Module 3: Lesson 1

Interest Rate Models

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Outline

- Importance of interest rates in risk-neutral valuation
- Revisiting Vasicek (1977) model
- Cox, Ingersoll and Ross (1985) model (CIR)

Importance of interest rates

So far, we have given little attention to an important part of any risk-neutral valuation: interest rates.

- We have considered interest rates as exogenously given to us, but is this the case?
  - No, this is actually determined by market forces (Zero-Coupon Bonds, ZCB).
- We have considered interest rates as constant over time, but is this the case?
  - No, since determined by market forces, they are constantly changing with expectations.
- We have considered interest rates as constant for all maturities, but is this the case?
  - No, there is a Term Structure of Interest Rates (TSIR).

Interest rates are central for risk-neutral valuation (after all, the risk-free rate is our discount rate).

It is quite important that we learn how we can model the behavior of interest rates, since...

- we need to know the discount rate to apply in the valuation of, for example, equity derivatives.
- interest rates can themselves constitute the underlying asset of a derivative product.

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Revisiting Vasicek (1977)

During the Derivative Pricing course, we dealt with one type of interest rate model: Vasicek (1977).

In short, Vasicek (1977) proposed the following mean-reverting risk-neutral process for short interest rate, r_t:

$$dr_t = k\,(\theta - r_t)\,dt + \sigma dZ_t$$

where $dZ_t = \sqrt{dt}z$, with $z \sim \mathcal{N}(0,1)$. Please, go back to Module 4 in Derivative Pricing for the interpretation of the other parameters.

Zero-Coupon Bond (ZCB) prices in Vasicek (1977) are therefore given by:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k}$$

and

$$A(t,T) = e^{\frac{(B(t,T)-T+t)(k^2\theta-\sigma^2/2)}{k^2} - \frac{\sigma^2 B(t,T)^2}{4k}}$$

CIR (1985)

The Cox-Ingersoll-Ross (CIR) model of 1985 enhances Vasicek (1977) model by including a term to make the standard deviation of short rate changes proportional to $\sqrt{r}$. This yields the following SDE:

$$dr_t = k_r\left(\theta_r - r_t\right)dt + \sigma_r\sqrt{r_t}dZ_t$$

- The presence of $\sqrt{r_t}$ means that when the short rate increases, its standard deviation becomes higher.
- ZCB prices in CIR85 are essentially the same as in Vasicek (note the change in notation, though), but function components are not:

$$B_0(T) = b_1(T)e^{-b_2(T)r_0}$$

$$b_1(T) = \left[\frac{2\gamma e^{((k_r+\gamma)T/2)}}{2\gamma + (k_r+\gamma)(e^{\gamma T}-1)}\right]^{\frac{2k_r\theta_r}{\sigma_r^2}}$$

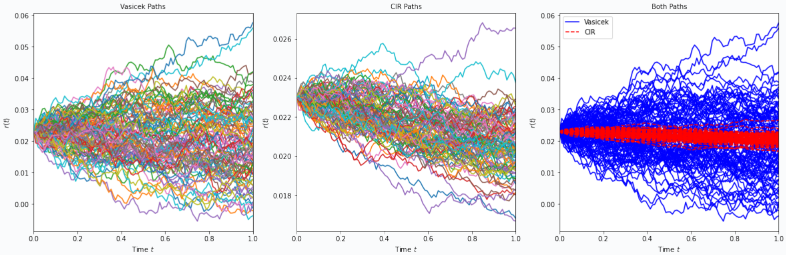
$$b_2(T) = \frac{2(e^{\gamma T}-1)}{2\gamma + (k_r+\gamma)(e^{\gamma T}-1)}$$

$$\gamma = \sqrt{k_r^2 + 2\sigma_r^2}$$

Monte Carlo on interest rates models

We know already that we can use Monte-Carlo methods to simulate the future behavior of interest rates under a certain model like Vasicek (1977) or CIR (1985).

Indeed, when you work on the Python notebook, you will see that, under certain given parameters, you would obtain the following interest rates paths for the Vasicek (1977) and CIR (1985) models:



- How come both models produce such different short rate paths?

  $\Rightarrow$ We need to calibrate model parameters properly.

Summary of Lesson 1

In Lesson 1, we have:

- Revisited the Vasicek (1977) model
- Introduced Cox-Ingersoll-Ross (1985) model
- Compared the different price paths for short rates generated by each model

⇒ **References:**

Cox, John C., et al. "An Intertemporal General Equilibrium Model of Asset Prices." *Econometrica: Journal of the Econometric Society*, 1985, pp. 363–384.

⇒ **TO DO NEXT**: In the notebook associated with this lesson, we will guide you through the process of generating interest rate paths using Monte-Carlo methods on both models.

⇒ In the next lesson, we will see how to calibrate interest rate models using market data.

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