COMP122/20 - Data Structures and Algorithms

08 Fundamentals of Algorithm Analysis

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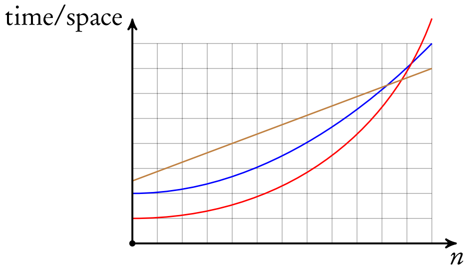
Outline

Textbook §3.1 – 3.4.

Complexity

- The complexity of an algorithm indicates how costly to apply the algorithm, in terms of execution *time* and memory *space*.
- Most algorithms transform input objects into output objects. The cost of an algorithm typically grows with the input size.
- We characterize the complexities as functions of the input size n.



Theoretical Analysis

- The absolute running time depends on computing and processing power of hardware, not only the algorithm.
- To focus on algorithms themselves, we count the *number* of overall primitive operations for time measurement.
  - Evaluating an arithmetic or logic expression
  - Assigning a value to a variable
  - Indexing into an array-based list
  - Entering a function or method
  - Returning from a function or method
- We count the total *number* of variables in use for space measurement, including those stored in collections and structures.
- Some input combinations may result shorter running time or less memory space than some others, we take account all possible input combinations.

Counting Primitive Operations

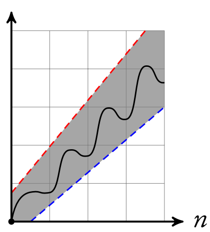
By inspecting the code, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size.

```
def list_max(a, n):        # operations
    m = a[0]               2
    i = 1                  1
    while i < n:           n
        if a[i] > m:       2(n−1)
            m = a[i]       2(n−1)
        i = i+1            2(n−1)
    return m               1
                     total 7n−2
```


Estimating Running Time

- Algorithm *list_max* executes $7n-2$ primitive operations in the worst case. We define:

$$a = \text{time taken by the fastest primitive operation}$$
$$b = \text{time taken by the slowest primitive operation}$$



- Let $T(n)$ be worst-case time of *list_max*. Then

$$a(7n-2) \leqslant T(n) \leqslant b(7n-2).$$

- Hence, the running time $T(n)$ is bounded by two linear functions.
- Changing the hardware/software environment affects $T(n)$ by a constant factor, but does not alter the ***growth rate*** of $T(n)$
- The *linear growth rate* of the running time $T(n)$ is an *intrinsic property* of algorithm *list_max*, it is not affected by *constant factors* or *lower-order terms*.

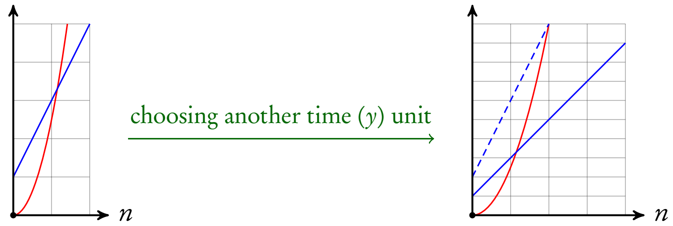
The Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $\mathcal{O}(g(n))$ if there are positive constants c and n_0 such that

$$f(n) \leqslant c \cdot g(n) \quad \text{for } n \geqslant n_0.$$

It merely says, for a sufficiently large input, and running on a sufficiently fast computer, f can be faster than g (running on a slower computer).

- Example: $2n+10$ is $\mathcal{O}(n)$. Because $2n+10 \leqslant c \cdot n \iff (c-2)n \geqslant 10 \iff n \geqslant \dfrac{10}{c-2}$. So, $2n+10 \leqslant 3n$ for $n \geqslant 10$.



Big-Oh Examples

- The function n^2 is not $\mathcal{O}(n)$.

$$n^2 \leqslant c \cdot n \iff n \leqslant c \quad \text{when } n > 0.$$

The above inequality cannot be satisfied since c must be a constant.

