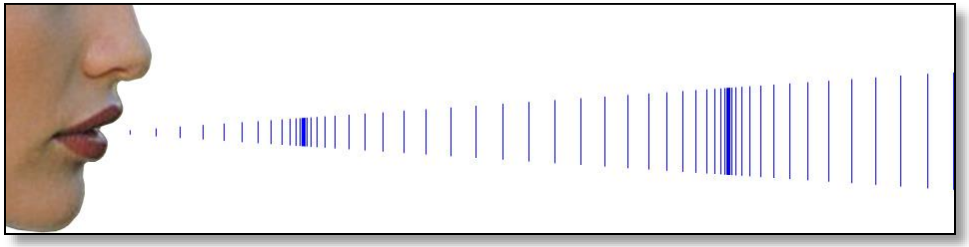
Fourier Transforms

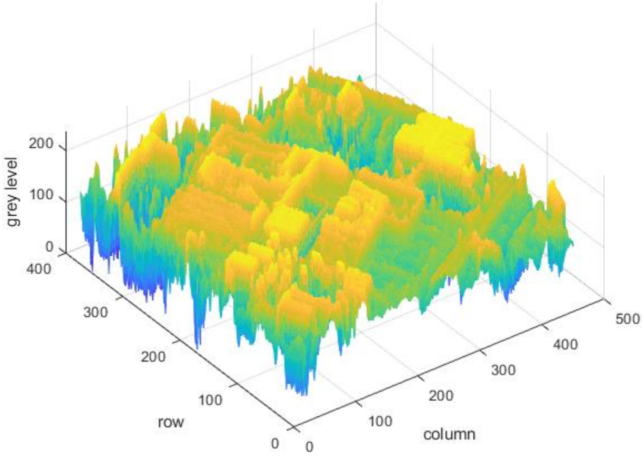
Signal

A measurable phenomenon that changes over time, throughout space, or both.

sound



image



code

01101000101101110110010110001

Space/Time vs. Frequency Domain Representation

Space/time representation: a graph of the measurements with respect to a point in time and/or positions in space.

Fact: signals undulate (otherwise they'd contain no information).

Frequency-domain representation: an exact description of a signal *in terms of* its undulations.

Space/Time vs. Frequency Domain Representation

Time domain

representation

Frequency domain

representation

The world is eternal in frequency domain!

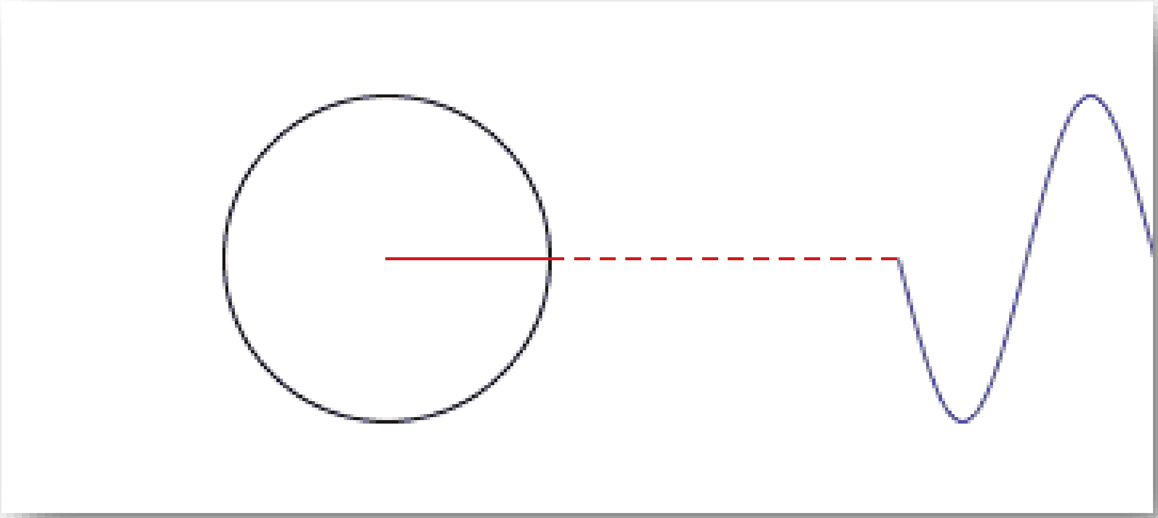
Fact: Any real signal has a Frequency-Domain Representation!

The generation of square wave

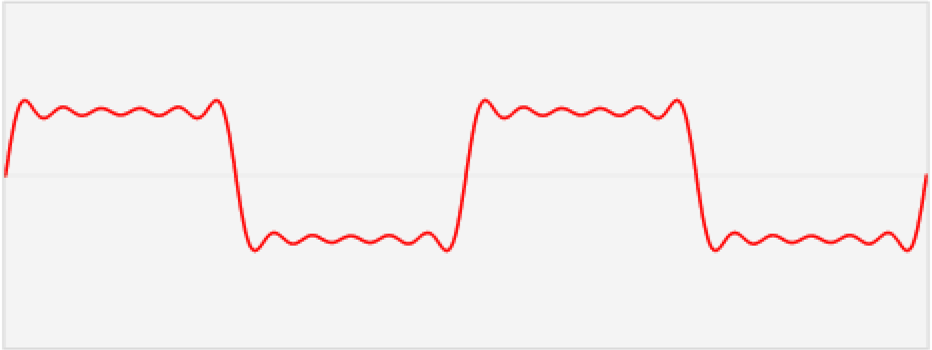
Any periodic signals can be described by a sum of sinusoids.

The sinusoids are called “basis functions”.

