1. (a) $f_X(x,\vartheta) = \frac{1}{\vartheta}\exp\left(-\frac{x}{\vartheta}\right) I_{(0,\infty)}(x)$

$$F_X(x) = \frac{1}{\vartheta}\int_{-\infty}^{x} \exp\left(-\frac{t}{\vartheta}\right) I_{(0,\infty)}(t)\, dt$$

$$= \begin{cases} 0 & x \leq 0 \\ \frac{1}{\vartheta}\int_0^x \exp\left(-\frac{t}{\vartheta}\right) dt & 0 < x \end{cases}$$

$$= \begin{cases} 0 & x \leq 0 \\ \frac{1}{\vartheta}(-\vartheta)\int_0^x \exp\left(-\frac{t}{\vartheta}\right)\left(-\frac{t}{\vartheta}\right) dt & x > 0 \end{cases}$$

$$= \begin{cases} 0 & x \leq 0 \\ -\exp\left(-\frac{t}{\vartheta}\right)\Big|_0^x & x > 0 \end{cases}$$

$$= \begin{cases} 0 & x \leq 0 \\ 1-\exp\left(-\frac{x}{\vartheta}\right) & x > 0 \end{cases}$$

Si $0 \leq x < \infty$, ent

$$0 \leq \underbrace{F_X(x) = 1-\exp\left(-\frac{x}{\vartheta}\right)}_{\text{dominio de la función cuantil } F_X^{-1}} < 1$$

$$\exp\left(-\frac{x}{\vartheta}\right) = 1-q$$

$$-\frac{x}{\vartheta} = \log(1-q)$$

$$x = -\vartheta \log(1-q)$$

$$F_X^{-1}(q) = -\vartheta \log(1-q)$$

(b). $F_Y(y) = P(Y \leq y)$

$= P(-\vartheta \log(1-U) \leq y)$

$= P(e^{-y/\vartheta} \leq 1-U)$ considerar $y<0$

$= P(U \leq 1-e^{-y/\vartheta})$

$= P(U \leq F_X(y))$

$= F_U(F_X(y))$

$$= \begin{cases} 0 & F_X(y) < 0 \\ F_X(y) & 0 \leq F_X(y) < 1 \\ 1 & 1 \leq F_X(y) \end{cases}$$

$= F_X(y)$

$\therefore F_Y = F_X \quad \therefore Y \sim X. \quad \square$

2.

Por el teorema central del límite,

$$\sqrt{n}\,\frac{(\bar{X}_n-\mu)}{\sigma} \xrightarrow[n\to\infty]{} N(0,1)$$

Sea $r_n(y) = \frac{\sqrt{n}\,(y-\mu)}{\sigma}$, de modo que $r_n(\bar{X}_n) \xrightarrow[n\to\infty]{} N(0,1)$.

Notemos que r es estrictamente creciente, así que, si $Z \sim N(0,1)$,

$$\begin{aligned} P(a \le \bar{X}_n \le b) &= P(r_n(a) \le r_n(\bar{X}_n) \le r_n(b)) \\ &\approx P(r_n(a) \le N \le r_n(b)) \\ &= \frac{1}{\sqrt{2\pi}} \int_{r_n(a)}^{r_n(b)} \exp\left(-\frac{z^2}{2}\right) dz \end{aligned}$$

Así, para $X_1, \dots, X_n$, tenemos

$$\mu = E[X_i] = \int_0^1 x\,dx = \frac{1}{2}$$

$$\sigma^2 = E[(X_i-\mu)^2] = \int_0^1 (x-\tfrac{1}{2})^2 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \left.\frac{x^3}{3}\right|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{12}.$$

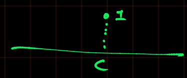
$$\sigma^2 = \frac{1}{\sqrt{12}}.$$

Así, $r_n(y) = \frac{\sqrt{n}\,(y-\frac{1}{2})}{\sqrt{1/12}} = \sqrt{12n}\,(y-\tfrac{1}{2})$

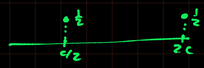
...

3. Suppose we play a game where we start with c dollars. On each play of the game you either double or halve your money, with equal probability. What is your expected fortune after n trials?

$$f_{X_0} = \chi_{\{c\}}$$



$$f_{X_1} = \tfrac{1}{2}\chi_{\{\frac{1}{2}c\}} + \tfrac{1}{2}\chi_{\{2c\}}$$



$$E(X_0) = c$$

$$\begin{aligned} E(X_1) &= \tfrac{1}{2}(\tfrac{1}{2}c) + \tfrac{1}{2}(2c) \\ &= \tfrac{1}{4}c + c \\ &= \tfrac{5}{4}c \end{aligned}$$

$$\begin{aligned} E(X_2) &= \tfrac{1}{2}(\tfrac{1}{2}\tfrac{5}{4}c) + \tfrac{1}{2}(2\cdot\tfrac{5}{4}c) \\ &= \tfrac{1}{4}\tfrac{5}{4}c + \tfrac{5}{4}c \\ &= (\tfrac{5}{4})^2 c \\ &\vdots \end{aligned}$$

$$E(X_n) = (\tfrac{5}{4})^n c$$

Demo.

$$\begin{aligned} E(X_{n+1}) &= \tfrac{1}{2}\,\tfrac{1}{2}E(X_n) + \tfrac{1}{2}\,2E(X_n) \\ &= (\tfrac{1}{4}+1)E(X_n) \\ &= \tfrac{5}{4}E(X_n) \\ &= \tfrac{5}{4}(\tfrac{5}{4})^n c = (\tfrac{5}{4})^{n+1} \end{aligned}$$