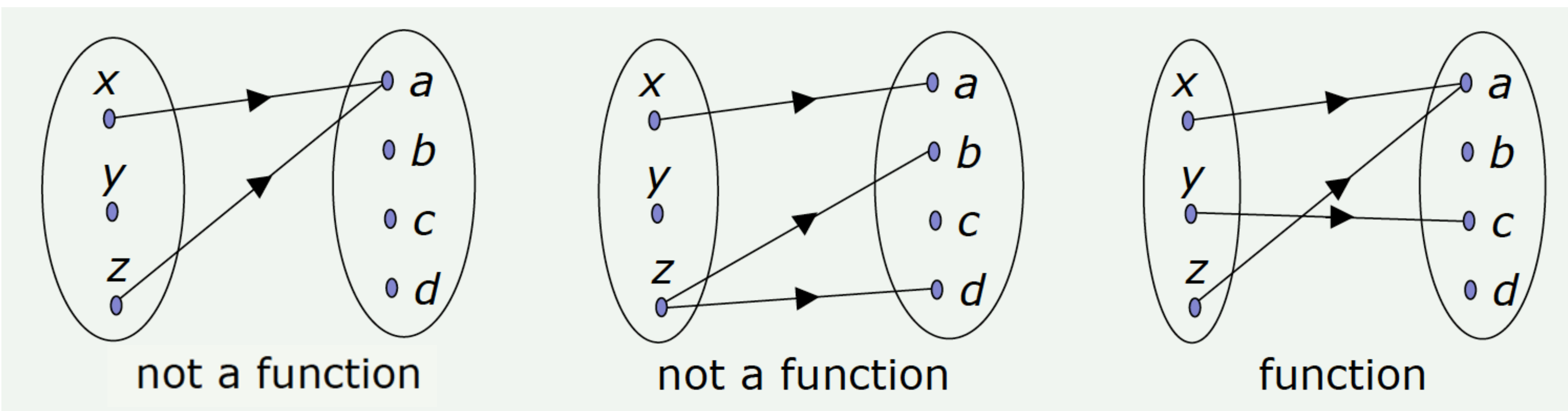
Matrix Transformation

COMP408 - Linear Algebra
Dennis Wong

Functions

A ***function*** *f* is a mapping between 2 sets A and B, denoted by *f: A -> B*, such that each $a \in A$ maps to exactly one element in B.

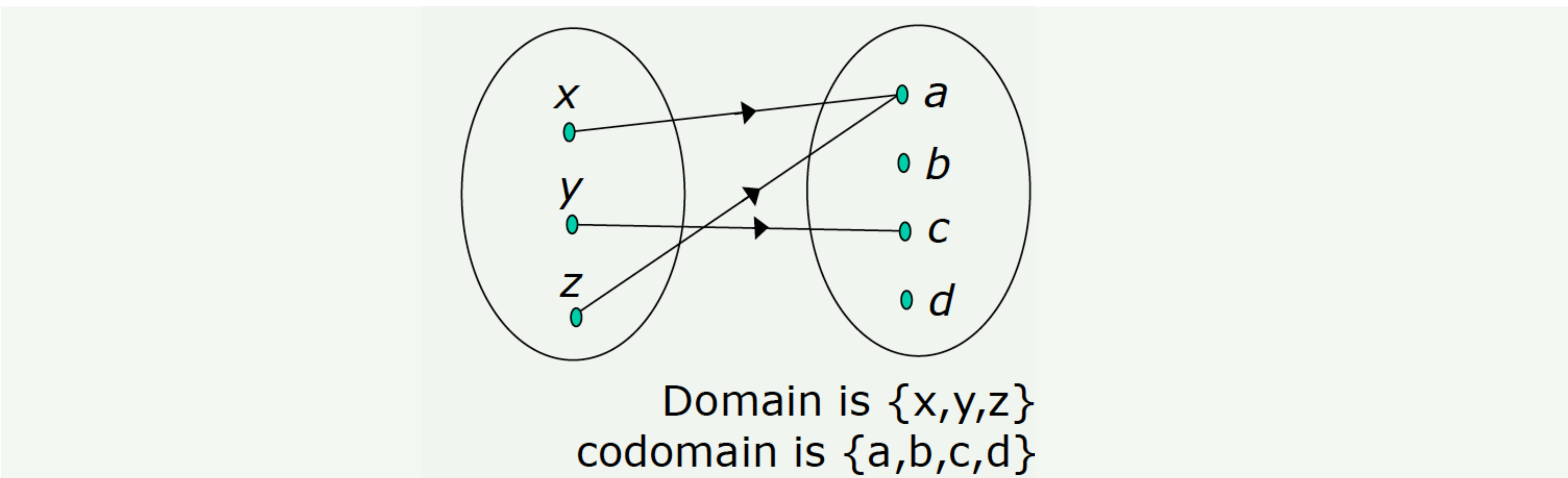


We write $f(a) = b$ if the function f maps the element $a \in A$ to the element $b \in B$.

Domain and codomain

Let *f* be a function from the sets A to B.

Then we say that A is the ***domain*** of the function *f* and B is the ***codomain*** of the function *f*.

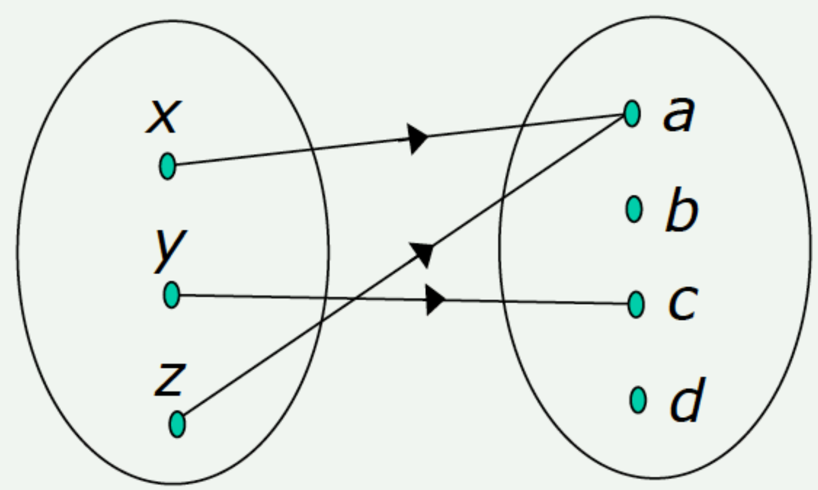


We also say *b* is an ***image*** of *a* (or *a* is a ***preimage*** of *b*) when *f(a)* = *b*.

