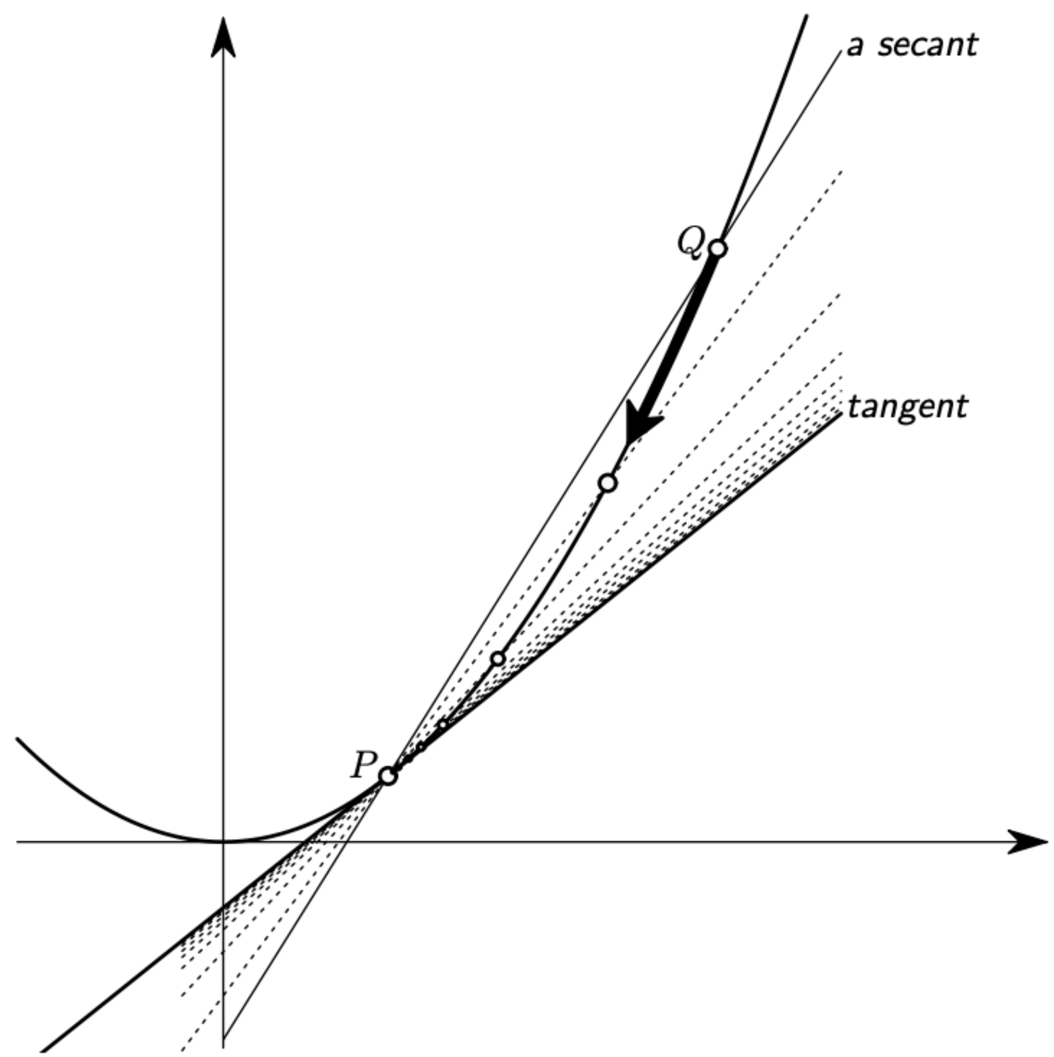
Limits

COMP406 - Calculus
Dennis Wong

Tangent lines

A ***tangent line*** to the function $f(x)$ at the point $x = a$ is a line that just touches the graph of the function at the point in question and is parallel to the graph at that point.



