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> **Question 1:** Calculate the gradient of the following multivariate function:
>
> (1) $u = xy + y^2 + 5$
>
> (2) $u = ln\sqrt{x^2+y^2+z^2}$, at (1, 2, -2)

(1) Calculate the gradient of the following multivariate function $u = xy + y^2 + 5$:

$$\because \frac{\partial u}{\partial x} = y, \frac{\partial u}{\partial y} = x + 2y$$
$$\therefore \nabla u = (y, x + 2y)$$

(2) Calculate the gradient of the following multivariate function $u = ln\sqrt{x^2+y^2+z^2}$:

$$\frac{\partial u}{\partial x} = \frac{1}{2} \times \frac{1}{x^2+y^2+z^2} \times 2x = \frac{x}{x^2+y^2+z^2}$$
$$\frac{\partial u}{\partial y} = \frac{1}{2} \times \frac{1}{x^2+y^2+z^2} \times 2y = \frac{y}{x^2+y^2+z^2}$$
$$\frac{\partial u}{\partial z} = \frac{1}{2} \times \frac{1}{x^2+y^2+z^2} \times 2z = \frac{z}{x^2+y^2+z^2}$$
$$\nabla u = (\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2})$$
$$\therefore \nabla u_{(1,2,-2)} = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})_{(1,2,-2)} = (\frac{1}{9}, \frac{2}{9}, -\frac{2}{9})$$

Code program for Question1:

```
import numpy as np
import sympy as sp

def calculate_gradient_1():
    """Calculate the gradient of the function u = xy + y^2 + 5"""
    x, y = sp.symbols('x y')
    u = x*y + y**2 + 5

    # Calculate partial derivatives
    du_dx = sp.diff(u, x)
    du_dy = sp.diff(u, y)

    print("Gradient of the function u = xy + y^2 + 5:")
    print(f"∂u/∂x = {du_dx}")
    print(f"∂u/∂y = {du_dy}")
    print(f"Gradient ∇u = ({du_dx}, {du_dy})")

def calculate_gradient_2():
    """Calculate the gradient of the function u = ln(sqrt(x^2 + y^2 + z^2)) at the
point (1, 2, -2)"""
    x, y, z = sp.symbols('x y z')
    u = sp.log(sp.sqrt(x**2 + y**2 + z**2))

    # Calculate partial derivatives
    du_dx = sp.diff(u, x)
    du_dy = sp.diff(u, y)
    du_dz = sp.diff(u, z)

    # Calculate the gradient at the point (1, 2, -2)
    point = {x: 1, y: 2, z: -2}
    grad_x_value = du_dx.subs(point)
    grad_y_value = du_dy.subs(point)
    grad_z_value = du_dz.subs(point)

    # Calculate the length of the vector
    vector_length = sp.sqrt(x**2 + y**2 + z**2).subs(point)
```

```
    print("\nGradient of the function u = ln(sqrt(x^2 + y^2 + z^2)) at the point (1, 2,
-2):")
    print(f"∂u/∂x at (1, 2, -2) = {grad_x_value}")
    print(f"∂u/∂y at (1, 2, -2) = {grad_y_value}")
    print(f"∂u/∂z at (1, 2, -2) = {grad_z_value}")
    print(f"Gradient ∇u at (1, 2, -2) = ({grad_x_value}, {grad_y_value},
{grad_z_value})")

    # Numerical results (converted to float)
    print("\nNumerical results:")
    print(f"∇u at (1, 2, -2) = ({float(grad_x_value)}, {float(grad_y_value)},
{float(grad_z_value)})")

if __name__ == "__main__":
    calculate_gradient_1()
    calculate_gradient_2()
```

Program output for Question1:

```
Gradient of the function u = xy + y^2 + 5:
∂u/∂x = y
∂u/∂y = x + 2*y
Gradient ∇u = (y, x + 2*y)

Gradient of the function u = ln(sqrt(x^2 + y^2 + z^2)) at the point (1, 2, -2):
∂u/∂x at (1, 2, -2) = 1/9
∂u/∂y at (1, 2, -2) = 2/9
∂u/∂z at (1, 2, -2) = -2/9
Gradient ∇u at (1, 2, -2) = (1/9, 2/9, -2/9)

Numerical results:
∇u at (1, 2, -2) = (0.1111111111111111, 0.2222222222222222, -0.2222222222222222)
```

> **Question 2:** As we all know, whether to sleep in is a complex question that depends on multiple variables. The following is a random selection of student A's 12-day data on sleeping in. Please build a decision tree based on this data, and use the information gain to divide the attributes. An illustration of the calculation process and the final decision tree is required. Hint: For some nodes, you may not need to calculate conditional entropy, but directly make decision by observing the data.

Season	After 8:00	Wind	Sleep in
spring	no	breeze	yes
winter	no	no wind	yes
autumn	yes	breeze	yes
winter	no	no wind	yes
summer	no	breeze	yes
winter	yes	breeze	yes
winter	no	gale	yes
winter	no	no wind	yes
spring	yes	no wind	no
summer	yes	gale	no
summer	no	gale	no
autumn	yes	breeze	yes

> (P.S. Sleeping in is not a good habit)

The two main formulas on the PPT:

$$\operatorname{Ent}(D) = -\sum_{k=1}^{|\mathcal{Y}|} p_k \log_2 p_k$$

$$\operatorname{Gain}(D, a) = \operatorname{Ent}(D) - \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right)$$

For the Node1, calculate:

$$Ent(D) = -\sum_{k=1}^{|\mathcal{Y}|} p_k \log_2 p_k = -\frac{3}{12} log_2(\frac{3}{12}) - \frac{9}{12} log_2(\frac{9}{12}) = 0.8113$$

For the attribute **season**:

$$\begin{aligned}
\sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) &= \frac{2}{12}(-\frac{1}{2}log_2(\frac{1}{2}) - \frac{1}{2}log_2(\frac{1}{2})) + \frac{3}{12}(-\frac{2}{3}log_2(\frac{2}{3}) - \frac{1}{3}log_2(\frac{1}{3})) \\
&+ \frac{2}{12}(-\frac{2}{2}log_2(\frac{2}{2})) + \frac{5}{12}(-\frac{5}{5}log_2(\frac{5}{5})) \\
&= \frac{2}{12} \times 1 + \frac{3}{12} \times 0.9183 + \frac{2}{12} \times 0 + \frac{5}{12} \times 0 = 0.3962 \\
gain(Season) &= \operatorname{Ent}(D) - \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) = 0.4151
\end{aligned}$$

