Column Space and Row Space

COMP408 - Linear Algebra
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Column space and row space

The ***column space (range space)***, col(*A*), of *A* is the subspace of R^m spanned by the columns of *A*.

The ***row space***, row(*A*), of *A* is the subspace of R^n spanned by the rows of *A*.

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

If *A* and *B* are matrices with A ~ B, then row(*A*) = row(*B*).

Bases of column space and row space

If R is a row-echelon matrix, then

1. The nonzero rows of R are a basis of row R.
2. The columns of R containing leading ones are a basis of col R.

Example: Consider the following matrix

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & -3 \\ 0 & -1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The basis of the column space are $(1, 2, 1, 1)^T$, $(3, 7, 5, 2)^T$ and $(4, 9, 1, 8)^T$.

Rank

Let A denote any $m \times n$ matrix of ***rank*** r. Then $dim(col\ A) = dim(row\ A) = r$.

Moreover, if A is carried to a row-echelon matrix R by row operations, then

1. The r nonzero rows of R are a basis of row A.
2. If the leading 1s lie in columns $j_1, j_2, \dots, j_r$ of R, then columns $j_1, j_2, \dots, j_r$ of A are a basis of $col\ A$.

If A is any matrix, then rank A = rank (A^T).

Rank

Example: Compute the rank and the bases of row A and col A for the following matrix:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank A = 2, and {[1, 2, 2, –1] , [0, 0, 1, –3]} is a basis of row A.

Furthermore, columns 1 and 3 of *A* are a basis $\{[1, 3, 1]^T, [2, 5, 1]^T\}$ of *col A*.

Null space and image space

The null space of A, denoted null(A), is defined by null(A) = $\{x \in R^n \mid Ax = 0\}$.

The image space of A, denoted $im(A)$, are defined by $im(A) = \{Ax \mid x \in R^n\}$

In other words, $null(A)$ consists of all solutions x in R^n of the homogeneous system $Ax = 0$, and $im(A)$ is the set of all vectors y in R^m such that $Ax = y$ has a solution x.