Tangent lines

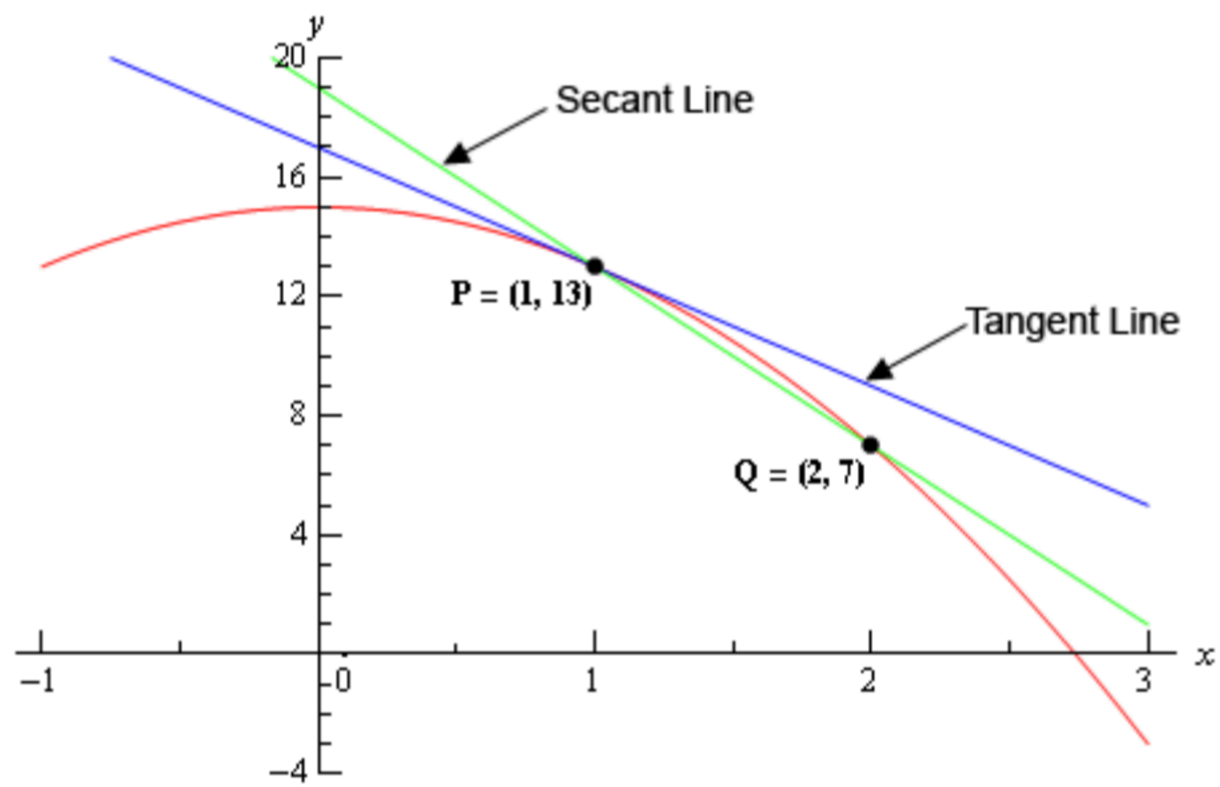
A tangent line for a function at a point x corresponds to the ***rate of change*** of $f(x)$ in terms of x, that is the ***slope*** of the function at the point x.

The tangent line of a linear function is the the slope m in its slope-intercept form $y = mx + c$.

What about the rate of change if the function is non-linear?

Tangent lines

Find the tangent line to $f(x) = 15 - 2x^2$ at x = 1.



The tangent line and the graph of the function must touch at $x = 1$ so the point $(1, f(1)) = (1, 13)$ must be on the line.

Tangent lines

Now to estimate of the slope of the tangent line we can use the slope of the secant line, let's call it m_{PQ} with x another point on the curve

$m_{PQ} = \Delta y / \Delta x = (f(x) - f(1)) / (x - 1)$

x	m_{PQ}	x	m_{PQ}
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

It appears that the slope of the secant lines appears to be approaching -4. In symbol, we write $\lim_{P \to Q} m_{PQ} = -4$

By substituting the point (1, 13) into the slope-intercept form $y = mx + c$. The equation of the tangent line is $y = -4x + 17$.