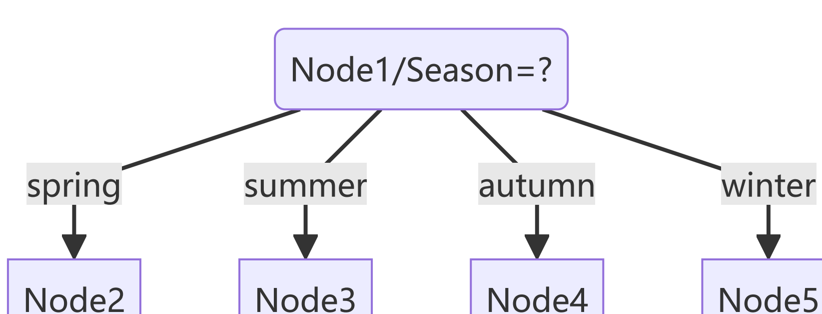
For the attribute **After 8:00**:

$$\begin{aligned}
\sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) &= \frac{7}{12}(-\frac{1}{7}log_2(\frac{1}{7}) - \frac{6}{7}log_2(\frac{6}{7})) + \frac{5}{12}(-\frac{2}{5}log_2(\frac{2}{5}) - \frac{3}{5}log_2(\frac{3}{5})) \\
&= \frac{7}{12} \times 0.5917 + \frac{5}{12} \times 0.9710 = 0.7497 \\
gain(After\ 8:00) &= \operatorname{Ent}(D) - \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) = 0.0616
\end{aligned}$$

For the attribute **Wind**:

$$\begin{aligned}
\sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) &= \frac{4}{12}(-\frac{1}{4}log_2(\frac{1}{4}) - \frac{3}{4}log_2(\frac{3}{4})) + \frac{5}{12}(-\frac{5}{5}log_2(\frac{5}{5})) + \frac{3}{12}(-\frac{2}{3}log_2(\frac{2}{3}) - \frac{1}{3}log_2(\frac{1}{3})) \\
&= \frac{4}{12} \times 0.8113 + \frac{5}{12} \times 0 + \frac{3}{12} \times 0.9183 = 0.5 \\
gain(Wind) &= \operatorname{Ent}(D) - \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}\left(D^v\right) = 0.3113
\end{aligned}$$

$0.4151 > 0.3113 > 0.0616$, so Season is chosen as root.



For the Node2, calculate:

$$Ent(D) = -\frac{1}{2}log_2(\frac{1}{2}) - \frac{1}{2}log_2(\frac{1}{2}) = 1$$

For the attribute **After 8:00**:

$$\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=\frac{1}{2}(-\frac{1}{1}log_2(\frac{1}{1}))+\frac{1}{2}(-\frac{1}{1}log_2(\frac{1}{1}))=0$$

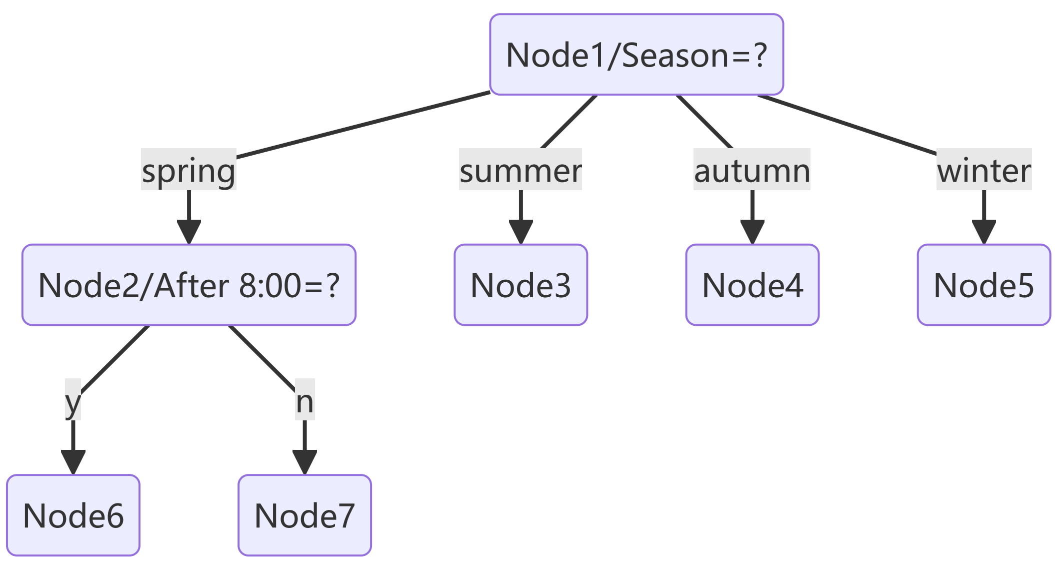
$$gain(After\ 8:00)=\text{Ent}(D)-\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=1.000$$

For the attribute **Wind**:

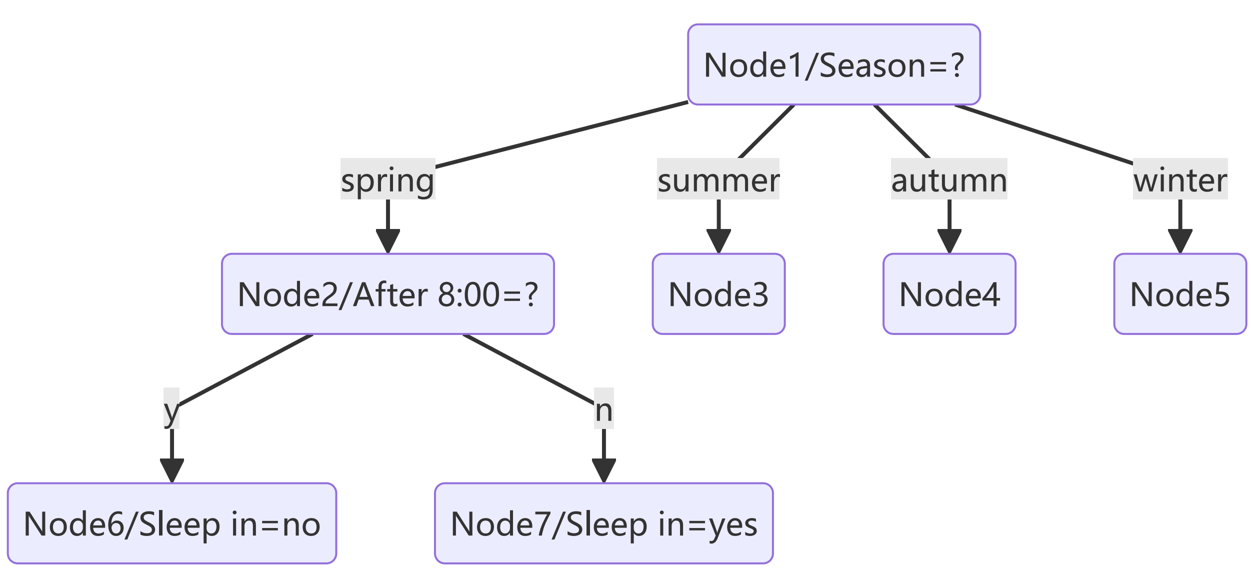
$$\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=\frac{1}{2}(-\frac{1}{1}log_2(\frac{1}{1}))+\frac{1}{2}(-\frac{1}{1}log_2(\frac{1}{1}))=0$$

