

module source where

open import lib

-- Operator precedence and associativity

infix 1 \leq : _

infix 2 \longrightarrow _

infix 4 \vdash _

infix 4 \in _

infixl 5 \dashv _

infixr 7 \Rightarrow _

-- Types

data Type : Set where

comm : Type

intexp : Type

intacc : Type

intvar : Type

\Rightarrow : Type \rightarrow Type \rightarrow Type

-- Subtype relation

data \leq : Type \rightarrow Type \rightarrow Set where

\leq :-refl : $\forall \{A\} \rightarrow A \leq A$

\leq :-trans : $\forall \{A A' A''\} \rightarrow A \leq A' \rightarrow A' \leq A'' \rightarrow A \leq A''$

\leq :-fn : $\forall \{A A' B B'\} \rightarrow A' \leq A \rightarrow B \leq B' \rightarrow A \Rightarrow B \leq A' \Rightarrow B'$

var- \leq :-exp : intvar \leq intexp

var- \leq :-acc : intvar \leq intacc

-- Contexts

data Context : Set where

\cdot : Context

$_,_$: Context \rightarrow Type \rightarrow Context

-- Variables and the lookup judgement

data \in : Type \rightarrow Context \rightarrow Set where

Zero : $\forall \{ \Gamma A \} \rightarrow A \in \Gamma , A$

Suc : $\forall \{ \Gamma A B \} \rightarrow B \in \Gamma \rightarrow B \in \Gamma , A$

-- Terms and the typing judgement

data \vdash : Context \rightarrow Type \rightarrow Set where

Var : $\forall \{ \Gamma A \} \rightarrow A \in \Gamma \rightarrow \Gamma \vdash A$

-- subtyping

Sub : $\forall \{ \Gamma A B \} \rightarrow \Gamma \vdash A \rightarrow A \leq B \rightarrow \Gamma \vdash B$

-- lambda function and application

Lambda : $\forall \{ \Gamma A B \} \rightarrow \Gamma , A \vdash B \rightarrow \Gamma \vdash A \Rightarrow B$

App : $\forall \{ \Gamma A B \} \rightarrow \Gamma \vdash A \Rightarrow B \rightarrow \Gamma \vdash A \rightarrow \Gamma \vdash B$

-- command

Skip : $\forall \{ \Gamma \} \rightarrow \Gamma \vdash \text{comm}$

Seq : $\forall \{ \Gamma \} \rightarrow \Gamma \vdash \text{comm} \rightarrow \Gamma \vdash \text{comm} \rightarrow \Gamma \vdash \text{comm}$

NewVar : $\forall \{ \Gamma \} \rightarrow \Gamma , \text{intvar} \vdash \text{comm} \rightarrow \Gamma \vdash \text{comm}$

Assign : $\forall \{ \Gamma \} \rightarrow \Gamma \vdash \text{intacc} \rightarrow \Gamma \vdash \text{intexp} \rightarrow \Gamma \vdash \text{comm}$

-- intexp

Lit : $\forall \{ \Gamma \} \rightarrow \mathbb{Z} \rightarrow \Gamma \vdash \text{intexp}$

Neg : $\forall \{ \Gamma \} \rightarrow \Gamma \vdash \text{intexp} \rightarrow \Gamma \vdash \text{intexp}$

Plus : $\forall \{ \Gamma \} \rightarrow \Gamma \vdash \text{intexp} \rightarrow \Gamma \vdash \text{intexp} \rightarrow \Gamma \vdash \text{intexp}$

-- Operational semantics

data Value : $\forall \{ \Gamma A \} \rightarrow \Gamma \vdash A \rightarrow$ Set where

V-Lambda : $\forall \{ \Gamma A B \} \{ F : \Gamma , A \vdash B \} \rightarrow \text{Value } (\text{Lambda } \{ \Gamma \} F)$

V-Lit : $\forall \{ \Gamma \} \{ i : \mathbb{Z} \} \rightarrow \text{Value } (\text{Lit } \{ \Gamma \} i)$

V-Skip : $\forall \{ \Gamma \} \rightarrow \text{Value } (\text{Skip } \{ \Gamma \})$

-- Renaming

ext : $\forall \{ \Gamma \Delta \} \rightarrow (\forall \{ A \} \rightarrow A \in \Gamma \rightarrow A \in \Delta)$

$\rightarrow (\forall \{ A B \} \rightarrow B \in \Gamma , A \rightarrow B \in \Delta , A)$

ext ρ Zero = Zero

ext ρ (Suc x) = Suc (ρ x)

rename : $\forall \{ \Gamma \Delta \} \rightarrow (\forall \{ A \} \rightarrow A \in \Gamma \rightarrow A \in \Delta)$

