Definite Integral

COMP406 - Calculus
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Definite integral

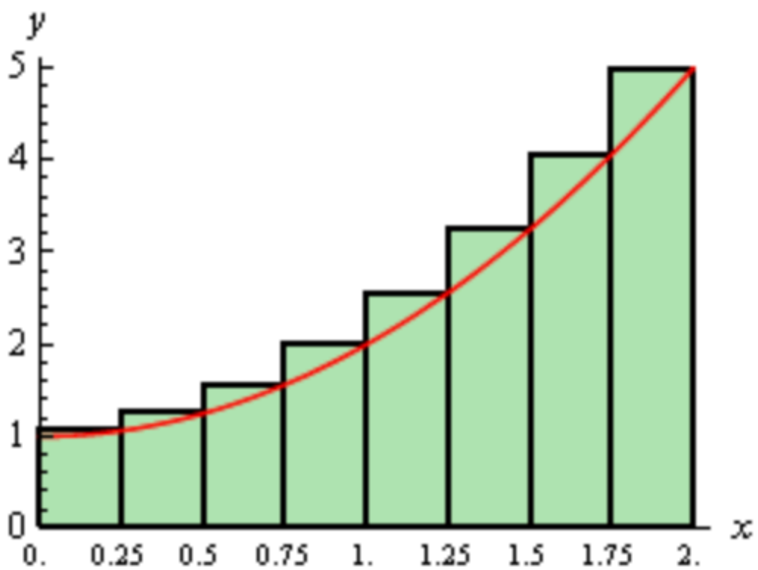
Given a function $f(x)$ that is continuous on the interval [a, b] we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i. Then the ***definite integral*** of $f(x)$ from a to b is

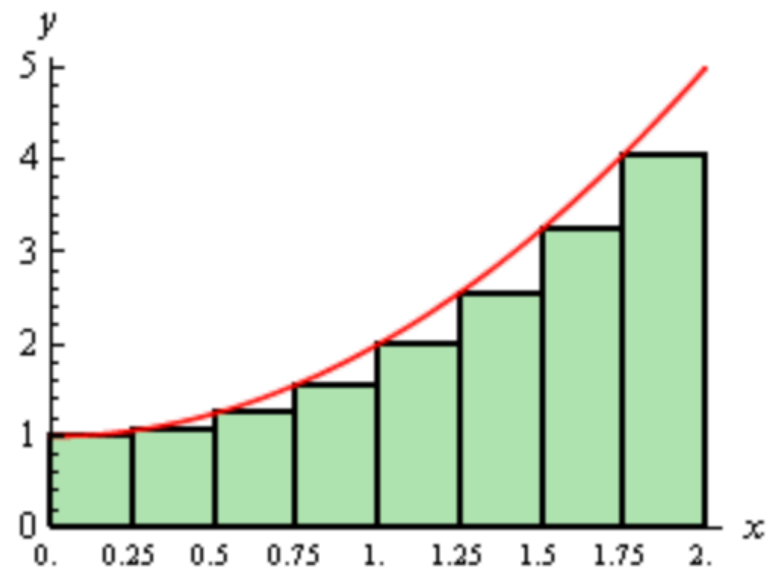
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

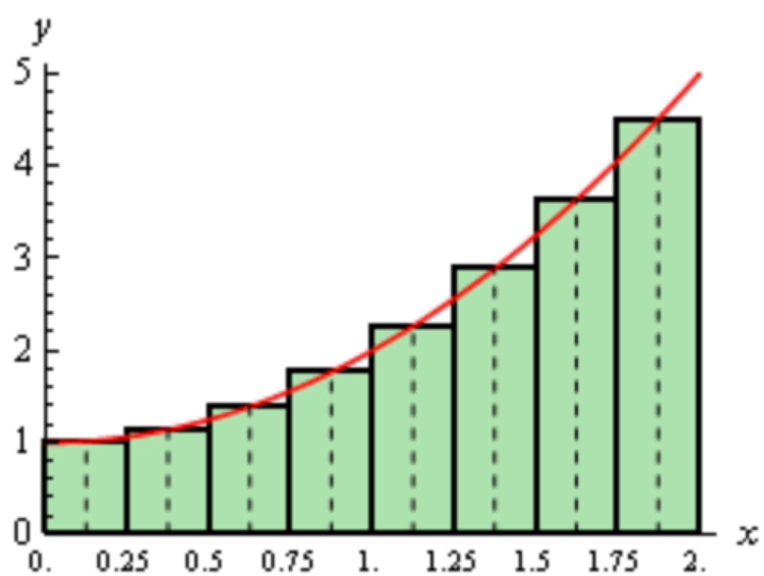
The number a at the bottom of the integral sign is called the ***lower limit*** of the integral.