Range

Let *f* be a function from the sets A to B.

The ***range*** of *f* is the subset of B defined as follows: $b \in B$ belongs to the range if and only if it has a preimage under *f.*



Range is {a,c}

Example

Consider the function f: $\mathbf{R}^+ \to \mathbf{R}$

$$x \to 2 - \sqrt{x}$$

Domain is $\mathbf{R}^+$ and codomain is $\mathbf{R}$.
Range is $]-\infty, 2[$.

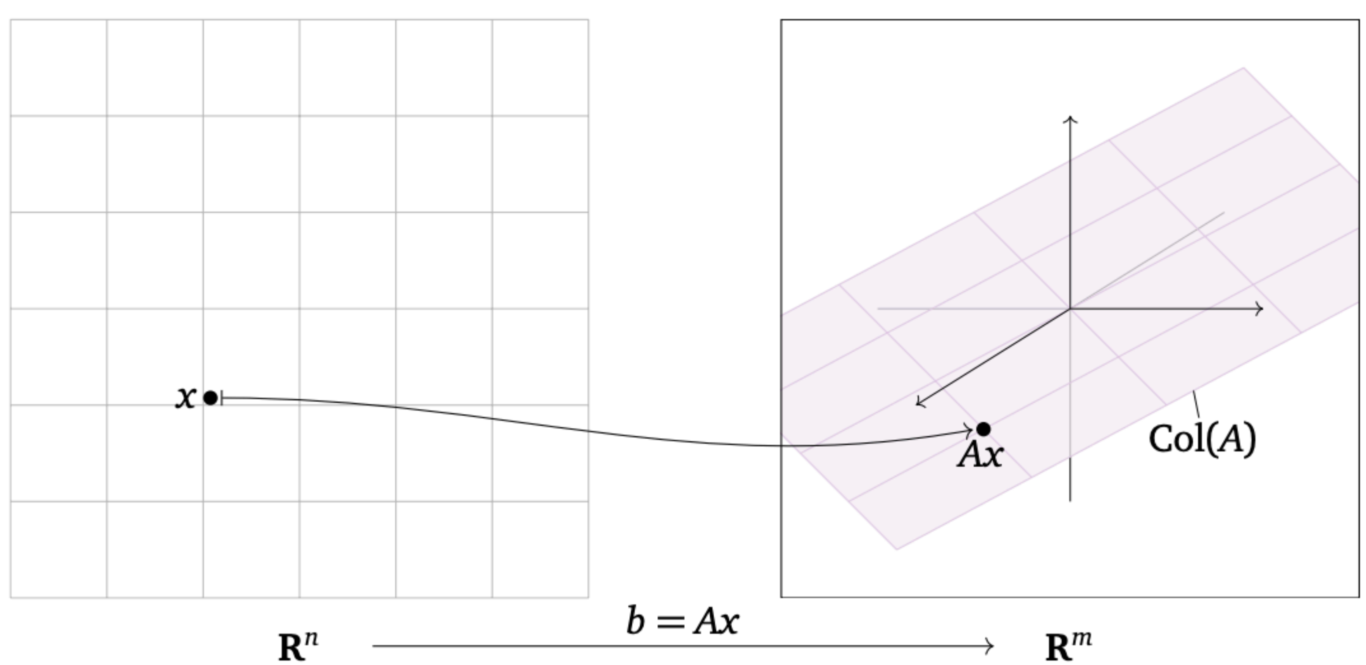
Question: If the domain of *f* is changed to **R**, is *f* still a function? Why?

Matrix as a function

Let A be an $m \times n$ matrix. The matrix transformation associated to A is the transformation

$$T: \mathbf{R}^n \rightarrow \mathbf{R}^m \text{ defined by } T(x) = Ax.$$

This is the transformation that maps a vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.



Matrix as a function

Let A be an $m \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation.

The ***domain*** of T is $\mathbf{R}^n$, where n is the number of columns of A.

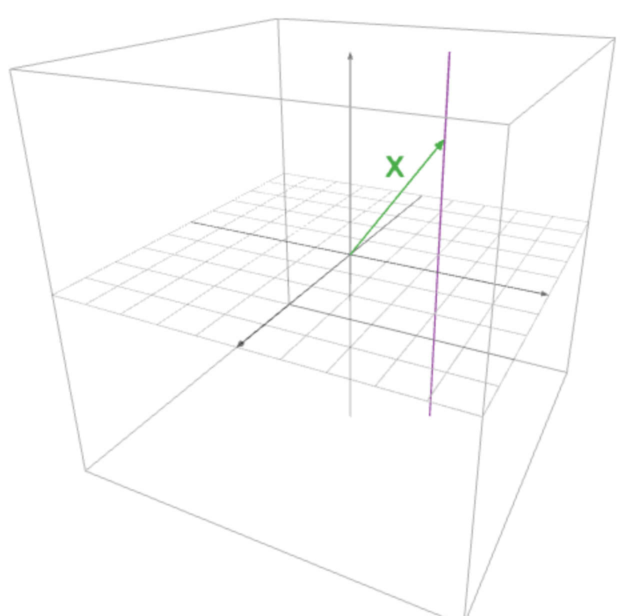
The ***codomain*** of T is $\mathbf{R}^m$, where m is the number of rows of A.

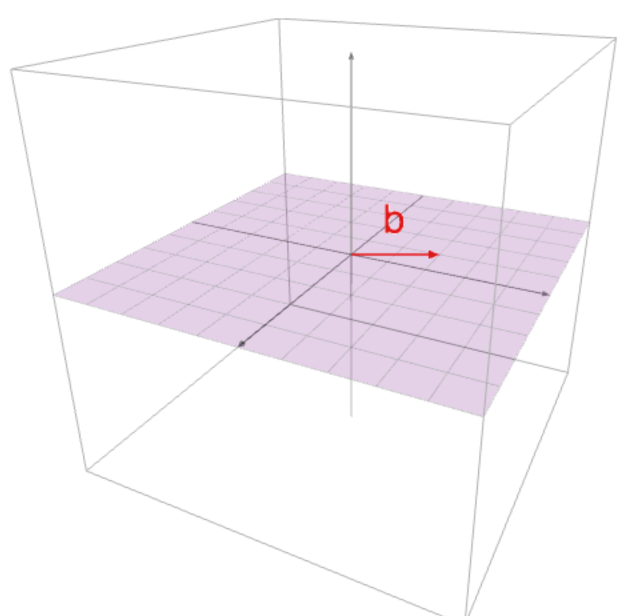
The ***range*** of T is the column space of A.

