Determinant

COMP408 - Linear Algebra

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Determinant

Let A be an $n \times n$ matrix. The ***determinant*** of A, written $\det(A)$, is a certain number associated to A which can be defined ***recursively***.

This number has some useful properties (we will discuss later).

Base case: 1 x 1 matrix and 2 x 2 matrix.

The determinant of a 1×1 matrix is the single entry itself: that is $\det(A) = a$.

The determinant of a 2×2 matrix is given by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Determinant

The ***(i, j)-minor*** of *A*, denoted m_{ij}, is the determinant of the matrix obtained from *A* by removing the *i*-th row and the *j*-th column.

The ***(i, j)-cofactor*** of *A*, denoted c_{ij}, is the corresponding minor m_{ij} multiplied by the number $(–1)^{I+j}$: that is $c_{ij} = (–1)^{I+j} m_{ij}$.

The determinant of the 3 × 3 matrix A is $\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$.

The determinant of an $n \times n$ matrix is defined just like the determinant of a 3 × 3 matrix: choose any row or column, multiply its entries by their corresponding cofactors, and add the results.

Determinant

Example: Find the determinant of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}.$$

Solution: The following are some examples of the (i, j)-minor of A:

$$m_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix}, \quad m_{12} = \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}, \quad m_{13} = \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix},$$

$$m_{21} = \begin{vmatrix} -1 & 0 \\ 1 & 6 \end{vmatrix}, \quad \text{etc.}$$

Determinant

Solution: (cont.) The (i, j)-cofactors of *A* are as follows.

$$c_{11} = (-1)^{1+1} m_{11} = (+1) \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = (+1)(16) = 16,$$

$$c_{12} = (-1)^{1+2} m_{12} = (-1) \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = (-1)(-4) = 4,$$

$$c_{13} = (-1)^{1+3} m_{13} = (+1) \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} = (+1)(-14) = -14,$$

$$c_{21} = (-1)^{2+1} m_{21} = (-1) \begin{vmatrix} -1 & 0 \\ 1 & 6 \end{vmatrix} = (-1)(-6) = 6,$$

etc.

Thus the determinant of *A* is as follows:

$$\begin{aligned} \det(A) &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\ &= (2)(16) + (-1)(4) + (0)(-14) = 28. \end{aligned}$$

Some properties of determinant

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \det(B)$.

If A is an $n \times n$ matrix, then A is invertible if and only if $\det(A) \neq 0$.

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

If A is an $n \times n$ matrix and either its rows or columns are linearly dependent, then $\det(A) = 0$.

Parallelepiped

Let c_1, c_2, and c_3 be vectors in $\mathbf{R}^3$. The volume of the ***parallelepiped*** determined by these three vectors is det(A), where A is the matrix having the three vectors as columns.