Module 4: Lesson 4 (Extra)

Estimation of Hidden Markov Models: A basic example

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Outline

1. Ingredients of a simple HMM model
2. Construction of likelihood function using Hamilton's filter
3. Compute the smoothed probabilities using Kim's filter
4. Updating of the model parameters using the EM algorithm

Ingredients of a simple HMM model

Suppose we have a time series with just three observations ($T = 3$):

$$\mathcal{Y} = (y_1, y_2, y_3) = (\text{-}0.85, 0.4, \text{-}0.2)$$

We assume that this time series is generated by a Hidden Markov process such that:

$$y_t = \varepsilon_t$$

where the distribution of ε_t is normal with mean μ_i and variance σ_i^2.

$$\mu = (\text{-}1, 1), \sigma = (0.8, 0.8)$$
$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$$
$$\pi = (0.2, 0.8)$$

Construction of likelihood function using Hamilton's filter

Log-likelihood function:

$$\mathcal{L}(\theta;\mathcal{Y}) = \log f(y_3|y_2, y_1;\theta) + \log f(y_2|y_1;\theta) + \log f(y_1;\theta)$$

where

$$f(y_1;\theta) = \mathbb{P}(s_1 = 1;\theta) \times \frac{1}{\sigma_1}\phi\left(\frac{y_1-\mu_1}{\sigma_1}\right) + \mathbb{P}(s_1 = 2;\theta) \times \frac{1}{\sigma_2}\phi\left(\frac{y_1-\mu_2}{\sigma_2}\right)$$

$$f(y_2|y_1;\theta) = \mathbb{P}(s_2 = 1|y_1;\theta) \times \frac{1}{\sigma_1}\phi\left(\frac{y_2-\mu_1}{\sigma_1}\right) + \mathbb{P}(s_2 = 2|y_1;\theta) \times \frac{1}{\sigma_2}\phi\left(\frac{y_2-\mu_2}{\sigma_2}\right)$$

$$f(y_3|y_2, y_1;\theta) = \mathbb{P}(s_3 = 1|y_2, y_1;\theta) \times \frac{1}{\sigma_1}\phi\left(\frac{y_3-\mu_1}{\sigma_1}\right) + \mathbb{P}(s_3 = 2|y_2, y_1;\theta) \times \frac{1}{\sigma_2}\phi\left(\frac{y_3-\mu_2}{\sigma_2}\right)$$

Construction of likelihood function using Hamilton's filter

$$\frac{1}{\sigma_1}\phi\left(\frac{y_1-\mu_1}{\sigma_1}\right)=0.49;\ \frac{1}{\sigma_2}\phi\left(\frac{y_1-\mu_2}{\sigma_2}\right)=0.034;$$
$$\frac{1}{\sigma_1}\phi\left(\frac{y_2-\mu_1}{\sigma_1}\right)=0.108;\ \frac{1}{\sigma_2}\phi\left(\frac{y_2-\mu_2}{\sigma_2}\right)=0.376;$$
$$\frac{1}{\sigma_1}\phi\left(\frac{y_3-\mu_1}{\sigma_1}\right)=0.302;\ \frac{1}{\sigma_2}\phi\left(\frac{y_3-\mu_2}{\sigma_2}\right)=0.162;$$

$$f(y_1;\theta)=\mathbb{P}(s_1=1;\theta)\times 0.49+\mathbb{P}(s_1=2;\theta)\times 0.034$$
$$f(y_2|y_1;\theta)=\mathbb{P}(s_2=1|y_1;\theta)\times 0.108+\mathbb{P}(s_2=2|y_1;\theta)\times 0.376$$
$$f(y_3|y_2,y_1;\theta)=\mathbb{P}(s_3=1|y_2,y_1;\theta)\times 0.302+\mathbb{P}(s_3=2|y_2,y_1;\theta)\times 0.162$$

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Construction of likelihood function using Hamilton's filter

We estimate the state probabilities using Hamilton's filter:

$$\mathbb{P}(s_1 = i; \theta) = \pi_i = \xi_{1|0}(i)$$
$$f(y_1; \theta) = \pi_1 \times 0.49 + \pi_2 \times 0.034 = 0.126$$

Using Bayes' Theorem, $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$:

$$\mathbb{P}(s_1 = i | y_1; \theta) = \frac{\mathbb{P}(s_1 = i; \theta) \times \mathbb{P}(y_1 | s_1 = i; \theta)}{\mathbb{P}(y_1; \theta)} = \frac{\pi_i \times \phi\left(\frac{y_1 - \mu_i}{\sigma_i}\right)/\sigma_i}{f(y_1; \theta)} = \xi_{1|1}(i)$$
$$\mathbb{P}(s_2 = i | y_1; \theta) = p_{1i}\xi_{1|1}(1) + p_{2i}\xi_{1|1}(2) = \xi_{2|1}(i)$$

We get:

$$\xi_{1|1}(1) = 0.781;\ \ \xi_{1|1}(2) = 0.219$$
$$\xi_{2|1}(1) = 0.668;\ \ \xi_{2|1}(2) = 0.332$$

