Gram-Schmidt

COMP408 - Linear Algebra

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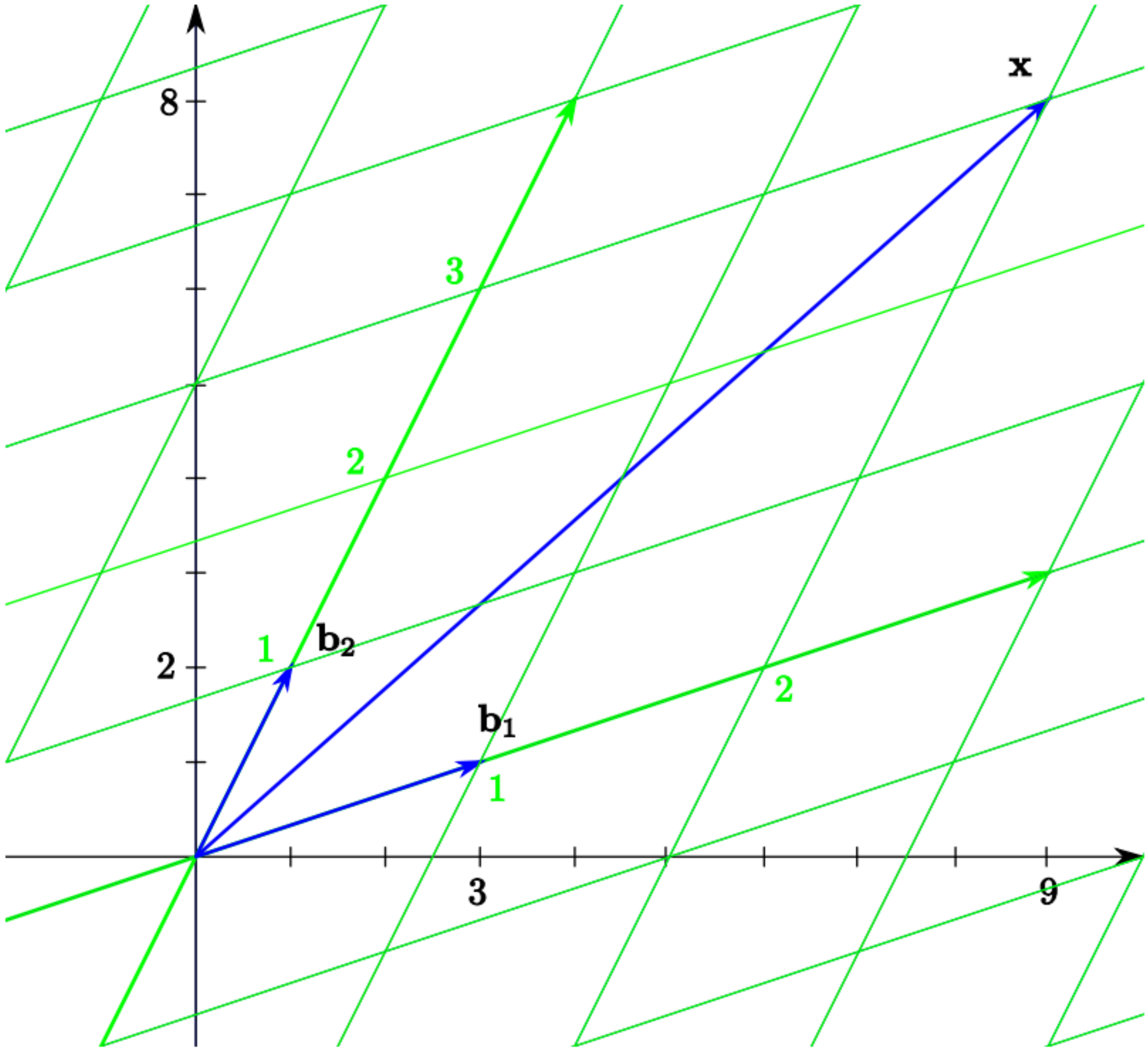
Remember last time…



Gram-Schmidt

Let S be a subspace of R^n and let b_1, b_2, … , b_s be vectors in S. The set $\{b_1, b_2, . . . , b_s\}$ is a ***basis*** for S if

1. Span$\{b_1, b_2, . . . , b_s\} = S$,
2. $b_1, b_2, . . . , b_s$ are linearly independent (***can be not*** $\perp$).

Suppose now we want to have a ***better*** basis where

1. the basis vectors are pairwise orthogonal,
2. each basis vector is a unit vector.