The number b at the top of the integral sign is called the ***upper limit*** of the integral.

Definite integral







Some properties of definite integral

$$\int_a^b f(x)\, \mathrm{dx} = - \int_b^a f(x)\, \mathrm{dx}$$

$$\int_a^a f(x)\, \mathrm{dx} = 0$$

$$\int_a^b cf(x)\, \mathrm{dx} = c \int_a^b f(x)\, \mathrm{dx}$$

$$\int_a^b f(x) + \mathrm{g}(x)\, \mathrm{dx} = \int_a^b f(x)\, \mathrm{dx} + \int_a^b \mathrm{g}(x)\, \mathrm{dx}$$

$$\int_a^b f(x)\, \mathrm{dx} = \int_a^c f(x)\, \mathrm{dx} + \int_c^b f(x)\, \mathrm{dx}$$

$$\int_a^b c\, \mathrm{dx} = c(b - a),\ c \text{ is a constant}$$

Computing definite integrals

Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,\mathrm{dx} = F(x)\,\Big|_a^b = F(b) - F(a).$$

Example: Evaluate $\int_0^2 x^2 + 1\,\mathrm{dx}$

First evaluate the indefinite integral $\int x^2 + 1\,\mathrm{dx}$:

$$\int x^2 + 1\,\mathrm{dx} = 1/3\, x^3 + x + c.$$

After that we substitute 2 and 1 into the equation to find their difference, we can also remove the constant c as it cancelled out each other:

$$\int_0^2 x^2 + 1\,\mathrm{dx} = (1/3\,(2)^3 + 2) - (1/3\,(0)^3 + 0) = 14/3.$$

Substitution rule for definite integral

The first part to find the indefinite integral is the same.

There are however, two ways to deal with the evaluation step.
1. Substitute the variable u with x and apply the evaluation.
2. Find the new evaluation values based on u.

Example (method 2): Evaluate $\int_{-2}^{0} 2x^2 \sqrt{(1 - 4x^3)}\, dx$

Let $u = 1 - 4x^3$, then we have $du = -12x^2\, dx$

Now when $x = -2$, then $u = 1 - 4(-2)^3 = 33$. Also when $x = 0$, then $u = 1 - 4(0)^3 = 1$. The integral thus becomes

$$\int_{-2}^{0} 2x^2 \sqrt{(1 - 4x^3)}\, dx = -1/6 \int_{33}^{1} u^{1/2}\, du = -1/9\, u^{3/2} \Big|_{33}^{1}$$

