Graph Sketching

COMP406 - Calculus

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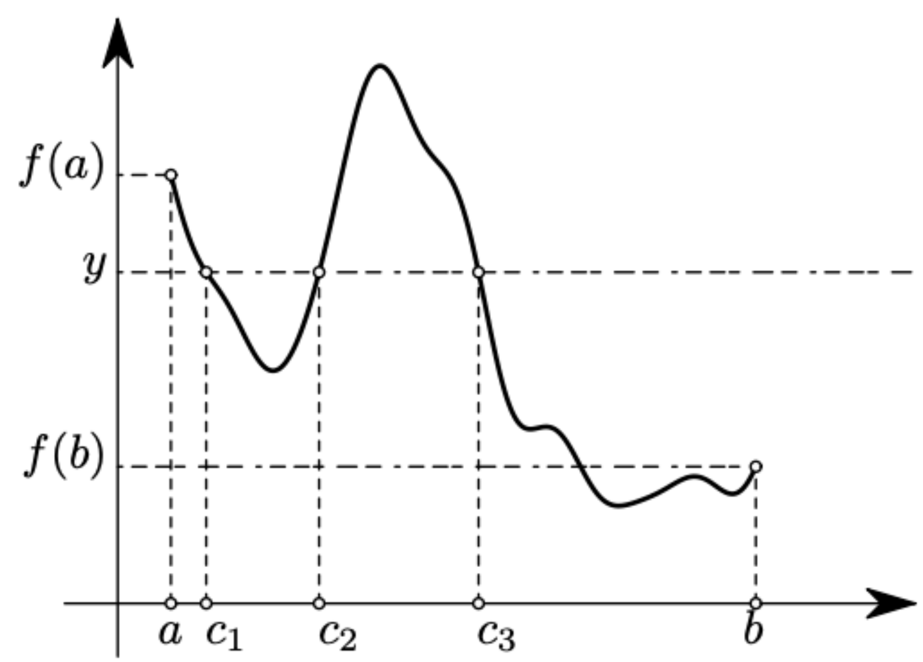
Intermediate value theorem

Intermediate value theorem: If f is a continuous function on an interval $a \leq x \leq b$, and if y is some number between $f(a)$ and $f(b)$, then there is a number c with $a \leq c \leq b$ such that $f(c) = y$.



If f is continuous function on some interval $a < x < b$, and if $f(x) \neq 0$ for all x in this interval, then $f(x)$ is either positive for all $a < x < b$ or else it is negative for all $a < x < b$.

Increasing and decreasing functions

A function is called ***increasing*** if $a < b$ implies $f(a) < f(b)$ for all numbers a and b in the domain of f.

A function is called ***decreasing*** if $a < b$ implies $f(a) > f(b)$ for all numbers a and b in the domain of f.

The function f is called ***non-decreasing*** if $a < b$ implies $f(a) \leq f(b)$ for all numbers a and b in the domain of f.