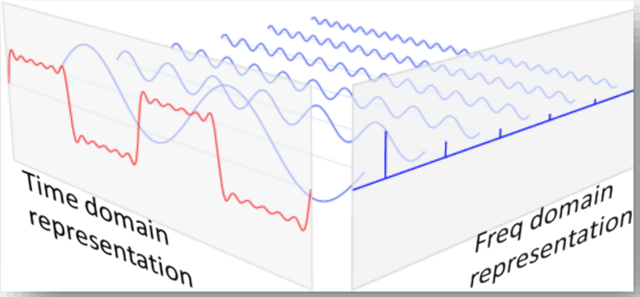
$$\mathrm{sq}(t)=\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}\sin\left[\frac{2\pi}{\lambda}(2n+1)\,t\right]$$



Time and Frequency representation of square wave



Through Fourier Transform equations, we can identify exactly which sine are used to compose a periodic signal.



More examples: http://www.falstad.com/fourier/j2/

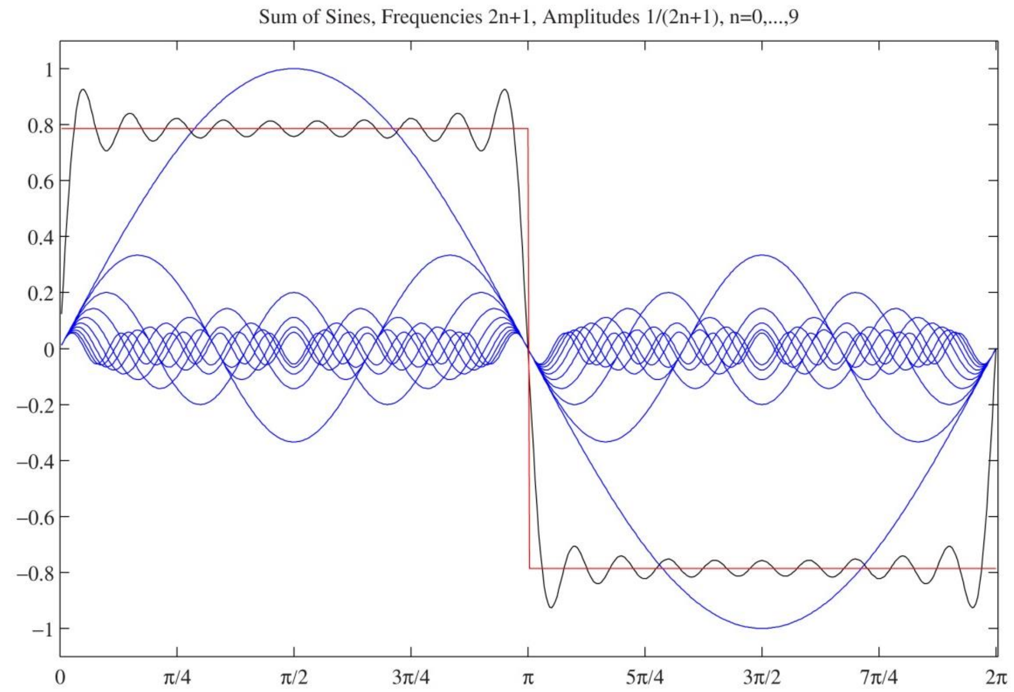
Fact: Any Real Signal has a Frequency-Domain Representation

Odd-order harmonics

$$\mathrm{sq}(t)=\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}\sin\left[\frac{2\pi}{\lambda}(2n+1)t\right]$$

The modes shown (blue) sum to the rippling square wave (black).

As the number of modes in the sum becomes large, it approaches a square wave (red).



The Fourier Transform

A transform turns one function (or signal) in the time/space domain into its frequency form. (The decomposition of a *nonperiodic* signal into a continuous integral of sinusoids.)

Fourier Transform (FT):

Frequency form

Frequency form $$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt$$

Time/space form

Inverse Fourier transform (IFT):

Time/Space form $$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df$$

The Discrete Fourier Transform (DFT)

A 'sampled version' of Fourier Transform. Used for *digital signals*.

For a discrete signal $\{ h_k \mid k = 0,\ 1,\ 2,\ \ldots,\ N-1 \}$

Discrete Fourier Transform (DFT):

Frequency form $$H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i k n / N}$$

Inverse Discrete Fourier transform (IDFT):

Time/Space form $$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n / N}$$

Fast Fourier Transform (FFT)

FFT is a very efficient algorithm for performing a DFT.

FFT principle first used by Gauss in 18??

FFT algorithm published by Cooley & Tukey in 1965

In 1969, the 2048-point analysis of a seismic trace took 13 ½ hours. Using the FFT, the same task on the same machine took 2.4 seconds!

Matlab functions:
1D: fft()/ifft() 2D:fft2()/ifft2()
Scilab functions:
1D: fft()/ifft() 2D:fft2()/NA

Two-Dimensional Fourier Transform

Why Fourier Transform on Image:

The image in the Fourier domain is decomposed into its sinusoidal components.
easy to examine or process certain frequencies of the image.
thus influencing the geometric structure in the spatial domain.

Two-Dimensional Fourier Transform

Primary Uses of the FT in Image Processing:

Explains why down-sampling can add distortion to an image and shows how to avoid it.

Useful for certain types of noise reduction, deblurring, and other types of image restoration.

For feature detection and enhancement, especially edge detection.

2D DFT on a Digital Image

Let **I**(*r, c*) be a single-band (intensity) digital image with *R* rows and *C* columns. Then, **I**(*r, c*) has Fourier representation

$$\mathbf{I}(r,c) = \frac{1}{RC}\sum_{u=0}^{R-1}\sum_{v=0}^{C-1}\mathfrak{I}(v,u)\,e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)},$$

these complex exponentials are 2D sinusoids.

