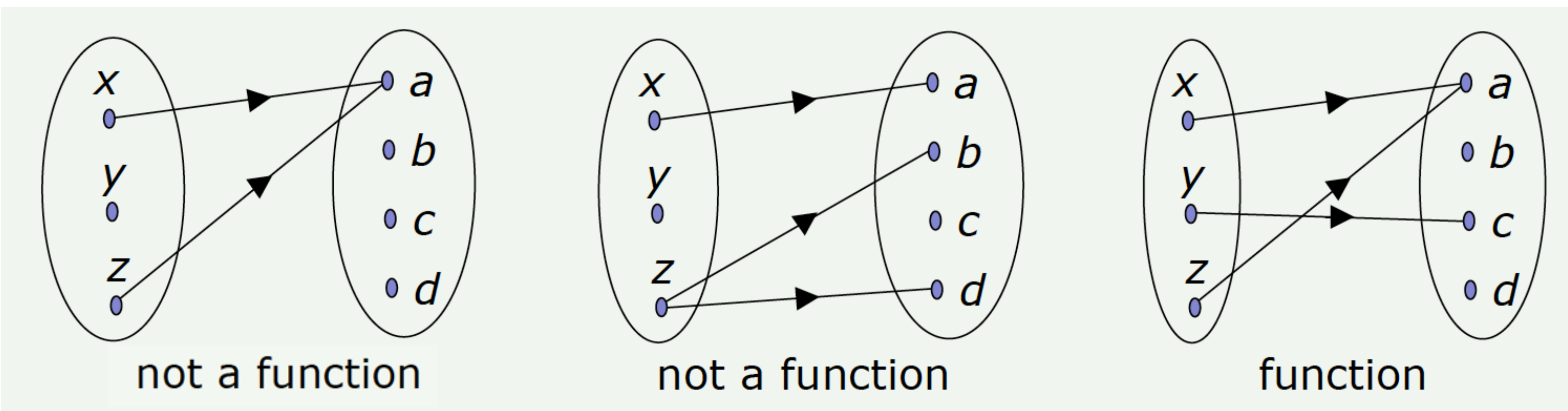
Functions

COMP416 - Calculus
Dennis Wong

Functions

A ***function*** *f* is a mapping between 2 sets A and B, denoted by *f: A -> B*, such that each $a \in A$ maps to exactly one element in B.

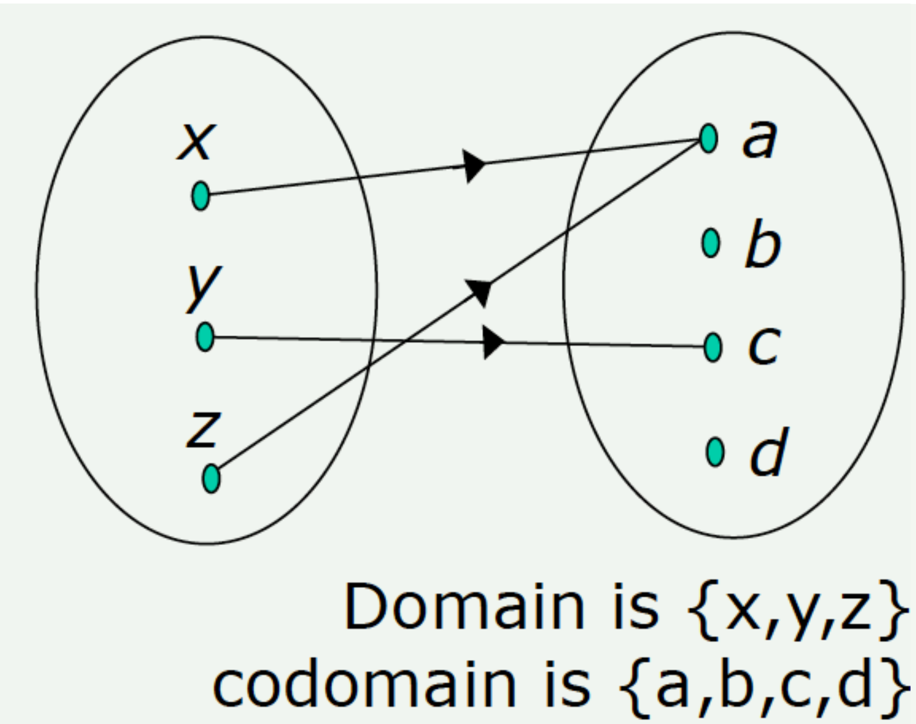


We write $f(a) = b$ if the function *f* maps the element $a \in A$ to the element $b \in B$.

Domain and codomain

Let *f* be a function from the sets A to B.

Then we say that A is the ***domain*** of the function *f* and B is the ***codomain*** of the function *f*.

