System of Linear Equations

COMP408 - Linear Algebra

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Linear equation

A ***linear equation*** is an equation that may be put in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + b = 0$$

where x_1, x_2, …, x_n are the ***variables*** (or unknowns), and b, a_1, a_2, …, a_n are the coefficients (***constants***).

A linear function is a linear equation. But a linear equation can be not a function. Why?

Example: x + 3y = 7, 2x - 5y + 8 = 0 are both linear equations and linear functions, while x = -2 is a linear equation but not a linear function.

System of linear equations

Given a system of n linear equations with n unknown ***variable*** $x_1, x_2, \ldots,$ x_n. A ***solution*** to the system is a tuple $(x_1, x_2, \ldots, x_n)$ such that it satisfies all the linear equations.

Example: Given the following two linear equations with two variables x and y:

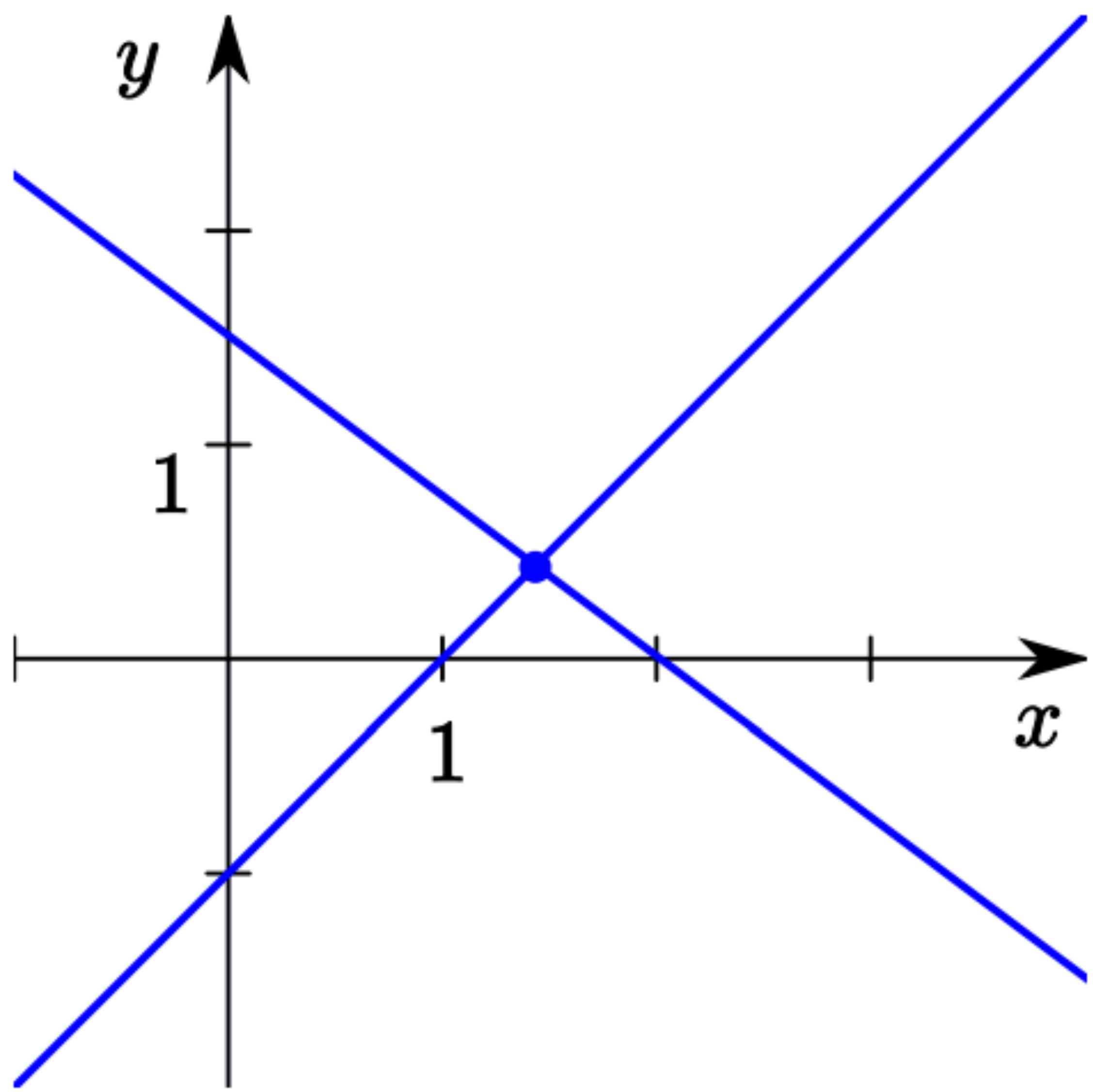
$$x - y = 1$$
$$3x + 4y = 6$$

$$-3(x - y = 1)$$
$$3x + 4y = 6$$

$$x - y = 1$$
$$7y = 3$$

There is only one solution that satisfies both equations, and that is (10/7, 3/7), that is $x = 10/7$ and $y = 3/7$.

System of linear equations



System of linear equations

Given a system of linear equations, there are three possibilities in terms of the number of solution.

It is possible the system of linear equations has one unique solution, just like the example we show.

If the lines that the equations represent are ***coincident*** (i.e., the same), then the solution includes every point on the line so there are infinitely many solutions.

If the equations represent ***parallel*** but not coincident lines, then there is no solution

System of linear equations

Exercise 1 (Infinitely many solutions): Consider the following system:

$$-x + 4y = 2$$
$$3x - 12y = -6$$

Solve the system of linear equations. How the equations appear in a 2-dimensional space?

Exercise 2 (No solution): Consider the following system:

