# Chapter 4 Relations and Functions

## §4.1. Overview

In many problems concerning discrete objects such as people, numbers, sets, and so on, it is often the case that there is some kind of relationship among the objects. It is called a relation.

We are going to discuss "binary relations". By a binary relation, we mean a relation between two objects. For instance, "father and son" is a binary relation between two people, "less than" is a binary relation between two numbers, and "subset" is a binary relation between two sets. In particular, we will focus on a special type of relations called equivalence relations.

At the end of this chapter, we will discuss functions. Here, a function will be defined as a special type of relation.

## §4.2. Product Set

### 4.2.1. *Ordered Pair*

Definition: An ordered pair of $a$ and $b$, denoted by $(a, b)$, is a listing of two objects $a$ and $b$ in a prescribed order with $a$ appearing first and $b$ appearing second. $a$ and $b$ are respectively called the first coordinate and second coordinate of the ordered pair $(a, b)$.

It follows from the above definition that two ordered pairs are equal iff their corresponding coordinates are equal, i.e. $(a, b)=(c, d)$ iff $a=c$ and $b=d$.

### 4.2.2. *Product Set*

Definition: Let $A$ and $B$ be nonempty sets. The set of **all** ordered pairs $(a, b)$, where $a\in A$ and $b\in B$, is called the product set, or Cartesian product, of $A$ and $B$. This product set is denoted by $A\times B$ (read as "$A$ cross $B$").

Using descriptive property method, we have $A\times B=\{(a, b)\,|\,a\in A \text{ and } b\in B\}$. When $B=A$, $A\times B$ becomes $A\times A$; in that case this product set is usually written as $A^2$.

Examples

1) $\{1, 2, 3\}\times\{4, 5\}=\{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$
2) $\{4, 5\}\times\{1, 2, 3\}=\{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

   Remark i/. If $|A|=m$ and $|B|=n$ ($m, n\in\mathbb{Z}^+$), then $|A\times B|=mn$.

   ii/. If $A$ and $B$ are two distinct nonempty sets, then $A\times B\neq B\times A$.

3) Let $A=B=\mathbb{R}$.

   Then $A\times B=\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$, which is the set of all ordered pairs of real numbers. Geometrically, $\mathbb{R}^2$ is represented by the $xy$-plane, as shown in Fig. 4.1.

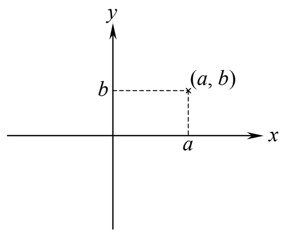



Fig. 4.1

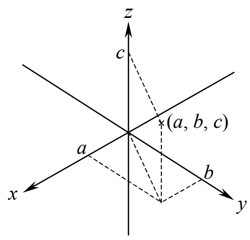



Fig. 4.2

4) Let $A=\mathbb{R}^2$ and $B=\mathbb{R}$.

   Then $A\times B=\mathbb{R}^2\times\mathbb{R}=\{((a, b), c)\,|\,a, b, c\in\mathbb{R}\}$.

   Usually, $((a, b), c)$ is simplified as $(a, b, c)$, and is called an ordered triple.

   Also, $\mathbb{R}^2\times\mathbb{R}$ will be written as $\mathbb{R}^3$. The geometrical representation of $\mathbb{R}^3$ is the set of all points in the 3-dimensional space, as shown in Fig. 4.2.

   Remark Similarly, we can extend the concept of ordered triple further to something like $(a, b, c, d)$ (called an ordered quadruple), $(a, b, c, d, e)$ (called an ordered quintuple), etc.

## §4.3. Partitions of Sets

Definition: A partition of a nonempty set $A$ is a collection of nonempty subsets of $A$, denoted by $A_i$, satisfying the following conditions:

(a) each element of $A$ belongs to one of the $A_i$'s,

(b) the sets $A_i$ are pairwise disjoint meaning that $A_i \cap A_j = \varnothing$ whenever $A_i \neq A_j$.

Each of the subsets $A_i$ is called a block (or cell) of the partition.

Notes

i/. Condition (a) can also be written as $\bigcup_i A_i = A$.

ii/. Condition (b) is **not the same** as $\bigcap_i A_i = \varnothing$.

Examples

1) Let $A = \{1, 2, 3\}$.

$\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1\}, \{2, 3\}\}$, and $\{\{1, 2, 3\}\}$ are four distinct partitions of $A$.

Among these partitions, the first one has three blocks, each of the second and the third has two blocks, and the fourth one has only one block.

2) Let $A = \{1, 2, 3\}$, $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, and $A_4 = \{1, 2\}$.

$\{A_1, A_3\}$ is not a partition of $A$ $\because$ condition (a) fails

$\{A_2, A_3, A_4\}$ is not a partition of $A$ $\because$ condition (b) fails. Note that condition (b) fails because of $A_2 \cap A_4 \neq \varnothing$. Note also that we have $A_2 \cap A_3 \cap A_4 = \varnothing$ here.

3) Let $A = \mathbb{Z}$, $A_1 = \{n \in \mathbb{Z} \mid n \text{ is even}\}$, and $A_2 = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$.

$\{A_1, A_2\}$ is a partition of $A$ with 2 blocks.

Remark In this example, we classify an integer as even (i.e. of the form $2k$) or odd (i.e. of the form $2k+1$) by using 2 as a divisor. This idea can be extended to any divisor $d$, where $d$ is an integer greater than 1. The next example is an illustration of this extension.