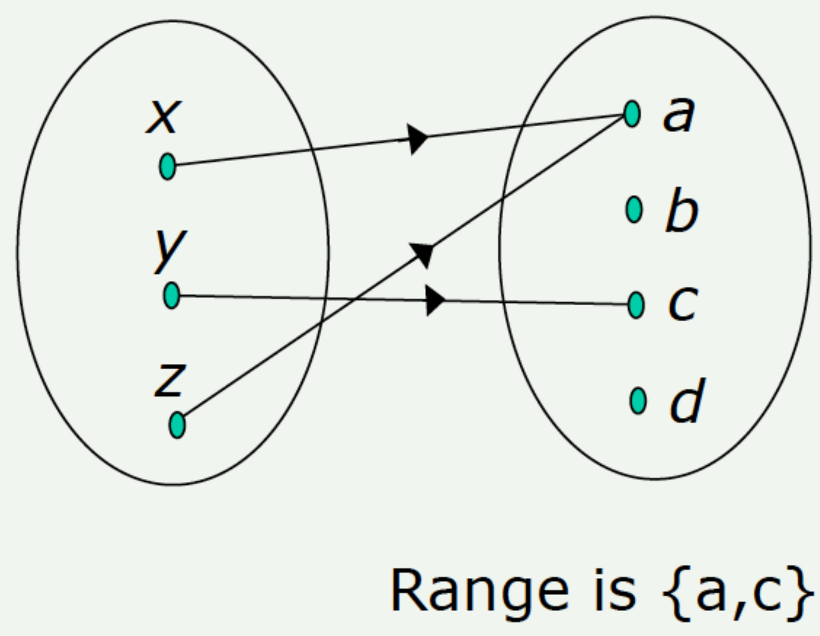


We also say *b* is an ***image*** of *a* (or *a* is a ***preimage*** of *b*) when *f(a)* = *b*.

Range

Let *f* be a function from the sets A to B.

The ***range*** of *f* is the subset of B defined as follows: $b \in B$ belongs to the range if and only if it has a preimage under *f*.



Example

Consider the function f: $\mathbf{R}^+ \to \mathbf{R}$

$$x \to 2 - \sqrt{x}$$

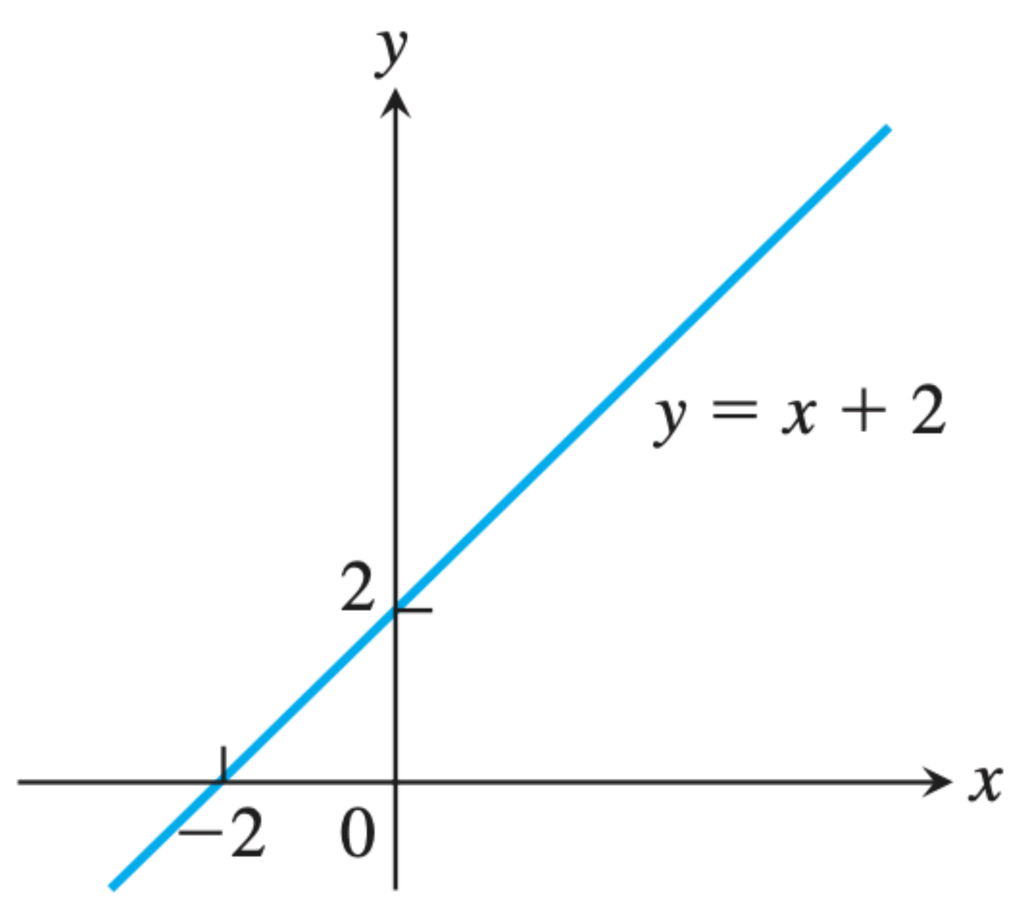
Domain is $\mathbf{R}^+$ and codomain is $\mathbf{R}$.
Range is $]-\infty, 2[$.

Question: If the domain of *f* is changed to **R**, is *f* still a function? Why?