$$gain(Wind)=\text{Ent}(D)-\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=1.000$$

$1.000 = 1.000$, We can choose both of them. Here I choose attribute **After 8:00** with gain 1.000.



For Node6 and Node7, we can observe that all target_values have the same value. So Node6 Sleep in = no and Node7 Sleep in = yes.



For the Node3, calculate:

$$Ent(D)=-\frac{2}{3}log_2(\frac{2}{3})-\frac{1}{3}log_2(\frac{1}{3})=0.9183$$

For the attribute **After 8:00**:

$$\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=\frac{2}{3}(-\frac{1}{2}log_2(\frac{1}{2})-\frac{1}{2}log_2(\frac{1}{2}))+\frac{1}{3}(-\frac{1}{1}log_2(\frac{1}{1}))=\frac{2}{3}\times 1+\frac{1}{3}\times 0=0.6667$$

$$gain(After\ 8:00)=\text{Ent}(D)-\sum_{v=1}^{V}\frac{|D^v|}{|D|}\text{Ent}\left(D^v\right)=0.2516$$

For the attribute **Wind**:

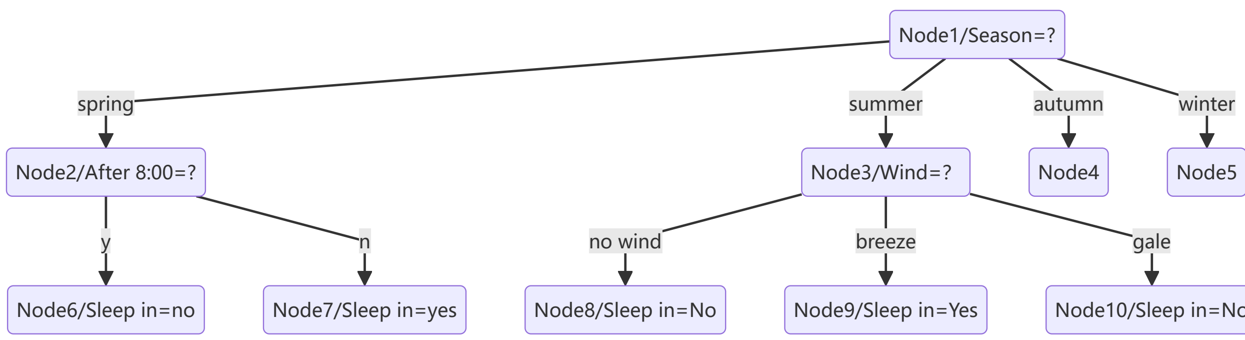
$$\sum_{v=1}^{V}\frac{|D^v|}{|D|}\mathrm{Ent}\left(D^v\right)=\frac{1}{3}(-\frac{1}{1}log_2(\frac{1}{1}))+\frac{2}{3}(-\frac{2}{2}log_2(\frac{2}{2}))=0$$

$$gain(Wind)=\mathrm{Ent}(D)-\sum_{v=1}^{V}\frac{|D^v|}{|D|}\mathrm{Ent}\left(D^v\right)=0.9183$$

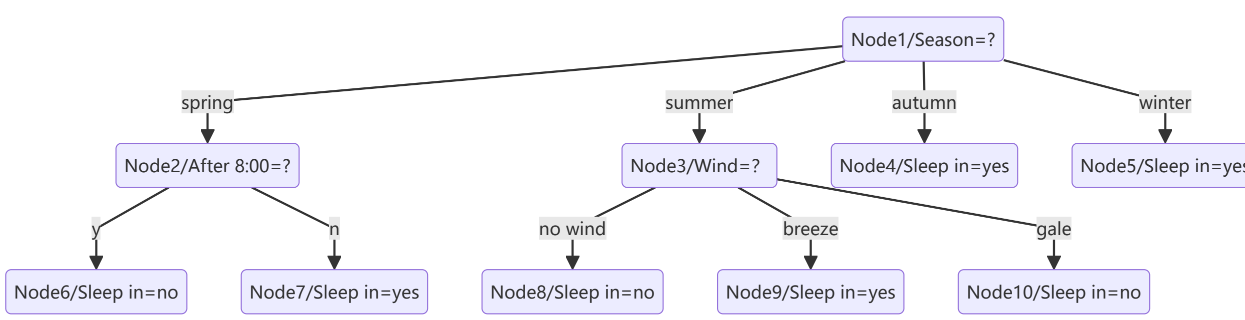
$0.9183 > 0.2516$ So I choose attribute **Wind** with gain 1.000.

Because $D_{No\ wind}$ is empty, Node8 is marked as the class in D with largest proportion: no.

For Node9 and Node10, we can observe that all target_values have the same value. So Node9 Sleep in = yes and Node10 Sleep in = no.