Projection on a plane

The following matrix projects a 3-dimensional point (vector in $\mathbf{R}^3$) to a plane (vector in $\mathbf{R}^2$).

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

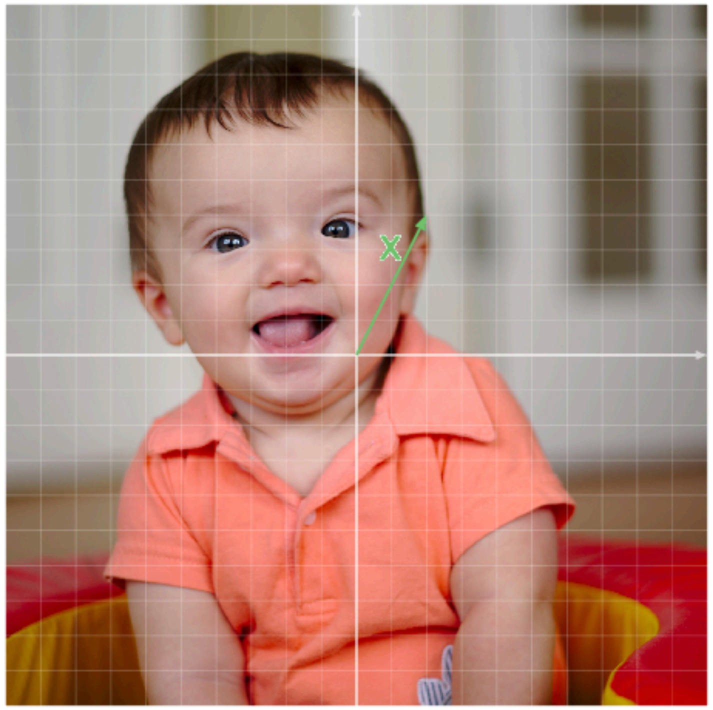


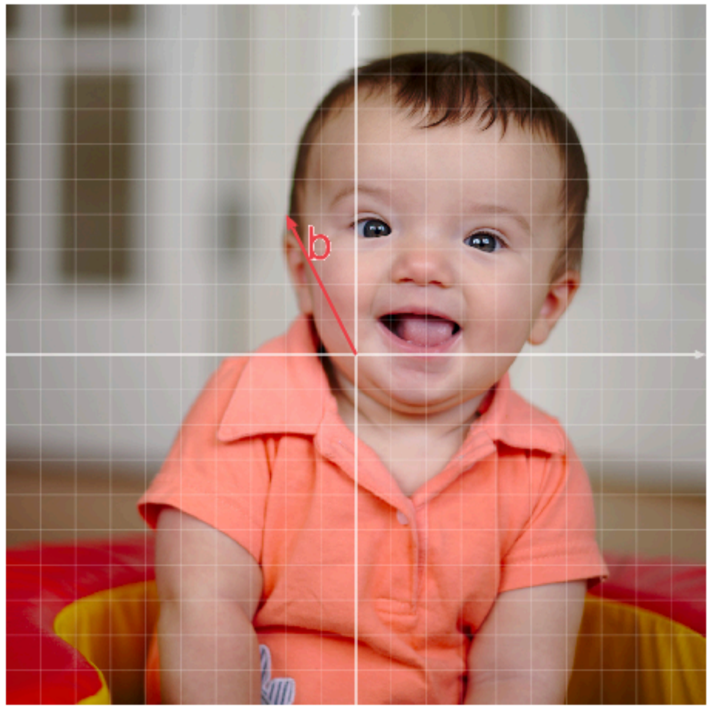


Reflection

The following matrix reflects a 2-dimensional point (vector in **R**^{2}) to its reflection (vector in **R**^{2}).

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

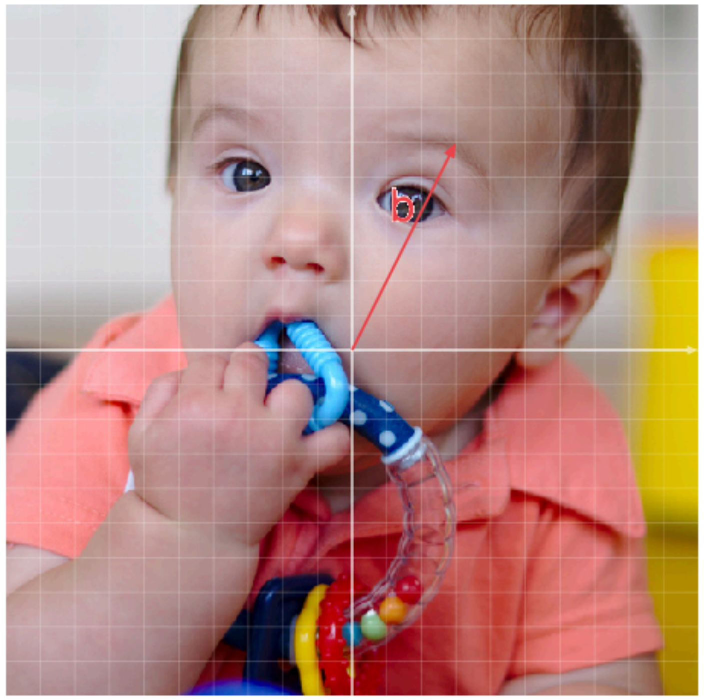




Dilation

The following matrix multiply a 2-dimensional point (vector in $\mathbf{R}^2$) by a scalar (vector in $\mathbf{R}^2$).