Graphs of functions

If f is a function with domain D, its ***graph*** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

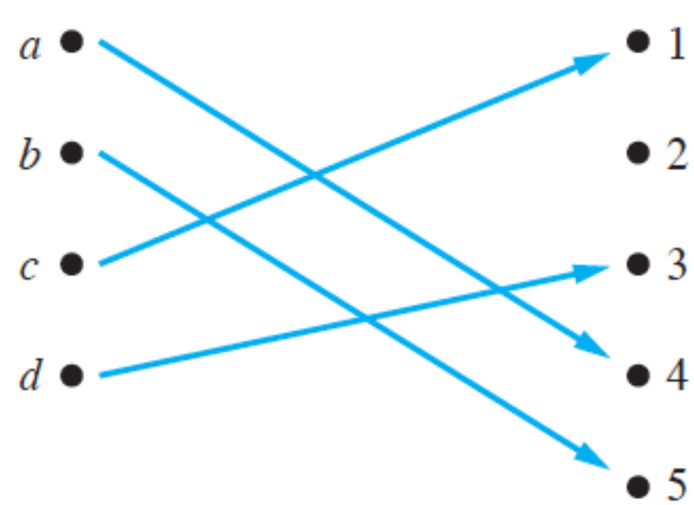
In set notation, the graph is $\{(x, f(x)) \mid x \in D\}$.



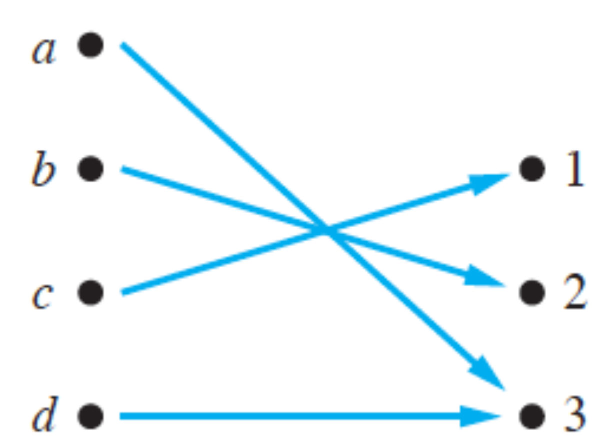
Injective (one-to-one)

A function *f* is said to be ***injective*** (or ***one-to-one***) if and only if *f(a)* = *f(b)* implies *a* = *b*.

That is, no two or more elements in the domain map to the same element in the codomain.



injective



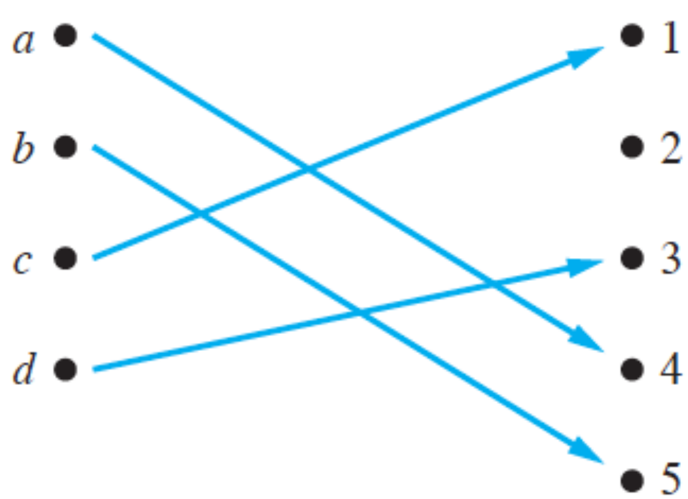
not injective

Is the function *f(x)* = *floor(x)* from **R** to **R** injective?

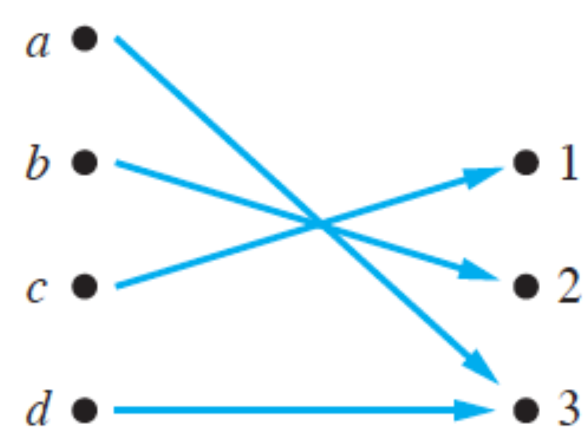
Surjective (onto)

A function *f: A -> B* is said to be ***surjective*** (or ***onto***) if and only if for every element $b \in B$, there is an element $a \in A$ such that $f(a) = b$.

