Linear Independence

COMP408 - Linear Algebra

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Linear independence

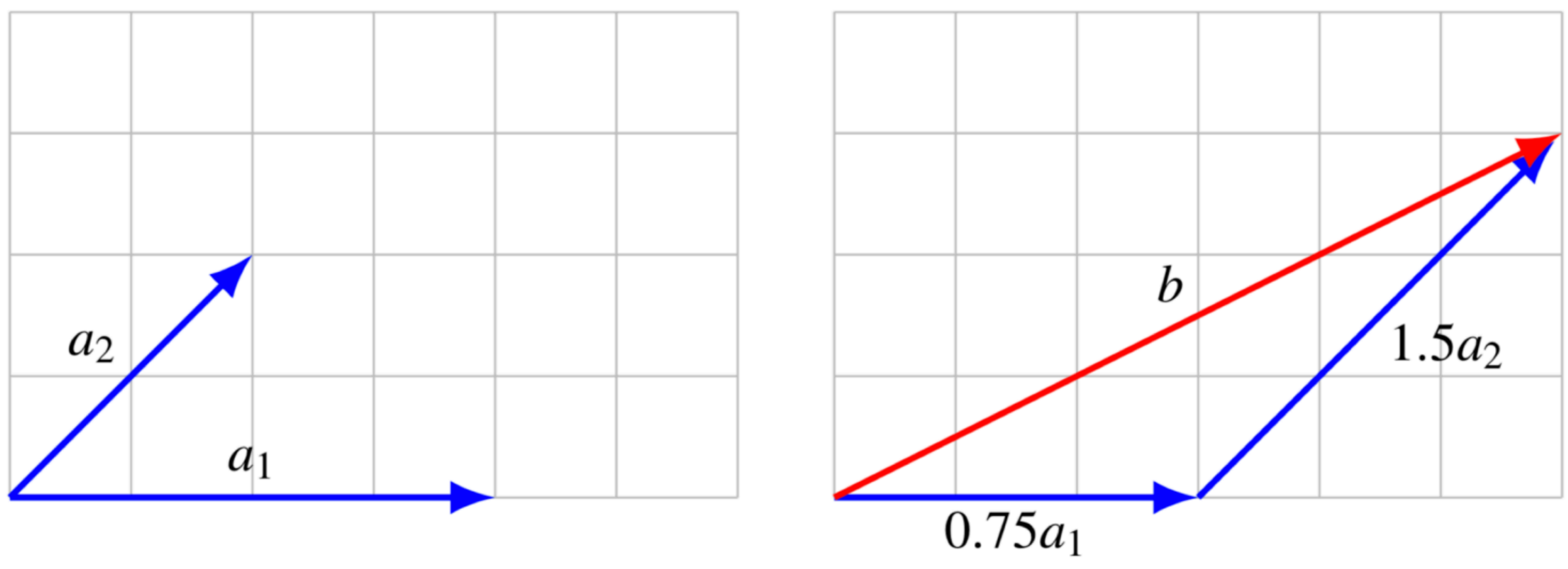
A family of vectors is ***linearly independent*** if no one of the vectors can be created by any linear combination of the other vectors in the family.

In other words, if two vectors point in different directions, they are said to be linearly independent.

If two vectors point in the same direction, then we can multiply one of the vector with a scalar to get the other vector, and the two vectors are said to be ***linearly dependent***.

Linear independence

Example: The below vector b is said to be linearly dependent to the vector a_1 and a_2.



Linear independence

We say that vectors $x_1, x_2, \ldots, x_s$ in R^n are ***linearly dependent*** if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_s$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_s x_s = 0.$$

We say that the vectors are ***linearly independent*** if they are not linearly dependent, that is, if

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_s x_s = 0 \text{ implies } \alpha_i = 0 \text{ for all } i.$$

The zero vector can never be on a list of independent vectors because $\alpha\vec{0} = \vec{0}$ for any scalar α.

Linear independence

We can use system of linear equations to show linear dependence of vectors.

Example: Determine whether the vectors x_1 = [-1, 3], x_2 = [5, 6] and x_3 = [1, 4] in R^2 are linearly dependent or linearly independent.

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 = \mathbf{0}$$

$$\alpha_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\alpha_1 + 5\alpha_2 + \alpha_3 \\ 3\alpha_1 + 6\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} -1 & 5 & 1 & 0 \\ 3 & 6 & 4 & 0 \end{array}\right] & \begin{array}{c} 3 \\ \end{array} \end{array} \sim \left[\begin{array}{ccc|c} -1 & 5 & 1 & 0 \\ 0 & 21 & 7 & 0 \end{array}\right].$$

Since there is a solution, the vectors are linearly dependent.