4) Let $A = \mathbb{Z}$. For $i = 0, 1, 2, \ldots, 6$, let

$$A_i = \{n \in \mathbb{Z} \mid \text{the remainder is } i \text{ when } n \text{ is divided by } 7\}$$

Note that
$$\begin{aligned} A_0 &= \{\ldots, -21, -14, -7, 0, 7, 14, 21, 28, \ldots\}, \\ A_1 &= \{\ldots, -20, -13, -6, 1, 8, 15, 22, 29, \ldots\}, \\ &\vdots \\ A_6 &= \{\ldots, -15, -8, -1, 6, 13, 20, 27, 34, \ldots\}. \end{aligned}$$

and $\{A_0, A_1, A_2, A_3, A_4, A_5, A_6\}$ is a partition of $A$ with 7 blocks.

## §4.4. Relations

Introduction

In real life, the notion of a relation between objects is very common. For example, let $A = \{a, b, c, d\}$ be a set of 4 students, and let $B = \{\text{CS123, CS221, CS264, CS273, CS281}\}$ be a set of 5 courses. Then a relation can be defined between $A$ and $B$. For instance, $a$ is related to CS123 if student $a$ takes the course CS123, $b$ is related to CS264 if student $b$ takes the course CS264, and so on. This can be regarded as a "take" relation.

The only thing that really matters about a relation is that we know precisely which elements in $A$ are related to which elements in $B$. In other words, it would be enough to be given the list of related pairs. The most direct way to express a relationship between two elements of two sets is to use an ordered pair made up of the two related elements. For example, the above "take" relation can be represented by the set of ordered pairs: $\{(a, \text{CS123}), (b, \text{CS264}), \ldots\}$.

Definition: Let $A$ and $B$ be nonempty sets. A (binary) relation $R$ from $A$ to $B$ is a subset of $A \times B$.

Notes

i/. If $R \subseteq A \times B$ (i.e. $R$ is a relation from $A$ to $B$) and $(a, b) \in R$, we say that $a$ is $R$-related to $b$, and is denoted by $aRb$. In other words, $aRb \Leftrightarrow (a, b) \in R$. If $a$ is not $R$-related to $b$, i.e. $(a, b) \notin R$, we write $a\not{R}b$.

ii/. A relation from a set $A$ to itself is simply called a relation on $A$.

Examples

1) Let $A=\{1, 2, 3\}$, $B=\{p, q\}$, and $R=\{(1, p), (2, q), (3, p)\}$.
   Then $R$ is relation from $A$ to $B$. Here, we have $1Rp$, $2Rq$, $3Rp$, and $1\not R q$.

2) Let $A=\{1, 2, 3, 4, 5\}$. Define a relation $R$ on $A$ by
   $$aRb \text{ iff } a<b.$$
   Then $R=\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.
   This is the "smaller than" relation.

3) Let $A=\{1, 2, 3, 4, 5\}$. Define a relation $R$ on $A$ by
   $$aRb \text{ iff } a=b.$$
   Then $R=\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.
   This is the "equality" relation.

4) Let $A=\mathbb{R}$. Define a relation $R$ on $A$ by
   $$xRy \text{ iff } x+2y=1.$$
   Using descriptive property method, we have $R=\{(x, y)\in\mathbb{R}^2 \,|\, x+2y=1\}$.
   Geometrically, $R$ is represented by a straight line on the $xy$-plane, as shown in Fig. 4.3.

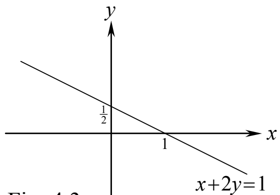


Fig. 4.3

5) $R=\{(x, y)\in\mathbb{R}^2 \,|\, x^2+y^2=1\}$ is another relation on $\mathbb{R}$. This relation can be represented by a circle with center at the origin (see Fig. 4.4). Since this circle has radius 1, it is called a unit circle.

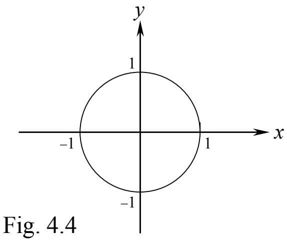


Fig. 4.4

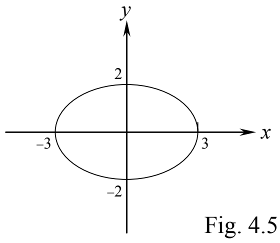


Fig. 4.5

6) The relation $R=\{(x, y)\in\mathbb{R}^2 \,|\, \frac{x^2}{9}+\frac{y^2}{4}=1\}$ can be represented by an ellipse (see Fig. 4.5).

7) The relation $R=\{(x, y)\in\mathbb{R}^2 \,|\, x^2-y^2=1\}$ can be represented by a hyperbola (see Fig. 4.6).

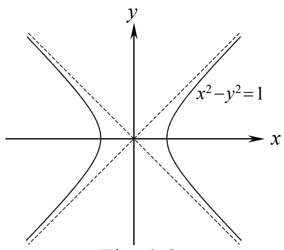


Fig. 4.6

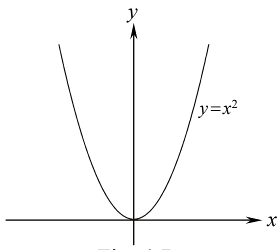


Fig. 4.7

8) The relation $R=\{(x, y)\in\mathbb{R}^2 \,|\, y=x^2\}$ can be represented by a parabola (see Fig. 4.7).

