STA 137—Applied Time Series Analysis   Solutions to Practice Midterm II

[1] Figure 1 displays the time series plot, the ACF and the qq plot of a series of residuals that has been obtained after detrending and deseasonalizing a data set of size $n = 100$.

(a) Based on the ACF alone would you suggest that the residuals are dependent?

(b) Based on the time series plot, would you trust your answer in (a)? Why or why not?

(c) Do the time series plot and the qq plot support the claim of normally distributed residuals?

Give precise arguments for each of your choices.

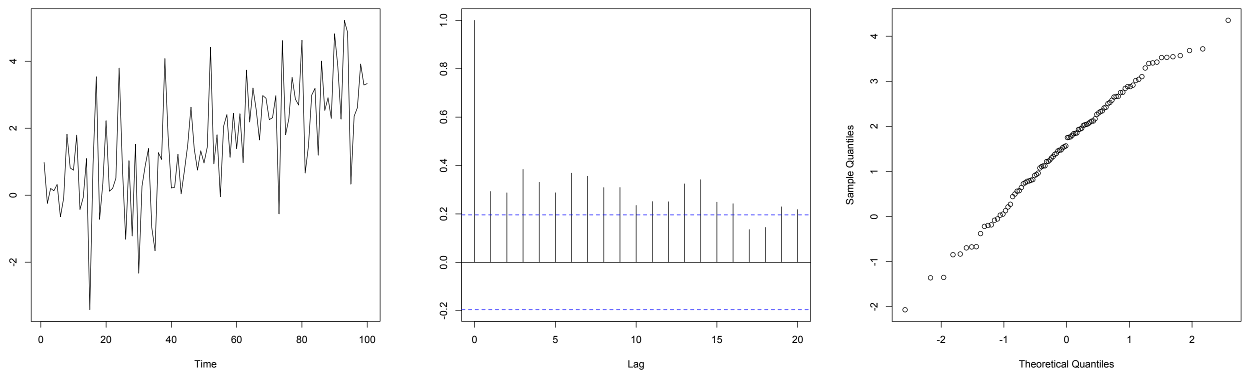

Figure 1: Time series plot, ACF and qq plot of a residual series.

(a) Yes, there are many significant lags showing in the ACF plot.

(b) The significant lags in the ACF plot seem to be coming from an upward trend still present in these residuals. This trend is an artifact that requires additional "mean modeling" of the deterministic trend component of the observations. An inspection of the fluctuations of the observations around this increasing mean seems to suggest that after this trend removal no correlation type dependence will be left in the data. (This correlation based dependence is utilized to fit ARMA models exploiting the conditional mean structure.)

(c) The qq plot looks reasonably normal with some deviations in the tail that are not all that unusual. The time series plot suggests that taking out the linear trend would lead to a new qq plot that would show heavier tails than normal.

Disclaimer: The observations were generated by superimposing t_6 distributed observations on a linear trend. If this linear trend is taken out and a proper residual series is available, ACF and qq plot look as suggested in (b) and (c). See Figure 2,

[2] Figure 3 on the next page shows the time series plot and the sample ACF of an AR(1) process $X_t = \phi X_{t-1} + Z_t$. Which parameter ϕ has been used to generate these observations?

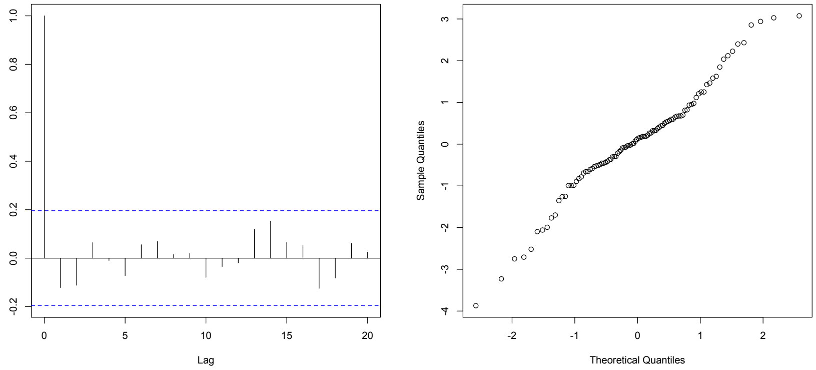

Figure 2: ACF and qq plot of the detrended residual series.

