Eigenvalue

COMP408 - Linear Algebra

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Eigenvalue

Let V be a vector space and let L: $V \to V$ be a ***linear function***.

- The scalar λ is an ***eigenvalue*** of L if $L(v) = \lambda v$ for some nonzero vector v in V.
- If λ is an eigenvalue of L, then any vector v in V for which $L(v) = \lambda v$ is an ***eigenvector*** of L corresponding to λ (or just λ-eigenvector).
- If λ is an eigenvalue of L, then the set of all λ-eigenvectors is the ***eigenspace*** of L corresponding to λ (or just λ-eigenspace).

The notions of eigenvalue and eigenvector are used in the study of differential equations, set up convenient coordinate systems, and they form the basis of quantum mechanics.

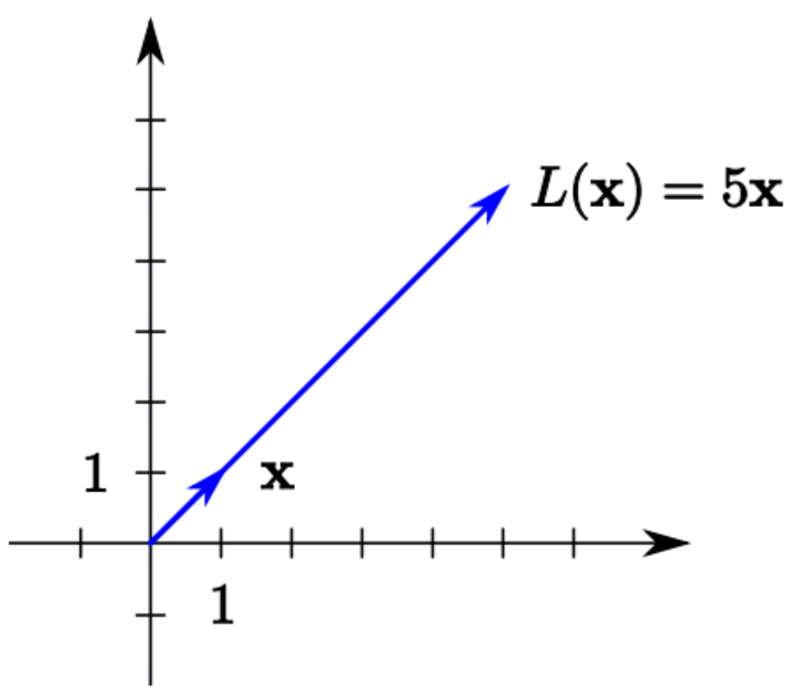
Eigenvalue

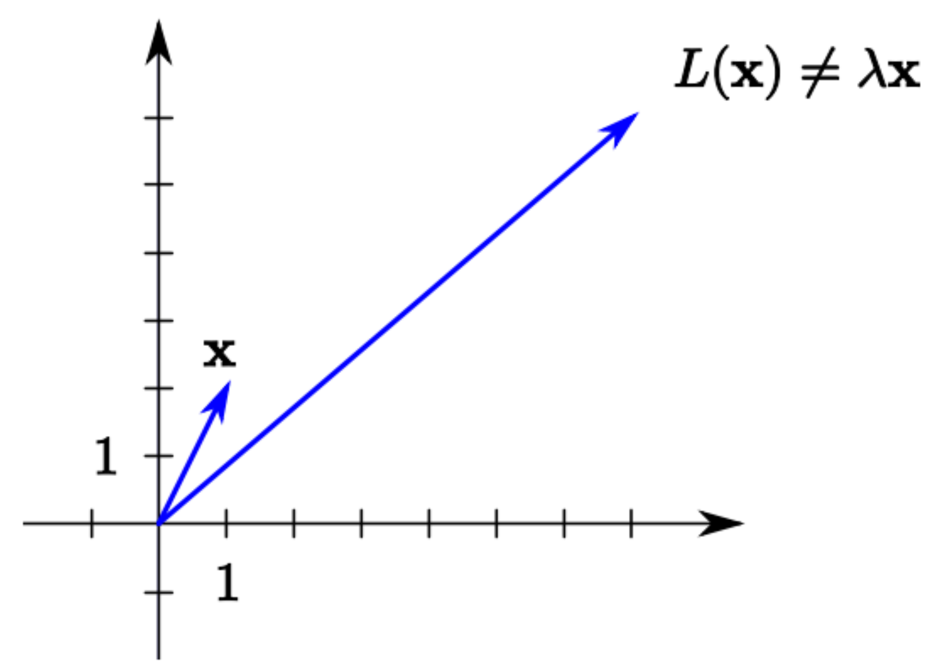
Example: Consider the linear function L: $\mathbf{R}^2 \to \mathbf{R}^2$

