Indefinite Integral

COMP406 - Calculus
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Indefinite Integral

Given a function $f(x)$, an ***anti-derivative*** of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an ***indefinite integral*** and denoted,

$$\int f(x)\, dx = F(x) + c, \text{ where } c \text{ is an arbitrary constant}$$

The symbol $\int$ is called the ***integral symbol***, $f(x)$ is called the ***integrand***, x is called the ***integration variable*** and the constant c is called the ***constant of integration***.

Properties of Integrals

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx\text{, } c \text{ is a constant}$$

$$\int k\,dx = k\,x + c$$

$$\int x^n\,dx = \tfrac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\int x^{-1}\,dx = \int \tfrac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b| + c$$

$$\int \ln u\,du = u\ln(u) - u + c$$

$$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$$

$$\int \cos u\,du = \sin u + c$$

$$\int \sin u\,du = -\cos u + c$$

$$\int \sec^2 u\,du = \tan u + c$$

$$\int \sec u \tan u\,du = \sec u + c$$

$$\int \csc u \cot u\,du = -\csc u + c$$

$$\int \csc^2 u\,du = -\cot u + c$$

$$\int \tan u\,du = \ln|\sec u| + c$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + c$$

$$\int \frac{1}{a^2+u^2}\,du = \tfrac{1}{a}\tan^{-1}\left(\tfrac{u}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\tfrac{u}{a}\right) + c$$

Computing Integrals

Example 1: Evaluate the indefinite integral $\int 5t^3 - 10t^{-6} + 4 \, dt$

Solution: $\int 5t^3 - 10t^{-6} + 4 \, dt$

$= 5\,(1/4)\, t^4 - 10\,(1/{-5})\, t^{-5} + 4t + c$

$= 5/4\ t^4 + 2t^{-5} + 4t + c$

Example 2: Evaluate the indefinite integral $\int x^8 + x^{-8} \, dx$

Solution: $\int x^8 + x^{-8} \, dx = (1/9)\, x^9 - (1/7)\, x^{-7} + c$

Substitution Rule

Substitution rule helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned} \int f(g(x))g'(x)\ \mathrm{dx} &= \int f(u)\ \mathrm{du} \\ &= F(u) + \mathrm{c} \\ &= F(g(x)) + \mathrm{c} \end{aligned}$$

Substitution rule is used to find the anti-derivative of functions formed by chain-rule.

Substitution Rule

Example: Evaluate the integral $\int (1 - 1/w) \cos(w - \ln w)\, dw$.

Solution: Let $u = w - \ln w$, then $du = (1 - 1/w)\, dw$.

$$
\begin{aligned}
\int (1 - 1/w) \cos(w - \ln w)\, dw &= \int \cos(u)\, du \\
&= \sin(u) + c \\
&= \sin(w - \ln w) + c
\end{aligned}
$$