Assignment1 Problem1–3

Problem 1:

We need to prove that $\{M_i\}$ satisfies the four properties of a group :

The closure, the associativity, the existence of an identity element, and the existence of an inverse element for each group element.

The closure:

$$\forall M_a = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}, M_b = \begin{bmatrix} R_b & \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}, M_a \cdot M_b = \begin{bmatrix} R_aR_b & R_a\mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

1. Prove $det(R_aR_b) = 1$:

   Known from the determinant of the upper triangle matrix,

   $det(M_a) = det(R_a) = 1$, $det(M_b) = det(R_b) = 1$,

   $\therefore det(M_a \cdot M_b) = det(R_aR_b) = det(R_a)det(R_b) = 1$.

2. Prove R_aR_b is orthonormal :

   R_a is an orthonormal matrix and R_b is also an orthonormal matrix,

   $\therefore (R_aR_b)^T(R_aR_b) = R_b^TR_a^TR_aR_b = R_b^TIR_b = I$ .Clearly R_aR_b is also an orthonormal matrix.

3. Prove $R_a\mathbf{t}_b + \mathbf{t}_a \in \mathbb{R}^{3\times1}$:

   $\because R_a \in \mathbb{R}^{3\times3}$, $\mathbf{t}_b \in \mathbb{R}^{3\times1}$, $\therefore R_a\mathbf{t}_b \in \mathbb{R}^{3\times1}$ .Clearly $R_a\mathbf{t}_b + \mathbf{t}_a \in \mathbb{R}^{3\times1}$.

   In summary, $M_a \cdot M_b \in \{M_i\}$, satisfied Closure.

The associativity:

$$\forall M_a = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}, M_b = \begin{bmatrix} R_b & \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}, M_c = \begin{bmatrix} R_c & \mathbf{t}_c \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}$$

$$(M_a \cdot M_b) \cdot M_c = \begin{bmatrix} R_aR_b & R_a\mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} R_c & \mathbf{t}_c \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_aR_bR_c & R_aR_b\mathbf{t}_c + R_a\mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix};$$

$$M_a \cdot (M_b \cdot M_c) = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} R_bR_c & R_b\mathbf{t}_c + \mathbf{t}_b \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_aR_bR_c & R_aR_b\mathbf{t}_c + R_a\mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Therefore $(M_a \cdot M_b) \cdot M_c = M_a \cdot (M_b \cdot M_c)$, satisfied Associativity.

The existence of an identity element:

Let $E = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$,and clearly $E \in \{M_i\}$.

$$M_a \cdot E = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = M_a;$$

$$E \cdot M_a = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} = M_a.$$

$\therefore M_a \cdot E = E \cdot M_a = M_a$, E is identity element, satisfied The existence of an identity element.

The existence of an inverse element for each group element:

$\forall M_a = \begin{bmatrix} R_a & \mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix} \in \{M_i\}$, we can find $M_a^{-1} = \begin{bmatrix} R_a^{-1} & -R_a^{-1}\mathbf{t}_a \\ \mathbf{0}^T & 1 \end{bmatrix}$, $M_a \cdot M_a^{-1} = E$.

Now we should prove that $M_a^{-1} \in \{M_i\}$:

1. Prove $det(M_a^{-1}) = 1$:

Known from the determinant of the upper triangle matrix,

$det(M_a^{-1}) = det(R_a^{-1}) = \frac{1}{det(R_a)} = 1$.

2. Prove R_a^{-1} is orthonormal :

$R_a^{-1}(R_a^{-1})^T = R_a^{-1}(R_a^T)^{-1} = (R_a^T R_a)^{-1} = I$, so R_a^{-1} is orthonormal.

3. Prove $-R_a^{-1}\mathbf{t}_a \in \mathbb{R}^{3\times1}$:

$\because R_a \in \mathbb{R}^{3\times3}$, $\therefore -R_a^{-1} \in \mathbb{R}^{3\times3}$, $\mathbf{t}_a \in \mathbb{R}^{3\times1}$ $\therefore -R_a^{-1}\mathbf{t}_a \in \mathbb{R}^{3\times1}$.

Therefore, satisfied The existence of an inverse element for each group element.

In conclusion, the set $\{M_i\}$ forms a group.