   Remark The curves mentioned in Examples 5 to 8, viz. circle, ellipse, hyperbola and parabola, are called conic sections because they can be obtained by the intersections of a plane with a conical surface.

9) The relation $R=\{(x, y)\in\mathbb{R}^2 \,|\, x>0,\ y>0, \text{ and } x+y<2\}$ can be represented by a triangular region, not including the boundary (see Fig. 4.8a).

10) The relation $R=\{(x, y)\in\mathbb{R}^2 \,|\, x\geq 0,\ y\geq 0, \text{ and } x+y\leq 2\}$ can be represented by a triangular region together with the boundary (see Fig. 4.8b).

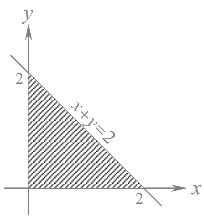


Fig. 4.8a

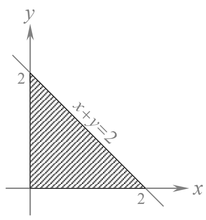


Fig. 4.8b

## §4.5. Domain and Range

Definition: The domain of a relation $R$, denoted by Dom($R$), is the set of all the first coordinates of the ordered pairs that belong to $R$. The range of a relation $R$, denoted by Ran($R$), is the set of all the second coordinates of the ordered pairs that belong to $R$.

Examples

1) Let $A=\{1, 2, 3\}$, $B=\{r, s, t\}$, and $R=\{(1, r), (3, r), (3, s)\}$.
   Then $\text{Dom}(R)=\{1, 3\}$ and $\text{Ran}(R)=\{r, s\}$.

2) Let $A=\{1, 2, 3, 4, 5\}$ and $R=\{(a, b)\in A^2 \mid a<b\}$.
   Using the enumeration method (see Example 2 of §4.4), we get $\text{Dom}(R)=\{1, 2, 3, 4\}$ and $\text{Ran}(R)=\{2, 3, 4, 5\}$.

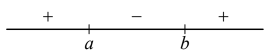
3) Let $R=\{(x, y)\in \mathbb{R}^2 \mid x^2+y^2=1\}$.
   From the graph of $R$ (i.e. Fig. 4.4), we see that $\text{Dom}(R)=\text{Ran}(R)=\{x\in\mathbb{R} \mid -1\le x\le 1\}=[-1, 1]$.

4) Let $R=\{(x, y)\in \mathbb{R}^2 \mid y=x^2\}$.
   From the graph of $R$ (i.e. Fig. 4.7), we see that $\text{Dom}(R)=\mathbb{R}$ and $\text{Ran}(R)=\{y\in\mathbb{R} \mid y\ge 0\}=[0, +\infty)$.

5) Let $R=\{(x, y)\in \mathbb{R}^2 \mid y=x^2-4x+6\}$.
   Since $y=x^2-4x+6=(x-2)^2+2$ (this is an example of "Completing the Square Method"), the graph of $R$ is a parabola with the minimum point at (2, 2). $\therefore$ $\text{Dom}(R)=\mathbb{R}$ and $\text{Ran}(R)=[2, +\infty)$.
   Remark The graph of this relation is a shift of the parabola of Fig. 4.7 to the point (2, 2).

6) Let $R=\{(x, y)\in \mathbb{R}^2 \mid x=\sqrt{y^2-6y-3}\}$. Determine Dom($R$) and Ran($R$).
   *Ans.*: (a) If $x\in\text{Dom}(R)$, then $x=\sqrt{y^2-6y-3}$ for some $y\in\mathbb{R} \Rightarrow x\ge 0 \Rightarrow x\in[0, +\infty)$. Thus $\text{Dom}(R)\subseteq[0, +\infty)$.
   On the other hand, if $x\in[0, +\infty)$ and if $y=3+\sqrt{x^2+12}$, then $(y-3)^2=x^2+12 \Rightarrow x=\sqrt{y^2-6y-3} \Rightarrow x\in \text{Dom}(R)$. This means that $[0, +\infty)\subseteq\text{Dom}(R)$.
   $\therefore$ $\text{Dom}(R)=[0, +\infty)$.
   (b) Observe that $y\in\text{Ran}(R) \Leftrightarrow y^2-6y-3\ge 0 \Leftrightarrow (y-3)^2-12\ge 0 \Leftrightarrow (y-3+\sqrt{12})(y-3-\sqrt{12})\ge 0 \Leftrightarrow y\le 3-\sqrt{12}$ or $y\ge 3+\sqrt{12} \Leftrightarrow y\in(-\infty, 3-\sqrt{12}]\cup[3+\sqrt{12}, +\infty)$. $\therefore$ $\text{Ran}(R)=(-\infty, 3-\sqrt{12}]\cup[3+\sqrt{12}, +\infty)$.

   Remark Let $a$ and $b$ be real numbers with $b>a$. To solve an inequality like $(x-a)(x-b)\ge 0$, we might consider the sign of $(x-a)(x-b)$ over the intervals $(-\infty, a)$, $(a, b)$, and $(b, +\infty)$, as shown in the right picture.



sign of $(x-a)(x-b)$

## §4.6. Special Relations

### 4.6.1. *Identity Relation*

Definition: Let $A$ be a nonempty set. The relation $\{(a, a) \mid a\in A\}$ is called the identity relation on $A$. This relation is denoted by $I_A$.

e.g. Let $A=\{1, 2, 3\}$. Then $I_A=\{(1, 1), (2, 2), (3, 3)\}$.