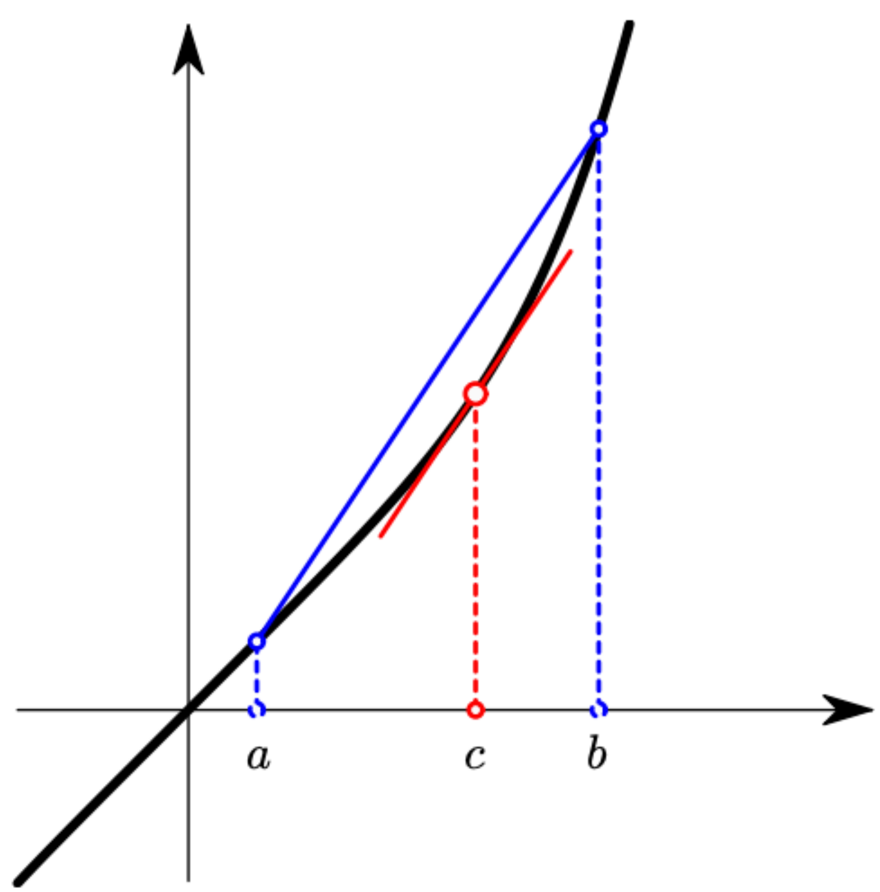
The function f is called ***non-increasing*** if $a < b$ implies $f(a) \geq f(b)$ for all numbers a and b in the domain of f.

Suppose f is a differentiable function on an interval (a, b). If $f'(x) > 0$ for all $a < x < b$, then f is increasing. If $f'(x) < 0$ for all $a < x < b$, then f is decreasing.

The mean value theorem

The Mean Value Theorem: If *f* is a differentiable function on the interval $a \leq x \leq b$, then there is some number *c*, with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Maxima and Minima

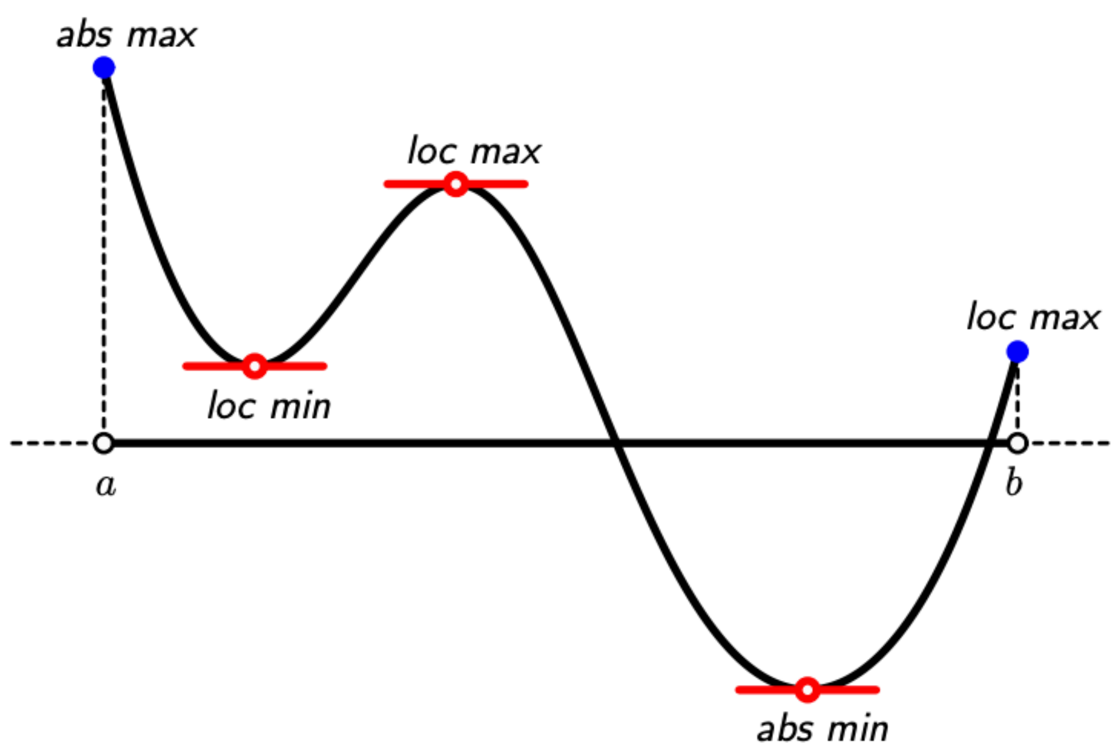
A function has a ***global maximum*** (***absolute maxima***) at some a in its domain if $f(x) \leq f(a)$ for all other x in the domain of f.

