

What are p -Adic Numbers? What are They Used for?

U A Rozikov

Abstract. In this short paper we give a popular introduction to the theory of p -adic numbers. We give some properties of p -adic numbers distinguishing them to "good" and "bad". Some remarks about applications of p -adic numbers to mathematics, biology and physics are given.

1. p -Adic Numbers

p -adic numbers were introduced in 1904 by the German mathematician K Hensel. They are used intensively in number theory. p -adic analysis was developed (mainly for needs of number theory) in many directions, see, for example, [20, 50].

When we write a number in decimal, we can only have finitely many digits on the left of the decimal, but we can have infinitely many on the right of the decimal. They might "terminate" (and become all zeros after some point) but they might not. The p -adic integers can be thought of as writing out integers in base p , but one can have infinitely many digits to the left of the decimal (and none on the right; but the rational p -adic numbers can have finitely many digits on the right of the decimal). For example, the binary expansion of 35 is $1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5$, often written in the shorthand notation 100011_2 . One has $1 = 0,11111111 \dots_2 = 0, (1)_2$. But what is $\dots 111111, 0_2 = (1), 0_2$? Compute $(1), 0_2 + 1$:

$$\begin{array}{r} \dots 111111, 0_2 \\ + \dots 000001, 0_2 \\ \hline \dots 000000, 0_2 \end{array}$$

Hence $(1), 0_2 = -1$. This equality can be written as

$$(1), 0_2 = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} 2^i = \lim_{n \rightarrow \infty} (2^n - 1) = -1. \quad (1)$$

This limit equivalent to $\lim_{n \rightarrow \infty} 2^n = 0$. In real case one has $\lim_{n \rightarrow \infty} q^n = 0$ if and only if absolute value $|q|$ is less than 1. Remember that to define real numbers one considers all limit points of sequences of rational numbers, using the absolute value as metric.

To give a meaning of the limit (1), one has to give a new absolute value $|\cdot|_*$, on the set of

rational numbers, such that $|2|_* < 1$. This is done as follows. Let \mathbb{Q} be the field of rational numbers. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number. The p -adic absolute value (norm) of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

The p -adic norm satisfies the so called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad (2)$$

and this is a non-Archimedean norm.

This definition of $|x|_p$ has the effect that high powers of p become "small", in particular $|2^n|_2 = 1/2^n$. By the fundamental theorem of arithmetic, for a given non-zero rational number x there is a unique finite set of distinct primes p_1, \dots, p_r and a corresponding sequence of non-zero integers a_1, \dots, a_n such that $x = p_1^{a_1} \dots p_r^{a_r}$. It then follows that $|x|_{p_i} = p_i^{-a_i}$ for all $i = 1, \dots, r$, and $|x|_p = 1$ for any other prime $p \notin \{p_1, \dots, p_r\}$.

For example, take $63/550 = 2^{-1} \cdot 3^2 \cdot 5^{-2} \cdot 7 \cdot 11^{-1}$ we have

$$\left| \frac{63}{550} \right|_p = \begin{cases} 2, & \text{if } p = 2, \\ 1/9, & \text{if } p = 3, \\ 25, & \text{if } p = 5, \\ 1/7, & \text{if } p = 7, \\ 11, & \text{if } p = 11, \\ 1, & \text{if } p \geq 13. \end{cases}$$

We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on \mathbb{Q} are equivalent if there exists $\alpha > 0$ such that

$$\|\cdot\|_1^\alpha = \|\cdot\|_2.$$

It is a theorem of Ostrowski (see [41]) that each absolute value on \mathbb{Q} is equivalent either to the Euclidean absolute value $|\cdot|$, the trivial absolute value, or to one of the p -adic absolute values for some prime p . So the only norms on

\mathbb{Q} modulo equivalence are the absolute value, the trivial absolute value and the p -adic absolute value which means that there are only as many completions (with respect to a norm) of \mathbb{Q} .

The p -adic absolute value defines a metric $|x - y|_p$ on \mathbb{Q} . Two numbers x and y are p -adically closer as long as r is higher, such that p^r divides $|x - y|_p$. Amazingly, for $p = 5$ the result is that 135 is closer to 10 than 35.