For Node4 and Node5, we can observe that all target_values have the same value. So Node4 Sleep in = yes and Node5 Sleep in = yes. So, the final decision tree structure is:



Code program for Question2:

```
import pandas as pd
import numpy as np
import math
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches

# Calculate information entropy
def calculate_entropy(data_labels):
    """Calculate information entropy
    Args:
        data_labels: List of sample labels
    Returns:
        entropy: Information entropy
    """
    # Get unique values and counts of labels
    unique_labels, counts = np.unique(data_labels, return_counts=True)
    # Calculate probabilities
    probabilities = counts / len(data_labels)
    # Calculate entropy
    entropy = 0
    for p in probabilities:
        entropy -= p * math.log2(p) if p > 0 else 0
    return entropy

# Calculate conditional entropy of a feature
def calculate_conditional_entropy(feature_values, labels):
    """Calculate conditional entropy
    Args:
        feature_values: List of feature values
```

```
        labels: Corresponding list of labels
    Returns:
        conditional_entropy: Conditional entropy
    """
    # Get unique feature values
    unique_values = np.unique(feature_values)
    conditional_entropy = 0

    # Calculate entropy for each feature value, weighted sum
    for value in unique_values:
        # Get indices of samples with current feature value
        indices = np.where(feature_values == value)[0]
        # Proportion of samples with current feature value
        proportion = len(indices) / len(labels)
        # Labels corresponding to current feature value
        subset_labels = [labels[i] for i in indices]
        # Calculate entropy for current feature value
        entropy = calculate_entropy(subset_labels)
        # Weighted sum
        conditional_entropy += proportion * entropy

    return conditional_entropy

# Calculate information gain
def calculate_gain(feature_values, labels):
    """Calculate information gain
    Args:
        feature_values: List of feature values
        labels: Corresponding list of labels
    Returns:
        gain: Information gain
    """
    # Calculate information entropy of dataset
    entropy = calculate_entropy(labels)
    # Calculate conditional entropy
    conditional_entropy = calculate_conditional_entropy(feature_values, labels)
    # Calculate information gain
    gain = entropy - conditional_entropy
    return gain, conditional_entropy

def process_data():
    """Process data, build decision tree as required"""
    # Create dataset
    data = {
        'Season': ['spring', 'winter', 'autumn', 'winter', 'summer', 'winter',
'winter', 'winter', 'spring', 'summer', 'summer', 'autumn'],
        'After 8:00': ['no', 'no', 'yes', 'no', 'no', 'yes', 'no', 'no', 'yes', 'yes',
'no', 'yes'],
        'Wind': ['breeze', 'no wind', 'breeze', 'no wind', 'breeze', 'breeze', 'gale',
'no wind', 'no wind', 'gale', 'gale', 'breeze'],
        'Sleep in': ['yes', 'yes', 'yes', 'yes', 'yes', 'yes', 'yes', 'yes', 'no',
'no', 'no', 'yes']
    }

    # Convert to DataFrame for easier operations
    df = pd.DataFrame(data)

    # Print dataset
    print("Dataset:")
    print(df)
    print("\n")

    # Extract labels
    labels = df['Sleep in'].values

    # Calculate information entropy of dataset
    dataset_entropy = calculate_entropy(labels)
```

```python
    print(f"Calculate information entropy of dataset:")
    print(f"Ent(D) = -{9/12}*log2({9/12}) - {3/12}*log2({3/12}) = 
{dataset_entropy:.4f}")

    # Calculate information gain for each feature, select best splitting feature
    feature_names = ['Season', 'After 8:00', 'Wind']
    gains = []
    conditional_entropies = []

    # Node 1 splitting calculation
    print("\nCalculate Node 1 splitting:")

    # Season feature
    season_values = df['Season'].values
    season_gain, season_conditional_entropy = calculate_gain(season_values, labels)
    gains.append(season_gain)
    conditional_entropies.append(season_conditional_entropy)

    # Detailed calculation process for Season
    print("\na=Season:")

    # Get unique values and counts for Season
    unique_seasons, season_counts = np.unique(season_values, return_counts=True)
    season_details = []

    for i, season in enumerate(unique_seasons):
        indices = np.where(season_values == season)[0]
        subset_labels = [labels[j] for j in indices]
        yes_count = subset_labels.count('yes')
        no_count = subset_labels.count('no')
        total = len(subset_labels)

        # Calculate entropy for current feature value
        if total > 0:
            if yes_count == total or no_count == total:  # Pure node
                entropy = 0
            else:
                p_yes = yes_count / total
                p_no = no_count / total
                entropy = -p_yes * math.log2(p_yes) - p_no * math.log2(p_no)
        else:
            entropy = 0

        season_details.append({
            'value': season,
            'count': total,
            'yes_count': yes_count,
            'no_count': no_count,
            'entropy': entropy
        })

        # Print calculation process
        print(f"  Season = {season}:")
        print(f"    Samples: {total}, yes: {yes_count}, no: {no_count}")
        if total > 0:
            if yes_count == total:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) = 
{entropy:.4f}")
            elif no_count == total:
                print(f"    Entropy = -({no_count}/{total})log2({no_count}/{total}) = 
{entropy:.4f}")
            else:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) - 
({no_count}/{total})log2({no_count}/{total}) = {entropy:.4f}")
        else:
            print(f"    Entropy = 0")
```

```python
    # Calculate conditional entropy
    season_conditional_entropy_calculation = ""
    for detail in season_details:
        value = detail['value']
        count = detail['count']
        entropy = detail['entropy']
        season_conditional_entropy_calculation += f"({count}/12) * {entropy:.4f} + "

    season_conditional_entropy_calculation =
season_conditional_entropy_calculation.rstrip(" + ")
    print(f"\n  Conditional Entropy = {season_conditional_entropy_calculation} =
{season_conditional_entropy:.4f}")
    print(f"  Information Gain = {dataset_entropy:.4f} -
{season_conditional_entropy:.4f} = {season_gain:.4f}")

    # After 8:00 feature
    after_8_values = df['After 8:00'].values
    after_8_gain, after_8_conditional_entropy = calculate_gain(after_8_values, labels)
    gains.append(after_8_gain)
    conditional_entropies.append(after_8_conditional_entropy)

    # Detailed calculation process for After 8:00
    print("\na=After 8:00:")

    # Get unique values and counts for After 8:00
    unique_after_8, after_8_counts = np.unique(after_8_values, return_counts=True)
    after_8_details = []

    for i, after_8 in enumerate(unique_after_8):
        indices = np.where(after_8_values == after_8)[0]
        subset_labels = [labels[j] for j in indices]
        yes_count = subset_labels.count('yes')
        no_count = subset_labels.count('no')
        total = len(subset_labels)

        # Calculate entropy for current feature value
        if total > 0:
            if yes_count == total or no_count == total:  # Pure node
                entropy = 0
            else:
                p_yes = yes_count / total
                p_no = no_count / total
                entropy = -p_yes * math.log2(p_yes) - p_no * math.log2(p_no)
        else:
            entropy = 0

        after_8_details.append({
            'value': after_8,
            'count': total,
            'yes_count': yes_count,
            'no_count': no_count,
            'entropy': entropy
        })

        # Print calculation process
        print(f"  After 8:00 = {after_8}:")
        print(f"    Samples: {total}, yes: {yes_count}, no: {no_count}")
        if total > 0:
            if yes_count == total:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) =
{entropy:.4f}")
            elif no_count == total:
                print(f"    Entropy = -({no_count}/{total})log2({no_count}/{total}) =
{entropy:.4f}")
            else:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) -
({no_count}/{total})log2({no_count}/{total}) = {entropy:.4f}")
```

```
        else:
            print(f"    Entropy = 0")

    # Calculate conditional entropy
    after_8_conditional_entropy_calculation = ""
    for detail in after_8_details:
        value = detail['value']
        count = detail['count']
        entropy = detail['entropy']
        after_8_conditional_entropy_calculation += f"({count}/12) * {entropy:.4f} + "

    after_8_conditional_entropy_calculation =
after_8_conditional_entropy_calculation.rstrip(" + ")
    print(f"\n  Conditional Entropy = {after_8_conditional_entropy_calculation} =
{after_8_conditional_entropy:.4f}")
    print(f"  Information Gain = {dataset_entropy:.4f} -
