AE 4803 Robotics and Autonomy
Professor Evangelos Theodorou
Homework 2

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Problem 1.

Problem 2.

2.1) For the derivation of the Reinforce Gradient we begin with the following cost function:

$$J(\boldsymbol{\theta}) = \int p(\boldsymbol{\tau})R(\boldsymbol{\tau})\, d\boldsymbol{\tau}$$

A trajectory can be expressed as $\boldsymbol{\tau} = (\boldsymbol{x_0}, \boldsymbol{u_0}, \ldots, \boldsymbol{x_{N-1}}, \boldsymbol{u_{N-1}}, \boldsymbol{x_N})$ with states $\boldsymbol{x} \in \mathbb{R}^{\ell}$ and controls $\boldsymbol{u} \in \mathbb{R}^{p}$ over the time horizon T = Ndt. R($\boldsymbol{\tau}$) is the accumalated cost over a trajectory and $p(\boldsymbol{\tau})$ represents the path probability of the trajectory, which using Bayesian and Markov properties can be expressed as:

$$p(\boldsymbol{\tau}) = p(\boldsymbol{x}_0) \prod_{i=0}^{N-1} p(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i, \boldsymbol{u}_i) p(\boldsymbol{u}_i|\boldsymbol{x}_i; \boldsymbol{\theta})$$

$$R(\boldsymbol{\tau}) = \sum_{t=0}^{N-1} r(\boldsymbol{x}_t, \boldsymbol{u}_t, t)$$

The $p(\boldsymbol{u}_i|\boldsymbol{x}_i; \boldsymbol{\theta})$ term in path probability represents the parametrized policy where $\boldsymbol{\theta} \in \mathbb{R}^n$. We begin our derivation by the gradient of the cost function with respect to $\boldsymbol{\theta}$, $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$.

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} \left(\int p(\boldsymbol{\tau})R(\boldsymbol{\tau})\, d\boldsymbol{\tau} \right)$$

$$\int \nabla_{\boldsymbol{\theta}} p(\boldsymbol{\tau})R(\boldsymbol{\tau})\, d\boldsymbol{\tau}$$

We use the following log property:

$$\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau})) = \frac{1}{p(\boldsymbol{\tau})} \nabla_{\boldsymbol{\theta}} p(\boldsymbol{\tau})$$

$$\nabla_{\boldsymbol{\theta}} p(\boldsymbol{\tau}) = p(\boldsymbol{\tau}) \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau}))$$

To make the substitution and rewrite the following as an expectation:

$$\int p(\boldsymbol{\tau}) \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau})) R(\boldsymbol{\tau})\, d\boldsymbol{\tau}$$

$$E_{p(\boldsymbol{\tau})} \left[\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau})) R(\boldsymbol{\tau}) \right]$$

We start by calculating $\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau}))$

$$\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{\tau})) = \nabla_{\boldsymbol{\theta}} log\Bigg(p(\boldsymbol{x}_0)\prod_{i=0}^{N-1} p(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i,\boldsymbol{u}_i)p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})\Bigg)$$

$$= \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{x}_0)) + \nabla_{\boldsymbol{\theta}} log\Bigg(\prod_{i=0}^{N-1} p(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i,\boldsymbol{u}_i)\Bigg) + \nabla_{\boldsymbol{\theta}} log\Bigg(\prod_{i=0}^{N-1} p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})\Bigg)$$

$$= \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{x}_0)) + \nabla_{\boldsymbol{\theta}} \sum_{i=0}^{N-1} log(p(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i,\boldsymbol{u}_i)) + \nabla_{\boldsymbol{\theta}} \sum_{i=0}^{N-1} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}))$$

$$= \sum_{i=0}^{N-1} \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}))$$

From this we rewrite our policy gradient as

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = E_{p(\boldsymbol{\tau})}\Bigg[\sum_{i=0}^{N-1} \nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}))R(\boldsymbol{\tau})\Bigg]$$

We can further simplify this by calculating $p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})$ given the parametrized policy with Gaussian noise $\epsilon \sim \mathcal{N}(0,\boldsymbol{I})$:

$$u(\boldsymbol{x},\boldsymbol{\theta},t_k) = \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{\theta} + \boldsymbol{B}(\boldsymbol{x})\epsilon(t_k)$$