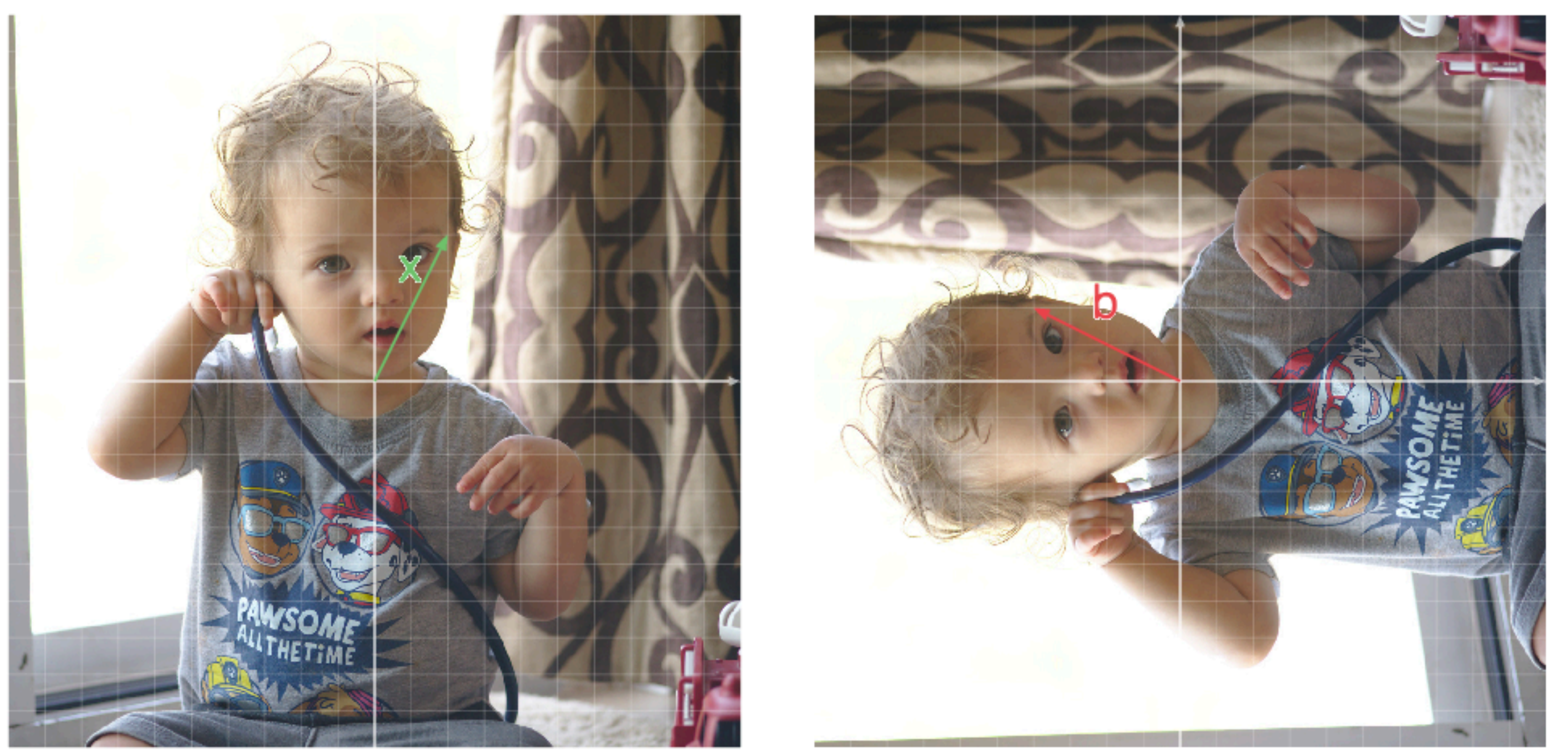
$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$



Rotation

The following matrix rotates a 2-dimensional point (vector in **R**^{2}) to its rotation (vector in **R**^{2}).

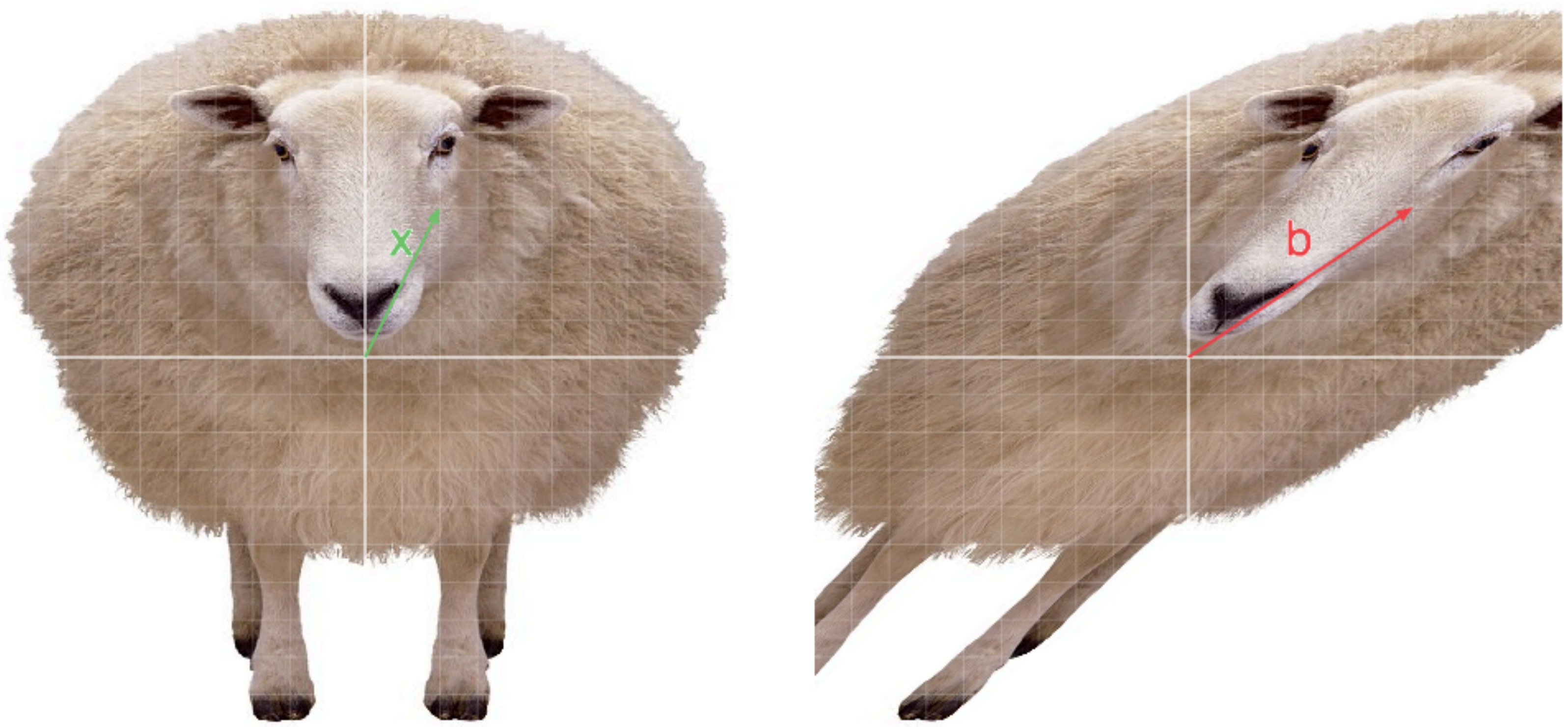
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$



