Diagonalization

COMP408 - Linear Algebra
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Change of basis

Let $B = (b_1, b_2, \ldots, b_n)$ and $C = (c_1, c_2, \ldots, c_n)$ be two ordered bases for $\mathbf{R}^n$, and put $B = b_1\ b_2 \ldots b_n$ and $C = c_1\ c_2 \ldots c_n$. For any vector x in $\mathbf{R}^n$, we have

$$B[x]_B = C[x]_C$$

so that

$$[x]_B = B^{-1}C[x]_C$$

The matrix $P = B^{-1}C$ is called the ***transformation matrix*** from C to B.

Change of basis

Example: Let $B = (b_1, b_2, \dots, b_n)$ and $C = (c_1, c_2, \dots, c_n)$ be the ordered bases for $\mathbf{R}^2$ with $b_1 = [2, 6]^T$, $b_2 = [1, 4]^T$, $c_1 = [0, -1]^T$, $c_2 = [1, 5]^T$.

(a). Find the transformation matrix P from C to B.

$$\mathbf{P} = \mathbf{B}^{-1}\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{bmatrix}.$$

(b). Given $[x]_C = [-3, 6]^T$, find $[x]_B$.

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{P}[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} \\ 15 \end{bmatrix}.$$

(c). Given $[y]_B = [7, 2]^T$, find $[y]_C$.

$$[\mathbf{y}]_{\mathcal{C}} = \mathbf{P}^{-1}[\mathbf{y}]_{\mathcal{B}} = \frac{1}{\frac{1}{2}}\begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 15 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 \\ 16 \end{bmatrix}.$$

Change of basis

Let L: $\mathbf{R}^n \to \mathbf{R}^n$ be a linear function, let B and C be ordered bases for $\mathbf{R}^n$, let P be the transformation matrix from C to B, and let A be the matrix of L relative to B. The matrix of L relative to C is $P^{-1}AP$, that is,

$$[L(x)]_C = P^{-1}AP[x]_C$$

for all $x \in \mathbf{R}^n$.

Method for diagonalization

A ***diagonal matrix*** is a matrix having the property that every entry not on the main diagonal is 0.

An $n \times n$ matrix A is ***diagonalizable*** if there exists an invertible $n \times n$ matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

Let A be an $n \times n$ matrix. The matrix A is ***diagonalizable*** if and only if there exists a basis for $\mathbf{R}^n$ consisting of eigenvectors of A. In this case, if P is the matrix with the eigenvectors as columns, then $P^{-1}AP = D$, where D is a diagonal matrix.

Method for diagonalization

Example: Find an invertible 2 x 2 matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix, for the following matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Solution: According to the theorem, such a matrix P exists if and only if there exists a basis for $\mathbf{R}^2$ consisting of eigenvectors of A. The characteristic polynomial of A is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3).$$

The eigenvalues of A are the zeros of this polynomial, namely, $\lambda = 2, 3$.

Method for diagonalization

Solution: (cont.) Next, the λ-eigenspace of A is the solution set of the equation $(A - \lambda I)x = 0$. When $\lambda = 2$ we have

$$[\mathbf{A} - 2\mathbf{I} \,|\, \mathbf{0}] \quad = \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

so the 2-eigenspace of A is $\{[t, t]^T \,|\, t \in \mathbf{R}\}$. Letting $t = 1$, we get a 2-eigenvector $[1, 1]^T$;

When $\lambda = 3$ we have

$$[\mathbf{A} - 3\mathbf{I} \,|\, \mathbf{0}] \quad = \quad \left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

so the 3-eigenspace of A is $\{[t/2, t]^T \,|\, t \in \mathbf{R}\}$. Letting $t = 2$ (to avoid fractions), we get a 3-eigenvector $[1, 2]^T$;

Method for diagonalization

Solution: (cont.) The eigenvectors $[1, 1]^T$ and $[1, 2]^T$ of A form a basis for $\mathbf{R}^2$ (neither is a multiple of the other so they are linearly independent; since dim $\mathbf{R}^2 = 2$, they form a basis). According to the theorem, the matrix P with these vectors as columns should have the indicated property:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Computing we get

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \mathbf{D}. \end{aligned}$$

Power of matrix

Computing high power of a square matrix can be much simplify if the matrix is diagonalizable.

Let A be a diagonalizable matrix. If $P^{-1}AP = D$, then $A^n = PD^nP^{-1}$.

Example: Compute A^{10} where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Solution: In previous slides we have already found P.

$$\begin{aligned}
\mathbf{A}^{10} = \mathbf{P}\mathbf{D}^{10}\mathbf{P}^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{10} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{10} & 3^{10} \\ 2^{10} & 2 \cdot 3^{10} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{11} - 3^{10} & -2^{10} + 3^{10} \\ 2^{11} - 2 \cdot 3^{10} & -2^{10} + 2 \cdot 3^{10} \end{bmatrix} \\
&= \begin{bmatrix} -57001 & 58025 \\ -116050 & 117074 \end{bmatrix}
\end{aligned}$$