The completion of \mathbb{Q} with respect to p -adic norm defines the p -adic field which is denoted by \mathbb{Q}_p . Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p - 1$, $x_0 > 0$, $j = 0, 1, 2, \dots$ (see more detail [31, 50, 54]). In this case $|x|_p = p^{-\gamma(x)}$. The set of p -adic numbers contains the field of rational numbers \mathbb{Q} but is different from it.

Using canonical form of p -adic numbers, similarly as real numbers, one makes arithmetic operations on p -adic numbers (see for example, [41]).

2. "Good" Properties of p -Adic Numbers

The ultra-metric triangle inequality, i.e. (2), underlies many of the interesting differences between real and p -adic analysis. The following properties of p -adic numbers make some directions of the p -adic analysis more simple than real analysis:

1. All triangles are isosceles.
 2. Any point of ball $D(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$ is center. Each ball has an empty boundary. Two balls are either disjoint, or one is contained in the other.
 3. $|\cdot|_{p_1} \not\sim |\cdot|_{p_2}$ if $p_1 \neq p_2$. This means that each prime number p generates its own field of p -adic numbers \mathbb{Q}_p .
 4. $x^2 = -1$ has a solution $x \in \mathbb{Q}_p$ if and only if $p \equiv 1 \pmod{4}$.
 5. A sequence $\{x_n\}$ in \mathbb{Q}_p is a Cauchy sequence if and only if $|x_{n+1} - x_n|_p \rightarrow 0$ as $n \rightarrow \infty$.
- This has the useful corollary that a sum converges if and only if the individual terms tend to zero:
6. (A student's dream) $\sum_{n=1}^{\infty} a_n < \infty$ if and only if $a_n \rightarrow 0$.

Since $|n|_p \rightarrow 0$ we have, for example,

$$\sum_{n=0}^{\infty} (-1)^n n!(n+2) = 1, \quad \sum_{n=0}^{\infty} (-1)^n n!(n^2-5) = -3.$$

The sum $\sum_{n=0}^{\infty} n!$ exists in every \mathbb{Q}_p . The following problem has been open since 1971.

Problem. Can $\sum_{n=0}^{\infty} n!$ be rational for some prime p ?

It is not known if $\sum_{n=0}^{\infty} n! \neq 0$ in every \mathbb{Q}_p .

7. For any $x \in \mathbb{Q}$, we have

$$|x| \prod_{p:\text{prime}} |x|_p = 1.$$

This formula have been used to solve several problems in number theory, many of them using Helmut Hasse's local-global principle, which roughly states that an equation can be solved over the rational numbers if and only if it can be solved over the real numbers and over the p -adic numbers for every prime p .

3. "Bad" Properties of p -Adic Numbers

1. \mathbb{Q}_p is not ordered.
2. \mathbb{Q}_p is not comparable with \mathbb{R} , for example $\sqrt{7} \notin \mathbb{Q}_5$, but $i = \sqrt{-1} \in \mathbb{Q}_5$.
3. \mathbb{Q}_p is not algebraically closed.

But $|\cdot|_p$ can be extended uniquely to the algebraic closure \mathbb{Q}_p^a and the completion of $(\mathbb{Q}_p^a, |\cdot|_p)$ is called \mathbb{C}_p , the field of the p -adic complex numbers. \mathbb{C}_p is no locally compact, but separable and algebraically closed.

Now define the functions $\exp_p(x)$ and $\log_p(x)$.

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

The p -adic logarithm is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$;

The p -adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Let $x \in B(0, p^{-1/(p-1)})$, then

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1 + x)|_p = |x|_p,$$

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.$$

4. Some "good" functions become "bad". For example $\exp(x)$ is very "good" function on \mathbb{R} , but as we seen above $\exp_p(x)$ is defined only on ball $B(0, p^{-1/(p-1)})$.

4. Remarks about Applications

When the p -adic numbers were introduced they considered as an exotic part of pure mathematics without any application (see for example [41, 42, 50, 56] for applications of p -adic numbers to mathematics). Since p -adic numbers have the interesting property that they are said to be close when their difference is divisible by a high power of p the higher the power the closer they are. This property enables p -adic numbers to encode congruence information in a way that turns out to have powerful applications in number theory including, for example, in the famous proof of Fermat's Last Theorem by Andrew Wiles (see [42, Chap. 7]).

