Module 1: Lesson 1

Fourier-based Option Pricing

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Outline

- Why Fourier-based option pricing?
- Fourier and Fast-Fourier transforms.
- Fourier option pricing: Lewis (2001) & Carr-Madan (1999).
- Application to Black-Scholes framework.

Why Fourier?

During the previous course on Derivative Pricing, we were able to obtain a closed-form expression for the price of an option under the Black-Scholes using Ito's Lemma.

However this became harder with more complex models (e.g., Heston), where **no closed-form solution** was attainable.

The **Fourier approach** will help us in this endeavour, as it conveys important advantages:

- Generality: it can be applied to the majority of existing (and yet to be created) models, with the only requirement of a **characteristic function**.
- Accuracy: while it does produce semi-analytical solutions that approximate the true one, these can be evaluated numerically, reaching a high degree of accuracy.
- Speed: this technique requires much lower computational costs than, for example, Monte-Carlo methods.

Fourier methods were firstly introduced in option pricing in Heston (1993). However, before jumping directly there, we'll look at the use of this technique in a setting much more familiar to us: Black-Scholes.

Fourier transforms

Now, we introduce some mathematical tools to use Fourier methods in option pricing. We will not show you here the detailed derivations nor proofs, but you can check them in the additional readings!

Fourier Transform. The Fourier transform of an integrable function $f(x)$ is:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{iux} f(x)\, dx$$

where u can be real or complex, and e^{iux} term is called *phase factor*.

Fourier Inversion. By Fourier inversion we can recover the original function, $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{f}(u)\, du$$

Parseval's Relation. Denote inner produce of two complex, square-integrable functions f and g:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \;= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}\, dk \;= \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$$

Fast Fourier Transform (FFT)

So far we have seen Fourier transforms in a continuous setting. But we can also look at a discrete-time setting:

Discrete Fourier Transform (DFT). DFT can be generally defined as:

$$\hat{x}(k) = \sum_{n=0}^{N} x(n) e^{\frac{-j2\pi kn}{N}}$$

The upper bound N will generally be a power of 2. Hence, the total number of operations to be computed will be N^2.

The **Fast Fourier Transform (FFT)** is an efficient algorithm to compute these kind of sums. Essentially, the algorithm computes 2 sums at the same time, reducing the number of operations from N^2 to $Nln_2(N)$:

$$\hat{x}[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r] e^{\frac{-j2\pi k(2r)}{N}} + \hat{x}[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] e^{\frac{-j2\pi k(2r+1)}{N}}$$

We will explore FFT in detail further in the Module. For now, take it just as an approach we could use.

Pricing problem and characteristic function

From the Derivative Pricing course we know that the no-arbitrage value of an European Call is denoted by:

$$C_0 = e^{-rT}\mathbf{E}^Q\left(S_T - K\right)^+ \;=\; e^{-rT}\int_0^{\infty} C_T(s)q(s)\,ds$$

where $q(s)$ is the risk-neutral Probability Density Function (PDF) of S_T.

$\Rightarrow$ Unfortunately, the PDF of S_T is not often known in closed-form, as it happened in Black-Scholes.

$\Rightarrow$ But there is something that is known to us in closed-form, which is the **characteristic function (CF) of** S_T:

- The CF of any real-valued random variable <u>completely</u> defines its probability distribution.
- Conveniently for us, the CF is the Fourier transform of its PDF.