Shear

The following matrix shears a 2-dimensional point (vector in **R**^{2} to a vector in **R**^{2}).

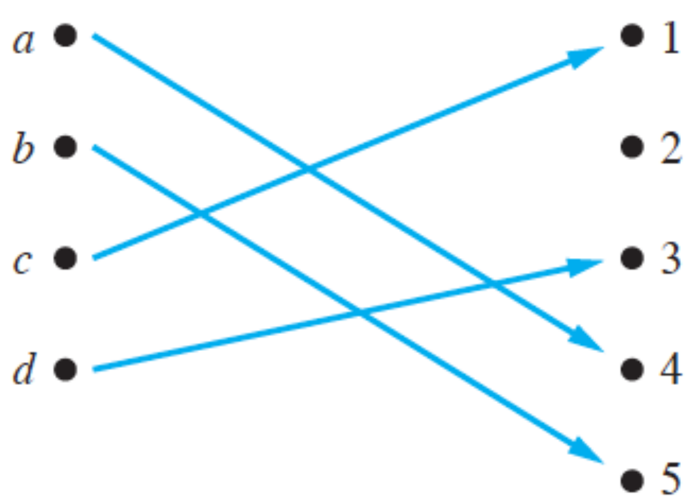
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$



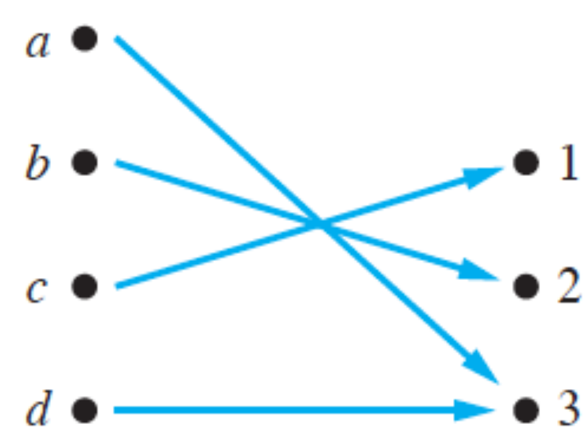
Injective (one-to-one)

A function *f* is said to be ***injective*** (or ***one-to-one***) if and only if *f(a)* = *f(b)* implies *a* = *b*.

That is, no two or more elements in the domain map to the same element in the codomain.



injective



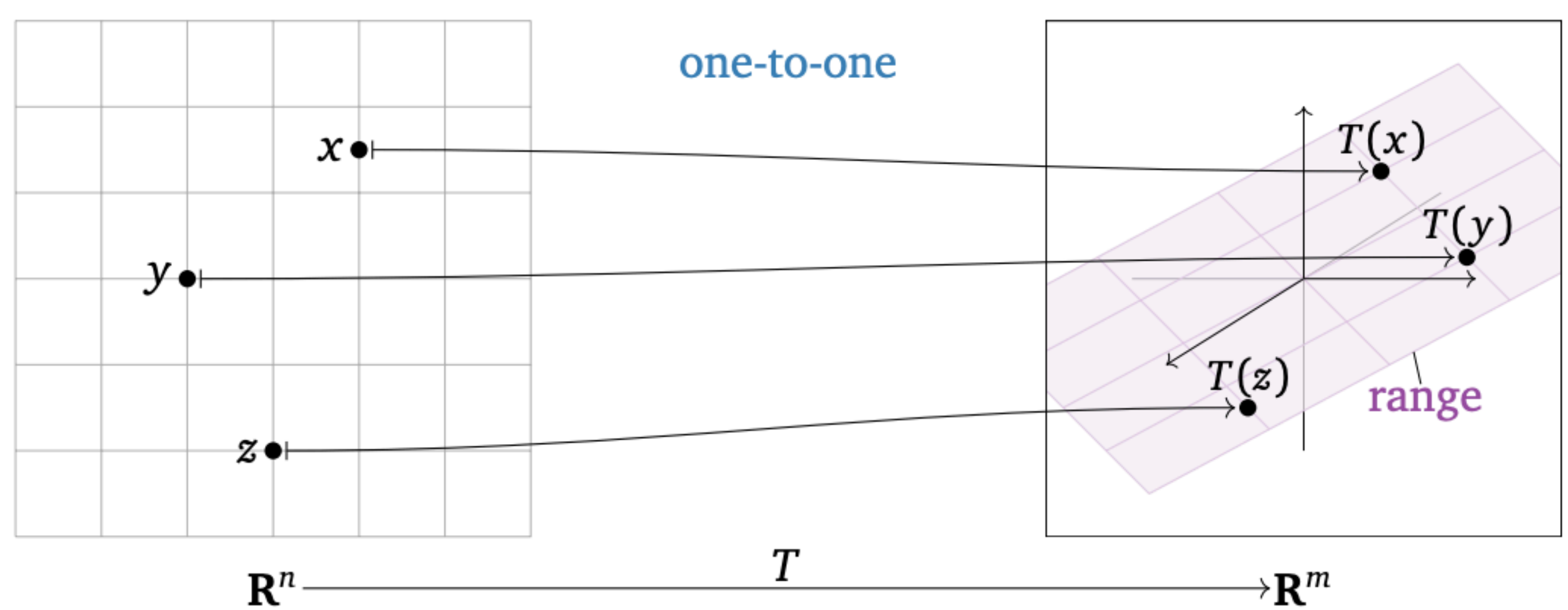
not injective

Is the function *f(x)* = *floor(x)* from **R** to **R** injective?