What is the main difference between real and p -adic space-time? It is the Archimedean axiom. According to this axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. This axiom is valid in the set of real numbers and is not valid in \mathbb{Q}_p . However, it is a physical axiom which concerns the process of measurement. To exchange a number field R to \mathbb{Q}_p is the same as to exchange axiomatics in quantum physics (see [31, 56]).

In 1968 two pure mathematicians, A Monna and F van der Blij, proposed to apply p -adic numbers to physics. In 1972 E Beltrametti and G Cassinelli investigated a model of p -adic valued quantum mechanics from the positions of quantum logic. Since 80th p -adic numbers are used in applications to quantum physics. p -adic strings and super strings were the first models of p -adic quantum physics (see, for example, [17, 29, 50, 54]). The interest of physicists to p -adic numbers is explained by the attempts to create new models of space-time for the description of (fantastically small) Planck distances.

There are some evidences that the standard model based on real numbers is not adequate to Planck's domain. On the other hand, some properties of fields of p -adic numbers seem to be closely related to Planck's domain. In particular, the fields of p -adic numbers have no order structure.

The pioneer investigations on p -adic string theory induced investigations on p -adic quantum mechanics and field theory (see the books [31, 54, 55]). This investigations induce a development of

p -adic mathematics in many directions: theory of distributions [6, 31], differential and pseudodifferential equations [32, 56], theory of probability [31, 56] spectral theory of operators in a p -adic analogue of a Hilbert space [7, 8, 33].

The representation of p -adic numbers by sequences of digits gives a possibility to use this number system for coding of information. Therefore p -adic models can be used for the description of many information processes. In particular, they can be used in cognitive sciences, psychology and sociology. Such models based on p -adic dynamical systems [3–5].

The study of p -adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in [21].

There most recent monograph on p -adic dynamics is Anashin and Khrennikov [9]; nearly a half of Silverman's monograph [52] also concerns p -adic dynamics.

Here are areas where p -adic dynamics proved to be effective: computer science (straight line programs), numerical analysis and simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers, T -functions), combinatorics (Latin squares), automata theory and formal languages, genetics. The monograph [9] contains the corresponding survey. For a newer results see recent papers and references therein: [10, 14, 15, 28, 36, 37, 38, 48, 51]. Moreover, there are studies in computer science and cryptography which along with mathematical physics stimulated in 1990th intensive research in p -adic dynamics since it was observed that major computer instructions (and therefore programs composed of these instructions) can be considered as continuous transformations with respect to the 2-adic metric, see [11, 12].

In [33, 53] p -adic field have arisen in physics in the theory of superstrings, promoting questions about their dynamics. Also some applications of p -adic dynamical systems to some biological, physical systems has been proposed in [3, 4, 5, 22, 23, 33, 35]. Other studies of non-Archimedean dynamics in the neighborhood of a periodic point and of the counting of periodic points over global fields using local fields appear in [39, 47]. It is known that the analytic functions play important role in complex analysis. In the p -adic analysis

the rational functions play a role similar to that of analytic functions in complex analysis [49]. Therefore, there naturally arises a question on study the dynamics of these functions in the p -adic analysis. On the other hand, these p -adic dynamical systems appear while studying p -adic Gibbs measures [26, 24, 44–46]. In [18, 19] dynamics on the Fatou set of a rational function defined over some finite extension of \mathbb{Q}_p have been studied, besides, an analogue of Sullivan’s no wandering domains theorem for p -adic rational functions which have no wild recurrent Julia critical points was proved. In [27] the behaviour and ergodicity of a p -adic dynamical system $f(x) = x^n$ in the fields of p -adic numbers \mathbb{Q}_p and complex p -adic numbers \mathbb{C}_p was investigated. Firstly, the problem of ergodicity of perturbed monomial dynamical systems which was posed in these papers and which stimulated intensive research, was solved in [13]. Secondly, quite recently a far-going generalisation of the problem for arbitrary 1-Lipschitz transformations of 2-adic spheres was also solved in [16]. Finally, we note that not only polynomial and rational p -adic dynamical systems has been studied: In past decade, a significant progress was achieved in a study of a very general p -adic dynamical systems like non-expansive, locally analytic, shift-like, etc.