Note: We also refer to $I_A$ as the "equality relation", which is usually denoted by "=".

### 4.6.2. *Universal Relation*

Definition: Let $A$ be a nonempty set. The relation $A\times A$ is called the universal relation (on $A$).

### 4.6.3. *Empty Relation*

Definition: Let $A$ be a nonempty set. The relation $\varnothing$ is called the empty relation (on $A$).

Note: Recall that $\varnothing$ is a subset of any set.

### 4.6.4. *Converse of Relations*

Definition: Let $R$ be a relation from $A$ to $B$, where $A$ and $B$ are nonempty sets. The converse of $R$, denoted by $R^{-1}$, is the relation from $B$ to $A$ defined by $R^{-1}=\{(b, a) \mid (a, b)\in R\}$.

e.g. Let $A=\{1, 2, 3\}$ and $R=\{(1, 2), (1, 3), (2, 3)\}$. Then $R^{-1}=\{(2, 1), (3, 1), (3, 2)\}$.

## §4.7. Pictorial Representation of Relations

Consider a relation from $A$ to $B$ or on $A$, where $A$ and $B$ are finite sets. There are several ways of picturing such relations. Here, we introduce two of these ways.

### 4.7.1. ***Matrix of a Relation***

Definition: Suppose $A=\{a_1, a_2, \ldots, a_m\}$, $B=\{b_1, b_1, \ldots, b_n\}$, and $R$ is a relation from $A$ to $B$. We can represent $R$ by the $m\times n$ matrix $\mathrm{M}_R=(m_{ij})$, where $m_{ij}$ denotes the entry in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column, and is defined by

$$m_{ij}=\begin{cases}1 & \text{if } a_i R b_j \text{ (i.e. } (a_i, b_j)\in R)\\ 0 & \text{if } a_i \not R b_j \text{ (i.e. } (a_i, b_j)\notin R)\end{cases}$$

The matrix $\mathrm{M}_R$ is called the matrix of $R$.

Examples

1) Let $A=\{1, 2, 3\}$, $B=\{r, s\}$, and $R=\{(1, r), (2, s), (3, r)\}$.
Then the matrix of $R$ can be written as

$$\mathrm{M}_R=\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix} \quad \text{or} \quad \begin{array}{c|cc} & r & s\\ \hline 1 & 1 & 0\\ 2 & 0 & 1\\ 3 & 1 & 0\end{array}$$

2) Consider the following matrix of a certain relation $R$:

$$\mathrm{M}_R=\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}.$$

Let $A=\{a_1, a_2, a_3\}$ and $B=\{b_1, b_2, b_3, b_4\}$.
Then $R=\{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$.

3) Let $A=\{1, 2, 3, 4\}$ and $B=\{x, y, z\}$. Consider the relation $R=\{(1, y), (1, z), (3, y), (4, x), (4, z)\}$. Determine the matrix of $R$.
*Ans.*:

$$\begin{array}{c|ccc} & x & y & z\\ \hline 1 & 0 & 1 & 1\\ 2 & 0 & 0 & 0\\ 3 & 0 & 1 & 0\\ 4 & 1 & 0 & 1\end{array} \quad \text{or} \quad \mathrm{M}_R=\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$$

### 4.7.2. ***Digraph of a Relation***

If $A$ is a finite set, say $A=\{a_1, a_2, \ldots, a_n\}$ ($n\in\mathbb{Z}^+$), and $R$ is a relation on $A$, we can also represent $R$ by drawing a digraph (also called directed graph).

We draw a small circle for each element of $A$ and label it, then draw an arrow (in the terminology of digraph, an "arrow" is called an arc) from the circle $a_i$ to the circle $a_j$ iff $a_i R a_j$.

Examples

1) Let $A=\{1, 2, 3, 4\}$ and $R_1=\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$.
The digraph of $R_1$ is shown in Fig. 4.9.
Here, $1R_1 1$ and $2R_1 2$. In that case, we have a loop at ① or ②.

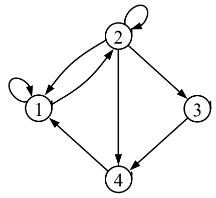



Fig. 4.9 Digraph of $R_1$

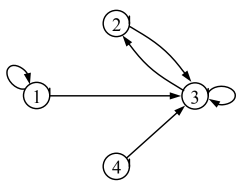



Fig. 4.10 Digraph of $R_2$

2) Let $A=\{1, 2, 3, 4\}$ and $R_2=\{(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)\}$.
The digraph of $R_2$ is shown in Fig. 4.10.

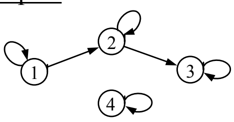
## §4.8. Properties of Relations

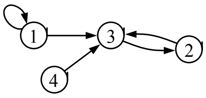
Here we concentrate on a relation defined on a given set $A$.

### 4.8.1. ***Reflexive***

Definition: A relation $R$ on $A$ is said to be reflexive iff $aRa$ for every element $a$ of $A$.

Examples

1)  is reflexive ∵ there is a loop at every vertex.

2)  is not reflexive ∵ there is no loop at ② (say).

3) The relation $R=\{(x, y)\in\mathbb{Z}^2 \mid x\geq y\}$ is reflexive ∵ $x\geq x\ \forall x\in\mathbb{Z}$.