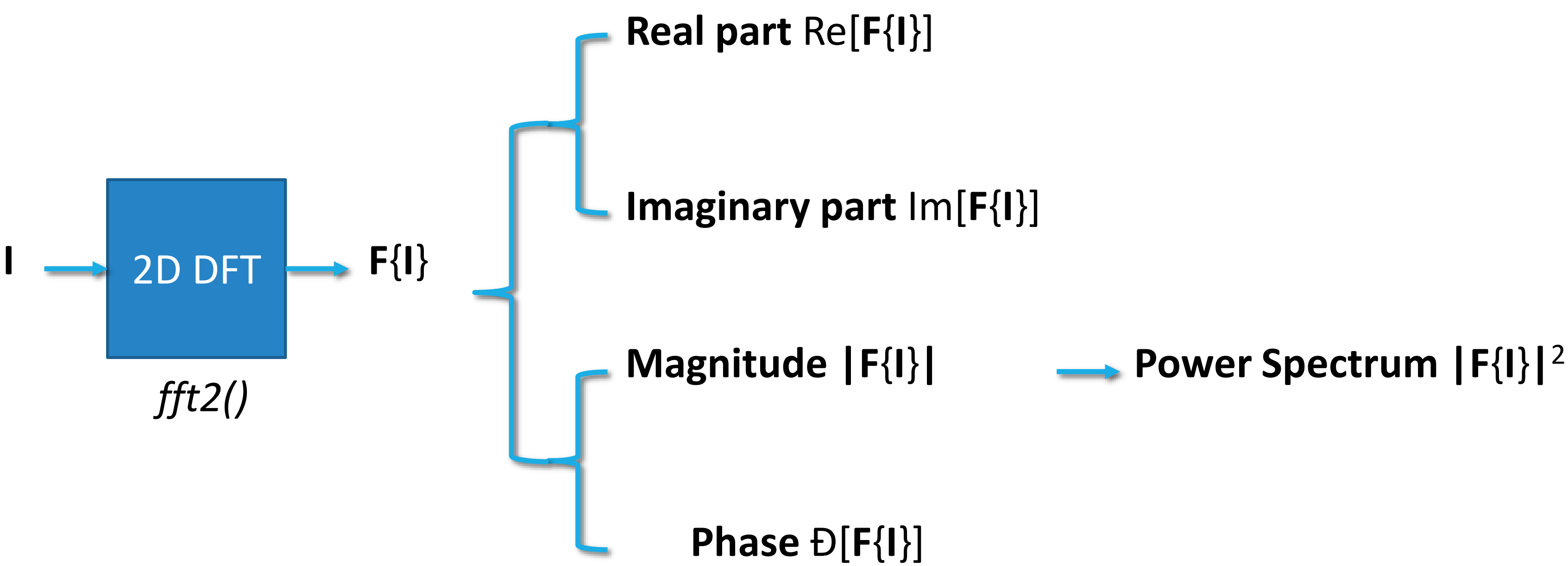
where

$$\mathfrak{I}(v,u) = \sum_{r=0}^{R-1}\sum_{c=0}^{C-1}\mathbf{I}(r,c)\,e^{-i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

are the *R* x *C* Fourier coefficients.

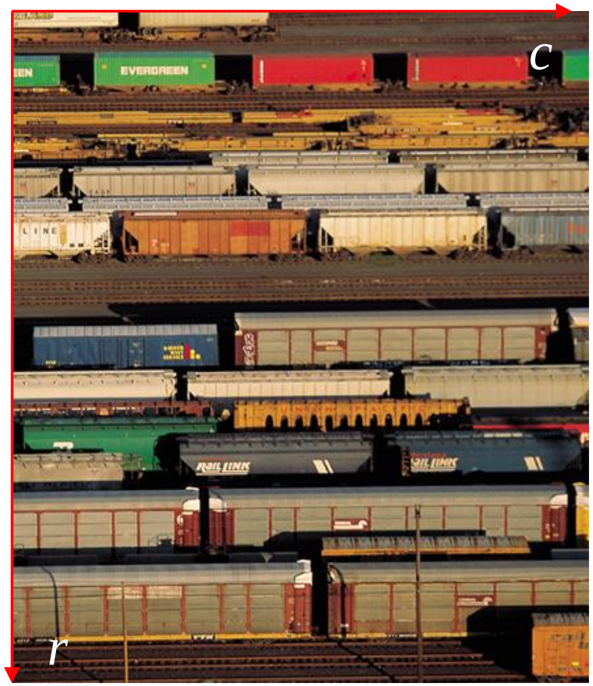
In matlab/scilab
fft2(I) is equivalent to fft(fft(I)')'

2D DFT on a Digital Image

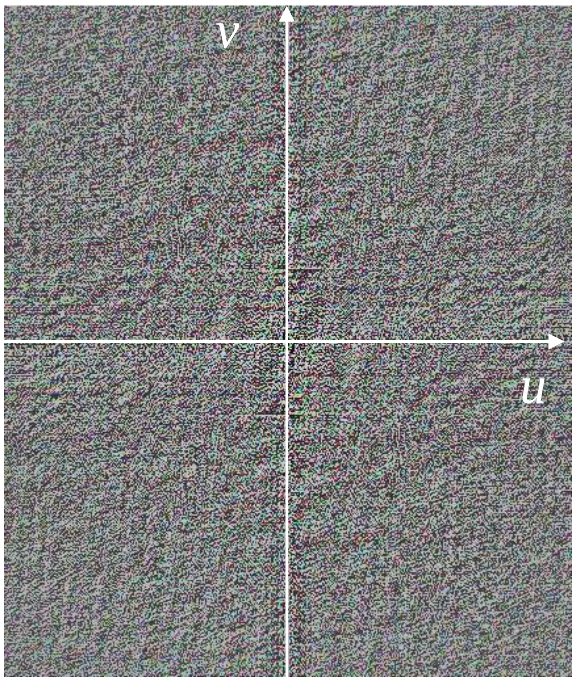


For better display, *fftshift(X)* and *log(1+X)* are used.