$$L(\mathbf{x}) = \begin{bmatrix} 3x_1 + 2x_2 \\ 4x_1 + x_2 \end{bmatrix}.$$

If $x = [1, 1]^T$, then $L(x) = [5, 5]^T = 5[1, 1]^T = 5x$. Therefore 5 is an eigenvalue of L and $[1, 1]^T$ is a corresponding eigenvector.

Now if $x = [1, 2]^T$, then $L(x) = [5, 5]^T = 5[1, 1]^T \neq \lambda x$. So $[1, 2]^T$ is not an eigenvector for L.





Eigenvalue

The idea of eigenvalue can be generalized to matrix.

Let A be an $n \times n$ matrix.

- The scalar λ is an ***eigenvalue*** of A if $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^n$.

- If λ is an eigenvalue of A, then any vector x in $\mathbf{R}^n$ for which $Ax = \lambda x$ is an ***eigenvector*** of A corresponding to λ (or just λ-eigenvector).

- If λ is an eigenvalue of A, then the set of all λ-eigenvectors is the ***eigenspace*** of A corresponding to λ (or just λ-eigenspace).

Eigenvalue

Example: Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

If x = $[7, 3, 1]^T$, then we have

$$\mathbf{Ax} = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = 2\mathbf{x}$$

Thus x is an eigenvector of *A* (namely, a 2-eigenvector).

If x = $[1, 2, 4]^T$, then we have

$$\mathbf{Ax} = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \neq \lambda\mathbf{x}$$

So in this case, x = $[1, 2, 4]^T$ is not a eigenvector of *A*.

Finding eigenvalues and eigenspace

Let A be an $n \times n$ matrix.

- The eigenvalues of A are the zeros of the ***characteristic polynomial*** $\det(A - \lambda I)$ of A.

- If λ is an eigenvalue of A, then the λ-eigenspace of A is the null space of the matrix $A - \lambda I$.

Finding eigenvalues and eigenspace

Example: Find the eigenvalues and corresponding eigenspaces of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}.$$

Solution: The characteristic function is as follows:

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \left(\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)(1-\lambda) - (2)(4) \\ &= \lambda^2 - 4\lambda - 5. \end{aligned}$$

The eigenvalues of *A* are found by setting this polynomial equal to zero and solving $\lambda^2 - 4\lambda - 5 = 0$. So the eigenvalues of *A* are 5 and -1.

Finding eigenvalues and eigenspace

Solution: (cont.) The λ-eigenspace is the null space of the matrix $A - \lambda I$, that is, the solution set of the equation $(A - \lambda I)x = 0$.

When $\lambda = 5$, we have

$$[\mathbf{A} - 5\mathbf{I}\,|\,\mathbf{0}] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 4 & -4 & 0 \end{array}\right]\begin{matrix} 2 \\ \end{matrix} \sim \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{matrix} -\frac{1}{2} \\ \end{matrix} \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

so the 5-eigenspace is $\{[t, t]^{\mathsf{T}} \mid t \in \mathbf{R}\}$.

Similarly when = -1, we have

$$[\mathbf{A} - (-1)\mathbf{I}\,|\,\mathbf{0}] = \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 4 & 2 & 0 \end{array}\right]\begin{matrix} -1 \\ \end{matrix} \sim \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{matrix} \frac{1}{4} \\ \end{matrix} \sim \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

so the -1-eigenspace is $\{[-t/2, t]^{\mathsf{T}} \mid t \in \mathbf{R}\}$