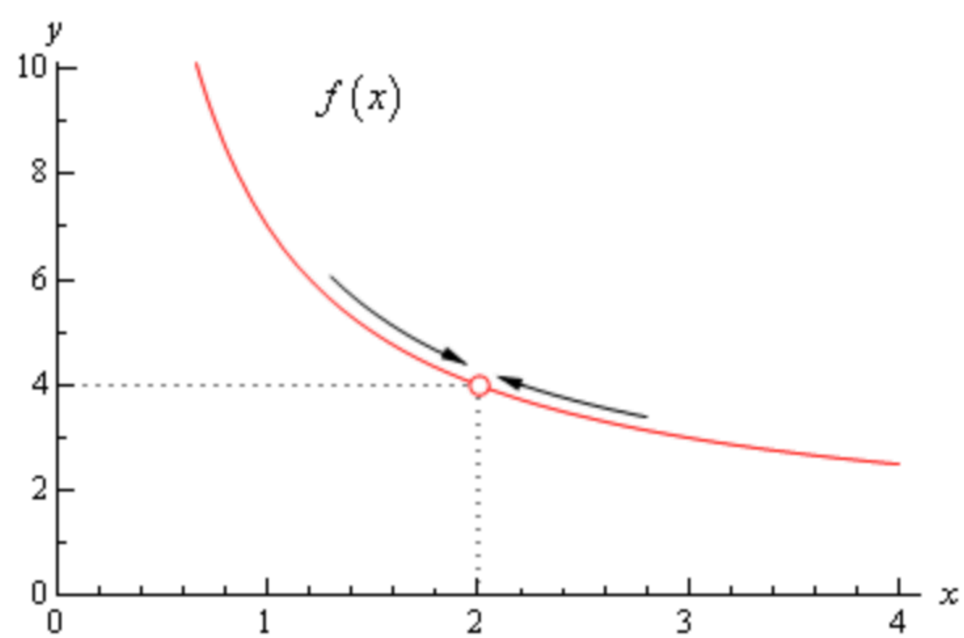
Limit

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x \to a} f(x) = L$$

if (a) $f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a, and (b) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of f one has

$$|x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$



Application - Instantaneous velocity

Average velocity is defined as total distance traveled divided by total travel time. That is

$$\frac{s(t+\Delta t)-s(t)}{\Delta t} \text{ miles per hour.}$$

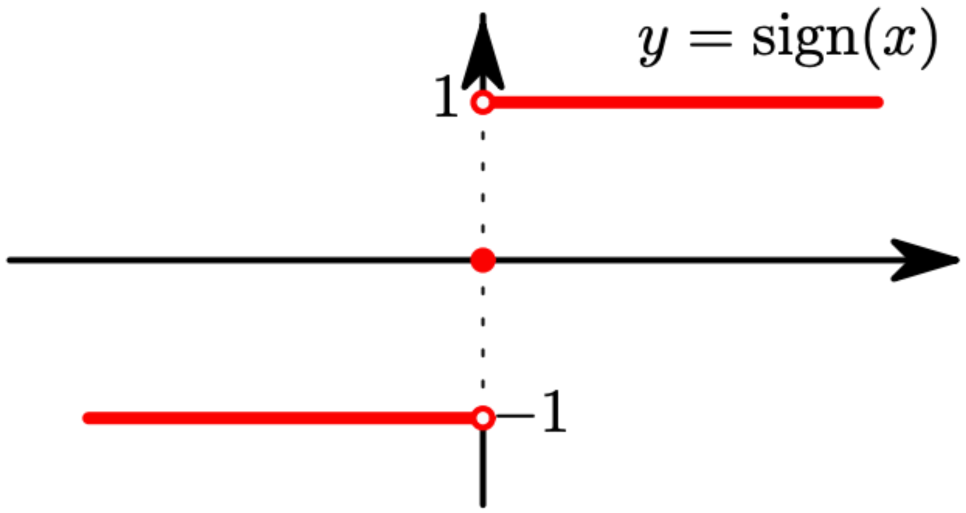
However, this is not the number the speedometer provides you. The speedometer in your car provides your velocity ***at the moment***, that is ***instantaneous velocity.*** That is

$$v(t)=\lim_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t}.$$

When limits fail to exist

Consider the following sign function. What is the limit of the function when *x* approaches 0?

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$



There is no limit when *x* approaches 0.