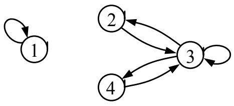
4) The relation $R=\{(x, y)\in\mathbb{Z}^2 \mid x<y\}$ is not reflexive ∵ $1 \nless 1\ \forall x\in\mathbb{Z}$.

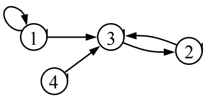
5) The relation $R=\{(x, y)\in\mathbb{Z}^2 \mid y \text{ is divisible by } x\}$ is reflexive ∵ $x=x\cdot 1\ \forall x\in\mathbb{Z}$.

### 4.8.2. ***Symmetric***

Definition: A relation $R$ on $A$ is said to be symmetric iff $bRa$ whenever $aRb$ ($a, b\in A$).

Examples

1)  is symmetric ∵ the arrows (not including loops) appear in pairs.

2)  is not symmetric ∵ $1R3$ but $3 \not R 1$.

3) The relation $R=\{(x, y)\in\mathbb{Z}^2 \mid x\geq y\}$ is not symmetric ∵ $2R1$ but $1 \not R 2$.

4) The relation $R=\{(x, y)\in\mathbb{Z}^2 \mid y \text{ is divisible by } x\}$ is not symmetric ∵ $1R2$ but $2 \not R 1$.

### 4.8.3. ***Transitive***

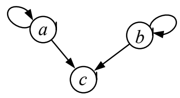
Definition: A relation $R$ on $A$ is said to be transitive iff $aRc$ whenever $aRb$ and $bRc$ ($a, b, c\in A$).

Examples

1) Let $A=\{a, b, c\}$ and $R=\{(a, a), (a, c), (b, b), (b, c)\}$. Is $R$ transitive?

Solution: Method I

From the digraph of $R$ (see the right picture), look at each pair of "connected arrows", say $x \to y \to z$, and see if there is an arrow from $x$ to $z$. For the digraph here, there are no connected arrows, and so $R$ is transitive.



Method II

Consider all those cases in which $xRy$ and $yRz$ ($x, y, z\in A$). There are four such cases (see the table). In each case, we also have $xRz$. Thus we conclude that $R$ is transitive.

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | ✓ |
| $a$ | $a$ | $c$ | ✓ |
| $b$ | $b$ | $b$ | ✓ |
| $b$ | $b$ | $c$ | ✓ |

2) Let $A=\{a, b, c\}$ and $R=\{(a, b), (b, b), (b, c)\}$.
Here $aRb$ and $bRc$ but $a \not R c$. ∴ $R$ is not transitive.

3) Let $A=\{a, b, c\}$ and $R=\{(a, a), (a, b), (a, c), (b, a), (b, c), (c, b), (c, c)\}$.
List all those elements $x, y, z$ of $A$ that satisfy $xRy$ and $yRz$.

Solution: There are 16 such cases, as listed in the right table.

Among those cases, there are three in which $x \not R z$. ∴ $R$ is not transitive.

Note also that $R$ is neither reflexive (∵ $b \not R b$) nor symmetric (∵ $aRc$ but $c \not R a$).

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | ✓ |
| $a$ | $a$ | $b$ | ✓ |
| $a$ | $a$ | $c$ | ✓ |
| $a$ | $b$ | $a$ | ✓ |
| $a$ | $b$ | $c$ | ✓ |
| $a$ | $c$ | $b$ | ✓ |
| $a$ | $c$ | $c$ | ✓ |
| $b$ | $a$ | $a$ | ✓ |

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| $b$ | $a$ | $b$ | × |
| $b$ | $a$ | $c$ | ✓ |
| $b$ | $c$ | $b$ | × |
| $b$ | $c$ | $c$ | ✓ |
| $c$ | $b$ | $a$ | × |
| $c$ | $b$ | $c$ | ✓ |
| $c$ | $c$ | $b$ | ✓ |
| $c$ | $c$ | $c$ | ✓ |

### 4.8.4. ***More Examples***

1) Let $R=\{(a,b)\in(\mathbb{Z}^+)^2 \mid 2a>b\}$. Is $R$ reflexive? Symmetric? Transitive?

   Solution: For any $a\in\mathbb{Z}^+$, $2a-a=a>0 \Rightarrow 2a>a$, i.e. $aRa$. $\therefore$ $R$ is reflexive.
   Since $3R1$ but $1\not R3$, $R$ is not symmetric.
   Since $2R3$ ($\because 2\cdot2>3$) and $3R5$ ($\because 2\cdot3>5$) but $2\not R5$ ($\because 2\cdot2<5$), $R$ is not transitive.

2) Let $R=\{(a,b)\in(\mathbb{Z}^+)^2 \mid a^2\geq b\}$. Is $R$ reflexive? Symmetric? Transitive?

   Solution: For any $a\in\mathbb{Z}^+$, $a^2-a=a(a-1)\geq0 \Rightarrow a^2>a$, i.e. $aRa$. $\therefore$ $R$ is reflexive.
   Since $3R1$ but $1\not R3$, $R$ is not symmetric.
   Since $2R3$ ($\because 2^2>3$) and $3R5$ ($\because 3^2>5$) but $2\not R5$ ($\because 2^2<5$), $R$ is not transitive.

3) Let $R=\{(a,b)\in(\mathbb{R}^+)^2 \mid a^2\geq 2b-1\}$. Is $R$ reflexive? Symmetric? Transitive?

