Derivatives

COMP406 - Calculus

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Derivative

The ***derivative*** of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

The following notations are equivalent which represent the derivative of $f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}(y)$$

The following notations denote the derivative at $x = a$:

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Derivative

A function $f(x)$ is called ***differentiable*** at $x = a$ if $f'(a)$ exists and $f(x)$ is called ***differentiable on an interval*** if the derivative exists for each point in that interval.

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Exercise: Find the derivative of the function $f(x) = 2x^2 - 16x + 35$ using the definition of the derivative.

Interpretation of Derivative

If $y = f(x)$ then m = $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.

$f'(a)$ can also be interpreted as the instantaneous rate of change of $f(x)$ at $x = a$.

If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object $x = a$.

Differentiation rules

If f has the constant value $f(x) = c$, then $f'(x) = 0$.

Power rule: If n is a real number and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

If u is a differentiable function of x, and c is a constant, then $f'(cu) = cf'(u)$.

If u and y are differentiable functions of x, then their sum $u + y$ is differentiable at every point where u and y are both differentiable. At such points, $f'(u + v) = f'(u) + f'(v)$.

Differentiation rules

Product rule: If u and v are differentiable at x, then so is their product uv, and $f'(uv) = uf'(v) + vf'(u)$.

Quotient rule: If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and $f'(u/v) = (vf'(u) - uf'(v)) / v^2$.

Derivative of natural exponential function: If $f(x) = e^x$ then $f'(e^x) = e^x$.

Derivative of natural logarithm function: If $f(x) = \ln x$ then $f'(\ln x) = 1/x$.

Differentiation rules

The derivatives of all six trigonometric functions:

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

Differentiation rules

The derivatives of all six inverse trigonometric functions:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Chain rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(\mathrm{x}))$ is differentiable at x, and $(f \circ g)'(x) = f'(g(\mathrm{x}))\, g'(x)$.

Example: Use the chain rule to differentiate $R(z) = \sqrt{(5z - 8)}$.

Let $\mathrm{f(x)} = \sqrt{z}$ and $\mathrm{g(z) = 5z - 8}$.

$$
\begin{aligned}
R'(z) &= f'(\mathrm{g(z)})\, g'(z) \\
&= f'(5z - 8)\, g'(z) \\
&= (1/2)(5z - 8)^{(-1/2)}\ (5) \\
&= 5/(2\sqrt{(5z - 8)})
\end{aligned}
$$

Higher-order derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by $f'' = (f')'$. The function f'' is called the ***second derivative*** of f because it is the derivative of the first derivative.

We can generalize the idea to the ***third derivative***, and the ***n*-th derivative** of y respect to x.

Example: The first four derivatives of $y = x^3 - 3x^2 + 2$ are:

First derivative: $y' = 3x^2 - 6x$
Second derivative: $y'' = 6x - 6$
Third derivative: $y''' = 6$
Fourth derivative: $y'''' = 0$