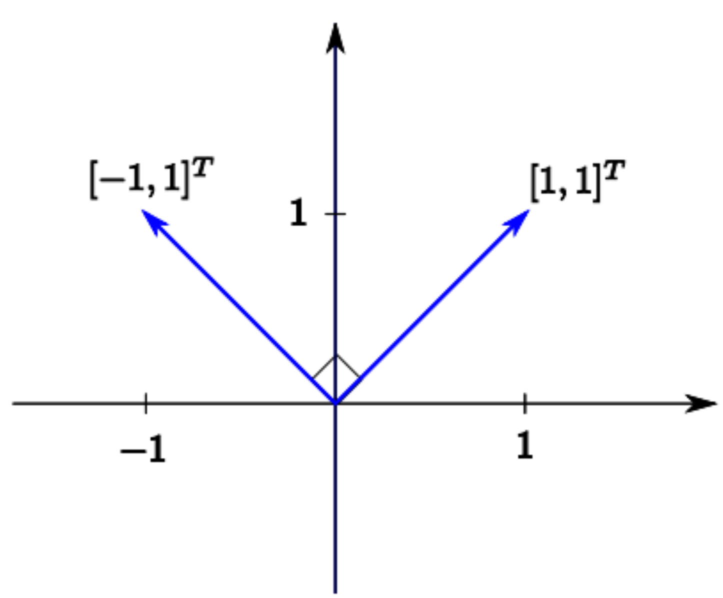
We can then use the ***Gram-Schmidt*** process to transform a basis into a basis with such properties.

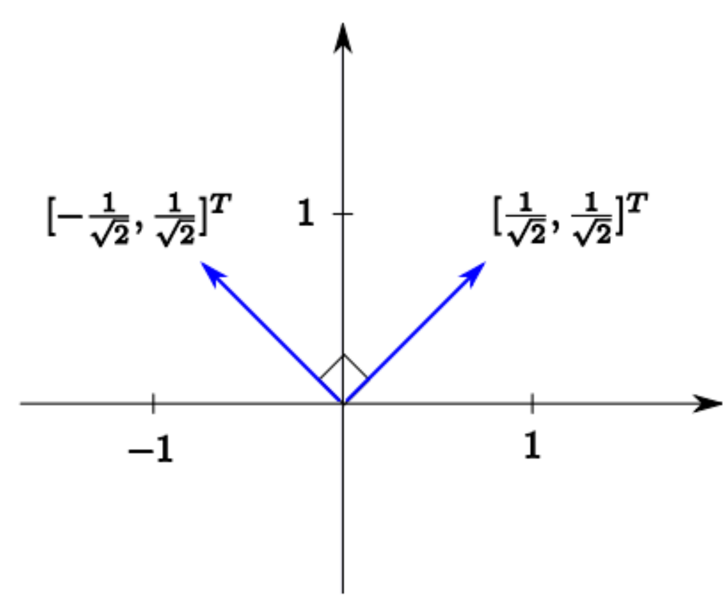
Orthogonal set

Let $\{b_1, b_2, \ldots, b_s\}$ be a set of vectors in the inner product space V. The set is ***orthogonal*** if $\{b_i, b_j\} = 0$ for all $i \neq j$ (the vectors are ***pairwise orthogonal***).

The set is ***orthonormal*** if it is orthogonal and each vector is a unit vector.

Any orthogonal set of nonzero vectors can be changed into an orthonormal set by dividing each vector by its norm.