Characteristic Function (CF). Let random variable X have PDF $q(x)$. Then, CF, $\hat{q}(u)$, will be:

$$\hat{q}(u) = \int_{-\infty}^{\infty} e^{iux}q(x)\,dx = \mathbf{E}^Q\left(e^{iuX}\right)$$

Fourier pricing: Lewis (2001)

There are different approaches that we can use to solve the option pricing problem using Fourier methods. One of the most common ones is in **Lewis (2001)**:

Consider a European Call option with payoff:

$$C_T = max[e^s - K, 0]\ ,\ where\ s = logS.$$

For $u = u_r + iu_i$, with $u_i > 1$, the **Fourier transform of** C_T is (you can check the additional references for proof):

$$\hat{C}(u) = -\frac{K^{iu+1}}{u^2 - iu}$$

Using **Fourier inversion** yields:

$$C_T(s) = \frac{1}{2\pi} \int_{-\infty+iu_i}^{\infty+iu_i} e^{-ius} \hat{C}_T(u)\, du$$

Fourier pricing: Lewis (2001)

Now remember that the price of the option at $t = 0$ is given by:

$$C_0 = e^{-rT}\mathbf{E}^Q(C_T) = \frac{e^{-rT}}{2\pi}\int_{-\infty+iu_i}^{\infty+iu_i}\mathbf{E}^Q(e^{i(-u)s})\hat{C}_T(u)\,du = \frac{e^{-rT}}{2\pi}\int_{-\infty+iu_i}^{\infty+iu_i}\hat{C}_T(u)\hat{q}(-u)\,du$$

If $S_t \equiv S_0 e^{-rt+X_t}$, with X_t a Lévy process and e^{X_t} a martingale such that $X_0 = 0$, then:

$$\hat{q}(-u) = e^{-iuy}\varphi(-u)$$

where φ is the characteristic function of X_t and $y \equiv logS_0 + rT$.

Further defining $k = log(S_0/K)$ and assuming $u_i = 0.5$, present value of option is

$$C_0 = S_0 - \frac{\sqrt{S_0K}e^{-rT}}{\pi}\int_0^{\infty}\mathbf{Re}[e^{izk}\varphi(z-i/2)]\frac{dz}{z^2+1/4}$$

where **Re**[**x**] denotes the real part of x.

$\Rightarrow$ Note that if you go to the original Lewis (2001) paper this expression is a little bit different. It is mathematically equivalent though!

Fourier pricing: Carr and Madan (1999)

While in most cases people rely on Lewis (2001) model for a semi-analytical solution to option pricing problem, there is another interesting approach by **Carr and Madan (1999)** that relied on FFT.

Carr and Madan (1999) define the European Call payoff $C_T \equiv max[S_T - K, 0]$, where $K \equiv e^k$ and $S_T \equiv e^s$:

$$C_0 \equiv e^{-rT} \mathbf{E}^Q \left(max[e^s - e^k, 0] \right) = e^{-rT} \int_k^{\infty} \left(e^s - e^k \right) q(s)\, ds$$

with $q(s)$ risk-neutral prob of $s_T = logS_T$. Defining $c_0 \equiv e^{\alpha k} C_0$, the Fourier transform:

$$\Psi(\nu) \equiv \int_{-\infty}^{\infty} e^{i\nu k} c_0\, dk \quad \rightarrow with\ inverse\ transform : C_0 = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-i\nu k} \Psi(\nu)\, d\nu$$

with C_0 call option value and $\Psi(\nu)$,

$$\Psi(\nu) = \frac{e^{-rT} \varphi\left(\nu - (\alpha + 1)i\right)}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)\nu}$$

for the case of **In-The-Money** options and characteristic function $\varphi(u) \equiv \mathbf{E}^Q(e^{ius_T})$, for $s_T \equiv logS_T$

Summary of Lesson 1

In Lesson 1 we have looked at:

- Fourier and Fast-Fourier methods.
- Carr-Madan (1999) and Lewis (2001) approaches to option pricing.

$\Rightarrow$ **References for this Lesson:**

Hilpisch, Yves. 'Derivatives analytics with Python: data analysis, models, simulation, calibration and hedging'. John Wiley & Sons, 2015. Chapter 6.

TO-DO NEXT: Now, please go to the associated Jupyter Notebook to this Lesson to see the practical use of Fourier methods for option pricing under a simple Black-Scholes framework.

In the Next Lesson we will take a closer look at the semi-analytical solutions for more complex models like Heston (1993).

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