That is, the range of *f* is equal to the codomain of *f*.



not surjective

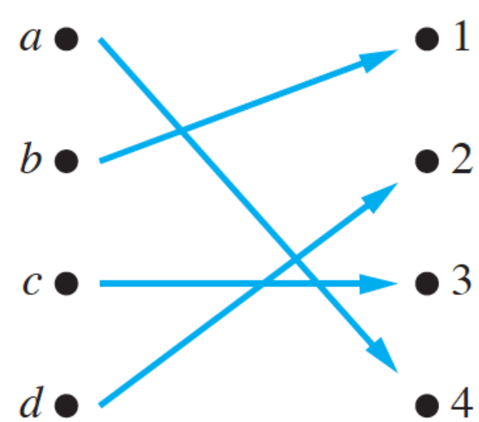


surjective

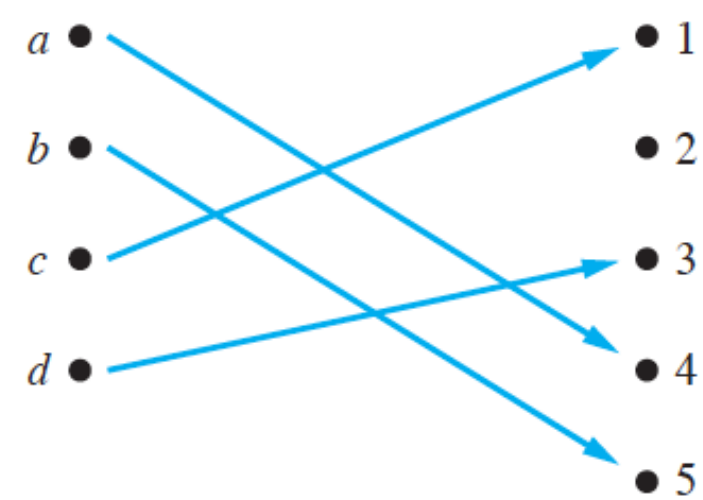
Is the function *f(x) = floor(x)* from **R** to **R** surjective?

Bijective (one-to-one correspondence)

A function *f* is said to be ***bijective*** (***one-to-one correspondence***) if and only if *f* is injective and surjective.



bijective



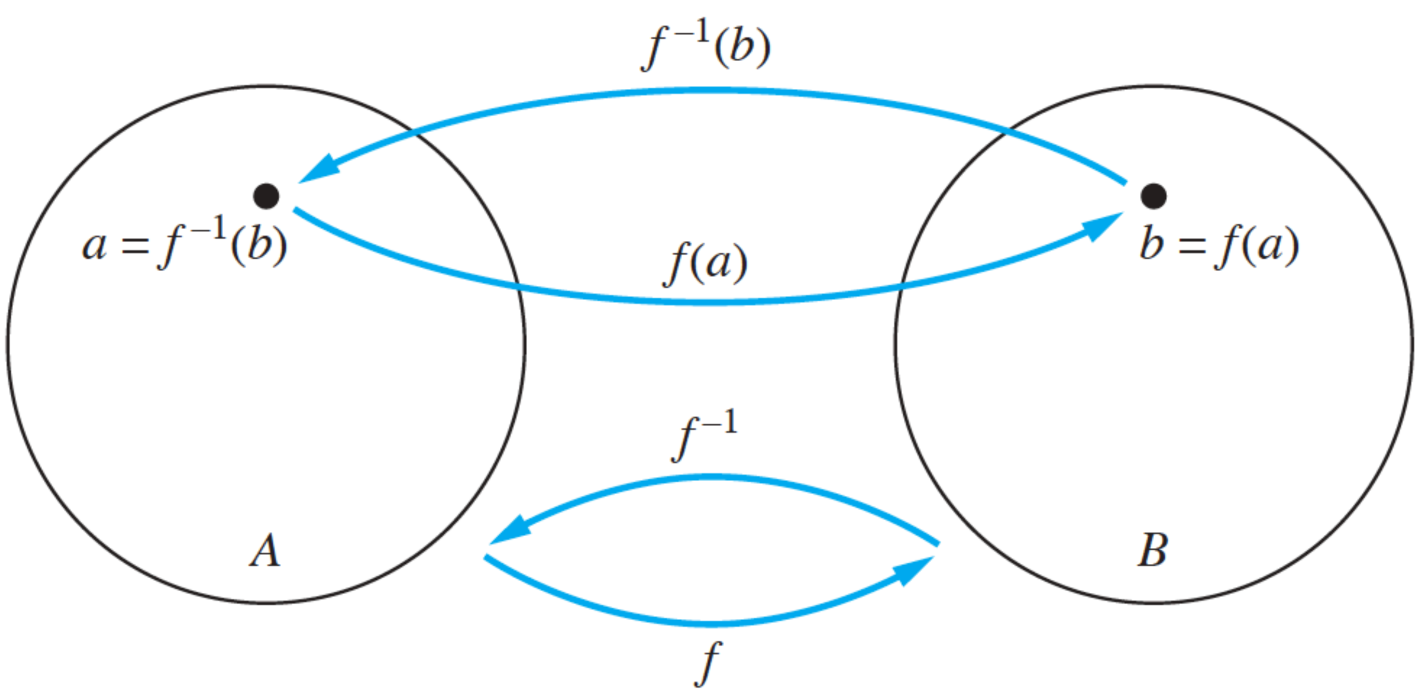
not bijective

Theorem: When a function is bijective, the domain, codomain and the range are of equal size.

Inverse function

An ***inverse function*** f^{-1} is a mapping between elements in codomain to the domain of the function *f*.

Theorem: The inverse function is a function if and only if *f* is bijective.