Average function value

Theorem: The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by,

$$f_{\text{avg}} = 1/(b - a) \int_a^b f(x)\, dx.$$

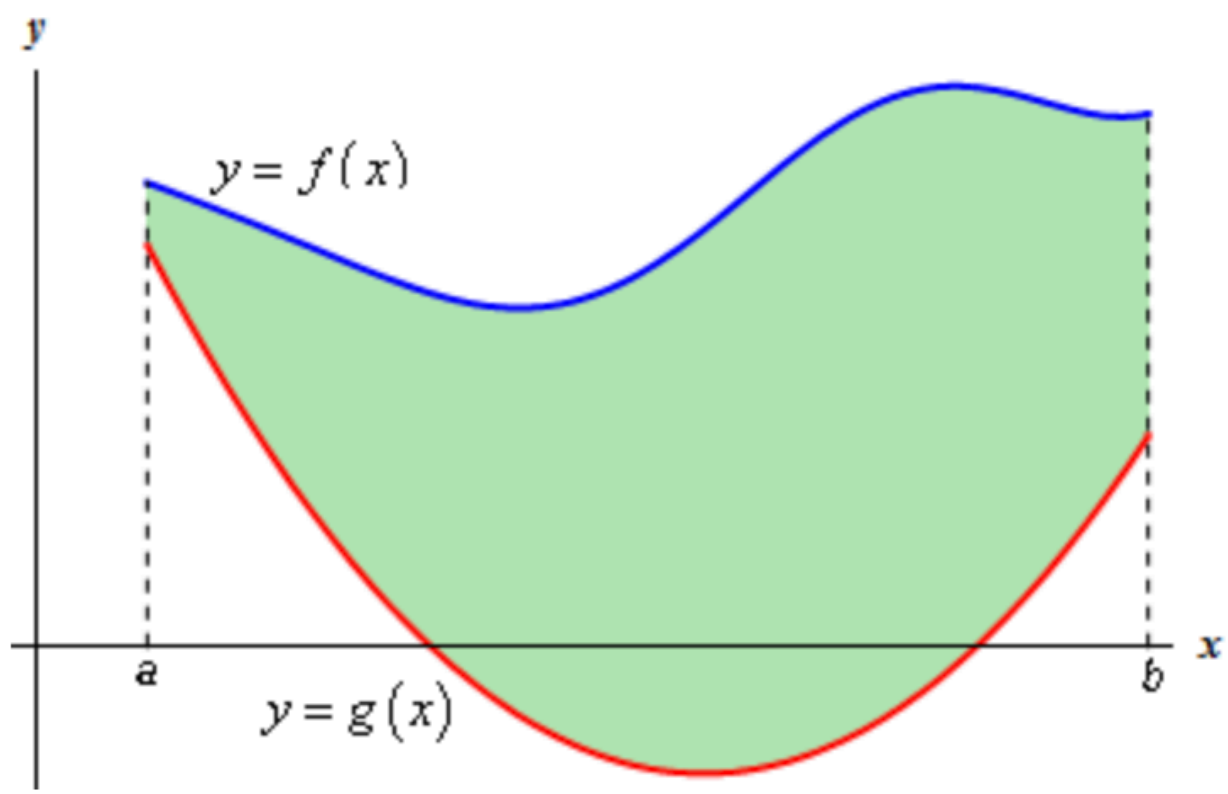
Theorem: If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x)\, dx = f(c)\,(b - a).$$

Application - Area between curves

The area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is given by the following formula:

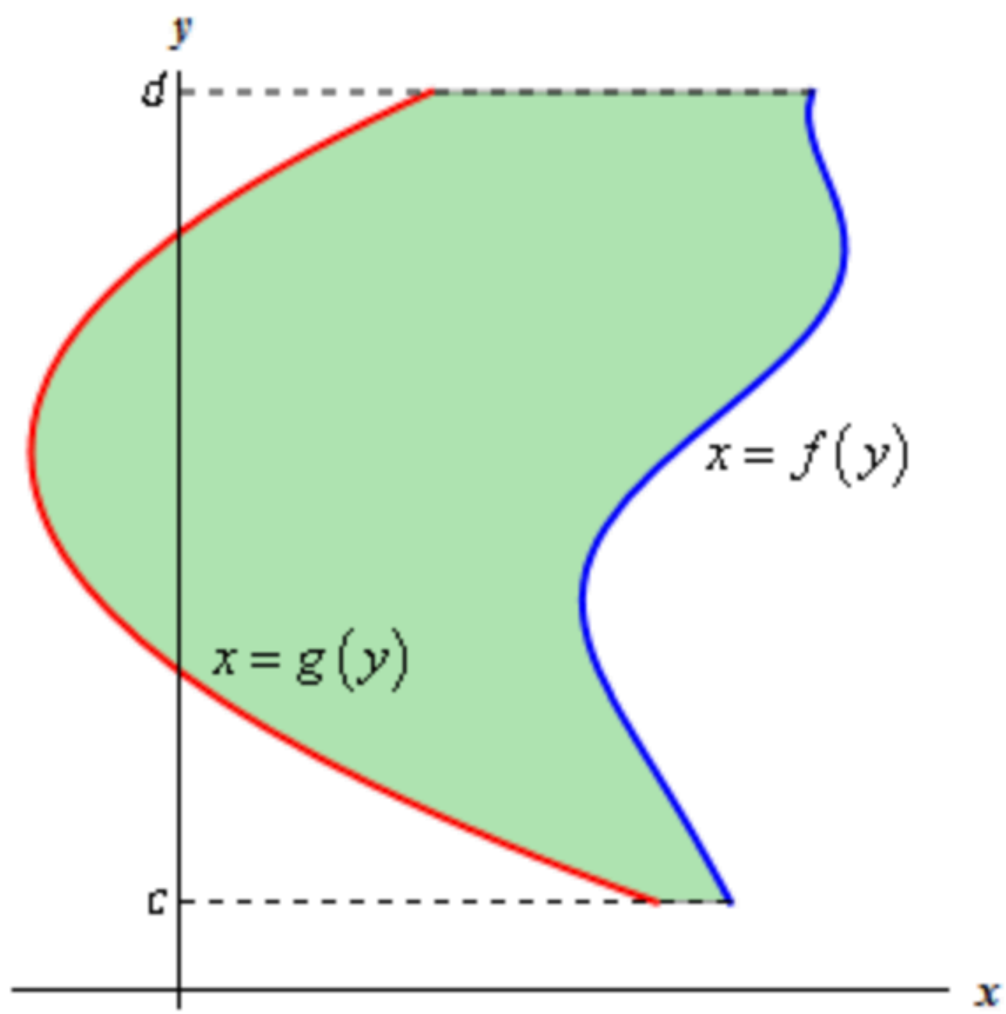
$$A = \int_a^b f(x) - g(x)\, dx.$$



Application - Area between curves

The area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ is given by the following formula:

$$A = \int_c^d f(y) - g(y)\, dy.$$



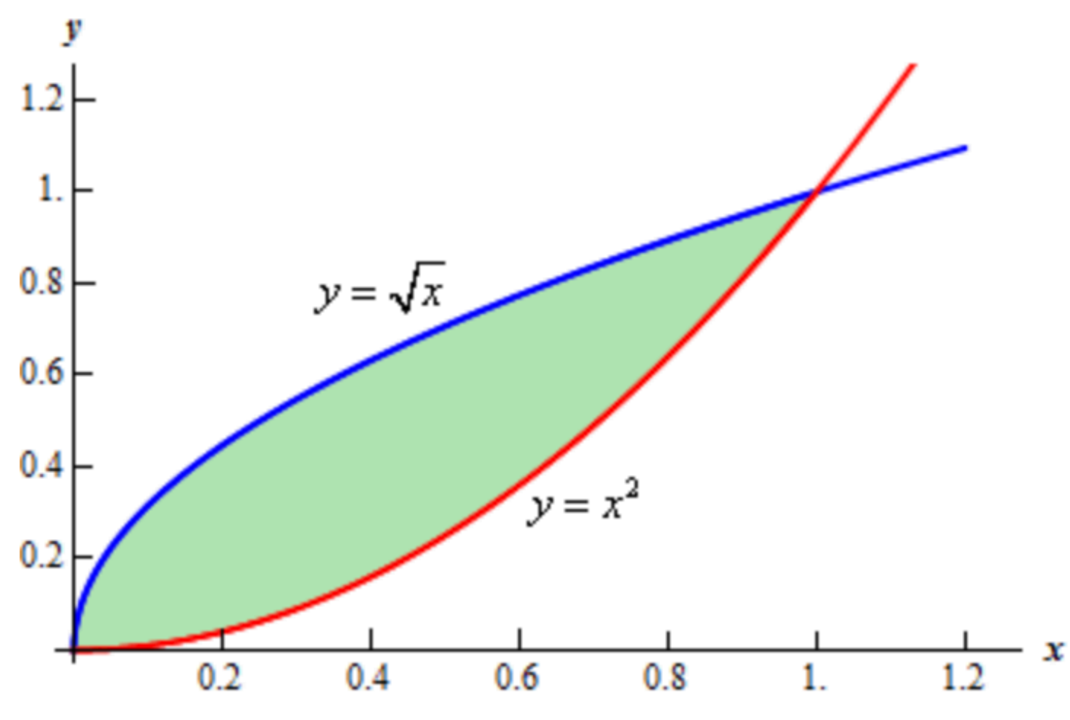
Application - Area between curves

Example: Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ on the interval [0, 1].

Solution: It is clear that $y = x^2$ is the top function while the bottom function is $y = \sqrt{x}$.