Construction of likelihood function using Hamilton's filter

In period $t = 2$ we repeat the process:

$$f(y_2|y_1;\theta) = \xi_{2|1}(1) \times 0.108 + \xi_{2|1}(2) \times 0.376 = 0.197$$

$$\mathbb{P}(s_2 = i|y_2, y_1;\theta) = \frac{\mathbb{P}(s_2 = i|y_1;\theta) \times \mathbb{P}(y_2|s_2 = i, y_1;\theta)}{\mathbb{P}(y_2|y_1;\theta)} = \frac{\xi_{2|1}(i) \times \phi\left(\frac{y_2-\mu_i}{\sigma_i}\right)/\sigma_i}{f(y_2|y_1;\theta)} = \xi_{2|2}(i)$$

$$\mathbb{P}(s_3 = i|y_2, y_1;\theta) = p_{1i}\xi_{2|2}(1) + p_{2i}\xi_{2|2}(2) = \xi_{3|2}(i)$$

We get:

$$\xi_{2|2}(1) = 0.366;\ \ \xi_{2|2}(2) = 0.634$$
$$\xi_{3|2}(1) = 0.42;\ \ \xi_{3|2}(2) = 0.58$$

Construction of likelihood function using Hamilton's filter

In period $t = 3$ we repeat again:

$$f(y_3|y_2, y_1; \theta) = \xi_{3|2}(1) \times 0.302 + \xi_{3|2}(2) \times 0.13 = 0.162$$
$$\mathbb{P}(s_3 = i|y_3, y_2, y_1; \theta) = p_{1i}\xi_{3|2}(1) + p_{2i}\xi_{3|2}(2) = \xi_{3|3}(i)$$

$$\xi_{3|3}(1) = 0.575; \;\; \xi_{3|3}(2) = 0.425$$

The log-likelihood is finally given by:

$$\mathcal{L}(\theta; \mathcal{Y}) = \log f(y_3|y_2, y_1; \theta) + \log f(y_2|y_1; \theta) + \log f(y_1; \theta) = -5.879$$

Kim's Filter

We can also use future information to assess the current state probabilities:

$$\begin{aligned}\mathbb{P}(s_2 = i | s_3 = j, y_3, y_2, y_1; \theta) &= \mathbb{P}(s_2 = i | s_3 = j, y_2, y_1; \theta) \\ &= p_{ij} \frac{\mathbb{P}(s_2 = i | y_2, y_1; \theta)}{\mathbb{P}(s_3 = j | y_2, y_1; \theta)} = p_{ij} \frac{\xi_{2|2}(i)}{\xi_{3|2}(j)}\end{aligned}$$

$$\mathbb{P}(s_2 = i | y_3, y_2, y_1; \theta) = \xi_{2|2}(i) \times \left[p_{i1} \frac{\xi_{3|3}(1)}{\xi_{3|2}(1)} + p_{i2} \frac{\xi_{3|3}(2)}{\xi_{3|2}(2)} \right] = \xi_{2|3}(i)$$

$$\mathbb{P}(s_1 = i | y_3, y_2, y_1; \theta) = \xi_{1|1}(i) \times \left[p_{i1} \frac{\xi_{2|3}(1)}{\xi_{2|2}(1)} + p_{i2} \frac{\xi_{2|3}(2)}{\xi_{2|2}(2)} \right] = \xi_{1|3}(i)$$

We get the following results:

$$\xi_{1|3}(1) = 0.682; \;\; \xi_{1|3}(2) = 0.318$$
$$\xi_{2|3}(1) = 0.455; \;\; \xi_{2|3}(2) = 0.545$$

Kim's Filter, estimating $\mathbb{I}(s_t = j; \theta)\mathbb{I}(s_{t-1} = i; \theta)$

If we want to estimate the joint probabilities $\mathbb{I}(s_t = j; \theta)\mathbb{I}(s_{t-1} = i; \theta)$, we need to consider the following updating rule:

$$\mathbb{P}(s_2 = i | s_3 = j, y_3, y_2, y_1; \theta) \times \mathbb{P}(s_3 = j | y_3, y_2, y_1; \theta) = p_{ij} \xi_{2|2}(i) \frac{\xi_{3|3}(j)}{\xi_{3|2}(j)}$$

$$\mathbb{P}(s_1 = i | s_2 = j, y_3, y_2, y_1; \theta) \times \mathbb{P}(s_2 = j | y_3, y_2, y_1; \theta) = p_{ij} \xi_{1|1}(i) \frac{\xi_{2|3}(j)}{\xi_{2|1}(j)}$$

We get the following matrices for the estimated joint probabilities:

$$\begin{pmatrix} 0.425 & 0.257 \\ 0.03 & 0.288 \end{pmatrix} \text{ for } t = 2$$

$$\begin{pmatrix} 0.401 & 0.054 \\ 0.174 & 0.372 \end{pmatrix} \text{ for } t = 3$$

Updating parameters in the EM algorithm

$$\mu_i^{(1)} = \frac{\xi_{1|3}(i)y_1 + \xi_{2|3}(i)y_2 + \xi_{3|3}(i)y_3}{\xi_{1|3}(i) + \xi_{2|3}(i) + \xi_{3|3}(i)}$$

$$\sigma_i^{(1)} = \sqrt{\frac{\xi_{1|3}(i)(y_1 - \mu_i)^2 + \xi_{2|3}(i)(y_2 - \mu_i)^2 + \xi_{3|3}(i)(y_3 - \mu_i)^2}{\xi_{1|3}(i) + \xi_{2|3}(i) + \xi_{3|3}(i)}}$$

$$\pi_i^{(1)} = \xi_{1|3}(i)$$

We get the following results:

$$\mu^{(1)} = (\text{-}0.3, \text{-}0.107)$$

$$\sigma^{(1)} = (0.504, 0.498)$$

$$\pi^{(1)} = (0.682, 0.318)$$