Problem 2:

$\boldsymbol{a)}$ To prove that M is positive semi-definite, equivalent to prove that all its eigenvalues λ_1, λ_2 satisfy: $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

$\lambda_1\lambda_2 = det(M) = \sum_{(x_i,y_i)\in w}(I_x)^2 \sum_{(x_i,y_i)\in w}(I_y)^2 - (\sum_{(x_i,y_i)\in w}(I_xI_y))^2$;

$\lambda_1 + \lambda_2 = trace(M) = \sum_{(x_i,y_i)\in w}(I_x)^2 + \sum_{(x_i,y_i)\in w}(I_y)^2$.

Obviously, $\sum_{(x_i,y_i)\in w}(I_x)^2 \geq 0$, $\sum_{(x_i,y_i)\in w}(I_y)^2 \geq 0$, so $\lambda_1 + \lambda_2 \geq 0$.

By the Cauchy–Schwarz Inequality:

$\sum_{(x_i,y_i)\in w}(I_x)^2 \sum_{(x_i,y_i)\in w}(I_y)^2 \geq (\sum_{(x_i,y_i)\in w}(I_xI_y))^2$, so $\lambda_1\lambda_2 \geq 0$.

$$\begin{cases} \lambda_1 + \lambda_2 \geq 0 \\ \lambda_1\lambda_2 \geq 0 \end{cases} \Longrightarrow \lambda_1 \geq 0, \lambda_2 \geq 0 .$$

So M is positive semi-definite.

$\boldsymbol{b)}$ We know that for a formulation $ax^2 + bxy + cy^2 + dx + ey + f = 0$,

if $\Delta = b^2 - 4ac < 0$, it represents an ellipse.

Let $a = \sum_{(x_i,y_i)\in w}(I_x)^2, b = \sum_{(x_i,y_i)\in w}(I_y)^2, c = \sum_{(x_i,y_i)\in w}(I_xI_y)$, so $M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$.

$$[x,\ y]M\begin{bmatrix} x \\ y \end{bmatrix} = [x,\ y]\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [ax\ + cy, cx\ + by]\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2cxy + by^2 = 1.$$

$\Delta = (2c)^2 - 4ab = 4c^2 - 4ab = 4(c^2 - ab)$.

$\because M$ is usually positive definite, $\therefore det(M) = ab - c^2 > 0,\ c^2 - ab < 0$.

$\therefore \Delta < 0$, $[x,\ y]M\begin{bmatrix} x \\ y \end{bmatrix}$ represents an ellipse.

c) We can diagonalize symmetric matrices :

$$M = P^{-1}\Lambda P = P^T \Lambda P\,,\ \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix},\ P \text{ is an orthogonal matrix.}$$

So $[x,\, y]M\begin{bmatrix} x \\ y \end{bmatrix} - 1 = [x,\, y]P^T\Lambda P\begin{bmatrix} x \\ y \end{bmatrix} - 1 = [x',\, y']\Lambda\begin{bmatrix} x' \\ y' \end{bmatrix} - 1 = \lambda_1 {x'}^2 + \lambda_2 {y'}^2 - 1$.

$P\begin{bmatrix} x \\ y \end{bmatrix}$ means to do a rotation transformation for the point (x, y) on the ellipse.

$\lambda_1 {x'}^2 + \lambda_2 {y'}^2 - 1 = \frac{x'^2}{\frac{1}{\lambda_1}} + \frac{y'^2}{\frac{1}{\lambda_2}} - 1$, and $\lambda_1 > \lambda_2 > 0$, $\therefore\ \frac{1}{\lambda_1} < \frac{1}{\lambda_2}$.

$\therefore$ The length of its semi–major axis is $(\frac{1}{\lambda_2})^{\frac{1}{2}}$, while the length of its semi–minor axis is $(\frac{1}{\lambda_1})^{\frac{1}{2}}$.

Problem 3:

$\because A^T A x = 0 \Longleftrightarrow x^T A^T A x = 0 \Longleftrightarrow (Ax)^T A x = 0 \Longleftrightarrow Ax = 0$

$\therefore$ The solutions of $Ax = 0$ are also the solutions of $A^T A x = 0$.

$\therefore dim\ ker(A) = dim\ ker(A^T A)$

$\because$ The size of A is $m \times n$, $\therefore$ The size of $A^T A$ is $n \times n$.

We know that for a matrix B of size $m \times n$, $dim\ ker(B) = n - rank(B)$.

So $rank(A) = n - dim\ ker(A) = n$, $rank(A^T A) = n - dim\ ker(A^T A)$,

and $dim\ ker(A) = dim\ ker(A^T A)$, $\therefore rank(A^T A) = n$.

Thus, $\therefore rank(A^T A) = n$, $det(A^T A) \neq 0$,

$\therefore$ It can prove that $A^T A$ is non–singular (or in other words, it is invertible).