Least squares:

Example: Find the parabola that best fits the data:

y	x
3	2
0	1
2	-1
5	-2

Theoretical model:

$$y = ax^2 + bx + c$$

Find a, b, c:

$$\begin{pmatrix} 3 \\ 0 \\ 2 \\ 5 \end{pmatrix} = a \begin{pmatrix} 4 \\ 1 \\ 1 \\ 4 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ 2 \\ 5 \end{pmatrix}}_{v}$$

$$M : \mathbb{R}^3 \mapsto \mathbb{R}^4$$

Least squares solution:

$$Mx = V$$

$$\Rightarrow M^T M x = M^T V$$

$$x = (M^T M)^{-1} M^T V$$

In MATLAB:

enter V, enter M

$\rightarrow$ colspace $(M) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 8 \end{pmatrix} \right\}$

$\rightarrow$ compute $(M^T M)$

$\rightarrow$ compute $(M^T M)^{-1}$

$$M^T M = \begin{pmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{pmatrix}$$

$$(M^T M)^{-1} M x = \begin{pmatrix} 1 \\ -3/5 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\text{cond}(M^T M) = 38.0849$$

$$= 53.78 \ (\infty \text{ norm})$$

$$Y_{th} = \frac{1}{} x^2 + \frac{-0.6}{} x + \frac{0}{}$$

x	Y	Y_{th}	$Y_{th} - Y$
2	3	2.8	-0.2
1	0	0.4	0.4
-1	2	1.6	-0.4
-2	5	5.2	0.2

Relative error $\rightarrow$ $$\frac{\|Y_{th} - Y\|_2}{\|Y\|_2} = 0.103$$

change the model:

$$Y = a(x^2 + x) + bx + c$$

$$\text{Cond}(M^T M) = 54.95$$

$$Y = \_\_\, x^2 + \_\_\, x + \_\_$$