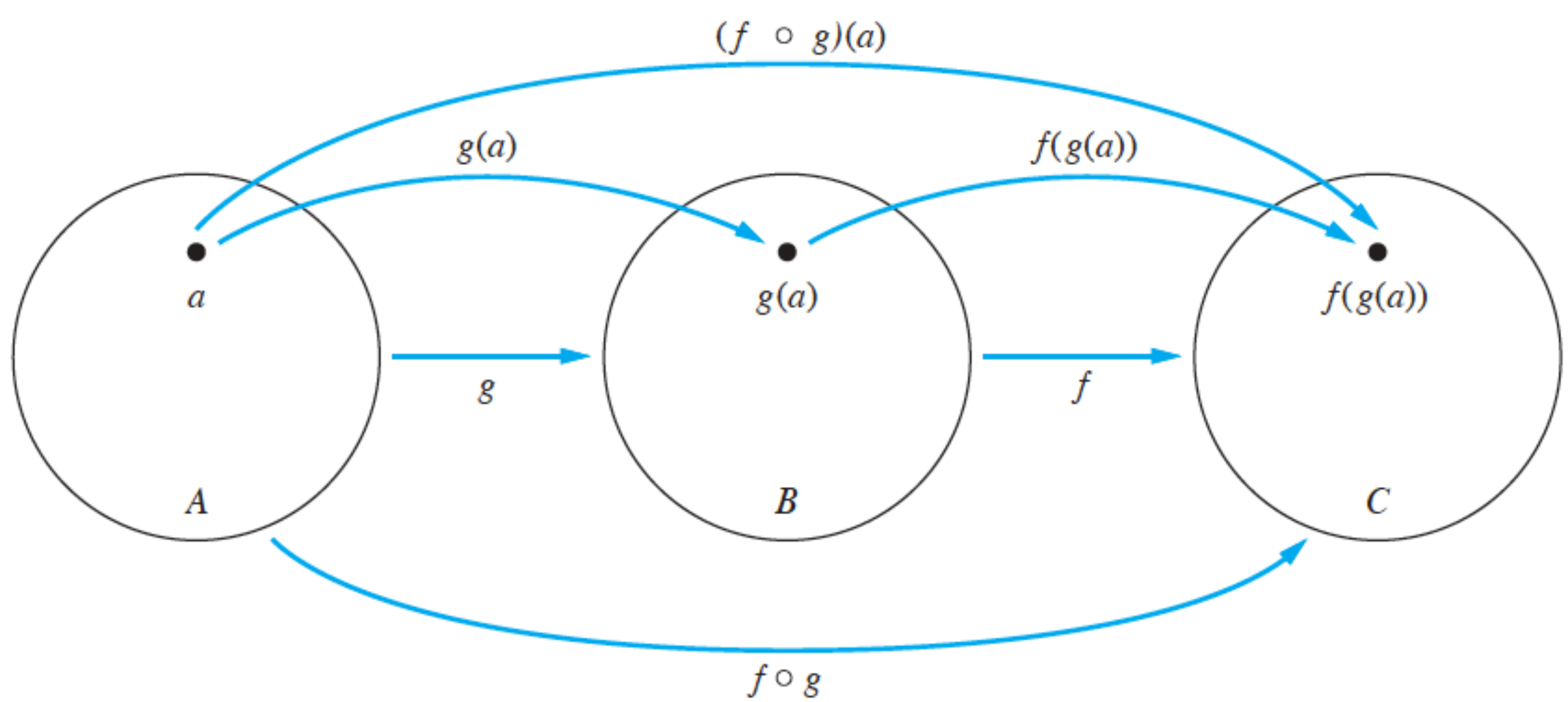


Example: Let *f*: **Z** -> **Z** be such that $f(x) = x + 1$, then $f^{-1}(x) = x - 1$.

Composition of functions

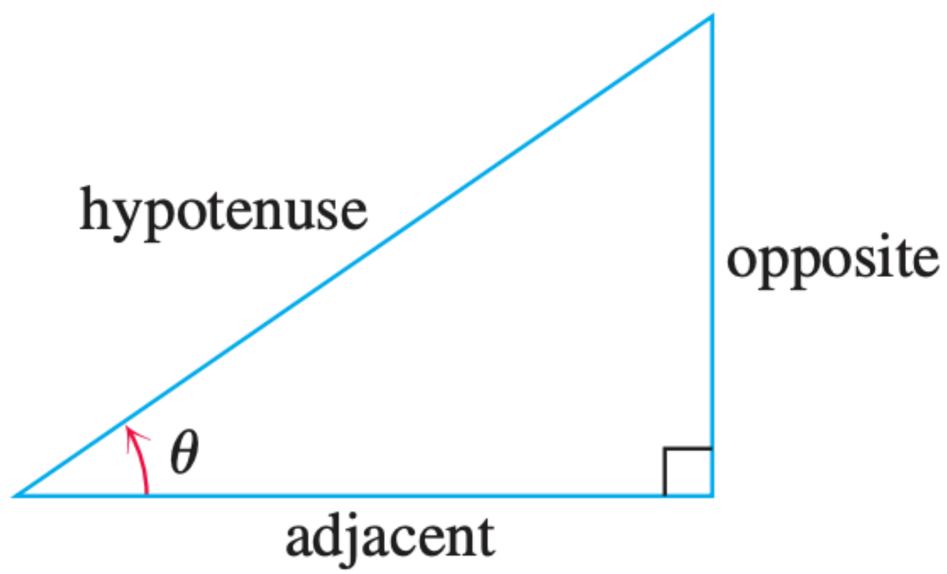
Let *g* be a function from A to B and let *f* be the function from B to C. The ***composition*** of the function *f* and *g*, denoted by $f \circ g$, is defined as $f \circ g(x) = f(g(x))$.