(a) $\phi = -0.9$ (b) $\phi = -0.3$ (c) $\phi = -0.1$ (d) $\phi = 0.3$ (e) $\phi = 0.9$

Circle the correct answer.

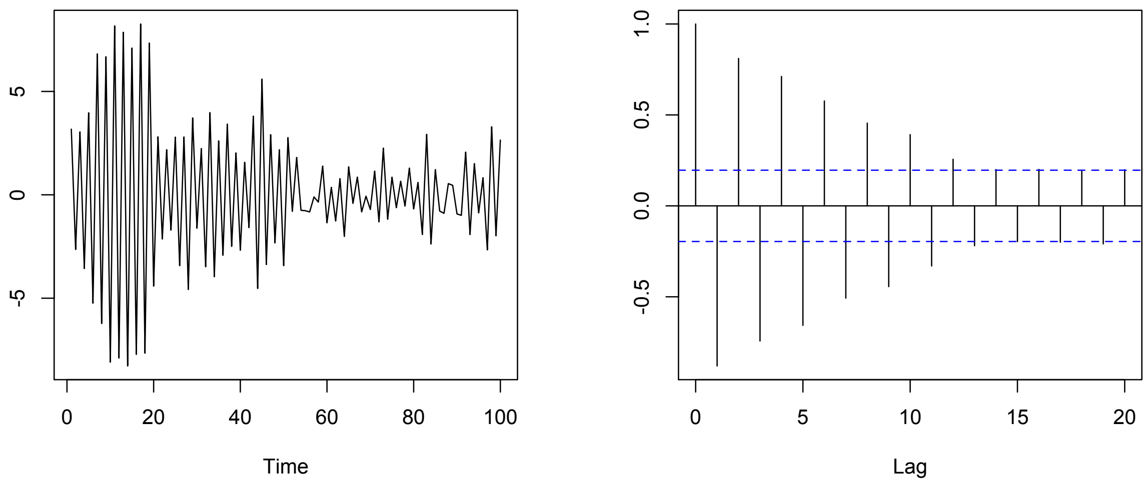

Figure 3: Time series plot and ACF of an AR(1) process.

[3] Give the conditions that make the ARMA(1,1) process

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$$

both causal and invertible.

The conditions are $|\phi| < 1$ and $|\theta| < 1$. Then the AR polynomial $\phi(z) = 1 - \phi z$ and the MA polynomial $\theta(z) = 1 + \theta z$ will not be zero for any $|z| \leq 1$.

[4] Let $(X_t \colon t \in \mathbb{Z})$ be stationary, causal and represented by the model

$$X_t = \phi X_{t-2} + Z_t,$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$. Find the variance of X_t.

Note that X_{t-2} and Z_t are uncorrelated because of causality. Therefore,

$$\begin{aligned}\mathrm{Var}(X_t) &= \mathrm{Var}(\phi X_{t-2} + Z_t)\\ &= \phi^2 \mathrm{Var}(X_{t-2}) + \mathrm{Var}(Z_t).\end{aligned}$$

By stationarity, $\gamma(0) = \phi^2\gamma(0) + \sigma^2$ and thus $\gamma(0) - \phi^2\gamma(0) = \sigma^2$. Consequently,

$$\gamma(0) = \mathrm{Var}(X_t) = \frac{\sigma^2}{1-\phi^2},$$

which coincides with the expression for variance of an AR(1) process.

[5] The process $(X_t \colon t \in \mathbb{Z})$ is expressed as

$$X_t = \beta_1 \sin\left(\frac{2\pi}{n}t\right) + \beta_2 \cos\left(\frac{2\pi}{n}t\right) + Z_t,$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$. Find the autocovariance function of X_t. Is the process stationary?

To compute the autocovariance function, note first that $\mathrm{Cov}(a+X, b+Y) = \mathrm{Cov}(X,Y)$ for any random variables X and Y and nonrandom terms a and b. Observe next that the sine and cosine terms in the definition of X_t are nonrandom. Therefore,

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(Z_{t+h}, Z_t) = \begin{cases}\sigma^2, & h = 0,\\ 0, & h \neq 0,\end{cases}$$

because $(Z_t \colon t \in \mathbb{Z})$ is white noise. The autocovariance function is independent of time t. But the mean function is not constant because

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}\left[\beta_1 \sin\left(\frac{2\pi}{n}t\right) + \beta_2 \cos\left(\frac{2\pi}{n}t\right) + Z_t\right]\\ &= \mathbb{E}\left[\beta_1 \sin\left(\frac{2\pi}{n}t\right) + \beta_2 \cos\left(\frac{2\pi}{n}t\right)\right] + \mathbb{E}[Z_t]\\ &= \beta_1 \sin\left(\frac{2\pi}{n}t\right) + \beta_2 \cos\left(\frac{2\pi}{n}t\right)\end{aligned}$$

depends on t. Hence the process is not stationary.