It is also known [33, 41, 43, 56] that a number of p -adic models in physics cannot be described using ordinary Kolmogorov’s probability theory. In [34] an abstract p -adic probability theory was developed by means of the theory of non-Archimedean measures. Applications of the non-Kolmogorov theory of probability can be considered not only in physics, but in many other sciences, especially in biology and possibly in sociology. The general principle of statistical stabilisation of relative frequencies is a new possibility to find a statistical information in the chaotic (from the real point of view) sequences of frequencies [31, 1, 2].

We refer the reader to [30, 24, 44–46] where various models of statistical physics in the context of p -adic fields are studied.

A non-Archimedean analogue of the Kolmogorov theorem was proved in [25]. Such a result allows to construct wide classes of stochastic processes and the possibility to develop statistical mechanics in the context of p -adic theory.

References

- [1] S. Albeverio and W. Karwowski, A random walk on p -adics — the generator and its spectrum, *Stochastic Processes Appl.* **53** (1994) 1–22.
- [2] S. Albeverio and X. Zhao, Measure-valued branching processes associated with random walks on p -adics, *Ann. Probab.* **28** (2000) 1680–1710.
- [3] S. Albeverio, U. A. Rozikov and I. A. Sattarov, p -adic $(2, 1)$ -rational dynamical systems, *J. Math. Anal. Appl.* **398**(2) (2013) 553–566.
- [4] S. Albeverio, A. Khrennikov and P. E. Kloeden, Memory retrieval as a p -adic dynamical system, *BioSys.* **49** (1999) 105–115.
- [5] S. Albeverio, A. Khrennikov, B. Tirozzi and S. De Smedt, p -adic dynamical systems, *Theor. Math. Phys.* **114** (1998) 276–287.
- [6] S. Albeverio, A. Y. Khrennikov and V. M. Shelkovich, *Theory of p -Adic Distributions: Linear and Nonlinear Models* (Cambridge University Press, 2010).
- [7] S. Albeverio and A. Yu. Khrennikov, Representation of the Weyl group in spaces of square integrable functions with respect to p -adic valued Gaussian distributions, *J. Phys. A* **29** (1996) 5515–5527.
- [8] S. Albeverio, R. Cianci and A. Yu. Khrennikov, A representation of quantum field Hamiltonians in a p -adic Hilbert space, *Theor. Math. Phys.* **112**(3) (1997) 355–374.
- [9] V. Anashin and A. Khrennikov, *Applied Algebraic Dynamics*, de Gruyter Expositions in Mathematics, Vol. 49 (Walter de Gruyter, Berlin, New York, 2009).
- [10] V. S. Anashin, A. Yu. Khrennikov and E. I. Yurova, Characterization of ergodicity of p -adic dynamical systems by using van der Put basis, *Doklady Math.* **83**(3) (2011) 306–308.
- [11] V. S. Anashin, Uniformly distributed sequences of p -adic integers, *Math. Notes* **55**(2) (1994) 109–133.
- [12] V. S. Anashin, Uniformly distributed sequences in computer algebra, or how to construct program generators of random numbers, *J. Math. Sci.* **89**(4) (1998) 1355–1390.
- [13] V. Anashin, Ergodic transformations in the space of p -adic integers, in *p -Adic Mathematical Physics. 2nd Inter. Conference* (Belgrade, Serbia and Montenegro 15–21 September 2005), eds. Andrei Yu. Khrennikov, Zoran Rakic, and Igor V. Volovich, Vol. 826 AIP Conference Proceedings (American Institute of Physics, Melville, New York, 2006), pp. 3–24.
- [14] V. Anashin, Non-Archimedean ergodic theory and pseudorandom generators, *The Computer J.* **53**(4) (2010) 370–392.
- [15] V. Anashin, Automata finiteness criterion in terms of van der Put series of automata functions, *p -Adic Numbers, Ultrametric Anal. Appl.* **4**(2) (2012) 151–160.
- [16] V. Anashin, A. Khrennikov and E. Yurova, *Ergodicity Criteria for Non-Expanding Transformations of 2-Adic Spheres*, preprint, arXiv:1205.0615.
- [17] L. Ya. Araf’eva, B. Dragovich, P. H. Frampton and I. V. Volovich, Wave function of the universe and p -adic gravity, *Mod. Phys. Lett. A* **6** (1991) 4341–4358.
- [18] R. Benedetto, Hyperbolic maps in p -adic dynamics, *Ergod. Theor. Dynam. Sys.* **21** (2001) 1–11.
- [19] R. Benedetto, p -Adic dynamics and Sullivan’s no wandering domains theorem, *Compositio Math.* **122** (2000) 281–298.