The composition $f \circ g$ is well-defined only if the range of *g* is a subset of the domain of *f*.



Trigonometric functions

Trigonometric functions are functions which relate an angle of a ***right-angled triangle*** to ratios of two side lengths.



There are six basic trigonometric functions with their definitions below:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Trigonometric functions

The functions are also related to each other as follows:

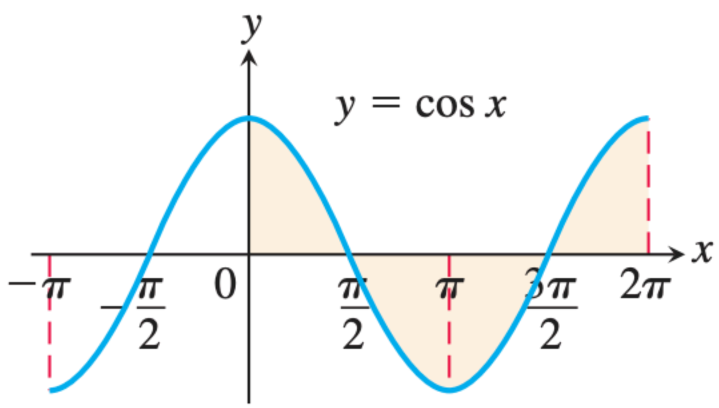
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

In Calculus, we usually measures the angle in ***radian***. The below table gives a translation between radian and degree.

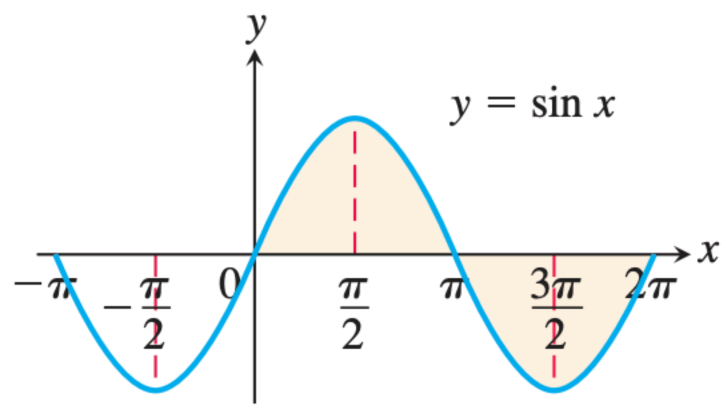
Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

Trigonometric functions



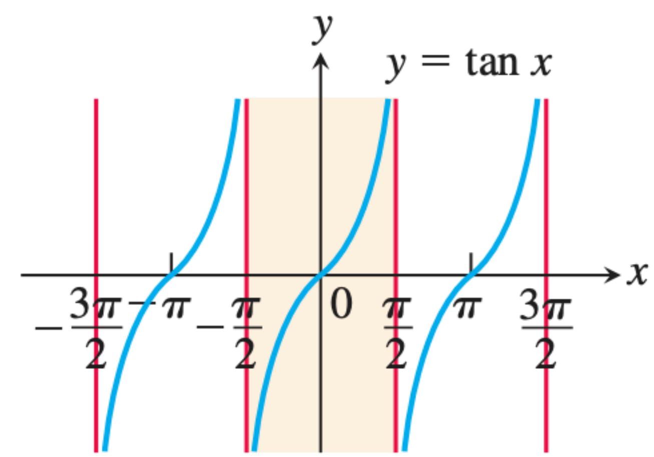
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



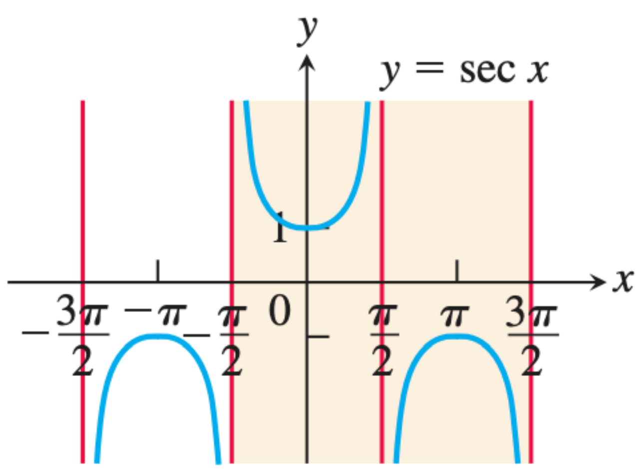
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