2D DFT on a Digital Image

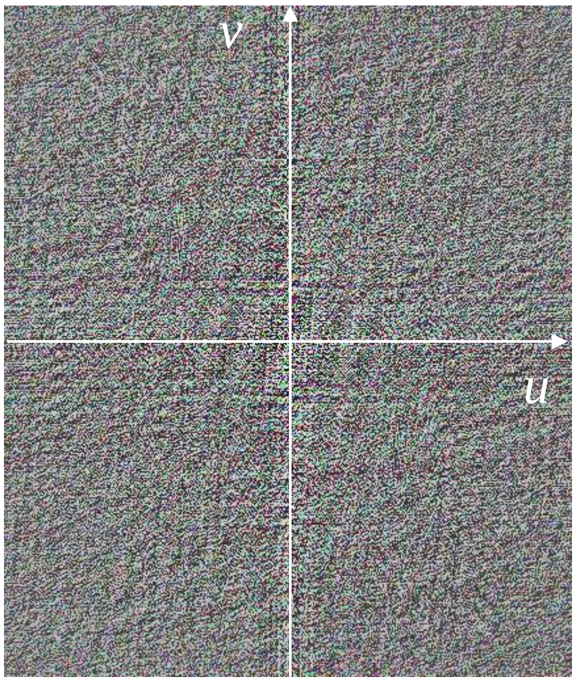


$\mathbf{I}$



$\mathrm{Re}[\mathfrak{F}\{\mathbf{I}\}]$

Real

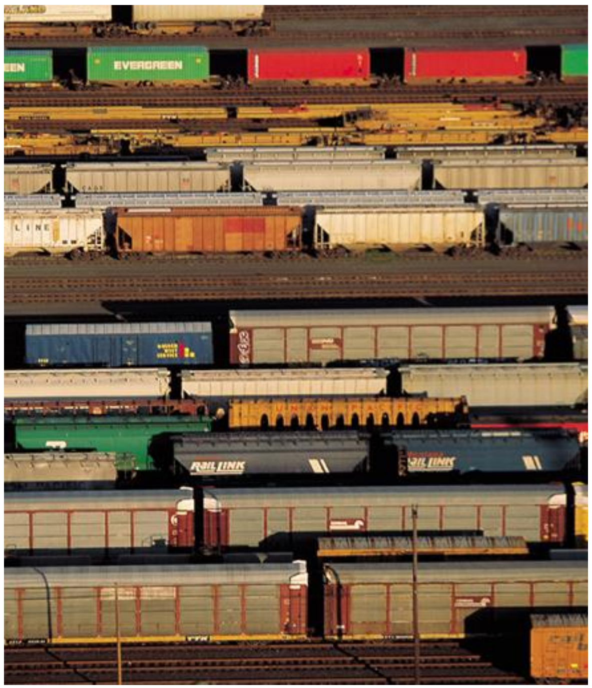


$\mathrm{Im}[\mathfrak{F}\{\mathbf{I}\}]$

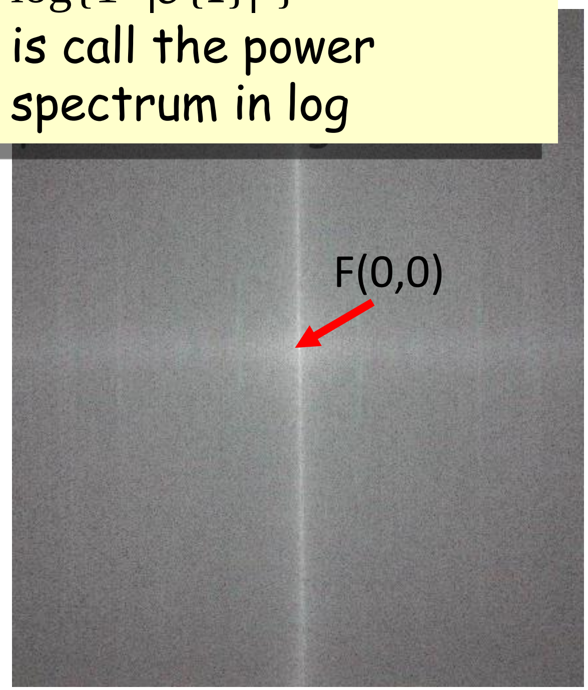
Imaginary

2D DFT on a Digital Image

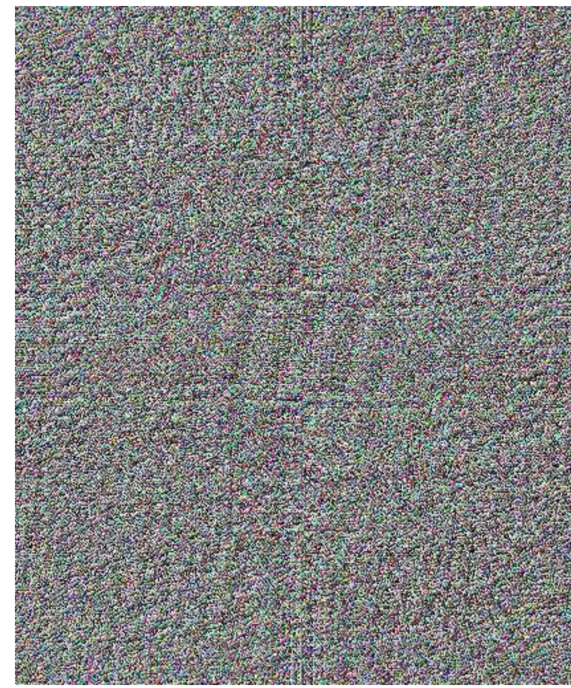
$\log\{1+|\mathfrak{F}\{\mathbf{I}\}|^2\}$
is call the power
spectrum in log



