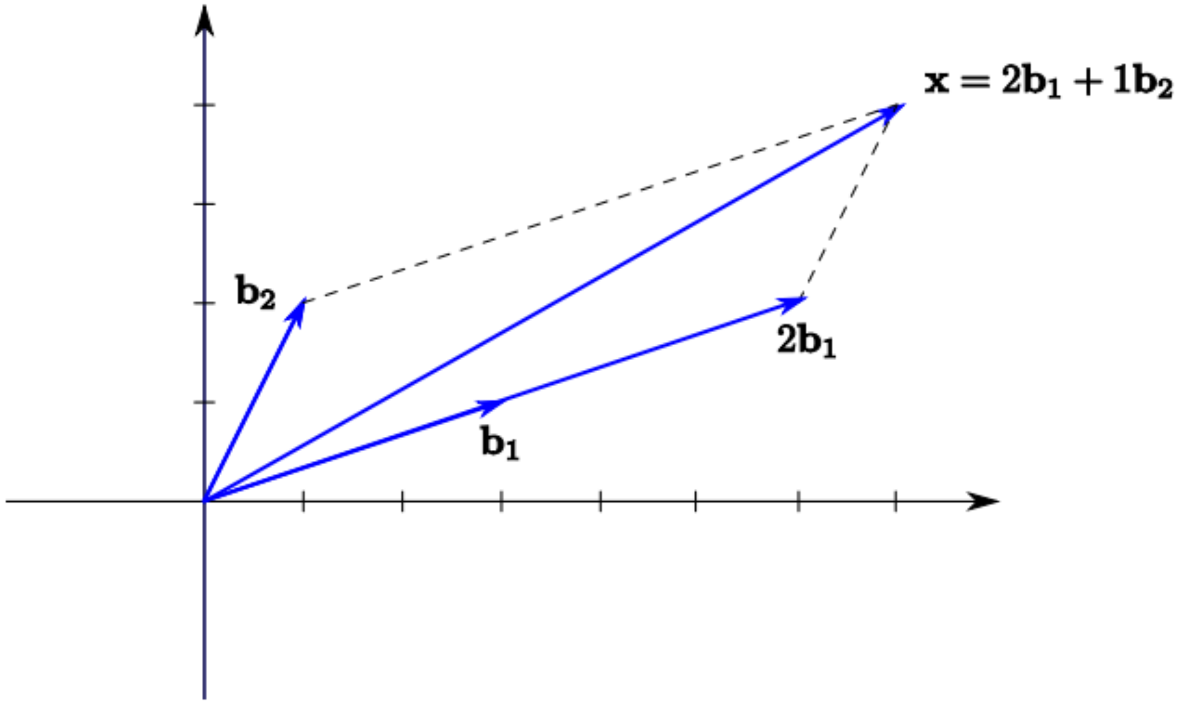
Basis

Let S be a subspace of R^n and let $b_1, b_2, \ldots, b_s$ be vectors in S. The set $\{b_1, b_2, \ldots, b_s\}$ is a ***basis*** for S if

1. $\text{Span}\{b_1, b_2, \ldots, b_s\} = S$.
2. $b_1, b_2, \ldots, b_s$ are linearly independent.



Any vector in S can be written ***uniquely*** as a linear combination of the vectors in basis.

Basis

Example: The vector $e_1 = [1, 0]$ and $e_2 = [0, 1]$ are basis of R^2.

Proof: (Span(e_1, e_2) = R^2?) we let $x = [x_1, x_2]$ be an arbitrary vector in R^2, we need to show that x = $\alpha_1 e_1 + \alpha_2 e_2$ for some scalar α_1 and α_2.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can let $\alpha_1 = x_1$ and $\alpha_2 = x_2$ and thus we have Span(e_1, e_2) = R^2.

(e_1 and e_2 linearly independent?) Neither vector is a linear combination of the other, so e_1 and e_2 are linearly independent.

Coordinate vector

Let S be a subspace of R^n, let B = (b_1, b_2, . . . , b_s) be an ordered basis for S, and let x be a vector in S. The ***coordinate vector of*** x ***relative to B*** is

$$[x]_B = [a_1, a_2, \dots, a_s],$$

where x = $a_1b_1 + a_2b_2 + \dots + a_sb_s$.

Coordinate vector

Example: Find the coordinate vector of x = [9, 8] relative to the basis (b_1, b_2) with b_1 = [3, 1], b_2 = [1, 2].

