Matrix

COMP408 - Linear Algebra

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Matrix

Let m and n be positive integers. An $m \times n$ ***matrix*** is a rectangular array of numbers having m rows and n columns. Such a matrix is said to have ***size*** $m \times n$.

A ***row matrix*** (or ***row***) is a $1 \times n$ matrix, and a ***column matrix*** (or ***column***) is an $m \times 1$ matrix.

A ***square matrix*** is an $n \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

Matrix

The ***(i, j)-entry of a matrix*** is the entry in row *i* and column *j*. For a matrix *A*, the (i, j)-entry of *A* is often written as a_{ij}.

$$\mathbf{A}_{\mathrm{mxn}} = \quad {}_{\mathrm{m}}\mathbf{A}^{\mathrm{n}} \quad \begin{bmatrix} a_{11} & a_{12}\cdots & a_{ij} & a_{in} \\ a_{21} & a_{22}\cdots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

Matrix is a ***vector space*** (How?), and thus most of the properties related to vector can be applied to matrix.

Addition

Two matrices of the same size are ***added*** by adding their corresponding entries.

Addition of two matrices that are not of the same size is undefined.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 6 \\ 4 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 9 \\ 0 & 1 & 6 \end{bmatrix}.$$

Scalar multiplication

A matrix is ***multiplied by a scalar*** (i.e., number) by multiplying each entry of the matrix by the scalar.

Example:

$$3\begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ -12 & 0 & 27 \end{bmatrix}.$$

Negative of a Matrix: for an $m \times n$ matrix A, its negative is denoted –A and –A = (–1)A.

Some properties of matrix

Two matrices are ***equal*** if and only if they have the same size and the corresponding entries are equal.

Zero Matrix: an $m \times n$ matrix with all entries equal to zero.

Subtraction: for $m \times n$ matrices A and B, $A - B = A + (-1)B$

Matrix-vector multiplication

Let $A = [a_1, a_2, \ldots, a_n]$ be an $m \times n$ matrix, written in terms of its columns $a_1, a_2, \ldots, a_n$. If $x = [x_1, x_2, \ldots, x_n]$ is any n-vector, the ***product*** Ax is defined to be the m-vector given by:

$$Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n.$$

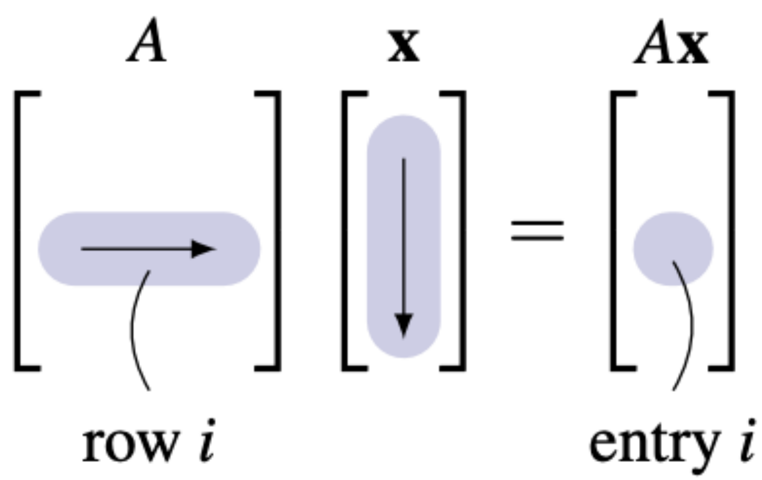
System of linear equations can be considered as a matrix-vector multiplication.

Example: (the middle steps are skipped)

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} (1)(5) + (-2)(6) \\ (-3)(5) + (4)(6) \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}.$$

Dot product

Let A be an $m \times n$ matrix and let x be an n-vector. Then each entry of the vector Ax is the ***dot product*** of the corresponding row of A with x.



The product is defined only if the number of columns of the first matrix is the same as the number of rows of the second matrix.

Example:

$$\underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 6 & 3 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ -2 & 5 & 0 & 2 \end{bmatrix}}_{3 \times 4} = \underbrace{\begin{bmatrix} 5 & 27 & -2 & 12 \\ -1 & 6 & 0 & 6 \end{bmatrix}}_{2 \times 4}$$

Identity matrix

The ***identity matrix*** $I_{n \times n}$ of size $n \times n$ is given by

$$\mathbf{I}_{1\times 1} = \begin{bmatrix} 1 \end{bmatrix}, \quad \mathbf{I}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and so forth.

If A is an $n \times n$ matrix and I is the identity matrix of the same size, then $IA = A$ and $AI = A$, so I is a ***multiplicative identity*** (it acts like the number 1).

Example:

$$\mathbf{IA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \mathbf{A}$$

Transpose

The ***transpose*** of an $m \times n$ matrix A (written A^{T}) is the $n \times m$ matrix obtained by writing the rows of A as columns.

The transpose of a matrix A can be visualized as the reflection of A through the 45 degree line starting from the first entry of the matrix and sloping downward to the right.

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$