Area enclosed by the function is thus:

$$A = \int_0^1 \sqrt{x} - x^2\, dx = (2/3\ x^{3/2} - 1/3\ x^3)\Big|_0^1 = 1/3.$$



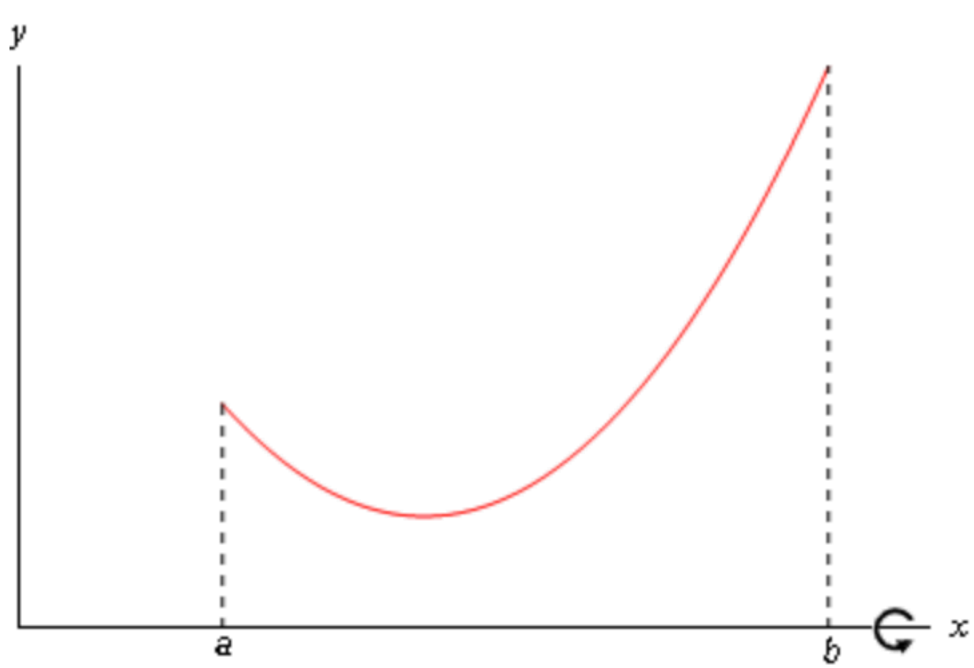
Application - Volume with rings

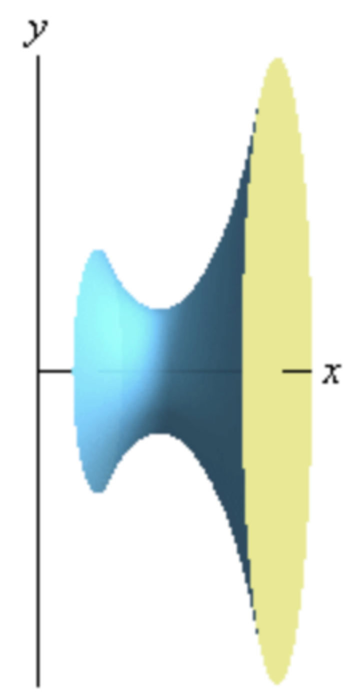
Given a function $y = f(x)$ on an interval $[a, b]$, the volume of a solid of revolution about a given axis is given as follows:

$$V = \int_a^b A(x)\,\mathrm{d}x, \text{ for rotating around } x\text{-axis;}$$

$$V = \int_c^d A(y)\,\mathrm{d}y, \text{ for rotating around } y\text{-axis.}$$

The terms $A(x)$ and $A(y)$ are the ***cross-sectional area*** functions of the solid. If the area is a ***solid disk***, the area is $A = \pi(\text{radius})^2$.





Application - Volume with rings

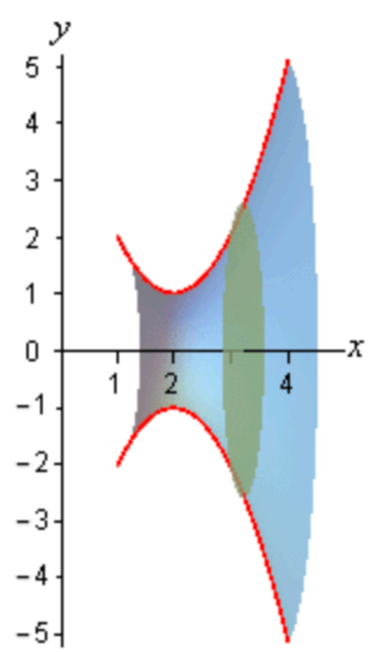
Example: Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