One-side limit

We say the ***right-hand limit*** of the function *f* on *a*, denoted by

$$\lim_{x \to a^+} f(x) = L,$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a with $\boldsymbol{x > a}$ without actually letting x be a.

We say the ***left-hand limit*** of the function *f* on *a*, denoted by

$$\lim_{x \to a^-} f(x) = L,$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a with $\boldsymbol{x < a}$ without actually letting x be a.

Example: The right-hand limit of the sign function is 1 when x approaches 0, and the left-hand limit of the sign function is -1 when x approaches 0.

Infinite limit

We say $\lim_{x \to \infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive. There is a similar definition by taking x large enough and negative.

We say $\lim_{x \to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large and positive taking x sufficiently close to a without letting x equal to a. There is a similar definition by taking x large enough and negative.

The function $f(x)$ will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x \to a^-} f(x) = \pm\infty; \lim_{x \to a^+} f(x) = \pm\infty; \lim_{x \to a} f(x) = \pm\infty.$$

Some properties of limits

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[cf(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

And below some properties for limit that involves infinity.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

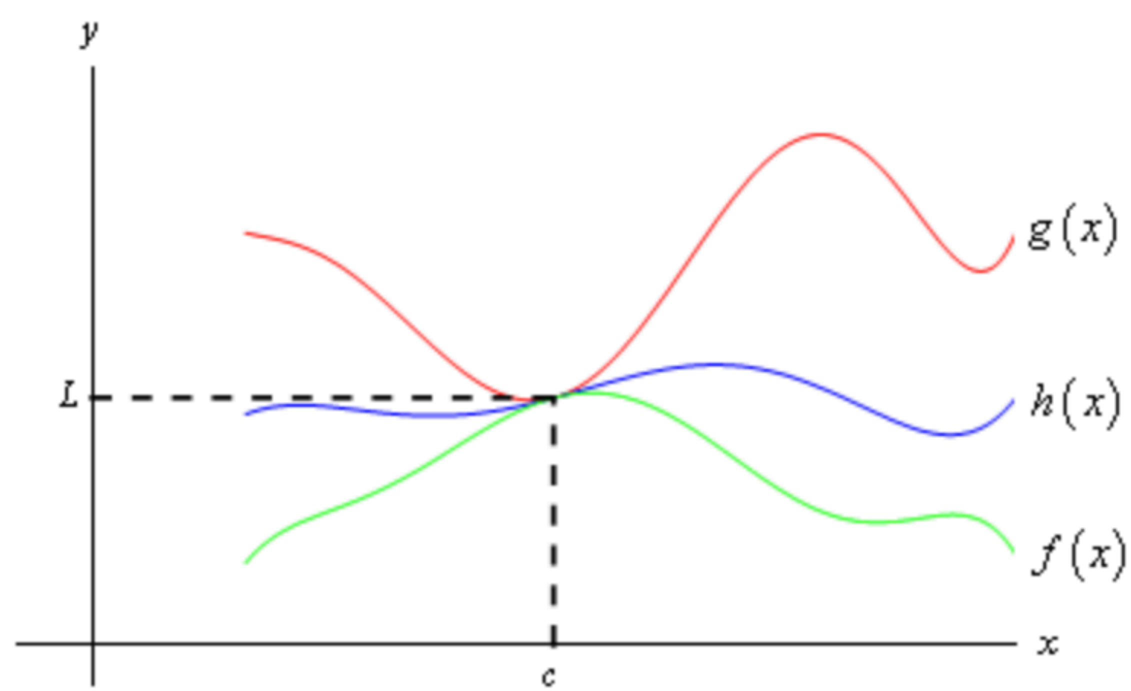
Sandwich theorem

Let f and g be functions whose limits for $x \to a$ exist, and assume that $f(x) \leq g(x)$ holds for all x. Then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Sandwich theorem: Suppose that $f(x) \leq g(x) \leq h(x)$ (for all x) and

$$\lim_{x\to a} \mathrm{f}(\mathrm{x}) = \lim_{x\to a} \mathrm{h}(\mathrm{x}).$$

Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$.



Continuity

A function g is ***continuous*** at a if

$$\lim_{x \to a} g(x) = g(a),$$

A function is continuous if it is continuous at every a in its domain.

(Immediate value theorem): Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.