{after_8_conditional_entropy:.4f} = {after_8_gain:.4f}")

    # Wind feature
    wind_values = df['Wind'].values
    wind_gain, wind_conditional_entropy = calculate_gain(wind_values, labels)
    gains.append(wind_gain)
    conditional_entropies.append(wind_conditional_entropy)

    # Detailed calculation process for Wind
    print("\na=Wind:")

    # Get unique values and counts for Wind
    unique_winds, wind_counts = np.unique(wind_values, return_counts=True)
    wind_details = []

    for i, wind in enumerate(unique_winds):
        indices = np.where(wind_values == wind)[0]
        subset_labels = [labels[j] for j in indices]
        yes_count = subset_labels.count('yes')
        no_count = subset_labels.count('no')
        total = len(subset_labels)

        # Calculate entropy for current feature value
        if total > 0:
            if yes_count == total or no_count == total:  # Pure node
                entropy = 0
            else:
                p_yes = yes_count / total
                p_no = no_count / total
                entropy = -p_yes * math.log2(p_yes) - p_no * math.log2(p_no)
        else:
            entropy = 0

        wind_details.append({
            'value': wind,
            'count': total,
            'yes_count': yes_count,
            'no_count': no_count,
            'entropy': entropy
        })

        # Print calculation process
        print(f"  Wind = {wind}:")
        print(f"    Samples: {total}, yes: {yes_count}, no: {no_count}")
        if total > 0:
            if yes_count == total:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) =
{entropy:.4f}")
            elif no_count == total:
                print(f"    Entropy = -({no_count}/{total})log2({no_count}/{total}) =
{entropy:.4f}")
```

```
            else:
                print(f"    Entropy = -({yes_count}/{total})log2({yes_count}/{total}) -
({no_count}/{total})log2({no_count}/{total}) = {entropy:.4f}")
        else:
            print(f"    Entropy = 0")

    # Calculate conditional entropy
    wind_conditional_entropy_calculation = ""
    for detail in wind_details:
        value = detail['value']
        count = detail['count']
        entropy = detail['entropy']
        wind_conditional_entropy_calculation += f"({count}/12) * {entropy:.4f} + "

    wind_conditional_entropy_calculation =
wind_conditional_entropy_calculation.rstrip(" + ")
    print(f"\n  Conditional Entropy = {wind_conditional_entropy_calculation} =
{wind_conditional_entropy:.4f}")
    print(f"  Information Gain = {dataset_entropy:.4f} - {wind_conditional_entropy:.4f}
= {wind_gain:.4f}")

    # Select best splitting feature
    best_feature_idx = np.argmax(gains)
    best_feature = feature_names[best_feature_idx]
    best_gain = gains[best_feature_idx]

    print(f"\nBest splitting feature: {best_feature}, Information Gain:
{best_gain:.4f}")

    # Split dataset based on best feature
    split_data = {}
    for value in np.unique(df[best_feature]):
        split_df = df[df[best_feature] == value]
        split_data[value] = split_df

    # Print child nodes after splitting
    print("\nChild nodes after splitting:")
    for value, subset in split_data.items():
        print(f"Node {best_feature} = {value}:")
        print(subset[['Season', 'After 8:00', 'Wind', 'Sleep in']])
        print()

    # Calculate information gain for spring node (Node 2)
    if 'spring' in split_data:
        spring_df = split_data['spring']
        spring_labels = spring_df['Sleep in'].values
        spring_entropy = calculate_entropy(spring_labels)

        print("\nCalculate Node 2 (Season = spring) splitting:")
        print(f"Ent(D) = {spring_entropy:.4f}")

        # Calculate information gain for each feature in spring subset
        spring_feature_names = ['After 8:00', 'Wind']
        spring_gains = []
        spring_conditional_entropies = []

        for feature in spring_feature_names:
            feature_values = spring_df[feature].values
            gain, cond_entropy = calculate_gain(feature_values, spring_labels)
            spring_gains.append(gain)
            spring_conditional_entropies.append(cond_entropy)

            print(f"\na={feature}:")
            print(f"  Conditional Entropy = {cond_entropy:.4f}")
            print(f"  Information Gain = {spring_entropy:.4f} - {cond_entropy:.4f} =
{gain:.4f}")
```

```
        best_spring_feature_idx = np.argmax(spring_gains)
        best_spring_feature = spring_feature_names[best_spring_feature_idx]
        best_spring_gain = spring_gains[best_spring_feature_idx]

        print(f"\nBest splitting feature for Node 2: {best_spring_feature}, Information
Gain: {best_spring_gain:.4f}")

    # Calculate information gain for summer node (Node 3)
    if 'summer' in split_data:
        summer_df = split_data['summer']
        summer_labels = summer_df['Sleep in'].values
        summer_entropy = calculate_entropy(summer_labels)

        print("\nCalculate Node 3 (Season = summer) splitting:")
        print(f"Ent(D) = {summer_entropy:.4f}")

        # Calculate information gain for each feature in summer subset
        summer_feature_names = ['After 8:00', 'Wind']
        summer_gains = []
        summer_conditional_entropies = []

        for feature in summer_feature_names:
            feature_values = summer_df[feature].values
            gain, cond_entropy = calculate_gain(feature_values, summer_labels)
            summer_gains.append(gain)
            summer_conditional_entropies.append(cond_entropy)

            print(f"\na={feature}:")
            print(f"  Conditional Entropy = {cond_entropy:.4f}")
            print(f"  Information Gain = {summer_entropy:.4f} - {cond_entropy:.4f} =
{gain:.4f}")

        best_summer_feature_idx = np.argmax(summer_gains)
        best_summer_feature = summer_feature_names[best_summer_feature_idx]
        best_summer_gain = summer_gains[best_summer_feature_idx]

        print(f"\nBest splitting feature for Node 3: {best_summer_feature}, Information
Gain: {best_summer_gain:.4f}")

    # Draw final decision tree
    plot_decision_tree()

def plot_decision_tree():
    """Draw the final decision tree"""
    plt.figure(figsize=(12, 8))
    ax = plt.gca()
    ax.set_axis_off()

    # Create simple node objects for drawing
    class Node:
        def __init__(self, id, feature=None, is_leaf=False, label=None):
            self.id = id
            self.feature = feature
            self.is_leaf = is_leaf
            self.label = label
            self.children = {}

    # Manually create decision tree structure based on reference answer
    root = Node(1, feature="Season")
    node2 = Node(2, feature="After 8:00")
    node3 = Node(3, feature="Wind")
    node4 = Node(4, is_leaf=True, label="yes")
    node5 = Node(5, is_leaf=True, label="yes")
    node6 = Node(6, is_leaf=True, label="no")
    node7 = Node(7, is_leaf=True, label="yes")
    node8 = Node(8, is_leaf=True, label="no")
    node9 = Node(9, is_leaf=True, label="yes")
```

```python
    node10 = Node(10, is_leaf=True, label="no")

    # Build relationships between nodes
    root.children["spring"] = node2
    root.children["summer"] = node3
    root.children["autumn"] = node4
    root.children["winter"] = node5

    node2.children["yes"] = node6
    node2.children["no"] = node7

    node3.children["no wind"] = node8
    node3.children["breeze"] = node9
    node3.children["gale"] = node10

    # Node positions
    node_positions = {
        1: (6, 7),
        2: (3, 5),
        3: (6, 5),
        4: (9, 5),
        5: (12, 5),
        6: (2, 3),
        7: (4, 3),
        8: (5, 3),
        9: (7, 3),
        10: (9, 3)
    }

    # Draw nodes and connections
    def draw_node(node, pos):
        x, y = pos

        # Set node style
        if node.is_leaf:
            color = 'lightgreen' if node.label == 'yes' else 'lightcoral'
            node_text = f"Node {node.id}\nSleep in: {node.label}"
        else:
            color = 'lightblue'
            node_text = f"Node {node.id}\n{node.feature}"

        # Draw node
        circle = plt.Circle((x, y), 0.5, fill=True, color=color, alpha=0.8)
        ax.add_patch(circle)
        ax.text(x, y, node_text, ha='center', va='center', fontsize=9)

        # Draw child nodes and edges
        for value, child in node.children.items():
            child_x, child_y = node_positions[child.id]

            # Draw edge
            ax.plot([x, child_x], [y - 0.5, child_y + 0.5], 'k-')

            # Add feature value text on edge
            mid_x = (x + child_x) / 2
            mid_y = (y - 0.5 + child_y + 0.5) / 2
            ax.text(mid_x, mid_y, str(value), ha='center', va='center',
                   bbox=dict(boxstyle="round,pad=0.3", facecolor='white', alpha=0.8),
fontsize=8)

    # Start drawing from root node
    for node_id, pos in node_positions.items():
        if node_id == 1:
            draw_node(root, pos)
        elif node_id == 2:
            draw_node(node2, pos)
        elif node_id == 3:
```