One-to-one transformation

A transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ is ***one-to-one*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution x in $\mathbf{R}^n$.

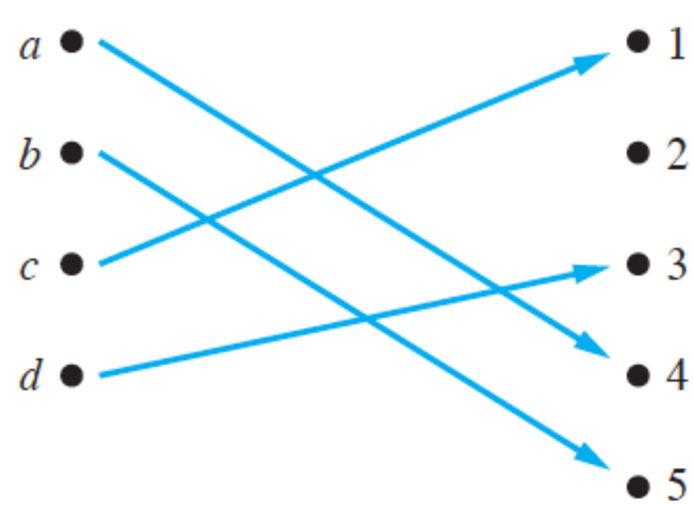
The equation $Ax = b$ is one-to-one if and only if the equation $Ax = b$ has a unique solution.



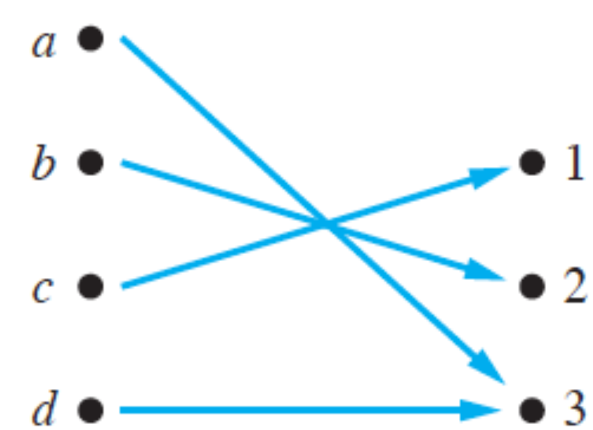
Surjective (onto)

A function *f: A -> B* is said to be ***surjective*** (or ***onto***) if and only if for every element $b \in B$, there is an element $a \in A$ such that $f(a) = b$.

That is, the range of *f* is equal to the codomain of *f*.



not surjective



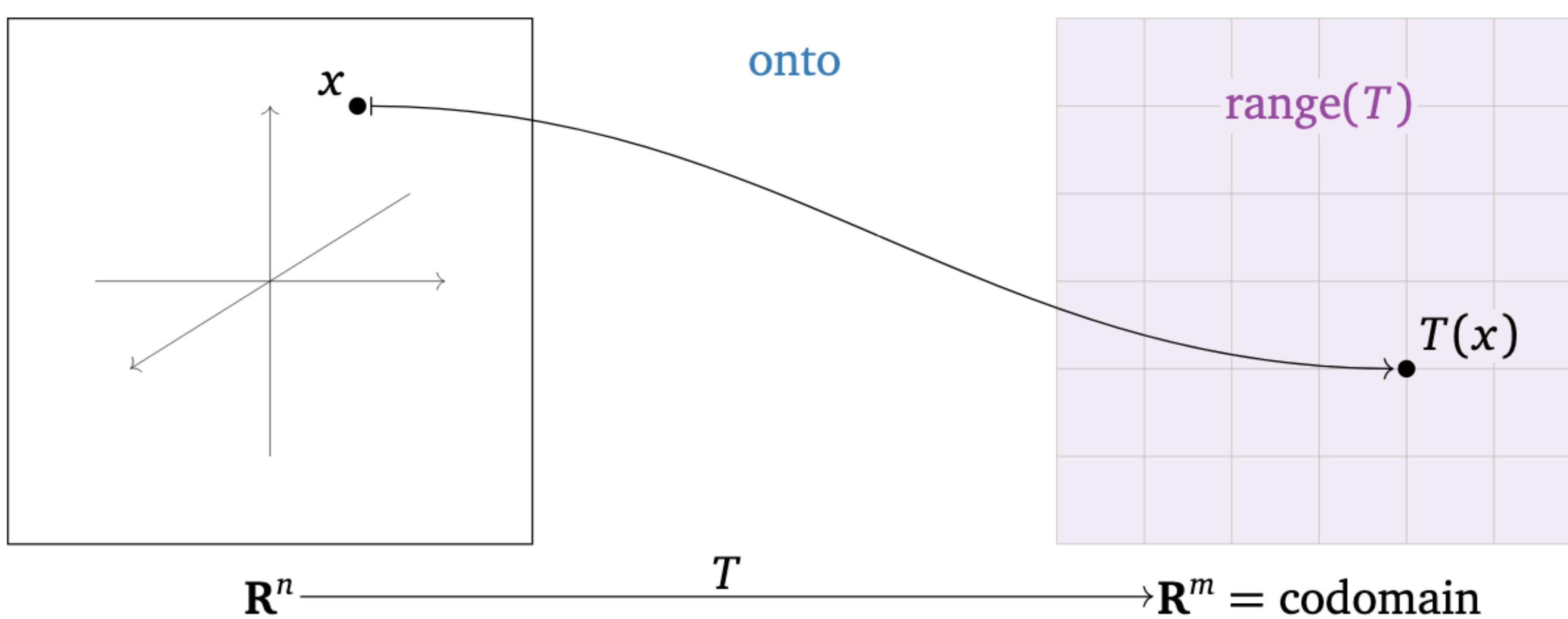
surjective

Is the function *f(x) = floor(x)* from **R** to **R** surjective?

Onto transformation

A transformation T: $\mathbf{R}^n \to \mathbf{R}^m$ is ***onto*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has at least one solution x in $\mathbf{R}^n$.