Solution: We need to write x as a linear combination of b_1 and b_2:

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2$$

$$\begin{bmatrix} 9 \\ 8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\alpha_1 + \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 3 & 1 & 9 \\ 1 & 2 & 8 \end{array}\right]_{-3} \sim \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 0 & -5 & -15 \end{array}\right]_{-\frac{1}{5}}$$

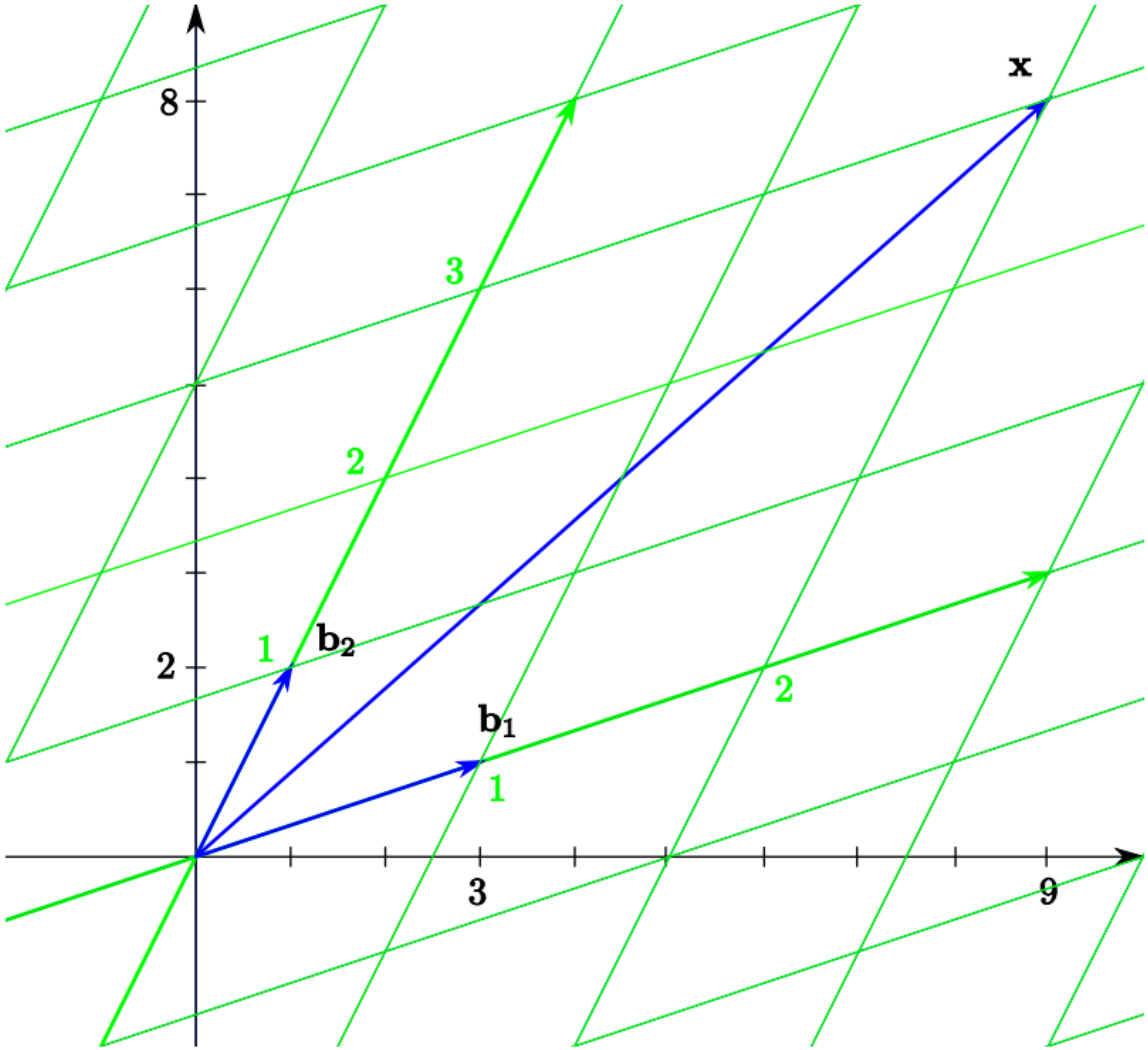
$$\sim \left[\begin{array}{cc|c} 3 & 1 & 9 \\ 0 & 1 & 3 \end{array}\right]_{-1}$$

$$\sim \left[\begin{array}{cc|c} 3 & 0 & 6 \\ 0 & 1 & 3 \end{array}\right]^{\frac{1}{3}}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

Therefore, x = $2b_1 + 3b_2$ so that $[x]_B$ = [2, 3].

Coordinate vector



Dimension

Let S be a subspace of R^n. If S has a basis consisting of s vectors, we say that S has ***dimension*** s and we write dim S = s.

Example: Find the dimension of S = Span{x_1, x_2} where x_1 = [1, 1, 0] and x_2 = [0, 1, 1].

We can prove that x_1 and x_2 are linearly independent and also thus x_1 and x_2 are basis of S. Therefore the dimension of S is 2.

Dimension

Let $y_1, y_2, \ldots, y_s$ be vectors in R^n and let S be their span. If $z_1, z_2, \ldots, z_t$ are vectors in S and $t > s$, then $z_1, z_2, \ldots, z_t$ are linearly dependent.

If $\{b_1, b_2, \ldots, b_s\}$ and $\{c_1, c_2, \ldots, c_t\}$ are both bases for a subspace S of R^n, then $s = t$.

The subspace $\{0\}$ has the empty set $\varnothing$ as basis and therefore dimension 0.