We start by expressing $p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})$ as the multi-variate Gaussian Distribution:

$$p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}) = \frac{1}{(2\pi)^{m/2}|\boldsymbol{B}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})^T|^{1/2}} exp\Bigg(-\frac{1}{2}(\boldsymbol{u}-\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{\theta})^T\Big(\boldsymbol{B}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})^T\Big)^{-1}(\boldsymbol{u}-\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{\theta})\Bigg)$$

We then express $log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}))$ as:

$$log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})) = log\Bigg(\frac{1}{(2\pi)^{m/2}|\boldsymbol{B}\boldsymbol{B}^T|^{1/2}} exp\Big(-\frac{1}{2}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})\Big)\Bigg)$$

$$= log\Bigg(\frac{1}{(2\pi)^{m/2}|\boldsymbol{B}\boldsymbol{B}^T|^{1/2}}\Bigg) + log\Bigg(exp\Big(-\frac{1}{2}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})\Big)\Bigg)$$

$$= -log\Big((2\pi)^{m/2}\boldsymbol{B}\boldsymbol{B}^T|^{1/2}\Big) - \frac{1}{2}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})$$

$$= -log\Big((2\pi)^{m/2}\boldsymbol{B}\boldsymbol{B}^T|^{1/2}\Big) - \frac{1}{2}(\boldsymbol{u}^T-\boldsymbol{\theta}^T\boldsymbol{\Phi}^T)\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})$$

$$= -log\Big((2\pi)^{m/2}\boldsymbol{B}\boldsymbol{B}^T|^{1/2}\Big) + \Bigg(-\frac{1}{2}\boldsymbol{u}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1} + \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\Bigg)(\boldsymbol{u}-\boldsymbol{\Phi}\boldsymbol{\theta})$$

$$= -log\Big((2\pi)^{m/2}\boldsymbol{B}\boldsymbol{B}^T|^{1/2}\Big) - \frac{1}{2}\boldsymbol{u}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{u} + \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

$$- \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

$$= -log\Big((2\pi)^{m/2}\boldsymbol{B}\boldsymbol{B}^T|^{1/2}\Big) - \frac{1}{2}\boldsymbol{u}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{u} + \boldsymbol{\theta}^T\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{u} - \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

We then compute the gradient of this expression with respect to $\boldsymbol{\theta}$ and substitute our parametrized policy $\boldsymbol{u} = \boldsymbol{\Phi}\boldsymbol{\theta} + \boldsymbol{B}\epsilon_k$:

$$\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})) = \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{u} - \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

$$= \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\Bigg(\boldsymbol{\Phi}\boldsymbol{\theta} + \boldsymbol{B}\epsilon_k\Bigg) - \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

$$= \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta} + \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{B}\epsilon_k - \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{\Phi}\boldsymbol{\theta}$$

$$= \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{B}\epsilon_k$$

Now given that $\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta})) = \boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{B}\epsilon_k$ we turn can rewrite our gradient policy:

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = E_{p(\boldsymbol{\tau})}\Bigg[\sum_{i=0}^{N-1}\nabla_{\boldsymbol{\theta}} log(p(\boldsymbol{u}_i|\boldsymbol{x}_i;\boldsymbol{\theta}))R(\boldsymbol{\tau})\Bigg]$$

$$= E_{p(\boldsymbol{\tau})}\Bigg[R(\boldsymbol{\tau})\sum_{i=0}^{N-1}\boldsymbol{\Phi}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{B}\epsilon_k\Bigg]$$

If we paramtrize the policy such that $\Phi = \boldsymbol{B}$ then the final form of the Reinforce Gradient can be written as:

$$= E_{p(\boldsymbol{\tau})}\Bigg[R(\boldsymbol{\tau})\sum_{i=0}^{N-1}\boldsymbol{B}^T\Big(\boldsymbol{B}\boldsymbol{B}^T\Big)^{-1}\boldsymbol{B}\epsilon_k\Bigg]$$