Span and Subspace

COMP408 - Linear Algebra
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Linear combination

Let $x_1, x_2, \dots, x_s$ be vectors in $\mathbf{R}^n$. A ***linear combination*** of $x_1, x_2, \dots, x_s$ is an expression of the form

$$a_1x_1 + a_2x_2 + \dots + a_sx_s,$$

where $a_1, a_2, \dots, a_s \in \mathbf{R}$.

Example: Let $x_1 = [2, -1, 3]^T$ and let $x_2 = [4, 2, 1]^T$, then $[22, 5, 13]^T$ is a linear combination of x_1 and x_2.

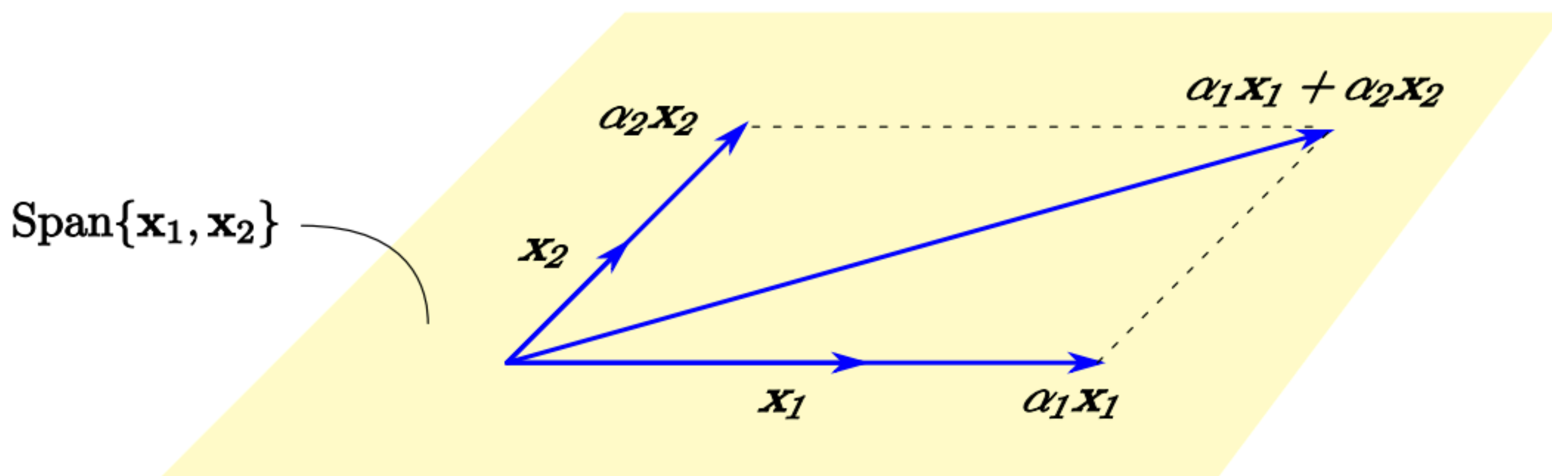
$$3\mathbf{x}_1 + 4\mathbf{x}_2 = 3\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \\ 13 \end{bmatrix}$$

Span

Let $\{x_1, x_2, \ldots, x_s\}$ be a set of vectors in $\mathbf{R}^n$. The ***span*** of $\{x_1, x_2, \ldots, x_s\}$, denoted by $\mathrm{Span}\{x_1, x_2, \ldots, x_s\}$, is the set of all linear combinations of $x_1, x_2, \ldots, x_s$:

$$\mathrm{Span}\{x_1, x_2, \ldots, x_s\} = \{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_s x_s \mid \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbf{R}\}.$$

If x_1 and x_2 are not parallel, then one can show that $\mathrm{Span}\{x_1, x_2\}$ is the ***plane*** determined by x_1 and x_2.



Span

We can use system of linear equations to determine if a vector is in a span or not.