   Solution: For any $a\in\mathbb{R}^+$, $a^2-(2a-1)=(a-1)^2\geq0 \Rightarrow a^2>2a-1$, i.e. $aRa$. $\therefore$ $R$ is reflexive.
   Since $3R1$ but $1\not R3$, $R$ is not symmetric.
   Since $4R5$ ($\because 4^2\geq 2\cdot5-1$) and $5R9$ ($\because 5^2\geq2\cdot9-1$) but $4\not R9$ ($\because 4^2<2\cdot9-1$), $R$ is not transitive.

4) Let $R=\{(a,b)\in(\mathbb{Z}^+)^2 \mid b^2+b \text{ is divisible by } a\}$. Is $R$ reflexive? Symmetric? Transitive?

   Solution: For any $a\in\mathbb{Z}^+$, $a^2+a=a(a+1)\geq0 \Rightarrow a^2+a$ is divisible by $a$, i.e. $aRa$. $\therefore$ $R$ is reflexive.
   Since $1R3$ but $3\not R1$, $R$ is not symmetric.
   Since $3R5$ ($\because 5^2+5=3\cdot10$) and $5R4$ ($\because 4^2+4=5\cdot4$) but $3\not R4$ ($\because 4^2+4=20$ which is not divisible by 3), $R$ is not transitive.

## §4.9. Equivalence Relations

Definition: A relation $R$ on a set is said to be an equivalence relation iff it is reflexive, symmetric, and transitive.

Examples

1) Let $A=\{1,2,3,4\}$ and $R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$. Is $R$ an equivalence relation?

   Solution: To determine whether $R$ is reflexive or symmetric, one way is to look at its matrix:

$$\mathrm{M}_R=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$$

main diagonal

Since the entries on the main diagonal are all 1's, $R$ is reflexive.
Also, the entries of this matrix are symmetrical about the main diagonal, and so $R$ is symmetric.
Now consider all those elements $x, y, z$ of $A$ that satisfy $xRy$ and $yRz$:

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| 1 | 1 | 1 | ✓ |
| 1 | 1 | 2 | ✓ |
| 1 | 2 | 1 | ✓ |
| 1 | 2 | 2 | ✓ |

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| 2 | 1 | 1 | ✓ |
| 2 | 1 | 2 | ✓ |
| 2 | 2 | 1 | ✓ |
| 2 | 2 | 2 | ✓ |

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| 3 | 3 | 3 | ✓ |
| 3 | 3 | 4 | ✓ |
| 3 | 4 | 3 | ✓ |
| 3 | 4 | 4 | ✓ |

| $x$ | $y$ | $z$ | $xRz$ |
|---|---|---|---|
| 4 | 3 | 3 | ✓ |
| 4 | 3 | 4 | ✓ |
| 4 | 4 | 3 | ✓ |
| 4 | 4 | 4 | ✓ |

For all cases, we have $xRz$. $\therefore$ $R$ is transitive.
Since $R$ is reflexive, symmetric, and transitive, $R$ is an equivalence relation.

2) Let $R=\{(a,b)\in\mathbb{Z}^2 \mid 2a+3b \text{ is a multiple of } 5\}$. Is $R$ an equivalence relation?

   Solution: For any $a\in\mathbb{Z}$, $2a+3a=5a$ which is a multiple of 5, i.e. $aRa$. $\therefore$ $R$ is reflexive.
   Suppose $aRb$ for some $a, b\in\mathbb{Z}$. Then $2a+3b=5c$ for some $c\in\mathbb{Z}$.
   Consider $2b+3a=(5-3)b+(5-2)a=5(a+b)-(2a+3b)=5(a+b)-5c=5(a+b-c)$. The last expression shows that $2b+3a$ is also a multiple of 5, and this means that $bRa$.
   $\therefore$ $R$ is symmetric.
   Next suppose $aRb$ and $bRc$ for some $a, b, c\in\mathbb{Z}$. Then $2a+3b=5x$ and $2b+3c=5y$ for some $x, y\in\mathbb{Z}$.
   It follows that $2a=5x-3b$ and $3c=5y-2b$, and hence $2a+3c=5x+5y-5b=5(x+y-b)$. $\therefore$ $aRc$.
   $\therefore$ $R$ is transitive.
   Since $R$ is reflexive, symmetric, and transitive, $R$ is an equivalence relation.

3) Let $R=\{(a, b)\in\mathbb{Z}^2 \mid a\le b\}$. Is $R$ an equivalence relation?

Solution: Since $1R2$ but $2\not R 1$, $R$ is not symmetric.

$\therefore$ $R$ is not an equivalence relation.

Remark To show that a given relation $R$ is not an equivalence relation, it is enough to point out just one of the properties (reflexive, symmetric, transitive) which is not satisfied by $R$. In that case, we do not have to consider the other properties.

## §4.10. Equivalence Classes

Definition: Suppose $R$ is an equivalence relation on $A$. For each $a\in A$, let $[a]$ denote the set of elements in $A$ to which $a$ is $R$-related. That is,

$$\underline{[a]=\{x\in A \mid aRx\}}.$$

$[a]$ is called an equivalence class of $a$ in $R$, and $a$ is called an representative of the equivalence class $[a]$.

Examples

1) From Example 1 of §4.9, we know that if

$$\mathrm{M}_R=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix} \qquad (A=\{1, 2, 3, 4\}),$$

then $R$ is an equivalence relation on $A$.

From $\mathrm{M}_R$, it can be readily seen that $[1]=[2]=\{1, 2\}$ and $[3]=[4]=\{3, 4\}$.