[6] Show that the AR(1) model

$$X_t = \phi X_{t-1} + Z_t,$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$ and $|\phi| > 1$, can be represented as

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$$

and find the autocovariance function of $(X_t \colon t \in \mathbb{Z})$.

See Example 3.2.2 of the Lecture Notes for the representation as infinite sum. The autocovariance function is computed as

$$\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t)$$

$$
\begin{aligned}
&= \mathrm{Cov}\left(-\sum_{j=1}^{\infty}\phi^{-j}Z_{t+h+j}, -\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\right)\\
&= \mathrm{Cov}\left(-\sum_{j=h}^{\infty}\phi^{-j+h}Z_{t+j}, -\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\right)\\
&= \phi^{h}\mathrm{Cov}\left(-\sum_{j=h}^{\infty}\phi^{-j}Z_{t+j}, -\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\right)\\
&= \phi^{h}\mathrm{Var}\left(-\sum_{j=h}^{\infty}\phi^{-j}Z_{t+j}\right)\\
&= \phi^{h}\sum_{j=h}^{\infty}\phi^{-2j}\mathrm{Var}(Z_{t+j})\\
&= \phi^{h}\sigma^{2}\sum_{j=h}^{\infty}\phi^{-2j}\\
&= \phi^{h}\sigma^{2}\left(\sum_{j=0}^{\infty}\phi^{-2j} - \sum_{j=0}^{h}\phi^{-2j}\right)\\
&= \phi^{h}\sigma^{2}\left(\frac{1}{1-\phi^{-2}} - \frac{1-\phi^{-2(h+1)}}{1-\phi^{-2}}\right)\\
&= \frac{\phi^{-h-2}\sigma^{2}}{1-\phi^{-2}}\\
&= \frac{\phi^{-h}\sigma^{2}}{\phi^{2}-1}.
\end{aligned}
$$

[7] Suppose the roots of the AR polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

from some ARMA process are $z_1 = 0.5 + i0.5$ and $z_2 = 0.5 + i0.5$. Is this process causal?

Noting that $|z| = \sqrt{0.5^2 + 0.5^2} = \sqrt{.5} < 1$, *the process is not causal.*

[8] Determine if the following ARMA process is invertible:

$$X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t - 0.04Z_{t-1} - 0.4Z_{t-2}.$$

The MA polynomial is

$$\theta(z) = 1 - 0.04z - 0.4z^2$$

the zeros (roots) of the polynomial are computed as follows:

$$\begin{aligned}\theta(z) = 0 \Rightarrow 1 - 0.04z - 0.4z^2 &= 0\\ \Rightarrow z &= \frac{-0.04 \pm \sqrt{0.04^2 + 4(0.4)}}{-2(0.4)}\\ \Rightarrow z &= \frac{-0.04 \pm \sqrt{0.0016 + 0.16}}{-0.8}\\ \Rightarrow z &= -1.632, 1.532.\end{aligned}$$

Since both of the zeros are greater than 1 in absolute value, the process is invertible.

[9] Consider the linear process

$$Y_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, 4)$, with $a_0 = 1$, $a_2 = -1$ and $a_j = 0$ for all $j \neq 0, 2$. What is the variance of of the output series?

The variance is

$$\begin{aligned}\gamma(0) &= \mathrm{Var}\Bigg(\sum_{j=-\infty}^{\infty} a_j Z_{t-j}, \sum_{j'=-\infty}^{\infty} a_{j'} Z_{t-j'}\Bigg)\\ &= \sum_{j=-\infty}^{\infty} \mathrm{Var}(a_j Z_{t-j})\\ &= \sum_{j=-\infty}^{\infty} a_j^2 \mathrm{Var}(Z_{t-j})\\ &= \sigma^2 \sum_{j=-\infty}^{\infty} a_j^2\\ &= \sigma^2(\cdots + 0 + 1^2 + 0 + (-1)^2 + 0 + \dots)\\ &= 2\sigma^2\\ &= 8.\end{aligned}$$

[10] Suppose

$$X_t = \mu + Z_t + \theta Z_{t-1}$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$.