- $7n-2$ is $\mathcal{O}(n)$.

$$7n-2 \leqslant 7n \quad \text{for } n \geqslant 1.$$

- $3n^3+20n^2+5$ is $\mathcal{O}(n^3)$.

$$3n^3+20n^2+5 \leqslant 4n^3 \quad \text{for } n \geqslant 21.$$

- $3\log n+5$ is $\mathcal{O}(\log n)$.

$$3\log n+5 \leqslant 8\log n \quad \text{for } n \geqslant 2.$$

Big-Oh Rules

- The big-Oh notation gives an *upper bound* on the growth rate of a function.
- The statement "$f(n)$ is $\mathcal{O}(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$.
- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $\mathcal{O}(n^d)$, i.e., we drop *lower-order terms* and *constant factors*.
- We use the smallest possible class of functions, say "$2n$ is $\mathcal{O}(n)$" instead of "$2n$ is $\mathcal{O}(n^2)$".
- We use the simplest expression of the class, say "$3n+5$ is $\mathcal{O}(n)$" instead of "$3n+5$ is $\mathcal{O}(3n)$".

Seven Important Functions

Seven functions that often appear in algorithm analysis as growth rates.

Contant	1	
Logarithmic	$\log n$	
Linear	n	
Linearithmic (N-log-N)	$n \log n$	
Quadratic	n^2	
Cubic	n^3	(tractable)
Exponential	2^n	

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation.
- To perform the asymptotic analysis:
  - We find the worst-case number of primitive operations executed as a function of the input size.
  - We express this function with big-Oh notation.
- Example:
  - We determine that algorithm *list_max* executes at most $7n - 2$ primitive operations.
  - We say that algorithm *list_max* "runs in $\mathcal{O}(n)$ time".
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations and focus on repeated operations.

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages.
- The i-th prefix average of a list v is the average of the first $i+1$ elements of v:

$$a[i] = \frac{v[0] + v[1] + \cdots + v[i]}{i+1}$$

- Computing the list a of prefix averages of another list v has applications to financial analysis

Prefix Averages — Quadratic

The following algorithm computes prefix averages in quadratic time by applying the definition.

```
def prefix_average_qua(v):           # operations
    n = len(v)                       1
    a = [None]*n                     n
    for i in range(n):               n
        s = v[0]                     n
        for j in range(1, i+1):      1+2+···+(n−1)
            s += v[j]                1+2+···+(n−1)
        a[i] = s/(i+1)               n
    return a                         1
```

The time complexity of *prefix_averages_qua* is $\mathcal{O}(1+2+\cdots+n)$, i.e., $\mathcal{O}(n^2)$.

Prefix Averages — Linear

The following algorithm computes prefix averages in linear time by keeping a running sum.

```
def prefix_average_lin(v):           # operations
    n = len(v)                       1
    a = [None]*n                     n
    s = 0                            1
    for i in range(n):               n
        s += v[i]                    n
        a[i] = s/(i+1)               n
    return a                         1
```

The time complexity of *prefix_averages_lin* is $\mathcal{O}(n)$.

Relatives of Big-Oh

- Big-Omega: $f(n)$ is $\Omega(g(n))$ if there is a constant $c>0$ and an integer constant $n_0 \geqslant 1$ such that
$$f(n) \geqslant c \cdot g(n),$$
for $n \geqslant n_0$.
- Big-Theta: $f(n)$ is $\Theta(g(n))$ if there are constants $c'>0$ and $c''>0$ and an integer constant $n_0 \geqslant 1$ such that
$$c' \cdot g(n) \leqslant f(n) \leqslant c'' \cdot g(n),$$
for $n \geqslant n_0$.
- Intuition for asymptotic notations:
  - $f(n)$ is $\mathcal{O}(g(n))$ if $f(n)$ is asymptotically **less than or equal to** $g(n)$.
  - $f(n)$ is $\Omega(g(n))$ if $f(n)$ is asymptotically **greater than or equal to** $g(n)$.
  - $f(n)$ is $\Theta(g(n))$ if $f(n)$ is asymptotically **equal to** $g(n)$.