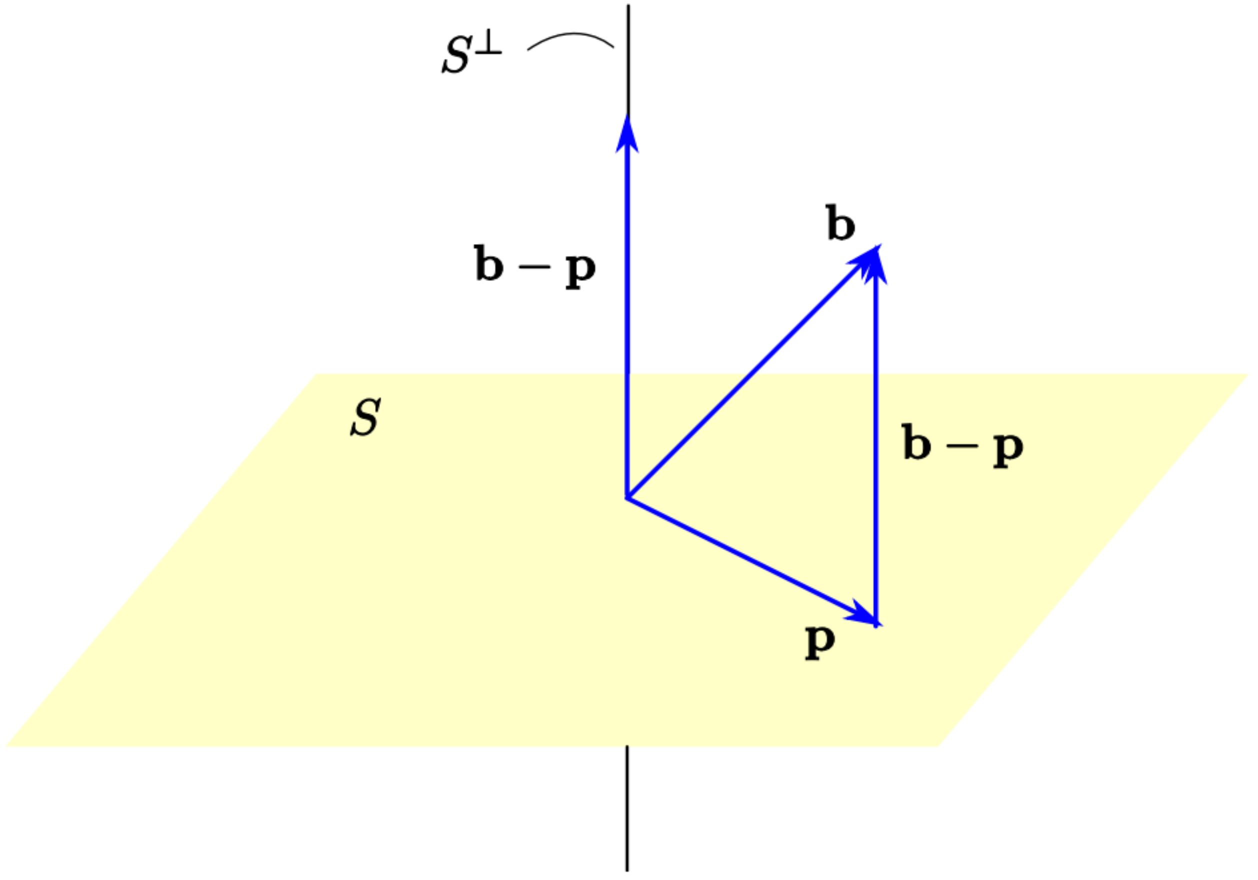


Orthogonal set

Let S be a subspace of V and let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal basis for S. Let b be a vector in V and let

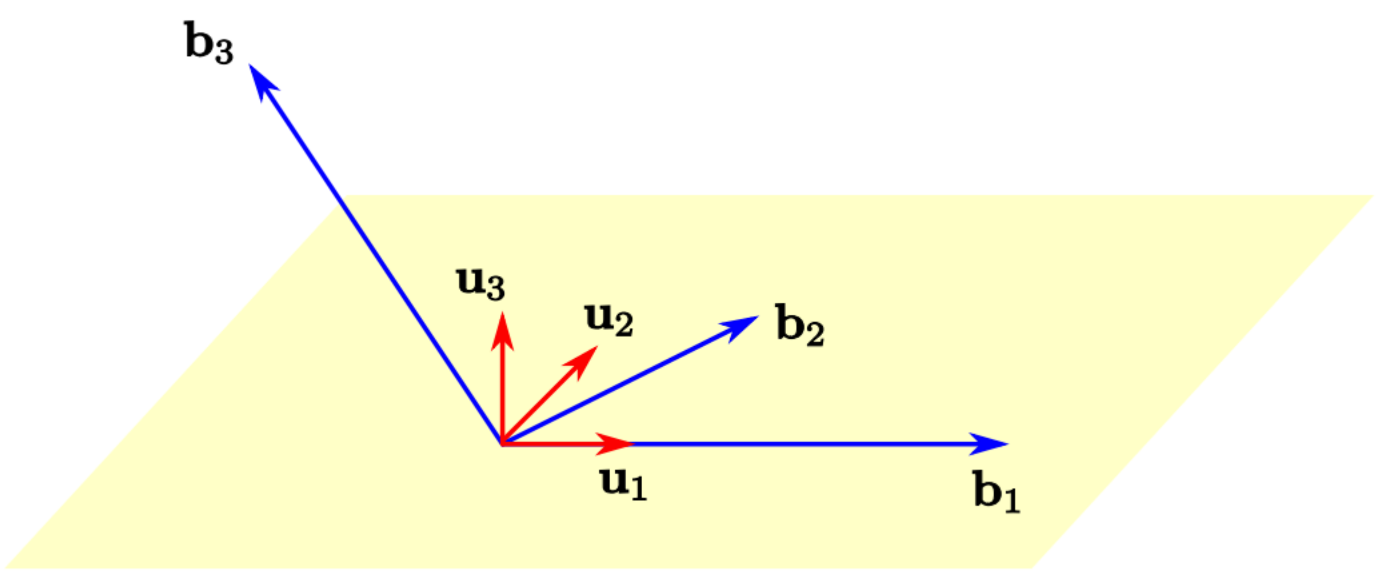
$$\mathbf{p} = \sum_{i=1}^{s} \langle \mathbf{b}, \mathbf{u}_i \rangle \mathbf{u}_i.$$