A function has a ***local maximum*** at some a in its domain if there is a small $\delta > 0$ such that $f(x) \leq f(a)$ for all x with $a - \delta < x < a + \delta$ which lie in the domain of f.

Any x value for which $f'(x) = 0$ is called a ***stationary point*** (***critical point***) for the function f.

Maxima and Minima

Suppose *f* is a differentiable function on some interval [*a*, *b*]. Every local maximum or minimum of *f* is either one of the end points of the interval [a, b], or else it is a stationary point for the function *f*.



Stationary point

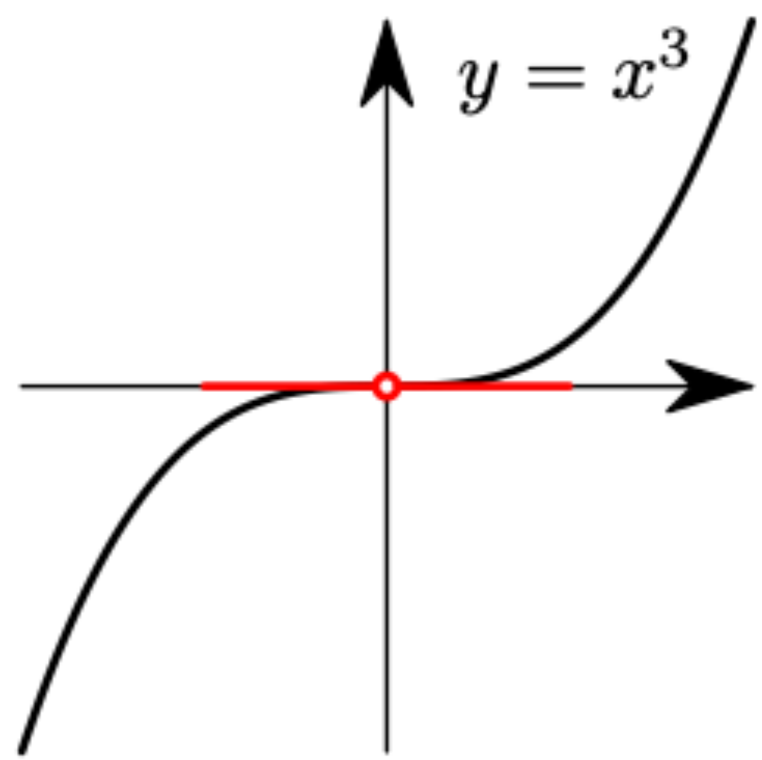
If $f'(c) = 0$ then c is a stationary point, and it might be local maximum or a local minimum. You can tell what kind of stationary point c is by looking at the signs of $f'(x)$ for x near c.

If in some small interval $(c - \delta, c + \delta)$ you have $f'(c) < 0$ for $x < c$ and $f'(c) > 0$ for $x > c$ then f has a local minimum at $x = c$.

If in some small interval $(c - \delta, c + \delta)$ you have $f'(c) > 0$ for x < c and $f'(c) < 0$ for x > c then f has a local maximum at x = c.

Stationary point

Note: there are cases where a stationary point is neither a maximum nor a minimum.