$$x + y = 1$$
$$2x - 2y = -2$$

Solve the system of linear equations. How the equations appear in a 2-dimensional space?

Gaussian elimination

Gaussian elimination, also known as ***row reduction***, is an algorithm for solving systems of linear equations.

Example: Solve the following system:

$$x_1 - 2x_2 + 3x_3 = 1$$
$$2x_1 - 3x_2 + 5x_3 = 0$$
$$-x_1 + 4x_2 - x_3 = -1$$

Add -2 times the first equation to the second equation in order to cancel the x_1-term (***operation type-2 and type-3***):

$$x_1 - 2x_2 + 3x_3 = 1$$
$$x_2 - x_3 = -2$$
$$-x_1 + 4x_2 - x_3 = -1$$

Gaussian elimination

Example: Solve the following system (cont.):

Then add the first equation to the third equation, again in order to cancel the x_1-term (***operation type-3***):

$$x_1 - 2x_2 + 3x_3 = 1$$
$$x_2 - x_3 = -2$$
$$2x_2 + 2x_3 = 0$$

Then add -2 times the second equation to the third equation in order to cancel the x_2-term (***operation type-2 and type-3***):

$$x_1 - 2x_2 + 3x_3 = 1$$
$$x_2 - x_3 = -2$$
$$4x_3 = 4$$

Gaussian elimination

Example: Solve the following system (cont.):

The last equation shows that $x_3 = 1$. The other unknowns are determined using a process called ***back substitution***: Now that we know that $x_3 = 1$, we use the second equation:

$$x_2 - x_3 = -2$$
$$x_2 - 1 = -2$$
$$x_2 = -1$$

And then the first equation:

$$x_1 - 2x_2 + 3x_3 = 1$$
$$x_1 - 2(-1) + 3(1) = 1$$
$$x_1 = -4$$

Therefore, the three planes intersect in the point (–4, –1, 1). The solution set is {(–4, –1, 1)}.

Augmented matrix

Let’s simplify the system of linear equations.

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 1 \\
2x_1 - 3x_2 + 5x_3 &= 0 \\
-x_1 + 4x_2 - x_3 &= -1
\end{aligned}
$$

We use a rectangular array of numbers to represent the coefficients of each variable, and the constant term of each linear equation is represented by the numbers on the right.

$$
\left[\begin{array}{ccc|c}
1 & -2 & 3 & 1 \\
2 & -3 & 5 & 0 \\
-1 & 4 & -1 & -1
\end{array}\right]
$$