This implies the columns of A span $\mathbf{R}^m$.



Linear transformation

A ***linear transformation*** is a transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$

$$T(u + v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

Dilation, rotation are examples of linear transformation.

Standard coordinate vectors

The ***standard coordinate vectors*** in $\mathbf{R}^n$ are the n vectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The *i*-th entry of e_i is equal to 1, and the other entries are 0.

Linear transformation

Let T: $\mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let A be the $m \times n$ matrix

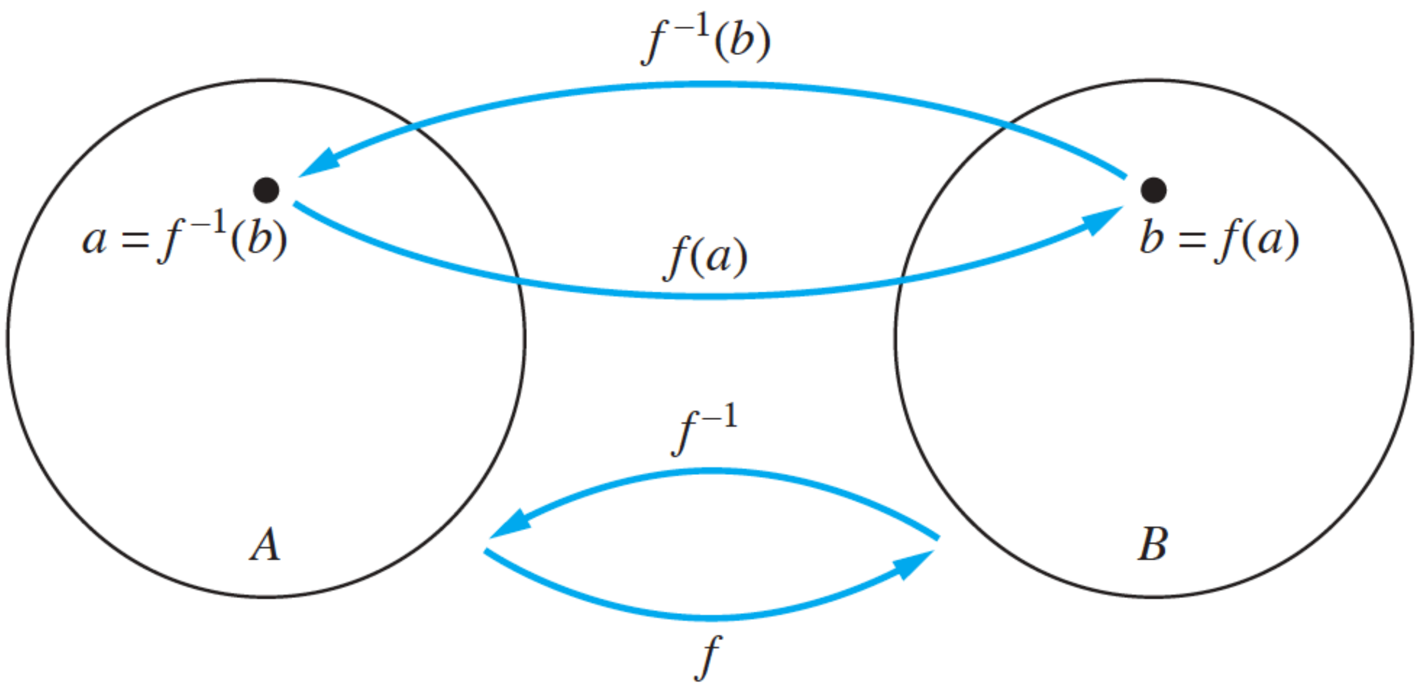
$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

Then T is the matrix transformation associated with A: that is $T(x) = Ax$.

Inverse function

An ***inverse function*** f^{-1} is a mapping between elements in codomain to the domain of the function *f*.

Theorem: The inverse function is a function if and only if *f* is bijective.



Example: Let *f:* **Z** -> **Z** be such that $f(x) = x + 1$, then $f^{-1}(x) = x - 1$.

Inverse matrix

Let A be an $n \times n$ matrix. An inverse of A is an $n \times n$ matrix A^{-1} such that

$$A^{-1} A = I \text{ and } A\, A^{-1} = I,$$

where I is the $n \times n$ identity matrix. The matrix A is invertible (or nonsingular) if it has an inverse.

Example: Given $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$ we have $\mathbf{A}^{-1} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$ since

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Finding inverse matrix

Let A be an $n \times n$ matrix. The inverse of A (if it exists) can be found by applying row operations to the augmented matrix $[A \mid I]$:

$$[A \mid I] \sim \cdots \sim [I \mid A^{-1}].$$

If A is not row equivalent to I, then A^{-1} does not exist.

In general, when A is a 2 x 2 matrix with $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and D = ad - bc, then

$$\mathbf{A}^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Finding inverse matrix

Example: Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution:

$$
\begin{array}{cc} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] & \begin{array}{c} -3 \\ \end{array} \end{array}
\sim
\begin{array}{cc} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] & \begin{array}{c} \\ 1 \end{array} \end{array}
$$

$$
\sim
\begin{array}{cc} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] & \begin{array}{c} \\ -\frac{1}{2} \end{array} \end{array}
$$