$\mathbf{I}$



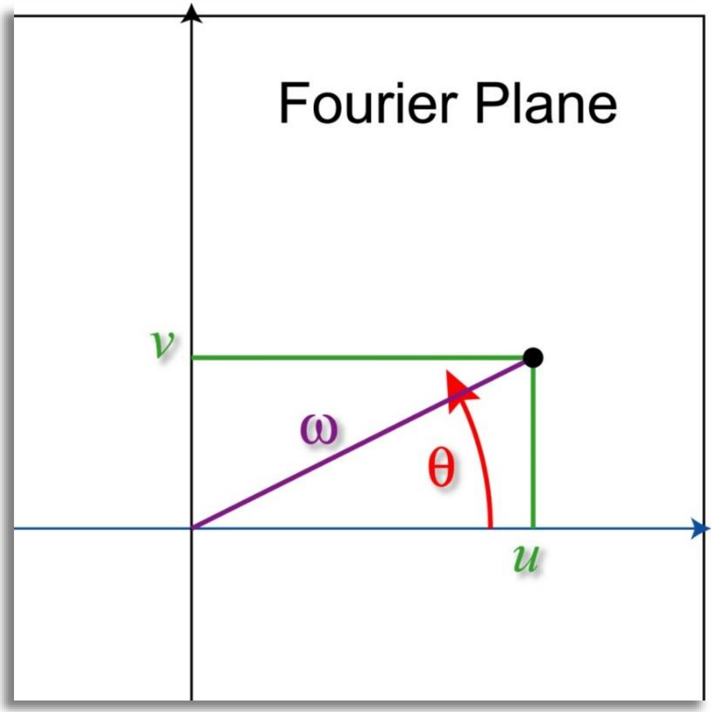
$\log\{1+|\mathfrak{F}\{\mathbf{I}\}|^2\}$
Magnitude^2 in log



$\angle[\mathfrak{F}\{\mathbf{I}\}]$
Phase

Magnitude

- The image contains components of all frequencies, each point indicates a frequency.
- F(0,0) located in the centre of Fourier plane is the DC-value (image mean).
- The further away from the origin a point is, the higher its frequency is.
- The magnitude gets smaller for higher frequencies.
- Low frequencies contain more image information than the higher ones.



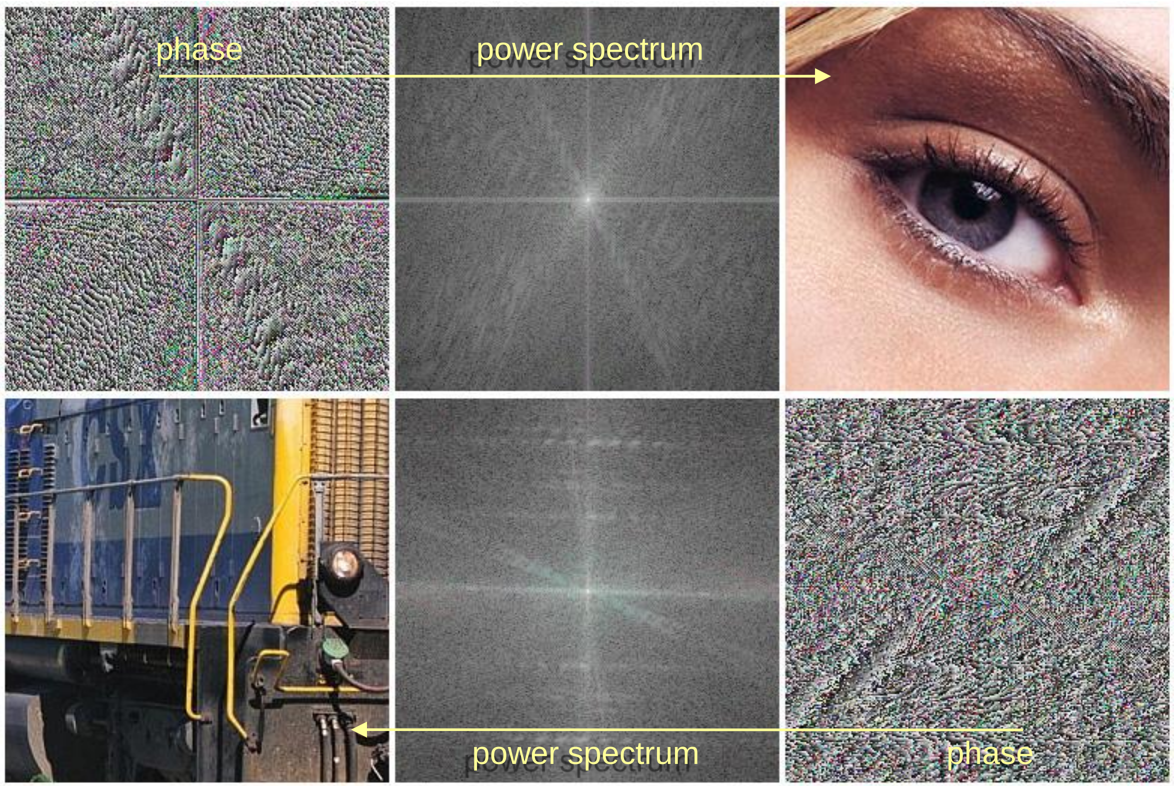
Magnitude

- There are two dominating directions in the Fourier image, one passing vertically and one horizontally through the centre.
- These originate from the regular patterns in the background of the original image.

Phase

- The value of each point determines the phase of the corresponding frequency.
- It reveals almost the same information about the structure of the spatial domain image as the magnitude image.
- The phase information is crucial to reconstruct the correct image in the spatial domain.