- [20] Z. I. Borevich and I. R. Shafarevich, *Number Theory* (Academic Press, New York, 1966).
- [21] G. Call and J. Silverman, Canonical height on varieties with morphisms, *Compositio Math.* **89** (1993) 163–205.
- [22] B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev and I. V. Volovich, On p -adic mathematical physics, *p-Adic Numbers, Ultrametric Anal. Appl.* **1**(1) (2009) 1–17.
- [23] D. Dubischer, V. M. Gundlach, A. Khrennikov and O. Steinkamp, Attractors of random dynamical system over p -adic numbers and a model of ‘noisy’ cognitive process, *Phys. D* **130** (1999) 1–12.
- [24] G. Gandolfo, U. A. Rozikov and J. Ruiz, On p -adic Gibbs measures for hard core model on a Cayley tree, *Markov Processes Related Fields* **18**(4) (2012) 701–720.
- [25] N. N. Ganikhodjaev, F. M. Mukhamedov and U. A. Rozikov, Phase transitions in the Ising model on \mathbb{Z} over the p -adic number field, *Uzb. Mat. Zh.* **4** (1998) 23–29.
- [26] N. N. Ganikhodjaev, F. M. Mukhamedov and U. A. Rozikov, Existence of phase transition for the Potts p -adic model on the set \mathbb{Z} , *Theor. Math. Phys.* **130** (2002) 425–431.
- [27] V. M. Gundlach, A. Khrennikov and K. O. Lindahl, On ergodic behavior of p -adic dynamical systems, *Infin. Dimen. Anal. Quantum Probab. Relat. Top.* **4** (2001) 569–577.
- [28] A.-H. Fan and L.-M. Liao, On minimal decomposition of p -adic polynomial dynamical systems, *Adv. Math.* **228** (2011) 2116–2144.
- [29] P. G. O. Freund and E. Witten, Adelic string amplitudes, *Phys. Lett. B* **199** (1987) 191–194.
- [30] M. Khamraev, F. M. Mukhamedov and U. A. Rozikov, On the uniqueness of Gibbs measures for p -adic non homogeneous λ -model on the Cayley tree, *Lett. Math. Phys.* **70** (2004) 17–28.
- [31] A. Yu. Khrennikov, *p-Adic Valued Distributions in Mathematical Physics* (Kluwer, Dordrecht, 1994).
- [32] A. Yu. Khrennikov, Mathematical methods of the non-Archimedean physics, *Uspekhi Mat. Nauk* **45**(4) (1990) 79–110.
- [33] A. Yu. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models* (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997).
- [34] A. Yu. Khrennikov, S. Yamada and A. van Rooij, The measure-theoretical approach to p -adic probability theory, *Ann. Math. Blaise Pascal* **6** (1999) 21–32.
- [35] A. Yu. Khrennikov, The description of Brain’s functioning by the p -adic dynamical system, preprint No. 355 (SFB-237), Ruhr Univ. Bochum, Bochum (1997).
- [36] J. Kingsbery, A. Levin, A. Preygel and C. E. Silva, On measure-preserving c_1 transformations of compact-open subsets of non-archimedean local fields, *Trans. Amer. Math. Soc.* **361**(1) (2009) 61–85.
- [37] J. Kingsbery, A. Levin, A. Preygel and C. E. Silva, Dynamics of the p -adic shift and applications, *Disc. Contin. Dyn. Syst.* **30**(1) (2011) 209–218.
- [38] D. Lin, T. Shi and Z. Yang, Ergodic theory over $\mathbb{F}_2[[X]]$, *Finite Fields Appl.* **18** (2012) 473–491.
- [39] J. Lubin, Nonarchimedean dynamical systems, *Composito Math.* **94**(3) (1994) 321–346.
- [40] E. Marinari and G. Parisi, On the p -adic five point function, *Phys. Lett. B* **203** (1988) 52–56.
- [41] N. Koblitz, *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions* (Springer, Berlin, 1977).
