Module 4: Lesson 3

Hidden Markov Processes

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Outline

- Hidden Markov Models
- Basics of estimation of Hidden Markov Models

Hidden Markov Models

A Hidden Markov Model (HMM) is a Markov process that is split into two components: An observable component and an unobservable, or "hidden" component that follows a Markov process.

HMMs naturally describe setups where a stochastic system is observed through noisy measurements, for instance, stock prices, that are affected by an unobserved economic factor.

A basic HMM contains the following components:

- A set of M states $\mathcal{S} = \{s_1, ..., s_N\}$
- A transition probability matrix P
- A sequence of T, possibly vector-valued, observations $\mathcal{Y}_T = \{y_1, ..., y_T\}$
- A sequence of observation marginal likelihoods $f(y_t|s_t = i)$ for each $i = 1, .., N$.
- An initial probability distribution $\pi = \{\pi_1, ..., \pi_N\}$.

An important assumption is that the hidden Markov process is independent of past observations $\mathcal{Y}_{t-1}$, i.e., $\mathbb{P}\{s_t = j|s_t = i, \mathcal{Y}_{t-1}\} = \mathbb{P}\{s_t = j|s_t = i\} = p_{ij}$.

An example of a Hidden Markov Model

The setup introduced above is sometimes also labeled as a Markov regime-switching model. The observable output y_t features a marginal distribution whose parameters change with the realization of an unobservable state.

Suppose that $y_t = \mu_t + \varepsilon_t$, where $\mu_t = \mu_j$ if $s_t = j$, $\mathcal{S} = \{1, ...N\}$, S_t is Markovian, and ε_t is i.i.d. $N(0, \sigma^2)$. The realization of the initial state s_0 arises from the distribution $\pi = \{\pi_1, ..., \pi_N\}$.

The conditional likelihood of an observation y_t is then $f(y_t|s_t = i) = \phi_t(i) = \phi\left(\frac{y_t - \mu_i}{\sigma}\right)$, and $\phi(.)$ denotes the standard normal probability density function.

Let's cover the basics of how to estimate HMMs by analyzing this simple process.

Estimation basics of a Hidden Markov Model

Our interest lies in making inferences about the probability of being in each state s_i at each date t, as well as estimating the model parameters of the transition matrix P, the initial distribution π, and the vector $(\mu_1, ..., \mu_N, \sigma)$.

Collecting all the parameters in a vector θ the log-likelihood function of the process becomes:

$$\mathcal{L}(\theta) = \sum_{t=1}^{T} log \ f(y_t|\mathcal{Y}_{t-1}; \theta) \tag{1}$$

where

$$f(y_t|\mathcal{Y}_{t-1}; \theta) = \sum_{i=1}^{N} \mathbb{P}(s_t = i|\mathcal{Y}_{t-1}; \theta) \times \phi_t(i) \tag{2}$$

The evaluation of $\mathbb{P}(s_t = i|\mathcal{Y}_{t-1}; \theta)$, which would usually require an overwhelming amount of computations, each for each potential path $\mathcal{Y}_{t-1}$, and for each period t. This rules out estimating directly θ by Maximum Likelihood.

Estimation basics of a Hidden Markov Model

Denote $\xi_{t|t}(j) = \mathbb{P}(s_t = j|\mathcal{Y}_t; \theta)$. To estimate the model, we can rely on a recursive algorithm that sets the optimal forecasts $\xi_{t+1|t}(j) = \mathbb{P}(s_{t+1} = j|\mathcal{Y}_t; \theta)$:

$$\xi_{t|t}(i) = \frac{\xi_{t|t-1}(i)\phi_t(i)}{f(y_t|\mathcal{Y}_{t-1}; \theta)} \tag{3}$$

$$\xi_{t+1|t}(j) = \sum_{i=1}^{N} p_{ij}\xi_{t|t}(i) \tag{4}$$

This procedure is named the Hamilton filter, due to Hamilton (1990). To initialize the recursion, we can set $\xi_{1|0}(i) = 1/N$, or with a guess, $\xi_{1|0}(i) = \pi_i$, or include it in the vector of parameters to estimate.

To estimate the model, we would set initial parameters θ^0 to recover each $\xi_{t+1|t}(j)$ and evaluate the log-likelihood function, then iterate with a new guess θ^1 and so on until convergence.

We can also make forecasts on the observable process y_{t+1} by exploiting the expressions for $\xi_{t+1|t}(j)$ once we have estimated the model parameters.

Summary of Lesson 3

In Lesson 3, we have looked at:

- Definition of a hidden Markov model
- Basics of estimation of hidden Markov models

⇒ **References for this lesson:**

Hamilton, James, D. "Analysis of Time Series Subject to Changes in Regime." *Journal of Econometrics,* vol. 45, 1990, pp. 39–70.

Hamilton, James D. *Time Series Analysis.* Princeton University Press, 1994. (see Chapter 22: "Modelling time series with changes in regime")

TO DO NEXT: Now, please go to the associated Jupyter notebook for this lesson to study the estimation techniques of a hidden Markov model.

In the next lesson, we will cover a more elaborated estimation method for HMMs.