$\rightarrow (\forall \{ A \} \rightarrow \Gamma \vdash A \rightarrow \Delta \vdash A)$

rename ρ (Var $A \in \Gamma$) = Var (ρ $A \in \Gamma$)

rename ρ (Lambda $\Gamma , A \vdash B$) = Lambda (rename (ext ρ) $\Gamma , A \vdash B$)

rename ρ (Sub $\Gamma \vdash A A \leq B$) = Sub (rename ρ $\Gamma \vdash A$) $A \leq B$

rename ρ (App $\Gamma \vdash A \Gamma \vdash B$) = App (rename ρ $\Gamma \vdash A$) (rename ρ $\Gamma \vdash B$)

rename ρ Skip = Skip

rename ρ (Seq $\Gamma \vdash c_1 \Gamma \vdash c_2$) = Seq (rename ρ $\Gamma \vdash c_1$) (rename ρ $\Gamma \vdash c_2$)

rename ρ (NewVar $\Gamma \vdash c$) = NewVar (rename (ext ρ) $\Gamma \vdash c$)

rename ρ (Assign $\Gamma \vdash i \Gamma \vdash e$) = Assign (rename ρ $\Gamma \vdash i$) (rename ρ $\Gamma \vdash e$)

rename ρ (Lit $\Gamma \vdash i$) = Lit $\Gamma \vdash i$

rename ρ (Neg $\Gamma \vdash i$) = Neg (rename ρ $\Gamma \vdash i$)

rename ρ (Plus $\Gamma \vdash i_1 \Gamma \vdash i_2$) = Plus (rename ρ $\Gamma \vdash i_1$) (rename ρ $\Gamma \vdash i_2$)

-- Simultaneous substitution

exts : $\forall \{ \Gamma \Delta \} \rightarrow (\forall \{ A \} \rightarrow A \in \Gamma \rightarrow \Delta \vdash A)$

$\rightarrow (\forall \{ A B \} \rightarrow B \in \Gamma , A \rightarrow \Delta , A \vdash B)$

exts σ Zero = Var Zero

exts σ (Suc x) = rename Suc (σ x)

subst : $\forall \{ \Gamma \Delta \} \rightarrow (\forall \{ A \} \rightarrow A \in \Gamma \rightarrow \Delta \vdash A)$

$\rightarrow (\forall \{ A \} \rightarrow \Gamma \vdash A \rightarrow \Delta \vdash A)$

subst σ (Var $A \in \Gamma$) = σ $A \in \Gamma$

subst σ (Sub $\Gamma \vdash A A \leq B$) = Sub (subst σ $\Gamma \vdash A$) $A \leq B$

subst σ (Lambda $\Gamma , A \vdash B$) = Lambda (subst (exts σ) $\Gamma , A \vdash B$)

subst σ (App $\Gamma \vdash A \Gamma \vdash B$) = App (subst σ $\Gamma \vdash A$) (subst σ $\Gamma \vdash B$)

subst σ Skip = Skip

subst σ (Seq $\Gamma \vdash c_1 \Gamma \vdash c_2$) = Seq (subst σ $\Gamma \vdash c_1$) (subst σ $\Gamma \vdash c_2$)

subst σ (NewVar $\Gamma \vdash c$) = NewVar (subst (exts σ) $\Gamma \vdash c$)

subst σ (Assign $\Gamma \vdash i \Gamma \vdash e$) = Assign (subst σ $\Gamma \vdash i$) (subst σ $\Gamma \vdash e$)

subst σ (Lit $\Gamma \vdash i$) = Lit $\Gamma \vdash i$

subst σ (Neg $\Gamma \vdash i$) = Neg (subst σ $\Gamma \vdash i$)

subst σ (Plus $\Gamma \vdash i_1 \Gamma \vdash i_2$) = Plus (subst σ $\Gamma \vdash i_1$) (subst σ $\Gamma \vdash i_2$)

-- Single substitution

$_ [_]$: $\forall \{ \Gamma A B \} \rightarrow \Gamma , B \vdash A \rightarrow \Gamma \vdash B \rightarrow \Gamma \vdash A$

$_ [_] \{ \Gamma \} \{ A \} \{ B \} N M = \text{subst } \{ \Gamma , B \} \{ \Gamma \} \sigma \{ A \} N$

where

$\sigma : \forall \{ A \} \rightarrow A \in \Gamma , B \rightarrow \Gamma \vdash A$

σ Zero = M

σ (Suc x) = Var x

-- Reduction

data \longrightarrow : $\forall \{ \Gamma A \} \rightarrow (\Gamma \vdash A) \rightarrow (\Gamma \vdash A) \rightarrow$ Set where

App-cong₁ : $\forall \{ \Gamma A B \} \{ F F' : \Gamma \vdash A \Rightarrow B \} \{ E : \Gamma \vdash A \}$

$\rightarrow F \longrightarrow F' \rightarrow \text{App } F E \longrightarrow \text{App } F' E$

App-cong₂ : $\forall \{ \Gamma A B \} \{ V : \Gamma \vdash A \Rightarrow B \} \{ E E' : \Gamma \vdash A \}$

$\rightarrow \text{Value } V \rightarrow E \longrightarrow E' \rightarrow \text{App } V E \longrightarrow \text{App } V E'$

Lambda- β : $\forall \{ \Gamma A B \} \{ F : \Gamma , A \vdash B \} \{ V : \Gamma \vdash A \}$

$\rightarrow \text{Value } V \rightarrow \text{App } (\text{Lambda } F) V \longrightarrow F [V]$