Graph sketching

Given a differentiable function *f* defined on some interval $a \leq x \leq b$, you can find the increasing and decreasing parts of the graph, as well as all the local maxima and minima by following this procedure:

1. find all solutions of $f'(x) = 0$ in the interval [a, b],
2. find the sign of $f'(x)$ at all other points,
3. Compute the function value $f(x)$ at each stationary point,
4. compute the function values at the endpoints of the interval, i.e. compute f(a) and f(b).
5. the absolute maximum is attained at the stationary point or the boundary point with the highest function value; the absolute minimum occurs at the boundary or stationary point with the smallest function value.

If the interval is unbounded, compute the limit of $f(x)$ goes to positive and negative infinity.

Graph sketching

Example: Let's sketch the graph for the function

$$f(x) = \frac{x(1-x)}{1+x^2}.$$

Solution: First compute the derivative of *f*:

$$f'(x) = \frac{1-2x-x^2}{\left(1+x^2\right)^2}.$$

Hence $f'(x) = 0$ holds if and only if $1 - 2x - x^2 = 0$. The roots for the derivative are $A = -1 - \sqrt{2}$ and $B = -1 - \sqrt{2}$ (stationary points).

Graph sketching

The denominator is always positive, and the numerator is

$$-x^2 - 2x + 1 = -(x^2 + 2x - 1) = -(x - A)(x - B).$$

Therefore we have

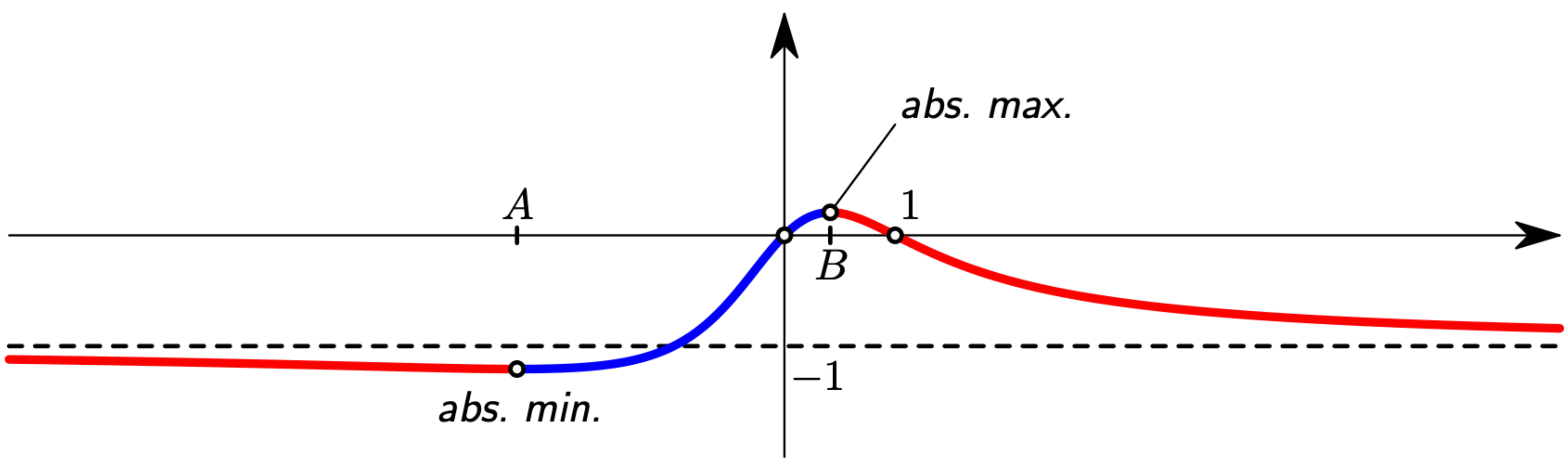
$$f'(x) \begin{cases} < 0 & \text{for } x < A \\ > 0 & \text{for } A < x < B \\ < 0 & \text{for } x > B \end{cases}$$

Therefore A is a local minimum, and B is a local maximum.

