Vector Space

COMP408 - Linear Algebra
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Vector space

A ***vector space*** is a set V (the elements of which are called vectors) with an addition and a scalar multiplication satisfying the following properties for all $u, v, w \in V$ and $\alpha, \beta \in R$:

(V1) $v + w = w + v$,
(V2) $(u + v) + w = u + (v + w)$,
(V3) there exists a vector 0 in V such that $v + 0 = v$,
(V4) for each vector v in V , there exists a vector $-v$ in V such that $v + (-v) = 0$,
(V5) $\alpha(v + w) = \alpha v + \alpha w$,
(V6) $(\alpha + \beta)v = \alpha v + \beta v$,
(V7) $(\alpha\beta)v = \alpha(\beta v)$,
(V8) $1v = v$.

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Example: (Euclidean space) The set $V = R^n$ is a vector space with usual vector addition and scalar multiplication.

Proof: Let $u = [u_1, u_2]^T$, $v = [v_1, v_2]^T$ and $w = [w_1, w_2]^T$.

(V1) $v + w = [v_1, v_2]^T + [w_1, w_2]^T = [v_1+w_1, v_2+w_2]^T$
$= [w_1+v_1, w_2+v_2]^T = w + v$;

(V2) $(u + v) + w = ([u_1, u_2]^T + [v_1, v_2]^T) + [w_1, w_2]^T$
$= [u_1, u_2]^T + ([v_1, v_2]^T + [w_1, w_2]^T)$
$= [u_1+v_1+w_1, u_2+v_2+w_2]^T = u + (v + w)$;

(V3) The vector $0 = [0, 0]^T$ satisfies the property as $u + 0 = u$;

(V4) The vector $-v = [-v_1, -v_2]^T$ satisfies the property since
$v + -v = [v_1, v_2]^T + [-v_1, -v_2]^T = 0$;

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Example: The set $V = R^n$ is a vector space with usual vector addition and scalar multiplication.

Proof (cont): Let u = $[u_1, u_2]^T$, v = $[v_1, v_2]^T$ and w = $[w_1, w_2]^T$.

(V5) $\alpha(v + w) = \alpha([v_1, v_2]^T + [w_1, w_2]^T) = \alpha[v_1+w_1, v_2+w_2]^T$
$= [\alpha(v_1+w_1), \alpha(v_2+w_2)]^T$
$= [\alpha v_1+\alpha w_1, \alpha v_2+\alpha w_2)]^T$
$= \alpha v + \alpha w$
(V6) $(\alpha + \beta)\mathrm{v} = (\alpha + \beta)([v_1, v_2]^T) = [(\alpha + \beta)v_1, (\alpha + \beta)v_2]^T$
$= \alpha v + \beta v$
(V7) $(\alpha\beta)\mathrm{v} = ([\alpha\beta v_1, \alpha\beta v_2]^T) = \alpha([\beta v_1, \beta v_2]^T) = \alpha(\beta \mathrm{v})$;
(V8) $1\mathrm{v} = ([1v_1, 1v_2]^T) = \mathrm{v}$.

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Other examples of vector space include:

1. Polynomial space (How?)
2. Function space
3. Matrix space (introduce in next chapter)
4. Many more actually…

If V is a vector space, then V satisfies most of the properties we discussed previously for vector (addition, scalar multiplication, dot product, basis, span, subspace, dimension, etc).