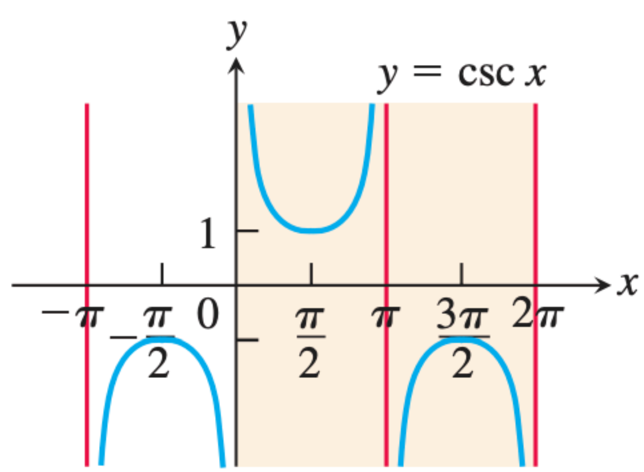
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



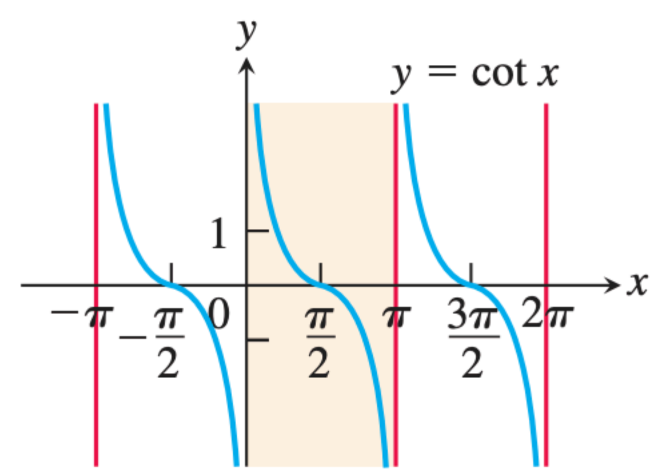
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

Exponential functions

An ***exponential function*** is a function in the form,

$$f(x) = b^{x},$$

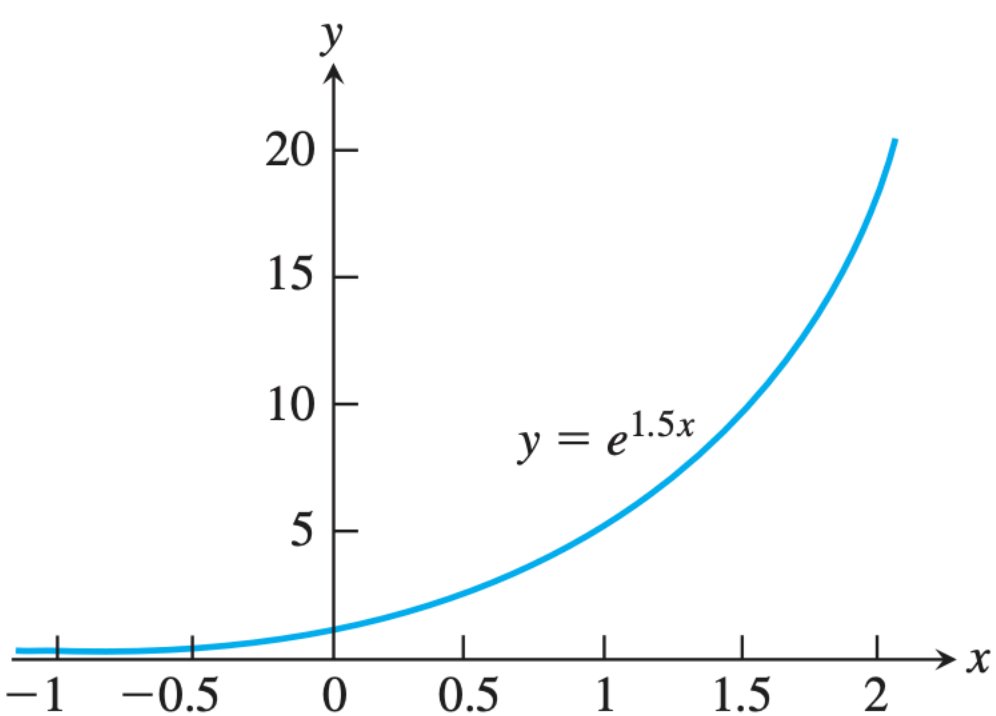
where b is some constant greater than 0.