$\therefore$ There are exactly 2 equivalence classes in $R$, namely [1] and [3].

Observe that the set of all equivalence classes, viz. $\{[1], [3]\}$, is a partition of $A$.

2) From Example 2 of §4.9, we know that $R=\{(a, b)\in\mathbb{Z}^2 \mid 2a+3b$ is a multiple of $5\}$ is an equivalence relation on $\mathbb{Z}$.

Note that $2a+3b=2(a-b)+5b$. It follows that $aRb$ iff $a-b$ is divisible by 5.

Consequently
$$\begin{aligned}
[0]&=\{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}=\{5k \mid k\in\mathbb{Z}\},\\
[1]&=\{\ldots, -14, -9, -4, 1, 6, 11, 16, \ldots\}=\{5k+1 \mid k\in\mathbb{Z}\},\\
[2]&=\{\ldots, -13, -8, -3, 2, 7, 12, 17, \ldots\}=\{5k+2 \mid k\in\mathbb{Z}\},\\
[3]&=\{\ldots, -12, -7, -2, 3, 8, 13, 18, \ldots\}=\{5k+3 \mid k\in\mathbb{Z}\},\\
[4]&=\{\ldots, -11, -6, -1, 4, 9, 14, 19, \ldots\}=\{5k+4 \mid k\in\mathbb{Z}\},\\
[5]&=[0],\ [6]=[1], \text{ etc.}
\end{aligned}$$

$\therefore$ There are exactly 5 equivalence classes in $R$, namely [0], [1], [2], [3], and [4].

Observe that the set of all equivalence classes, viz. $\{[0], [1], [2], [3], [4]\}$, is a partition of $A$.

## §4.11. Correspondence Between Partitions and Equivalence Relations

**Theorem 1**

Given a partition of a nonempty set $A$. Define a relation $R$ on $A$ by

$$aRb \text{ iff } a \text{ and } b \text{ belong to } A_i \text{ for some block } A_i \text{ of the partition.}$$

Then $R$ is an equivalence relation on $A$.

Lemma

Let $R$ be an equivalence relation on $A$, and let $a, b\in A$. Then

(i) $a\in[a]$

(ii) $aRb$ iff $[a]=[b]$

(iii) $[a]\cap[b]=\varnothing$ or $[a]=[b]$

Remark (iii) is equivalent to "any two distinct equivalence classes must be disjoint".

**Theorem 2**

Let $R$ be an equivalence relation on $A$. Then the set of all equivalence classes is a partition of $A$.

## §4.12. Functions

Definition

Let $A$ and $B$ be nonempty sets. A binary relation $R$ from $A$ to $B$ is said to be a function iff for every element $a \in A$, there is a unique element $b \in B$ such that $aRb$.

Remark For the element $b$ mentioned in the above definition, we have to consider two things: existence and uniqueness (see examples below).

Notation and Terminology

A function $f$ from $A$ to $B$ is usually denoted by $f: A \to B$. The set $A$ is called the domain of $f$, and the set $B$ is called the codomain of $f$. For each $a \in A$, the unique element of $B$ that is related by $a$ will be denoted by $f(a)$. $f(a)$ is called the image of $a$ under $f$. Moreover, for a given element $b$ of $B$, if $a \in A$ satisfies $f(a)=b$, then $a$ is called a preimage of $b$.

Notes

i/. A function is also called a mapping. If $f(a)=b$, we say that $a$ is mapped to $b$ by $f$ (or "$f$ maps $a$ to $b$").

ii/. According to the definition of a function, "domain of function $f$" is consistent with "domain of $f$" regarding $f$ as a relation.

iii/. The meaning of "range of $f$", where $f$ is a function, remains the same as that for a relation, i.e.

$$\text{Ran}(f) = \{f(a) \mid a \in A\}.$$

In other words, Ran($f$) is the set of all images under $f$.

iv/. For any $b \in B$, $b$ may have no preimage, exactly one preimage, or more than one preimage (see examples below).

Examples

1) Let $A=\{1, 2, 3, 4\}$, $B=\{a, b, c, d\}$, and $f=\{(1, a), (2, a), (3, d), (4, c)\}$.
$f$ is a function $\because$ each element of $A$ is uniquely related to an element of $B$.
This function can be represented by the picture in Fig. 4.11.
Here, $f(1)=f(2)=a, f(3)=d, f(4)=c$, and Ran($f$)$=\{a, c, d\}$.
Moreover, $a$ has exactly two preimages, viz. 1 and 2; $b$ has no preimage; $c$ has exactly one preimage, viz. 4; and $d$ has exactly one preimage, viz. 3.

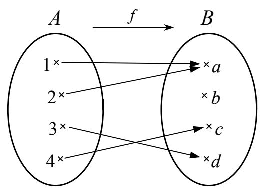



Fig. 4.11

2) Let $A=\{1, 2, 3, 4\}$, $B=\{a, b, c, d\}$, and $f=\{(1, a), (3, d), (4, c)\}$.
$f$ is not a function $\because$ the element 2 of $A$ is not $f$-related to any element of $B$. That is, the existence condition is not satisfied here.

3) Let $A=\{1, 2, 3, 4\}$, $B=\{a, b, c, d\}$, and $f=\{(1, a), (1, b), (2, a), (3, d), (4, c)\}$.
$f$ is not a function $\because$ $1fa$ and $1fb$. That is, the uniqueness condition is not satisfied here.