Example: Determine whether $[2, -5, 8]^T$ is in Span$\{x_1, x_2\}$.

$$\begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 4\alpha_2 \\ -\alpha_1 + 2\alpha_2 \\ 3\alpha_1 + \alpha_2 \end{bmatrix}.$$

Equating components leads to the following augmented matrix:

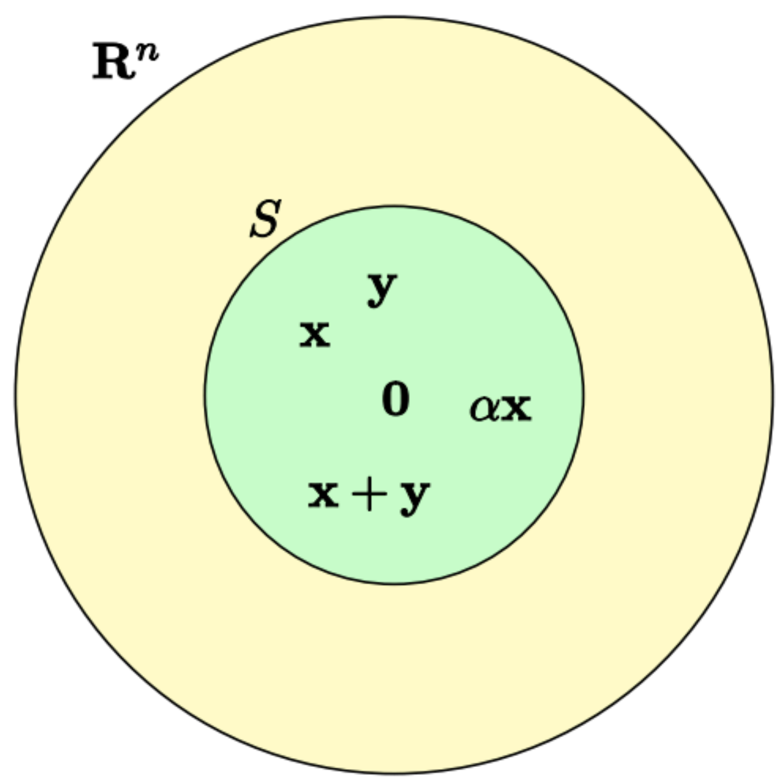
$$\begin{array}{c} \left[\begin{array}{cc|c} 2 & 4 & 2 \\ -1 & 2 & -5 \\ 3 & 1 & 8 \end{array}\right] \end{array} \begin{array}{cc} 1 & -3 \\ 2 & \\ & 2 \end{array} \sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 8 & -8 \\ 0 & -10 & 10 \end{array}\right] \begin{array}{c} \\ \frac{1}{8} \\ \frac{1}{10} \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Subspace

A subset S of $\mathbf{R}^n$ is called a ***subspace*** if
(a) $0 \in S$ (***origin***),
(b) x, y $\in S$ implies x + y $\in S$ (***addition***), and
(c) x $\in S$, $\alpha \in \mathbf{R}$ implies $\alpha x \in S$ (***scalar multiplication***).



Subspace

Example: Let S be the subset of $\mathbf{R}^2$ given by

$$S = \{ \begin{bmatrix} 2t \\ -t \end{bmatrix} \mid t \in \mathbf{R} \}.$$

Show that S is a subspace of $\mathbf{R}^2$.

Solution: First we have $0 = [0, 0]^T$ and thus it contains the origin.

Secondly, let $x, y \in S$ with $x = (2t, -t)$ and $y = (2s, -s)$ for some $t, s \in R$, we have

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 2t \\ -t \end{bmatrix} + \begin{bmatrix} 2s \\ -s \end{bmatrix} = \begin{bmatrix} 2t + 2s \\ -t - s \end{bmatrix} = \begin{bmatrix} 2(t + s) \\ -(t + s) \end{bmatrix} \in S.$$

Finally, let $x = (2t, -t)$ and $\alpha \in \mathbf{R}$, we have

$$\alpha \mathbf{x} = \alpha \begin{bmatrix} 2t \\ -t \end{bmatrix} = \begin{bmatrix} 2(\alpha t) \\ -(\alpha t) \end{bmatrix} \in S.$$

Subspace

If x_1, x_2, ... , x_s are vectors in $\mathbf{R}^n$ and S is their span, then S is a subspace of $\mathbf{R}^n$.

The subspaces of $\mathbf{R}^2$ are (a). {0}, (b). ***lines through origin***, and (c). $\mathbf{R}^2$.

The subspaces of $\mathbf{R}^3$ are (a). {0}, (b). ***lines through origin***, (c). ***planes through origin***, and (d). $\mathbf{R}^3$.

If L is a linear function on $\mathbf{R}^n$, then L(x) is a subspace of $\mathbf{R}^n$ if $x \in \mathbf{R}^n$.