Relationship between Image and FT



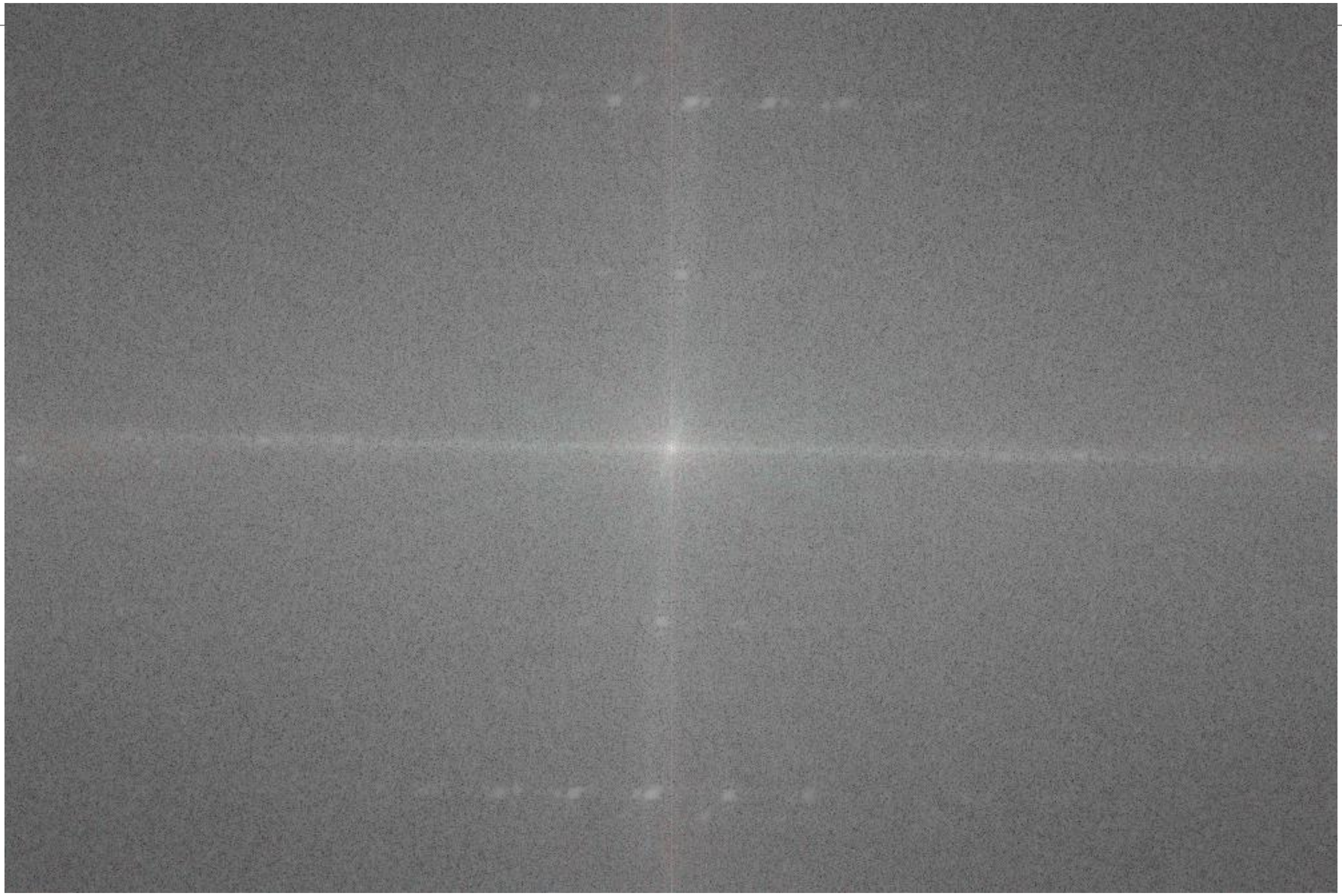
*The power spectrum of a signal is the square of the magnitude of its Fourier Transform. $|\mathfrak{F}\{\mathbf{I}\}|^2$