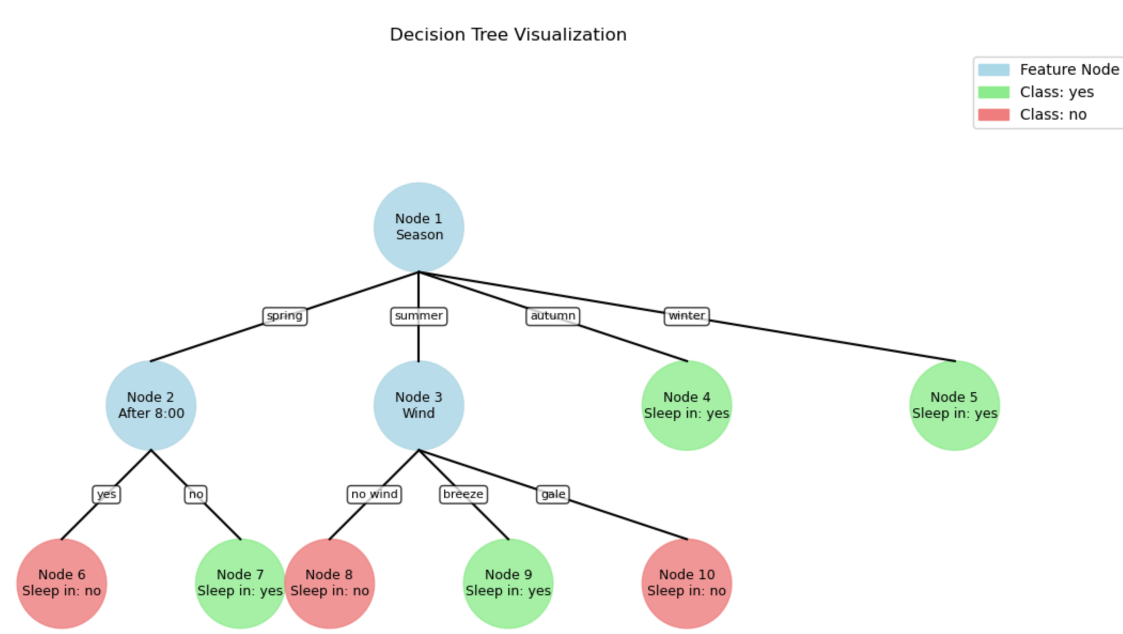
```
            draw_node(node3, pos)
        elif node_id == 4:
            draw_node(node4, pos)
        elif node_id == 5:
            draw_node(node5, pos)
        elif node_id == 6:
            draw_node(node6, pos)
        elif node_id == 7:
            draw_node(node7, pos)
        elif node_id == 8:
            draw_node(node8, pos)
        elif node_id == 9:
            draw_node(node9, pos)
        elif node_id == 10:
            draw_node(node10, pos)

    # Add legend
    legend_patches = [
        mpatches.Patch(color='lightblue', label='Feature Node'),
        mpatches.Patch(color='lightgreen', label='Class: yes'),
        mpatches.Patch(color='lightcoral', label='Class: no')
    ]
    plt.legend(handles=legend_patches, loc='upper right')

    plt.title('Decision Tree Visualization')
    plt.xlim(0, 14)
    plt.ylim(0, 9)
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    process_data()
```