Then $p \in S$ and $b - p \in S^{\perp}$. Also, the vector p is the projection of b on S.



Gram-Schmidt

Suppose that $\{b_1, b_2, b_3\}$ is a basis for an inner product space V. The GramSchmidt process uses these vectors to produce an orthonormal basis $\{u_1, u_2, u_3\}$ for V.



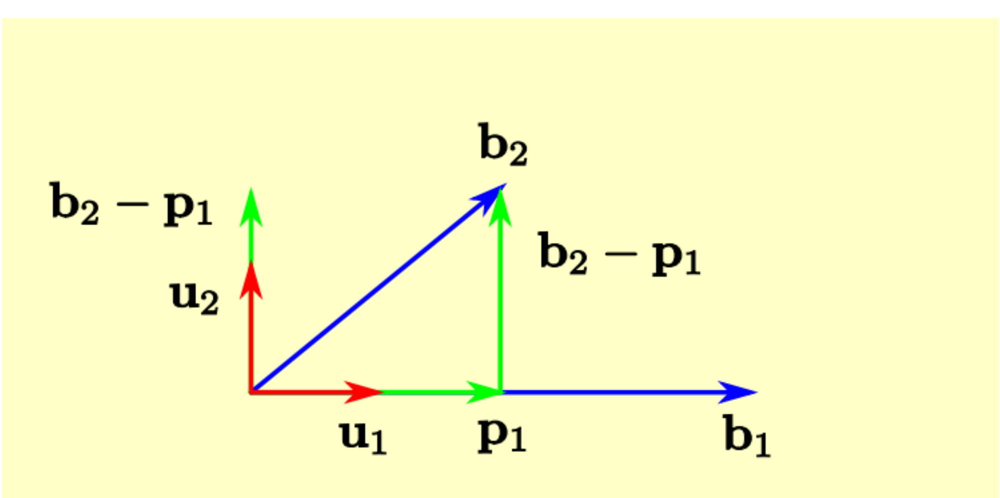
Gram-Schmidt

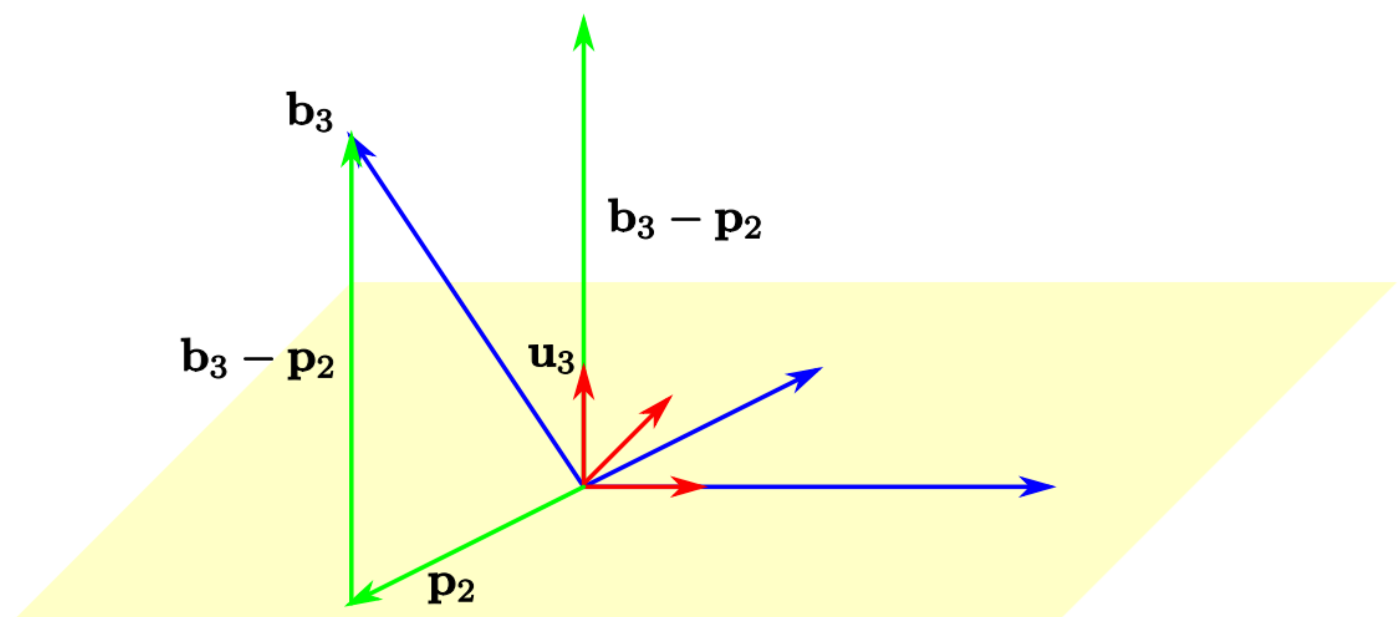
Let $\{b_1, b_2, \ldots, b_s\}$ be a basis for the inner product space V. Define vectors $u_1, u_2, \ldots, u_2$ recursively by

$$\mathbf{u}_1 = \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|}$$

$$\mathbf{u}_k = \frac{\mathbf{b}_k - \mathbf{p}_{k-1}}{\|\mathbf{b}_k - \mathbf{p}_{k-1}\|}, \quad \text{where } \mathbf{p}_{k-1} = \sum_{i=1}^{k-1} \langle \mathbf{b}_k, \mathbf{u}_i \rangle \mathbf{u}_i \quad (k > 1)$$

Then $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis for V. Moreover, $\mathrm{Span}\{u_1, u_2, \ldots, u_k\} = \mathrm{Span}\{b_1, b_2, \ldots, b_k\}$ for each k.





Gram-Schmidt