Since we are dealing with an unbounded interval we must compute the limits of f(x) as x → ±∞. We have

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = -1.$$

Graph sketching



Second Derivative

A function f is ***convex*** on some interval $a < x < b$ if the line segment connecting any pair of points on the graph lies above the piece of the graph between those two points.

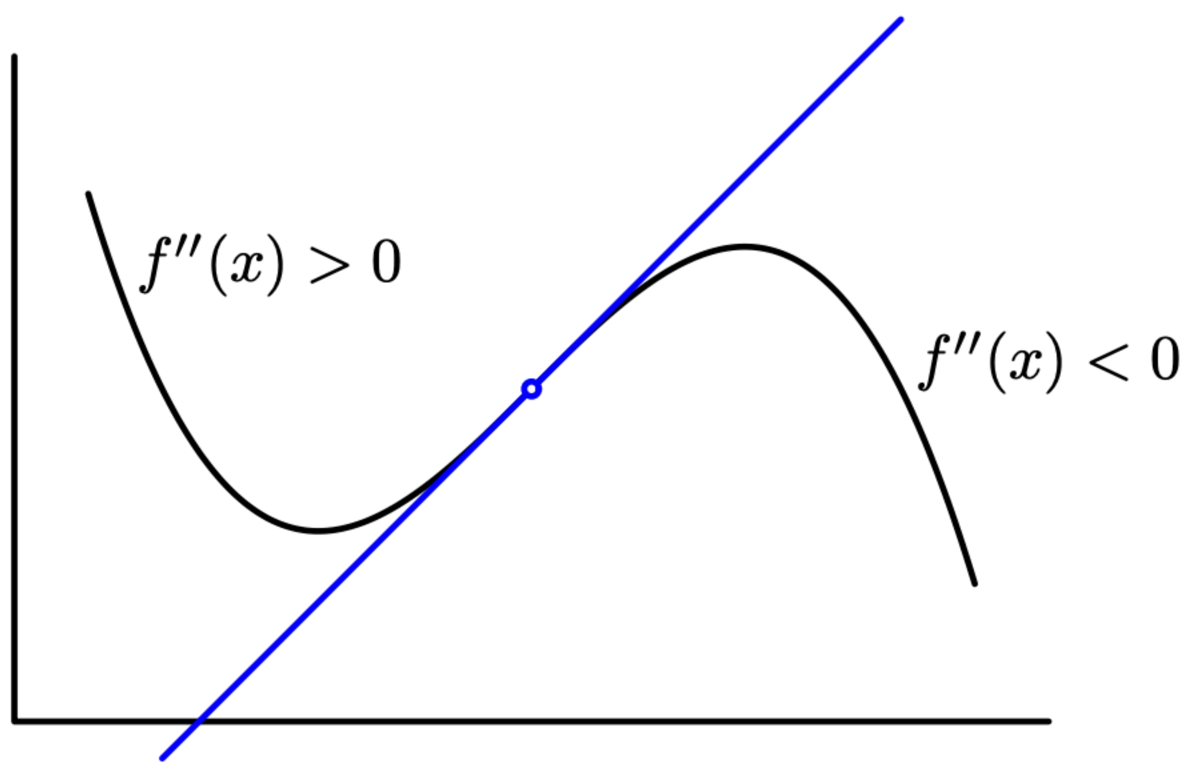
A function f is called ***concave*** if the line segment connecting any pair of points on the graph lies below the piece of the graph between those two points.

A point on the graph of f where $f''(x)$ changes sign is called an ***inflection point***.

Second Derivative

A function *f* is convex on some interval $a < x < b$ if and only if $f''(x) > 0$ for all *x* on that interval.

A function *f* is concave on some interval $a < x < b$ if and only if $f''(x) < 0$ for all *x* on that interval.



Second Derivative

If *c* is a stationary point for a function *f*, and if $f''(c) < 0$ then *f* has a local maximum at *c*.

If *c* is a stationary point for a function *f*, and if $f''(c) > 0$ then f has a local minimum at c.

In case $f''(c) = 0$, we have to go back and check the signs near the stationary point (see below).