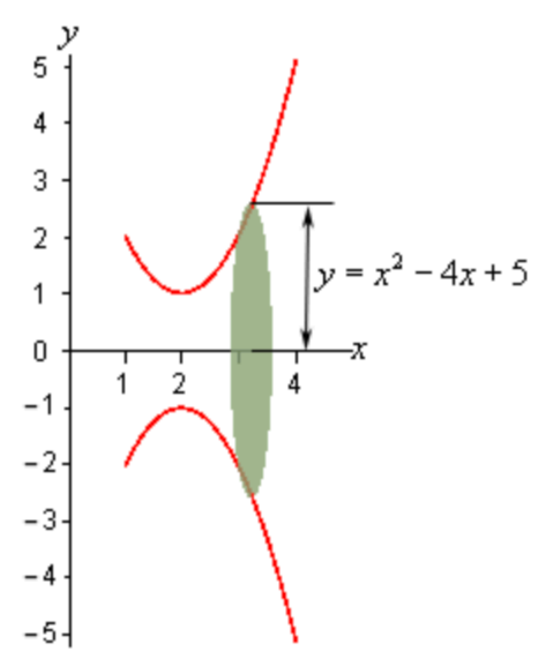
The cross-sectional area is given by $A = \pi(\text{radius})^2$ and thus:

$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

The volume of this solid is thus

$$V = \int_a^b A(x)\, dx = \pi \int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\, dx$$
$$= \pi \left(1/5\, x^5 - 2x^4 + 26/3\, x^3 - 20x^2 + 25x\right) \Big|_1^4 = 78\, \pi / 5$$

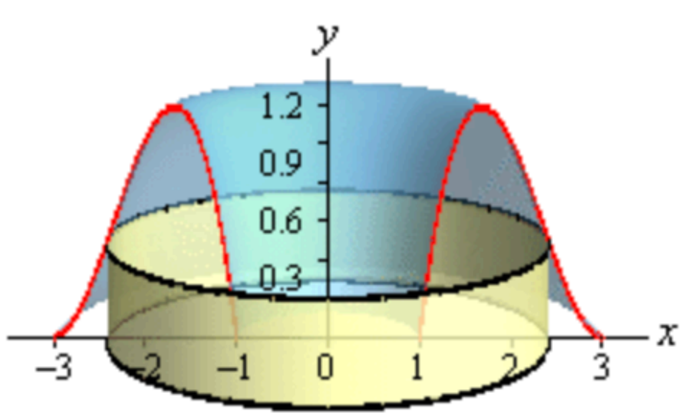


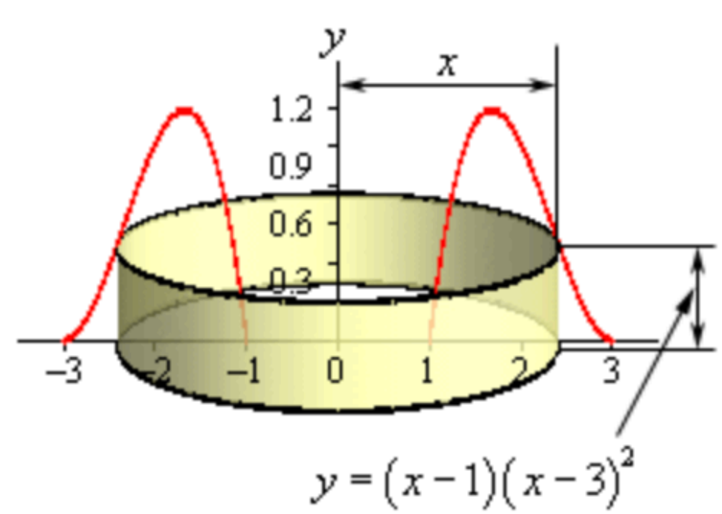


Application - Volume with cylinders

Sometimes we might have to use cylinder instead of solid disk to compute the volume.

The surface area is a ***cylinder*** is A = 2π(radius)(height).





Application - Volume with cylinders

Example: Determine the volume of the solid obtained by rotating the region bounded by $y = (x - 1)(x - 3)^2$ and the x-axis about the y-axis.

The surface area of the cylinder is $A = 2\pi(\text{radius})(\text{height})$ and thus:

$A(x) = 2\pi(x)(x - 1)(x - 3)^2 = 2\pi(x^4 - 7x^3 + 15x^2 - 9x)$.

The volume of this solid is thus

$$V = \int_a^b A(x)\,dx = 2\pi \int_1^3 x^4 - 7x^3 + 15x^2 - 9x\,dx$$
$$= \pi\,(1/5\,x^5 - 7/4\,x^4 + 5\,x^3 - 9/2\,x^2)\,\Big|_1^3 = 24\,\pi / 5.$$