Invertible Matrices

Definition If A is a square matrix, and if a matrix B of the same size can be found such that $AB=BA=I$ (I is the identity matrix of the same size), then A is said to be <u>invertible</u> and B is called an <u>inverse of A</u>. If no such matrix B can be found, then A is said to be <u>singular</u>.

<u>Examples</u>

1) $B=\begin{pmatrix}3&5\\1&2\end{pmatrix}$ is an inverse of $A=\begin{pmatrix}2&-5\\-1&3\end{pmatrix}$ because $AB=BA=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ (verify!).

2) The matrix $A=\begin{pmatrix}1&4&0\\2&5&0\\3&6&0\end{pmatrix}$ is singular because for any 3×3 matrix B, we have $BA=\begin{pmatrix}*&*&0\\*&*&0\\*&*&0\end{pmatrix}\neq\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, where $*$ denotes an entry of which the actual value is not important.

<u>Theorem 1</u> If B and C are both inverses of the matrix A, then $B=C$.

Proof: $C=IC=(BA)C=B(AC)=BI=B$.

As a consequence of this theorem, we can speak of "the" inverse of A. This inverse, if exists, will be denoted by <u>A^{-1}</u>.

<u>Theorem 2</u> Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$.

(i) A is invertible iff $ad-bc\neq0$.

(ii) If $ad-bc\neq0$, then $A^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$.

<u>Examples</u>

1) $\begin{pmatrix}1&0\\0&1\end{pmatrix}^{-1}=\dfrac{1}{1\cdot1-0\cdot0}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

2) $\begin{pmatrix}3&2\\7&5\end{pmatrix}^{-1}=\dfrac{1}{3\cdot5-2\cdot7}\begin{pmatrix}5&-2\\-7&3\end{pmatrix}=\begin{pmatrix}5&-2\\-7&3\end{pmatrix}$.

3) $\begin{pmatrix}3&2\\6&4\end{pmatrix}$ has no inverse $\because\ 3\cdot4-2\cdot6=0$.

4) Let $S=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix}\middle|\, a,b,c\in\mathbb{Z}\right\}$. Determine all those matrices A of S that are invertible and that A^{-1} is also in S.

Solution The problem is the same as to determine the following set:

$$T=\{A\in S\mid A\text{ is invertible and }A^{-1}\in S\}.$$

Let $A=\begin{pmatrix}a&b\\0&c\end{pmatrix}\in T$. Then, by Theorem 2, $ac\neq0$ and $A^{-1}=\begin{pmatrix}\frac{1}{a}&-\frac{b}{ac}\\0&\frac{1}{c}\end{pmatrix}$.

Since $A^{-1}\in S$, $\frac{1}{a}$ and $\frac{1}{c}$ must be integers. This implies that $a, c\in\{-1, 1\}$, or equivalently $a^2=c^2=1$. In that case,

$$A^{-1}=\begin{pmatrix}a&-abc\\0&c\end{pmatrix},$$

where b could be any integer.

Therefore, $T=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix}\middle|\, a^2=c^2=1\text{ and }b\in\mathbb{Z}\right\}$.

Theorem 3 If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Remark Let M_n denote the set of all invertible $n \times n$ matrices, and let $\bullet$ denote matrix multiplication. It follows from this theorem that, for each $n \in \mathbb{Z}^+$, $\bullet$ is closed on M_n.

Exercises

1. Let $A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 \\ 6 & 0 \end{pmatrix}$, $c = \begin{pmatrix} -7 & 8 \\ 0 & 0 \end{pmatrix}$. Evaluate the following:

   (a) AB (b) CA (c) A^2B

2. Let $A = \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 0 \\ 6 & 3 \end{pmatrix}$, $c = \begin{pmatrix} -7 & 8 \\ 0 & 0 \end{pmatrix}$.

   (a) If possible, find A^{-1}, B^{-1}, C^{-1}, and $(AB)^{-1}$.

   (b) Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \middle| \, a, b \in \mathbb{Z} \right\}$. Determine all those matrices A of S that are invertible and that A^{-1} is also in S.