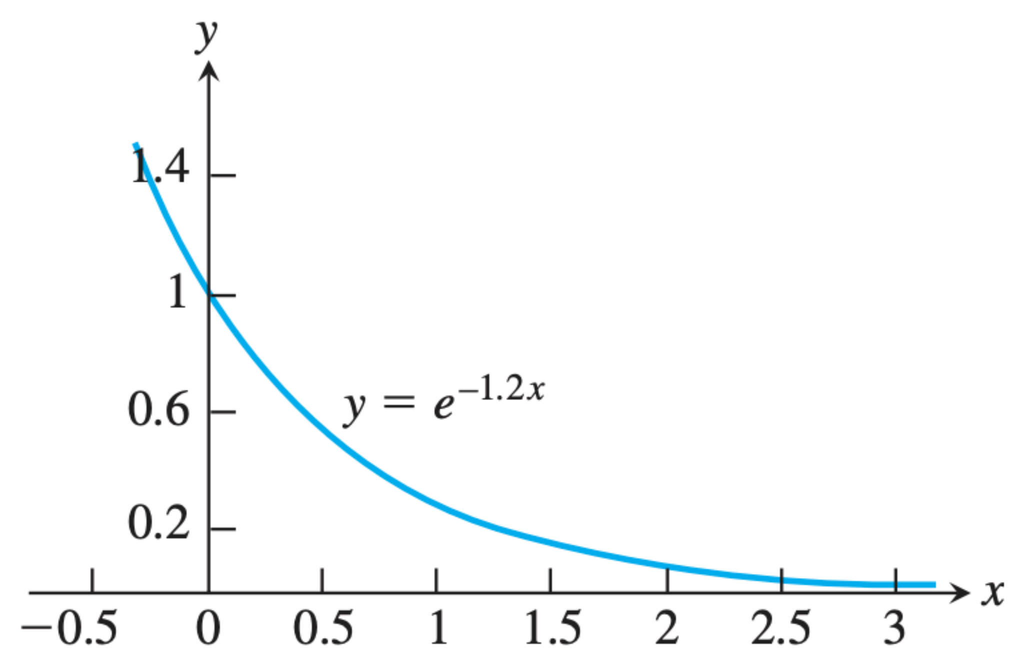
The most important exponential function used for modeling natural, physical, and economic phenomena is the ***natural exponential function***, whose base is the special number e.

The number e is irrational, and its value is 2.718281828 to nine decimal places

Exponential functions

The exponential functions $y = e^{kx}$, where k is a nonzero constant, are frequently used for modeling ***exponential growth or decay***. The function is a model for ***exponential growth*** if $k > 0$, and a model for ***exponential decay*** if $k < 0$.



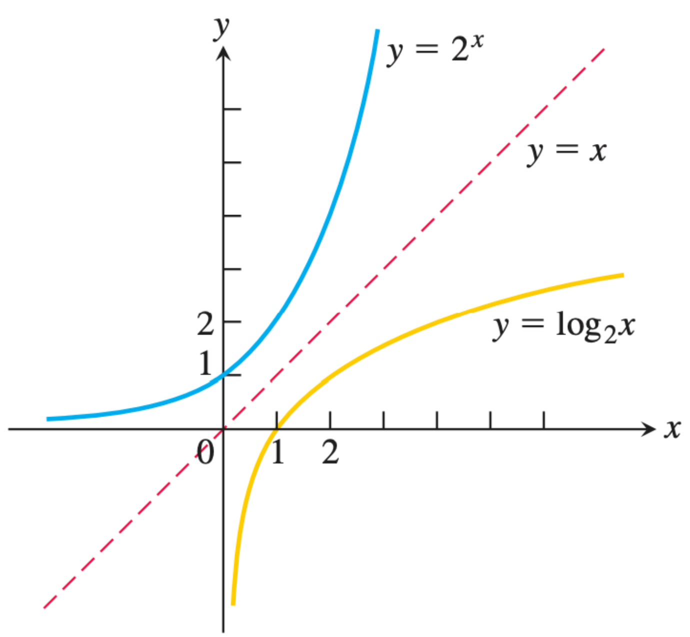


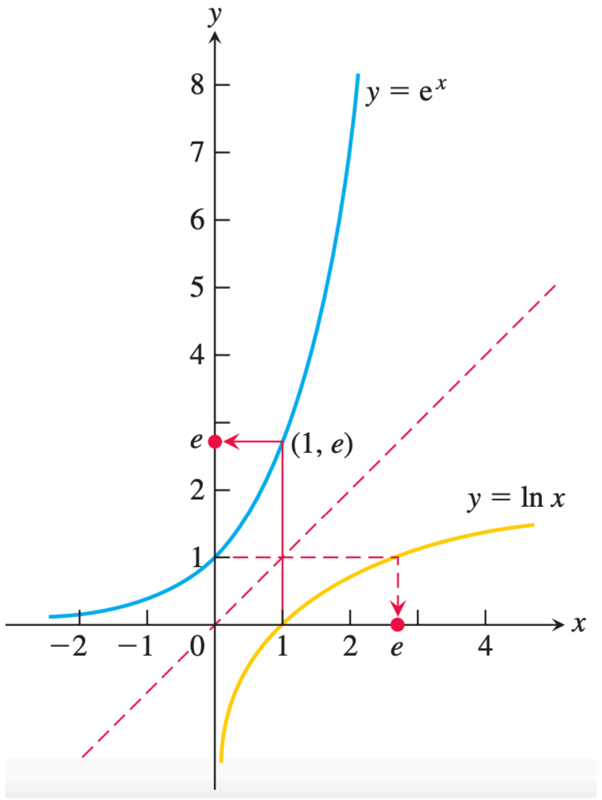
Logarithmic functions

The ***logarithm function*** with base b, $y = \log_b x$, is the inverse of the base b exponential function $y = b^x$ ($b > 0$ and $b \neq 1$).

Logarithms with base 2 are commonly used in computer science. Logarithms with base e have many important in applications in mathematics and simulation.

The function $y = \ln x$ is called the natural logarithm function, that is $y = \ln x$ implies $e^y = x$.





Logarithmic functions

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

Every logarithmic function can be expressed as a constant multiple of the natural logarithm *ln x*.

$$\log_a x = \frac{\ln x}{\ln a} \qquad (a > 0, a \neq 1)$$