4) Let $A=\{1, 2, 3, 4\}$. Is $f=\{(a, b) \in A^2 \mid a=b^2\}$ a function?

Solution: Since $2=b^2$ has no solution in $A$, 2 is not $f$-related to any element of $A$.
$\therefore f$ is not a function.

5) Let $A=\{-1, 0, 1\}$. Is $f=\{(a, b) \in A^2 \mid a=b^2\}$ a function?

Solution: Since $1=1^2=(-1)^2$, we have $1f1$ and $1f{-1}$.
$\therefore f$ is not a function.

Remark The existence condition also fails here $\because$ the equation $-1=b^2$ has no solution in $A$. This is another way to show that $f$ is not a function.

6) Is $f=\{(a, b) \in (\mathbb{Z}^+)^2 \mid b=a-1\}$ a function from $\mathbb{Z}^+$ to itself?

Solution: Since $1-1=0 \notin \mathbb{Z}^+$, 1 is not $f$-related to any element of $\mathbb{Z}^+$.
$\therefore f$ is not a function.

7) Is $f=\{(a, b) \in \mathbb{Z}^2 \mid b=a-1\}$ a function from $\mathbb{Z}$ to itself?

Solution: Clearly, for any given number $a$, $a-1$ determines a unique value. Furthermore, if $a \in \mathbb{Z}$, then $a-1$ is also an integer. Summing up, each element $a$ of $\mathbb{Z}$ is uniquely $f$-related to $a-1$ ($\in \mathbb{Z}$).
$\therefore f$ is a function.

## §4.13. Types of Functions

<u>One-to-one</u>

<u>Definition</u>: A function $f$: $A\to B$ is said to be <u>one-to-one</u> (also called <u>injective</u>) iff no two distinct elements of $A$ have the same image.

<u>Remark</u> An equivalent way of saying "$f$ is one-to-one" is that "$a_1=a_2$ whenever $f(a_1)=f(a_2)$". Usually, in order to prove that $f$ is one-to-one, we start with the supposition "suppose $f(a_1)=f(a_2)$ for some $a_1$, $a_2\in A$".

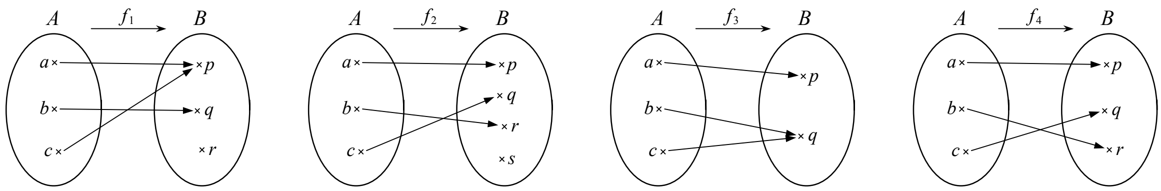
<u>Onto</u>

<u>Definition</u>: A function $f$: $A\to B$ is said to be <u>onto</u> (also called <u>surjective</u>) iff every element of $B$ is an image of one or more elements of $A$.

<u>Remark</u> An equivalent way of saying "$f$ is onto" is that "Ran($f$)$=B$".

<u>Examples</u>

1) Consider the following functions:




$f_1$ is neither one-to-one nor onto.
$f_2$ is one-to-one but not onto.
$f_3$ is onto but not one-to-one.
$f_4$ is both one-to-one and onto.

<u>Remark</u> A function (like $f_4$) that is both one-to-one and onto is said to be <u>bijective</u>.

2) Is the function $f=\{(a, b)\in\mathbb{Z}^2\,|\,b=a^2-a\}$ one-to-one? Onto?

Solution: Since $f(1)=1^2-1=0=f(0)$, $f$ is not one-to-one.
Consider the equation $x^2-x=1$. This equation corresponds to finding a preimage of the element 1.
This quadratic equation has roots $x=\frac{1\pm\sqrt{5}}{2}$, but these roots are not integers.
$\therefore$ 1 has no preimage, and so $f$ is not onto.
Conclusion: $f$ is neither one-to-one nor onto.

3) Is the function $f=\{(a, b)\in(\mathbb{R}^+)^2\,|\,b=a^2+a\}$ one-to-one? Onto?

Solution: Suppose $f(a_1)=f(a_2)$ for some $a_1$, $a_2\in\mathbb{R}^+$. Then

$$a_1^2+a_1=a_2^2+a_2$$
$$\Rightarrow\quad a_2^2-a_1^2+a_2-a_1=0$$
$$\Rightarrow\quad (a_2-a_1)(a_2+a_1+1)=0 \quad \text{----------------------} \quad (*)$$

Since $a_1$ and $a_2$ are positive, we have $a_2+a_1+1>0$. $\therefore$ $(*)\Rightarrow a_2-a_1=0\Rightarrow a_2=a_1$.
$\therefore f$ is one-to-one.
Let $b\in\mathbb{R}^+$ (note that $\mathbb{R}^+$ is the codomain here). Consider the equation $x^2+x=b$.
This quadratic equation has roots $x=\frac{-1\pm\sqrt{1+4b}}{2}$.
Since $b>0$, $\sqrt{1+4b}$ is a real number and $-1+\sqrt{1+4b}>0$. This means that the equation $x^2+x=b$ has a positive root whenever $b>0$.
$\therefore$ $\forall b\in\mathbb{R}^+$, $b$ has a preimage.
$\therefore f$ is onto.
Conclusion: $f$ is bijective.