Fundamental Algorithms, Sample Midterm, Fall 2022

Answer all questions. The exam, including uploading it on Gradescope, is to be completed in two hours. 1 hour 50 minutes are for taking the exam; the remaining 10 minutes to upload your exam. Please hand in your written work too. The actual exam will have more whitespace so you can write your answers on the exam paper if you wish.

1.a. (**5 pts.**) For each pair of functions, circle the one that grows to be the asymptotically larger as n tends to infinity. If they remain within constant factors of each other, circle both.

i.	$10n^5$	$1,000,000n^4 + 7n^2 \log n$
ii.	$n^{1/\log n}$	1
iii.	3^n	2^n
iv.	$n^{\sqrt{n}}$	$\sqrt{n}^{\,n}$
v.	$\log n$	$\log \log n$

b. (**5 pts.**) Suppose that f, g and h are positive functions on the positive integers. Suppose further that $f = o(g)$ and $h = \Omega(g)$. Show that $f = o(h)$.

2. Use the recursion tree method to solve, as best you can, the following recurrence equations. It is OK to write $T(n) =$ some specific sum of n or $\log n$ terms or whatever if you do not recall how to add up the sequence you obtain (this will result in a small reduction in score).

a. (**5 pts.**)

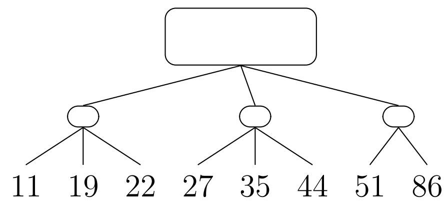
$$R(1) = 1$$
$$R(n) = 2 + R(n/4) \qquad n > 1 \text{ and } n \text{ is an integer power of } 4.$$

b. (**5 pts.**)

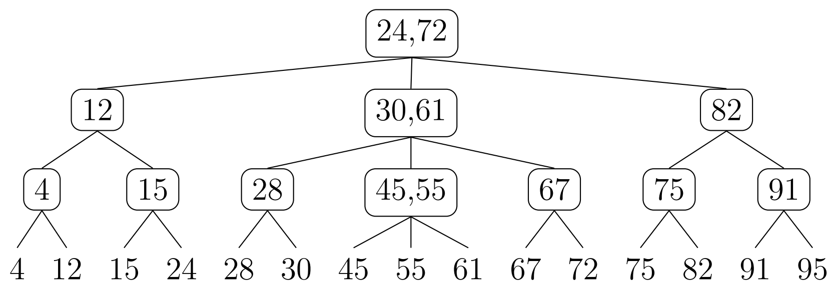
$$S(0) = 1$$
$$S(n) = 3^n + 9S(n-1) \qquad n > 0$$

3. This question concerns 2–3 trees.

a. (**2 pts.**) What are the guides at the root for the 2–3 tree shown below?



b. (**6 pts**.) Show the effect of the operations Delete(24) on the 2–3 tree shown below. Show the sequence of steps performed in carrying out this operation.

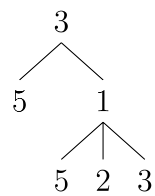


c. (**2pts.**) What are the exact minimum and exact maximum heights of a 2–3 tree storing 100 items (where the height of a single node tree is defined to be zero)?

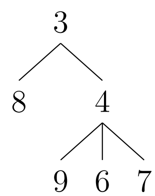
4.a. (**5 pts.**) Let T be an arbitrary tree so each vertex can have any number of children.

Suppose that there is a field $v.\mathrm{val}$ which already holds a non-negative integer value for each vertex. You are to write an algorithm which will use an additional field, $v.\mathrm{PthTo}$. For each node v your task is to compute the sum of the values on the path from the tree root to v including the value at node v; this value should be stored in the field $v.\mathrm{PthTo}$.

The following example tree shows the values to be computed.



Input tree showing $.\mathrm{val}$ field



output tree showing $.\mathrm{PthTo}$ field

Please complete the following procedure. Remember to make an initial call.

PathCalc(v,);

for each child w of v **do**

PathCalc(w,);

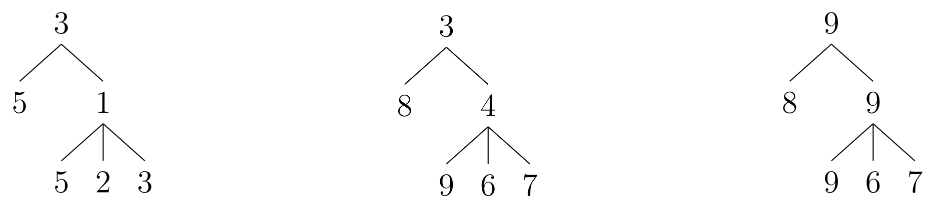
end (* **for** *)

end

Initial call (**answer here too**):

b. (**5 pts.**) Now suppose that we want to also compute in field $v.\mathrm{PthToLf}$ the maximum $x.\mathrm{PthTo}$ value over all the leaves in v's subtree. Augment your program for part (a) so as to compute this value too.

The following example tree shows the values to be computed.



Input tree showing $.\mathrm{val}$ field | output tree showing $.\mathrm{PthTo}$ field | output tree showing $.\mathrm{PthToLf}$ field

PathCalc(v,);

for each child w of v **do**

PathCalc(w,);

end (* **for** *)

end

Initial call (**answer here too**):

5. (**10 pts.**) (**10 pts.**) Let $S = \{e_1, e_2, \ldots, , e_n\}$ be a set of n distinct integers in a range large enough to mean that radix sort does not run in linear time. Give an expected $O(n)$ time algorithm to report all pairs (e, f) of integers in S such that $e = 2f$.

6.a. (**4 pts.**) Suppose that you are maintaining some statistics regarding tennis players, identified by name. For each player, you need to store the number of games won. Suppose that there are n players. There are three basic operations: add a player, delete a player, and for a given player, increase its score (of games won). These all need to run in $O(\log n)$ time. What data structure would you use? Explain how it supports these three operations in $O(\log n)$ time.

b. (**6 pts.**) Suppose that you want to be able to determine in $O(\log n)$ time how many players won at least x games, given a query x. Explain how to modify your solution to part (a) so as to support this operation in addition to the three operations specified in part (a).