- [42] Yu. I. Manin and A. A. Panchishkin, *Introduction to Modern Number Theory* (Springer, Berlin, 2007).
- [43] E. Marinary and G. Parisi, On the p -adic five point function, *Phys. Lett. B* **203** (1988) 52–56.
- [44] F. M. Mukhamedov and U. A. Rozikov, On rational p -adic dynamical systems, *Methods Func. Anal. Topology* **10**(2) (2004) 21–31.
- [45] F. M. Mukhamedov and U. A. Rozikov, On Gibbs measures of p -adic Potts model on Cayley tree, *Indag. Math. (N.S.)* **15**(1) (2004) 85–100.
- [46] F. M. Mukhamedov and U. A. Rozikov, On inhomogeneous p -adic Potts model on a Cayley tree, *Inf. Dim. Anal. Quant. Prob. Rel. Fields* **8**(2) (2005) 277–290.
- [47] T. Pezda, Polynomial cycles in certain local domains, *Acta Arith.* **66** (1994) 11–22.
- [48] J.-E. Pin, Profinite methods in automata theory, in *Symposium on Theoretical Aspects of Computer Science — STACS 2009*, (Freiburg, 2009), pp. 31–50.
- [49] A. M. Robert, *A Course of p-Adic Analysis* (Springer, New York, 2000).
- [50] W. Schikhof, *Ultrametric Calculus* (Cambridge University, Cambridge, 1984).
- [51] T. Shi, V. Anashin and D. Lin, Linear weaknesses in T -functions, in *SETA 2012*, eds. T. Helleseth and J. Jedwab, *Lecture Notes Comp. Sci.*, Vol. 7280 (Springer-Verlag, Berlin, Heidelberg, 2012), pp. 279–290.
- [52] J. Silverman, *The Arithmetic of Dynamical Systems*, *Graduate Texts in Mathematics*, No. 241 (Springer-Verlag, New York, 2007).
- [53] E. Thiran, D. Verstegen and J. Weters, p -adic dynamics, *J. Stat. Phys.* **54**(3/4) (1989) 893–913.
- [54] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics* [in Russian], Nauka, Moscow (1994); English transl., World Scientific, Singapore (1994).
- [55] I. V. Volovich, p -adic strings, *Class. Quantum Grav.* **4** (1987) L83–L87.
- [56] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, The spectral theory in the p -adic quantum mechanics, *Izvestia Akad. Nauk SSSR, Ser. Mat.* **54**(2) (1990) 275–302.



U A Rozikov

Institute of Mathematics, Tashkent, Uzbekistan
rozikovu@yandex.ru

U A Rozikov is a professor in Institute of Mathematics, Tashkent, Uzbekistan. He graduated from the Samarkand State University (1993). He obtained PhD (1995) and Doctor of Sciences in physics and mathematics (2001) degrees from the Institute of Mathematics, Tashkent. He is known for his works on the theory of Gibbs measures of models on trees of statistical mechanics (recently he published a book: *Gibbs Measures on Cayley Trees* (World Scientific, 2013)). He developed a contour method to study the models on trees and described complete set of periodic Gibbs measures. Rozikov has (with F Mukhamedov) an important contribution to non-Archimedean theory of phase transitions and dynamical systems. He has important results on non-Volterra quadratic operators. N Ganikhodjaev and Rozikov gave a construction of a quadratic operator which connects phases of the models of statistical mechanics with models of genetics. His most recent works are devoted to evolution algebras of sex-linked populations. He was invited to several leading universities and research centres in the UK, France, Italy, Germany, Spain etc. His (more than 100) papers are published in front-line journals.