(a) Find the mean function $\mathbb{E}[X_t]$;

(b) Show that the autocovariance function of $(X_t \colon t \in \mathbb{Z})$ is given by $\gamma(0) = \sigma^2(1+\theta^2)$, $\gamma(\pm 1) = \sigma^2\theta$, and $\gamma(h) = 0$ otherwise;

(c) Show that $(X_t \colon t \in \mathbb{Z})$ is stationary for all values of $\theta \in \mathbb{R}$;

(d) Calculate $\mathrm{Var}(\bar{X})$ for estimating μ when (i) $\theta = 1$, (ii) $\theta = 0$, and (iii) $\theta = -1$.

(a) *The mean function is* $\mathbb{E}[X_t] = \mathbb{E}[\mu + Z_t + \theta Z_{t-1}] = \mu$.

(b) *Since the process is an MA(1) process with mean* μ *the ACVF is*

$$\begin{aligned}\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t)\\ &= \begin{cases}\sigma^2(1+\theta^2), & h = 0.\\ \sigma^2\theta, & h = \pm 1.\\ 0, & h \neq 0, \pm 1.\end{cases}\end{aligned}$$

(c) *Since mean and ACVF are independent of* t*, the process is weakly stationary.*

(d) *Using Chapter 2 of the Lecture Notes and part (b) gives that the variance of the sample mean is equal to*

$$\mathrm{Var}(\bar{X}) = \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma(h) = \frac{1}{n}\sum_{h=-1}^{1}\left(1 - \frac{|h|}{n}\right)\gamma(h).$$

(i) *If* $\theta = 1$*, then* $\gamma(\pm 1) = \sigma^2$ *and* $\gamma(0) = 2\sigma^2$*. Therefore,*

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= \frac{1}{n}\left\{\left(1 - \frac{|-1|}{n}\right)\gamma(-1) + \left(1 - \frac{|0|}{n}\right)\gamma(0) + \left(1 - \frac{|1|}{n}\right)\gamma(1)\right\}\\ &= \frac{1}{n}\left\{\left(1 - \frac{|-1|}{n}\right)\sigma^2 + \left(1 - \frac{|0|}{n}\right)2\sigma^2 + \left(1 - \frac{|1|}{n}\right)\sigma^2\right\}\\ &= \frac{2\sigma^2}{n}\left(2 - \frac{1}{n}\right).\end{aligned}$$

(ii) *If* $\theta = 0$*, then* $\gamma(\pm 1) = 0$ *and* $\gamma(0) = \sigma^2$*. Therefore, as in the independent case,*

$$\mathrm{Var}(\bar{X}) = \frac{1}{n}\left(1 - \frac{|0|}{n}\right)\gamma(0) = \frac{\sigma^2}{n}.$$

(iii) *If* $\theta = -1$*, then* $\gamma(\pm 1) = -\sigma^2$ *and* $\gamma(0) = 2\sigma^2$*. Therefore,*

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= \frac{1}{n}\left\{\left(1 - \frac{|-1|}{n}\right)\gamma(-1) + \left(1 - \frac{|0|}{n}\right)\gamma(0) + \left(1 - \frac{|1|}{n}\right)\gamma(1)\right\}\\ &= \frac{1}{n}\left\{\left(1 - \frac{|-1|}{n}\right)(-\sigma^2) + \left(1 - \frac{|0|}{n}\right)2\sigma^2 + \left(1 - \frac{|1|}{n}\right)(-\sigma^2)\right\}\\ &= \frac{1}{n}\left(-\sigma^2 + \frac{\sigma^2}{n} + 2\sigma^2 - \sigma^2 + \frac{\sigma^2}{n}\right)\\ &= \frac{2\sigma^2}{n^2}.\end{aligned}$$

which is an order n^{-1} *smaller than the variances in (i) and (ii). This is because for* $\theta = -1$*, the sample mean is a telescoping sum:*

$$\bar{X} = \frac{1}{n}\sum_{t=1}^{n}X_t = \frac{1}{n}\sum_{t=1}^{n}(Z_t - Z_{t-1}) = \frac{1}{n}(Z_n - Z_0).$$