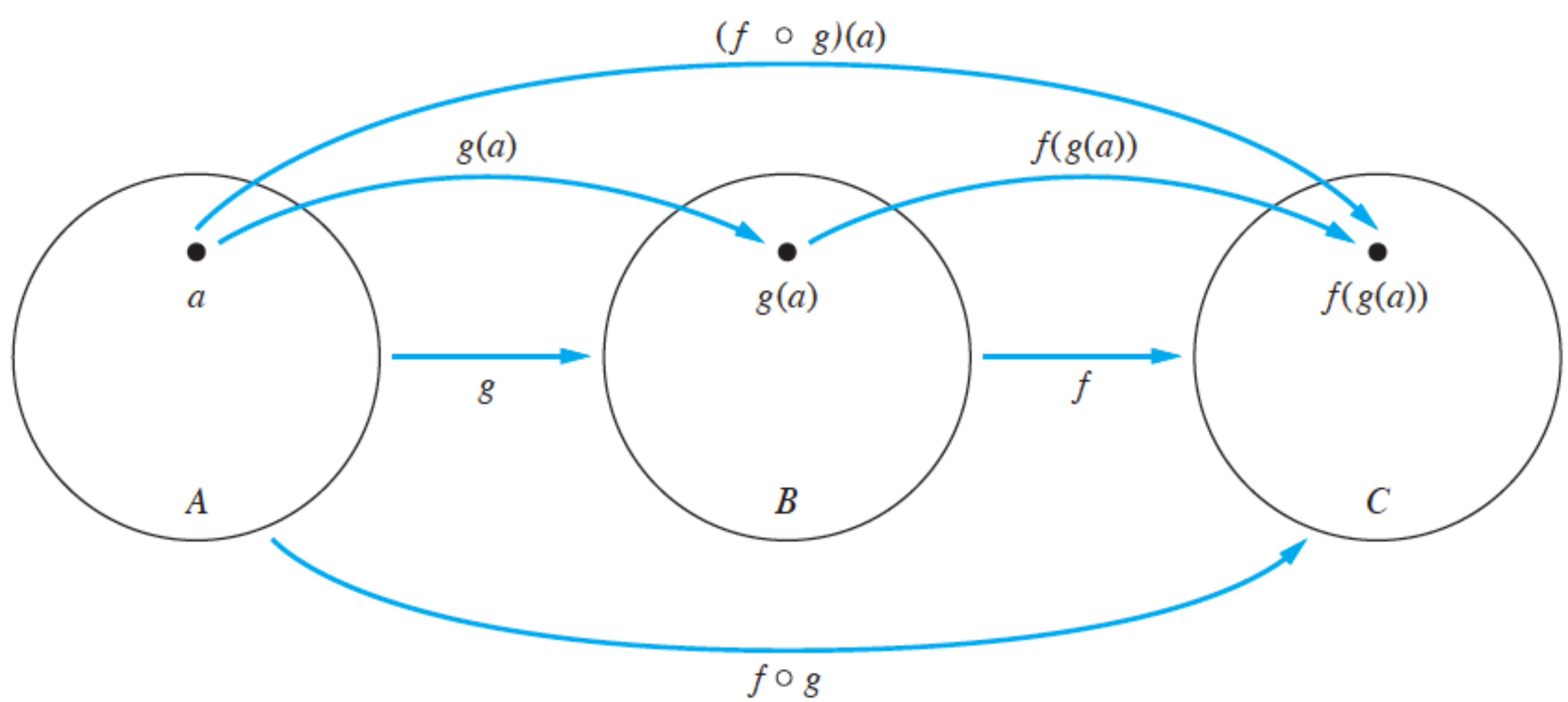
$$
\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right].
$$

Therefore, the inverse of the matrix is $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Composition of functions

Let *g* be a function from A to B and let *f* be the function from B to C. The ***composition*** of the function *f* and *g*, denoted by $f \circ g$, is defined as $f \circ g(x) = f(g(x))$.

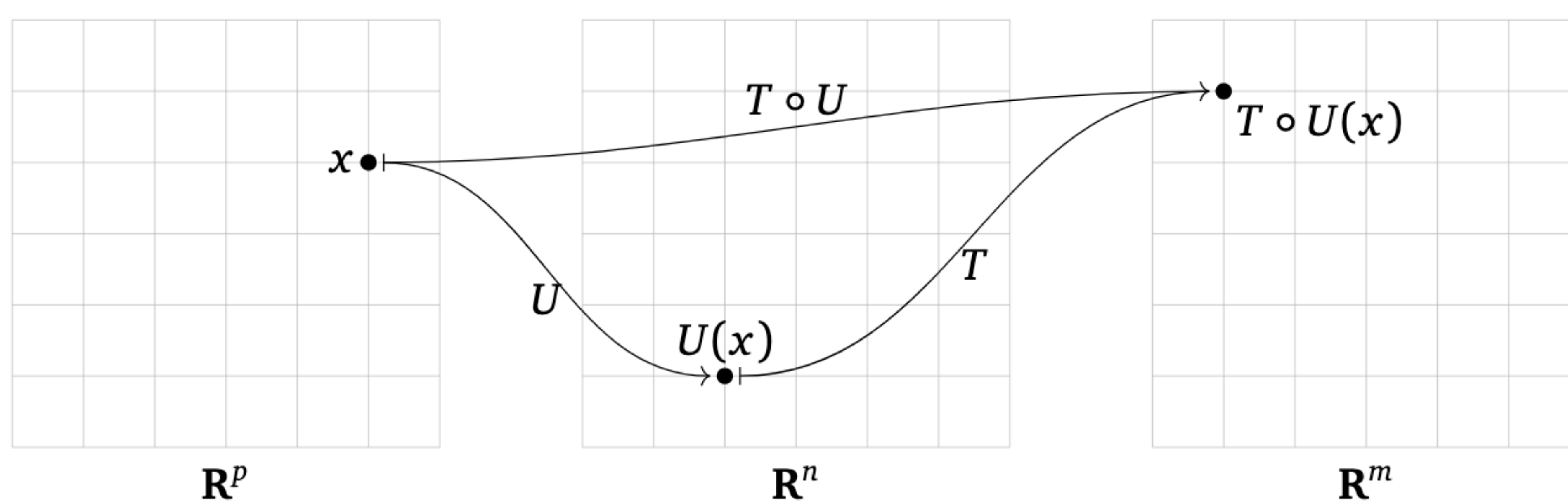
The composition $f \circ g$ is well-defined only if the range of *g* is a subset of the domain of *f*.



Composition of linear transformations

Let T: $\mathbf{R}^n \to \mathbf{R}^m$ and U : $\mathbf{R}^p \to \mathbf{R}^n$ be transformations. Their ***composition*** is the transformation $T \circ U : \mathbf{R}^p \to \mathbf{R}^m$ defined by

$$(T \circ U)(x) = T(U(x)).$$



Matrix multiplication

Let A be an $m \times n$ matrix, let B be an $n \times p$ matrix, and let C = AB . Then the ij entry of C is the i-th row of A times the j-th column of B:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

$$\begin{array}{c} \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \\ \end{array} \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} \\ j\text{th column} \end{array} = \begin{array}{c} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix} \\ ij \text{ entry} \end{array}$$