Program output for Question2:



> **Question 3:** Given the following data: where *x* is a 2D vector, the first dimension takes values in (1, 2, 3), the second dimension takes values in (S, M, L), and *y* takes values in (-1, 1). Given new data *x* = (2, S), try the Naive Bayes method to predict the value of *y* at this time.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x(1)	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
x(2)	S	M	M	S	S	S	M	M	L	L	L	M	M	L	L
y	-1	-1	1	1	-1	-1	1	1	1	1	1	1	1	1	-1

First of all, we conduct category statistics. The prior probability:

$$P(y=-1)=\frac{5}{15}=\frac{1}{3}$$
$$P(y=1)=\frac{10}{15}=\frac{2}{3}$$

Then we calculate the conditional probability:

$$P(x_1=2|y=-1)=\frac{P(x_1=2,y=-1)}{P(y=-1)}=\frac{\frac{1}{15}}{\frac{1}{3}}=\frac{1}{5}$$
$$P(x_1=2|y=1)=\frac{P(x_1=2,y=1)}{P(y=1)}=\frac{\frac{4}{15}}{\frac{2}{3}}=\frac{4}{10}=\frac{2}{5}$$
$$P(x_2=S|y=-1)=\frac{P(x_2=S,y=-1)}{P(y=-1)}=\frac{\frac{3}{15}}{\frac{1}{3}}=\frac{3}{5}$$
$$P(x_2=S|y=1)=\frac{P(x_2=S,y=1)}{P(y=1)}=\frac{\frac{1}{15}}{\frac{2}{3}}=\frac{1}{10}$$

So, next we can calculate:

$$P(y=-1|x_1=2,x_2=S)=\frac{P(y=-1)\times P(x_1=2|y=-1)\times P(x_2=S|y=-1)}{P(x_1=2,x_2=S)}$$
$$=\frac{\frac{1}{3}\times\frac{1}{5}\times\frac{3}{5}}{\frac{1}{15}}=\frac{\frac{1}{25}}{\frac{1}{15}}=\frac{3}{5}$$
$$P(y=1|x_1=2,x_2=S)=\frac{P(y=1)\times P(x_1=2|y=1)\times P(x_2=S|y=1)}{P(x_1=2,x_2=S)}$$
$$=\frac{\frac{2}{3}\times\frac{2}{5}\times\frac{1}{10}}{\frac{1}{15}}=\frac{\frac{2}{75}}{\frac{1}{15}}=\frac{2}{5}$$
$$\frac{3}{5}>\frac{2}{5},\therefore P(y=-1|x_1=2,x_2=S)>P(y=1|x_1=2,x_2=S)$$
$$So\ we\ predict\ the\ value\ of\ y\ is\ -1, y=-1.$$