Example: Let $b_1 = [1, 2, 2, 4]^T$, $b_2 = [-2, 0, -4, 0]^T$, and $b_3 = [-1, 1, 2, 0]^T$, and let S be the span of these vectors. Apply the Gram-Schmidt process to $\{b_1, b_2, b_3\}$ to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ for S.

Solution: First we compute u_1 and p_1:

$$u_1 = b_1 / \| b_1 \| = [1, 2, 2, 4]^T / \| [1, 2, 2, 4]^T \| = 1/5[1, 2, 2, 4]^T$$

$$\begin{aligned} p_1 = \langle b_2, u_1 \rangle u_1 &= \langle [-2, 0, -4, 0]^T, 1/5[1, 2, 2, 4]^T \rangle u_1 \\ &= -2/5[1, 2, 2, 4]^T \end{aligned}$$

Gram-Schmidt

Solution (cont): Then, we compute b_2 - p_1 and u_2:

$b_2 - p_1 = [-2, 0, -4, 0]^T + 2/5[1, 2, 2, 4]^T = 4/5[-2, 1, -4, 2]^T$

$$u_2 = (b_2 - p_1)\ ||b_2 - p_1|| = 4/5[-2, 1, -4, 2]^T \,/\, ||\ 4/5[-2, 1, -4, 2]^T\ ||$$
$$= 1/5[-2, 1, -4, 2]^T$$

Finally we compute p_2, b_3 - p_2, and u_3:

$$p_2 = <b_2, u_1>\ u_1 + <b_3, u_2>\ u_2$$
$$= <[–1, 1, 2, 0]^T, 1/5[1, 2, 2, 4]^T>u_1 + <[–1, 1, 2, 0]^T, 1/5[-2, 1, -4, 2]^T>u_2$$
$$= 1/5[3, 1, 6, 2]^T$$

$b_3 - p_2 = [-1, 1, 2, 0]^T - 1/5[3, 1, 6, 2]^T = 2/5[-4, 2, 2, -2]^T$

$$u_3 = (b_3 - p_2)\ ||b_3 - p_2|| = 2/5[-4, 2, 2, -1]^T \,/\, ||\ 2/5[-4, 2, 2, -1]^T\ ||$$
$$= 1/5[-4, 2, 2, -1]^T$$