Fourier Magnitude and Phase

I

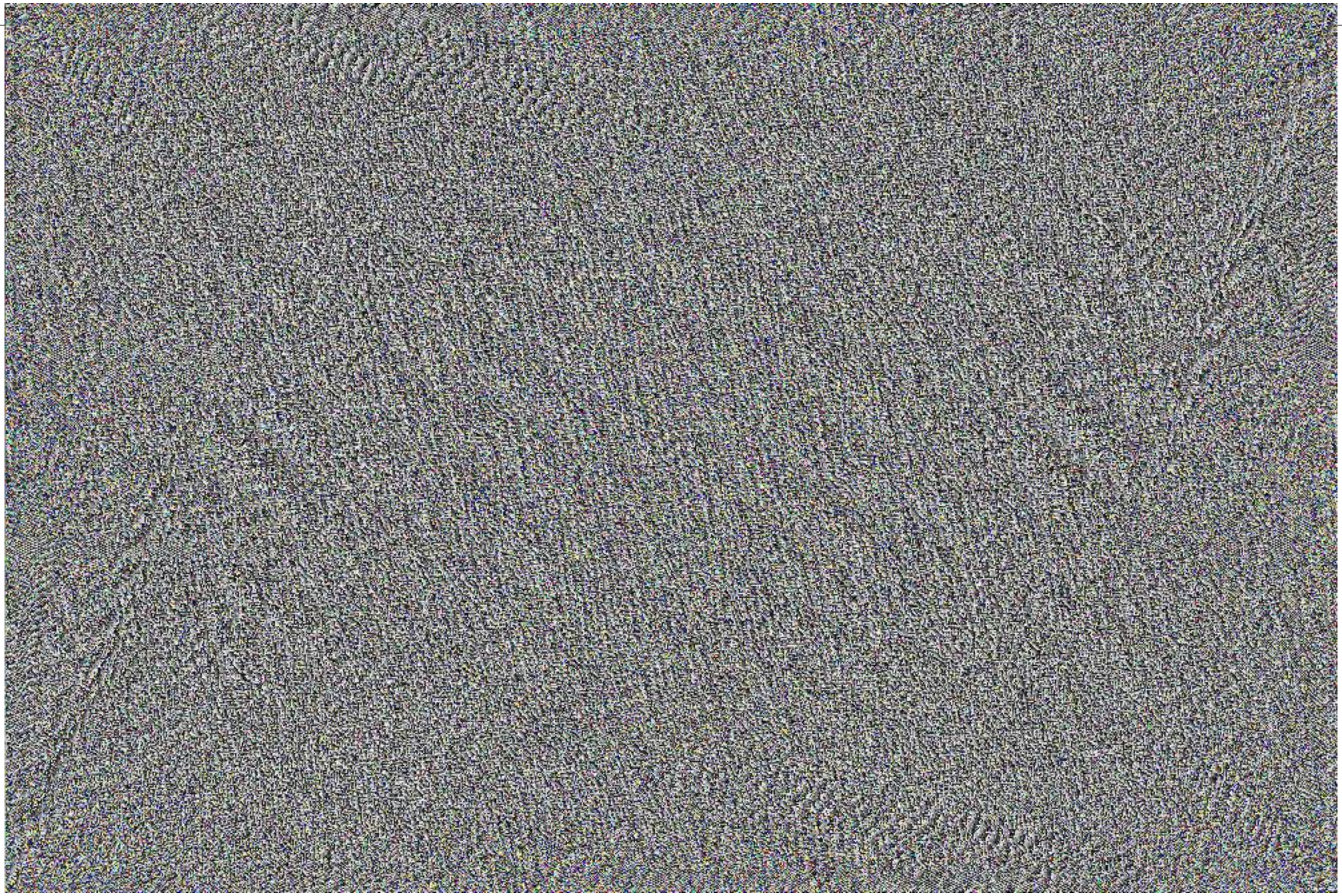
Fourier Magnitude

$\log(1 + |\mathfrak{F}\{\mathbf{I}\}|)$



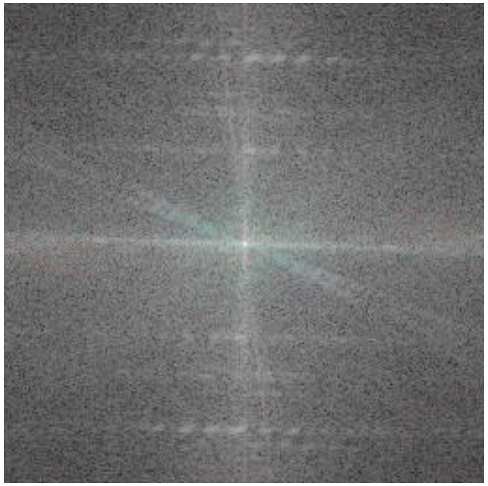
Fourier Phase

$\angle \mathfrak{F}\{\mathbf{I}\}$



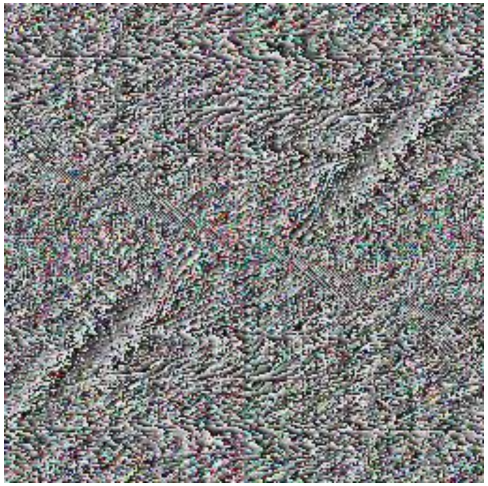
Q: Which contains more visually relevant information; magnitude or phase?

original image



Fourier magnitude in log

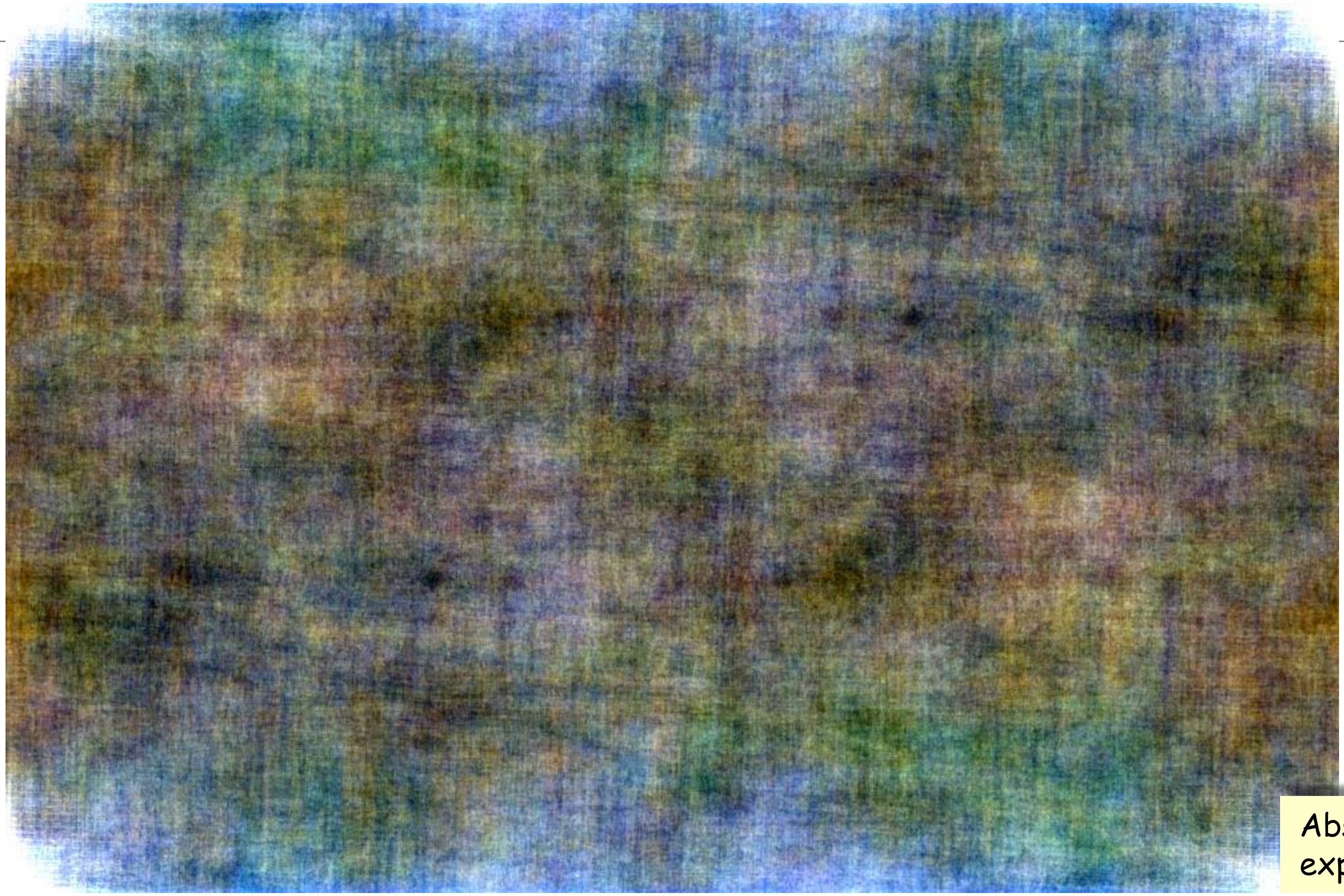
$$\log(1+\left|\mathfrak{F}\{\mathbf{I}\}\right|)$$