Code program for Question3:

```
import numpy as np
import pandas as pd
from fractions import Fraction

def naive_bayes_calculation():
    # Create dataset
    data = {
        'x1': [1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3],
        'x2': ['S', 'M', 'M', 'S', 'S', 'S', 'M', 'M', 'L', 'L', 'L', 'M', 'M', 'L',
'L'],
        'y': [-1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1, -1]
    }

    df = pd.DataFrame(data)

    # New data point
    new_x = (2, 'S')

    # 1. Calculate prior probabilities P(y=-1) and P(y=1)
    total_samples = len(df)
    negative_samples = len(df[df['y'] == -1])
```

```python
    positive_samples = len(df[df['y'] == 1])

    p_y_neg = negative_samples / total_samples
    p_y_pos = positive_samples / total_samples

    # Express as fractions
    p_y_neg_fraction = Fraction(negative_samples, total_samples)
    p_y_pos_fraction = Fraction(positive_samples, total_samples)

    print("Step 1: Calculate Prior Probabilities")
    print(f"P(y=-1) = {negative_samples}/{total_samples} = {p_y_neg_fraction} =
{p_y_neg:.4f}")
    print(f"P(y=1) = {positive_samples}/{total_samples} = {p_y_pos_fraction} =
{p_y_pos:.4f}")
    print("\n")

    # 2. Calculate conditional probabilities
    # 2.1 Calculate P(x1=2|y=-1)
    x1_neg_count = len(df[(df['y'] == -1) & (df['x1'] == new_x[0])])
    p_x1_given_y_neg = x1_neg_count / negative_samples
    p_x1_given_y_neg_fraction = Fraction(x1_neg_count, negative_samples)

    # 2.2 Calculate P(x1=2|y=1)
    x1_pos_count = len(df[(df['y'] == 1) & (df['x1'] == new_x[0])])
    p_x1_given_y_pos = x1_pos_count / positive_samples
    p_x1_given_y_pos_fraction = Fraction(x1_pos_count, positive_samples)

    # 2.3 Calculate P(x2=S|y=-1)
    x2_neg_count = len(df[(df['y'] == -1) & (df['x2'] == new_x[1])])
    p_x2_given_y_neg = x2_neg_count / negative_samples
    p_x2_given_y_neg_fraction = Fraction(x2_neg_count, negative_samples)

    # 2.4 Calculate P(x2=S|y=1)
    x2_pos_count = len(df[(df['y'] == 1) & (df['x2'] == new_x[1])])
    p_x2_given_y_pos = x2_pos_count / positive_samples
    p_x2_given_y_pos_fraction = Fraction(x2_pos_count, positive_samples)

    print("Step 2: Calculate Conditional Probabilities")
    print(f"P(x1={new_x[0]}|y=-1) = P(x1={new_x[0]},y=-1)/P(y=-1) =
({x1_neg_count}/{total_samples})/({negative_samples}/{total_samples}) =
{x1_neg_count}/{negative_samples} = {p_x1_given_y_neg_fraction} =
{p_x1_given_y_neg:.4f}")
    print(f"P(x1={new_x[0]}|y=1) = P(x1={new_x[0]},y=1)/P(y=1) =
({x1_pos_count}/{total_samples})/({positive_samples}/{total_samples}) =
{x1_pos_count}/{positive_samples} = {p_x1_given_y_pos_fraction} =
{p_x1_given_y_pos:.4f}")
    print(f"P(x2={new_x[1]}|y=-1) = P(x2={new_x[1]},y=-1)/P(y=-1) =
({x2_neg_count}/{total_samples})/({negative_samples}/{total_samples}) =
{x2_neg_count}/{negative_samples} = {p_x2_given_y_neg_fraction} =
{p_x2_given_y_neg:.4f}")
    print(f"P(x2={new_x[1]}|y=1) = P(x2={new_x[1]},y=1)/P(y=1) =
({x2_pos_count}/{total_samples})/({positive_samples}/{total_samples}) =
{x2_pos_count}/{positive_samples} = {p_x2_given_y_pos_fraction} =
{p_x2_given_y_pos:.4f}")
    print("\n")

    # 3. Calculate the posterior probabilities
    p_y_neg_given_x_numerator = p_y_neg * p_x1_given_y_neg * p_x2_given_y_neg
    p_y_pos_given_x_numerator = p_y_pos * p_x1_given_y_pos * p_x2_given_y_pos

    # Calculate the denominator (evidence)
    p_x = p_y_neg_given_x_numerator + p_y_pos_given_x_numerator

    # Calculate full posterior probabilities
    p_y_neg_given_x = p_y_neg_given_x_numerator / p_x
    p_y_pos_given_x = p_y_pos_given_x_numerator / p_x
```

```
    # Express as fractions
    p_y_neg_given_x_fraction_numerator = p_y_neg_fraction * p_x1_given_y_neg_fraction * p_x2_given_y_neg_fraction
    p_y_pos_given_x_fraction_numerator = p_y_pos_fraction * p_x1_given_y_pos_fraction * p_x2_given_y_pos_fraction
    p_x_fraction = p_y_neg_given_x_fraction_numerator + p_y_pos_given_x_fraction_numerator

    p_y_neg_given_x_fraction = p_y_neg_given_x_fraction_numerator / p_x_fraction
    p_y_pos_given_x_fraction = p_y_pos_given_x_fraction_numerator / p_x_fraction

    print("Step 3: Calculate Posterior Probabilities")
    print("Numerator part:")
    print(f"P(y=-1) × P(x1={new_x[0]}|y=-1) × P(x2={new_x[1]}|y=-1) = {p_y_neg_fraction} × {p_x1_given_y_neg_fraction} × {p_x2_given_y_neg_fraction} = {p_y_neg_given_x_fraction_numerator} = {p_y_neg_given_x_numerator:.6f}")
    print(f"P(y=1) × P(x1={new_x[0]}|y=1) × P(x2={new_x[1]}|y=1) = {p_y_pos_fraction} × {p_x1_given_y_pos_fraction} × {p_x2_given_y_pos_fraction} = {p_y_pos_given_x_fraction_numerator} = {p_y_pos_given_x_numerator:.6f}")

    print("\nDenominator part (Evidence):")
    print(f"P(x) = {p_y_neg_given_x_fraction_numerator} + {p_y_pos_given_x_fraction_numerator} = {p_x_fraction} = {p_x:.6f}")

    print("\nComplete posterior probabilities:")
    print(f"P(y=-1|x) = {p_y_neg_given_x_fraction_numerator}/{p_x_fraction} = {p_y_neg_given_x_fraction} = {p_y_neg_given_x:.6f}")
    print(f"P(y=1|x) = {p_y_pos_given_x_fraction_numerator}/{p_x_fraction} = {p_y_pos_given_x_fraction} = {p_y_pos_given_x:.6f}")
    print("\n")

    # 4. Compare posterior probabilities and predict result
    if p_y_neg_given_x > p_y_pos_given_x:
        prediction = -1
    else:
        prediction = 1

    print("Step 4: Compare Posterior Probabilities and Predict Result")
    if p_y_neg_given_x > p_y_pos_given_x:
        print(f"P(y=-1|x) > P(y=1|x)")
        print(f"{p_y_neg_given_x_fraction} > {p_y_pos_given_x_fraction}")
        print(f"{p_y_neg_given_x:.6f} > {p_y_pos_given_x:.6f}")
    else:
        print(f"P(y=-1|x) < P(y=1|x)")
        print(f"{p_y_neg_given_x_fraction} < {p_y_pos_given_x_fraction}")
        print(f"{p_y_neg_given_x:.6f} < {p_y_pos_given_x:.6f}")

    print(f"\nSince {('P(y=-1|x) > P(y=1|x)' if p_y_neg_given_x > p_y_pos_given_x else 'P(y=-1|x) < P(y=1|x)')}, the predicted result is y = {prediction}")

    # 5. Summary
    print("\nSummary:")
    print(f"For the new data point x = {new_x}, the Naive Bayes method predicts y = {prediction}")

    return prediction

if __name__ == "__main__":
    naive_bayes_calculation()
```

Program output for Question3:

```
Step 1: Calculate Prior Probabilities
P(y=-1) = 5/15 = 1/3 = 0.3333
P(y=1) = 10/15 = 2/3 = 0.6667
```

```
Step 2: Calculate Conditional Probabilities
P(x1=2|y=-1) = P(x1=2,y=-1)/P(y=-1) = (1/15)/(5/15) = 1/5 = 1/5 = 0.2000
P(x1=2|y=1) = P(x1=2,y=1)/P(y=1) = (4/15)/(10/15) = 4/10 = 2/5 = 0.4000
P(x2=S|y=-1) = P(x2=S,y=-1)/P(y=-1) = (3/15)/(5/15) = 3/5 = 3/5 = 0.6000
P(x2=S|y=1) = P(x2=S,y=1)/P(y=1) = (1/15)/(10/15) = 1/10 = 1/10 = 0.1000

Step 3: Calculate Posterior Probabilities
Numerator part:
P(y=-1) × P(x1=2|y=-1) × P(x2=S|y=-1) = 1/3 × 1/5 × 3/5 = 1/25 = 0.040000
P(y=1) × P(x1=2|y=1) × P(x2=S|y=1) = 2/3 × 2/5 × 1/10 = 2/75 = 0.026667

Denominator part (Evidence):
P(x) = 1/25 + 2/75 = 1/15 = 0.066667

Complete posterior probabilities:
P(y=-1|x) = 1/25/1/15 = 3/5 = 0.600000
P(y=1|x) = 2/75/1/15 = 2/5 = 0.400000

Since P(y=-1|x) > P(y=1|x), the predicted result is y = -1

Summary:
For the new data point x = (2, 'S'), the Naive Bayes method predicts y = -1
```