Fourier phase

$$\angle\mathfrak{F}\{\mathbf{I}\}$$

Magnitude Only Reconstruction



Phase of FT set to 0 before inverse.

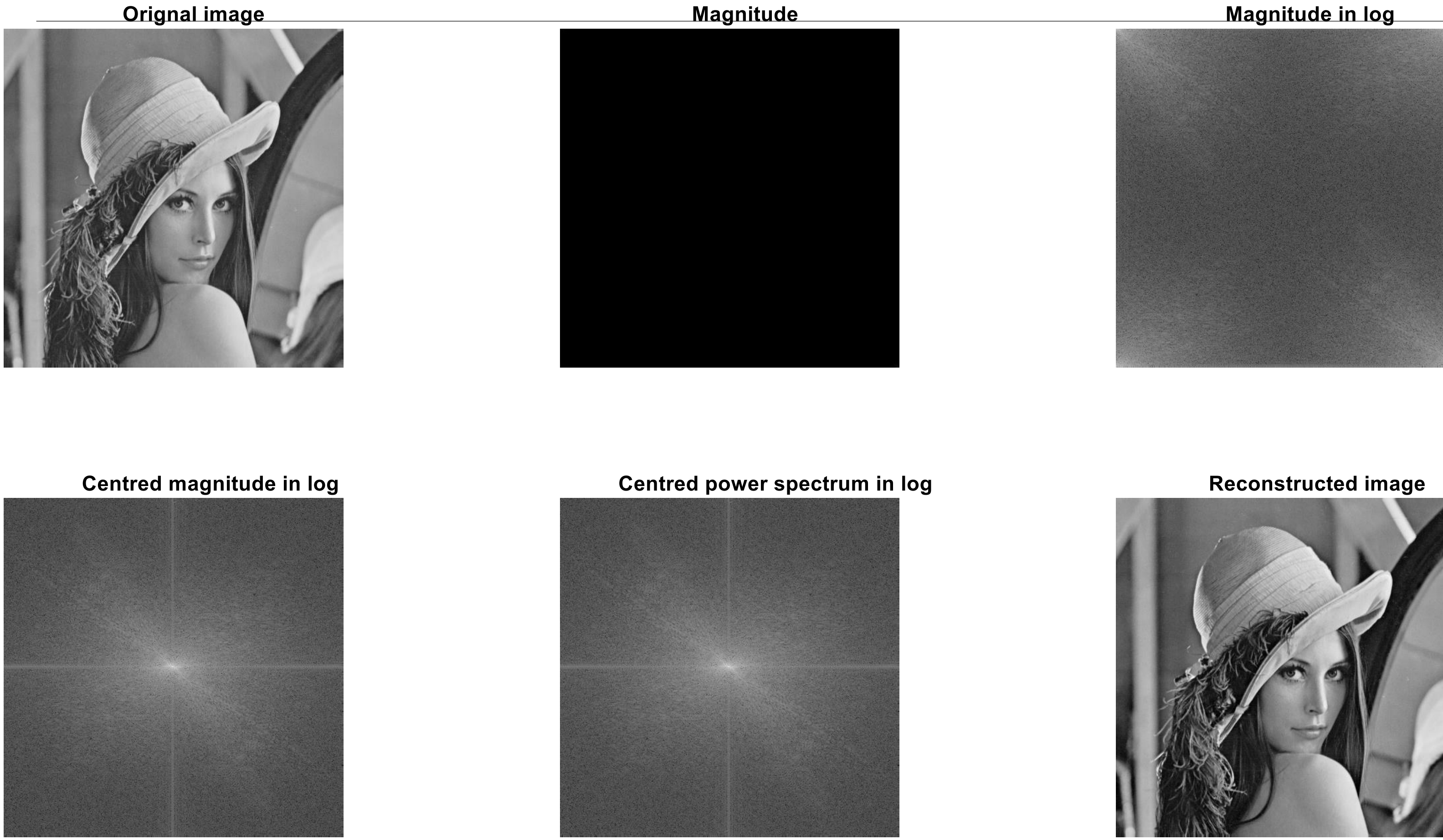
Abstract expressionism?

Phase Only Reconstruction

M'tude of FT set to 1 before inverse.

The phase information is crucial to reconstruct the correct image in the spatial domain.

Demo on Lena


Orignal image
Magnitude
Magnitude in log
Centred magnitude in log
Centred power spectrum in log
Reconstructed image

Some useful links

http://www.falstad.com/fourier/
- Fourier series java applet

http://www.jhu.edu/~signals/
- Collection of demonstrations about digital signal processing

http://www.ni.com/events/tutorials/campus.htm
- FFT tutorial from National Instruments

http://www.cf.ac.uk/psych/CullingJ/dictionary.html
- Dictionary of DSP terms

http://jchemed.chem.wisc.edu/JCEWWW/Features/McadInChem/mcad008/FT4FreeIndDecay.pdf
- Mathcad tutorial for exploring Fourier transforms of free-induction decay

Q&A