<Machine Learning HW#2>

21300008 강민수

1.32. Consider a vector x of continuous variables with distribution $p(x)$ and corresponding entropy $H(x)$. Suppose that we make a nonsingular linear transformation of x obtain a new variable $\mathrm{y} = \mathrm{Ax}$. Show that the corresponding entropy is given by $H[y] = H[x] + ln\,|A|$ where $|A|$ denotes determinant of A.

Proof)

Vector x가 distribution p(x)를 갖는 continuous variables이므로, vector x의 entropy는 아래와 같이 정의된다.

$$H(x) = -\int p(x) ln\, p(x) dx$$

위 식을 토대로, $H[y] = H[x] + ln\,|A|$를 증명하면, 아래와 같다.

$H(y) = -\int p(y) ln\, p(y) dy$ (y에 대한 entropy)

$p(x) = p(y)\left|\frac{dy_i}{dx_j}\right| = p(y)|A|$에 의해, 위 식을 변형하면 아래와 같다.

$$\begin{aligned}
(\text{준 식}) &= -\int p(y)\, ln\, p(y) dy = -\int p(x)\, ln\{p(x)\,|A|^{-1}\} dx \\
&= -\int p(x)\{ln\, p(x) + ln\ |A|^{-1}\}\, dx \\
&= -\int p(x)\, ln\, p(x) + p(x)\, ln\ |A|^{-1}\, dx \\
&= -\int p(x)\, ln\, p(x)\, dx - \int p(x)\, ln\ |A|^{-1}\, dx \\
&= -\int p(x)\, ln\, p(x)\, dx - ln\ |A|^{-1} \int p(x) dx \\
&= -\int p(x)\, ln\, p(x)\, dx + ln\ |A| \\
&= \mathrm{H}[x] + ln\ |A|
\end{aligned}$$

따라서 $\boldsymbol{H[y] = H[x] + ln\,|A|}$는 참이다.

(증명 끝)

1.35. Use the results (1.106) and (1.107) to show that the entropy of the univariate Gaussian (1.109) is given by (1.110).

$$\int_{-\infty}^{\infty} \boldsymbol{xp(x)dx = \mu} \qquad (\mathbf{1.106})$$

$$\int_{-\infty}^{\infty} \boldsymbol{(x-\mu)^2 p(x)dx = \sigma^2} \qquad (\mathbf{1.107})$$

$$\boldsymbol{p(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\, exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}} \qquad (\mathbf{1.109})$$

$$\boldsymbol{H(x) = \frac{1}{2}\{1 + ln(2\pi\sigma^2)\}} \qquad (\mathbf{1.110})$$

Proof)

$$H(x) = -\int p(x) ln\, p(x) dx$$

$$= -\int p(x) \times ln\left[\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\, exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}\right] dx$$

$$= -\int p(x) \times \left[ln\left\{\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\right\} - \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= -\int p(x) \times \left[-ln\,(2\pi\sigma^2)^{\frac{1}{2}} - \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \int p(x) \times \left[ln\,(2\pi\sigma^2)^{\frac{1}{2}} + \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \int p(x) \times \left[\frac{1}{2}\ln(2\pi\sigma^2) + \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \int p(x) \times \frac{1}{2}\ln(2\pi\sigma^2)\, dx + \int p(x) \times \frac{(x-\mu)^2}{2\sigma^2} dx$$

$$= \frac{1}{2}\int p(x) \times \ln(2\pi\sigma^2)\, dx + \int p(x) \times \frac{(x-\mu)^2}{2\sigma^2} dx$$

$$= \frac{1}{2}\int p(x) \times \ln(2\pi\sigma^2)\, dx + \frac{1}{2\sigma^2}\int p(x)(x-\mu)^2\, dx$$

(1.107 식에 의해 $\int p(x)(x-\mu)^2\, dx = \sigma^2$이므로,)

$$= \frac{1}{2}\int p(x) \times ln(2\pi\sigma^2)\, dx + \frac{1}{2\sigma^2} \times \sigma^2$$

$$= \frac{1}{2}\int p(x) \times ln(2\pi\sigma^2)\, dx + \frac{1}{2}$$

$$= \frac{1}{2} ln(2\pi\sigma^2) \int p(x)\, dx + \frac{1}{2}$$

(확률공리에 의해, $\int p(x)\, dx = 1$이므로,)

$$= \frac{1}{2} ln(2\pi\sigma^2) + \frac{1}{2}$$

$$= \frac{1}{2}\{ln(2\pi\sigma^2) + 1\}$$

$$\therefore\ \boldsymbol{H(x) = \frac{1}{2}\{1 + ln(2\pi\sigma^2)\}}$$

(증명 끝)

1.37. Using the definition (1.111) together with the product rule of probability, prove the result (1.112).

$$\mathbf{H}[\boldsymbol{y}\,|\boldsymbol{x}] = -\iint \boldsymbol{p(y,x)\ ln\,p(y\mid x)\ dy\ dx} \qquad (\mathbf{1.111})$$

$$\mathbf{H}[\boldsymbol{x},\boldsymbol{y}] = \mathbf{H}[\mathbf{y}\mid \boldsymbol{x}] + \mathbf{H}[\boldsymbol{x}] \qquad (\mathbf{1.112})$$

Proof)

$$\mathrm{H}[x,y] = -\iint p(x,y) ln\, p(x,y)\ dx\ dy$$

Continuous variable의 product rule of probability, $p(x,y) = p(y \mid x)p(x) = p(x \mid y)p(y) = p(y,x)$ 이용하면, 아래와 같이 식을 전개할 수 있다.

$$\left(\text{준 식}\right) = -\iint p(x,y)\ ln\, p(y\,|x)p(x)\ dx\ dy$$

$$= -\iint p(x,y)\ ln\, p(y\,|x) + p(x,y) ln\, p(x)\ dx\ dy$$

$$= -\iint p(x,y)\ \ln p(y\,|x)\ dx\ dy - \iint p(x,y) ln\, p(x)\ dx\ dy$$

$$= -\iint p(y,x)\ \ln p(y\,|x)\ dx\ dy - \iint p(x,y) ln\, p(x)\ dx\ dy$$

$$= \mathrm{H}[y \mid x] - \iint p(x,y) ln\, p(x)\ dx\ dy$$

$$= \mathrm{H}[y \mid x] - \int\ ln\, p(x)\left\{\int p(x,y)\ dy\right\} dx$$

$$= \mathrm{H}[y \mid x] - \int\ p(x) ln\, p(x) dx = \mathrm{H}[y \mid x] + \mathrm{H}[x]$$

$$\therefore\ \mathbf{H}[\boldsymbol{x},\boldsymbol{y}] = \mathbf{H}[\mathbf{y}\mid \boldsymbol{x}] + \mathbf{H}[\boldsymbol{x}]$$

(증명 끝)

1.40. By applying Jensen's inequality (1.115) with $f(x) = ln\, x$, show that the arithmetic mean of a set of real numbers is never less than their geometrical mean.

$$f\left(\sum_{i=1}^{M} \lambda_i x_i\right) \le \sum_{i=1}^{M} \lambda_i f(x_i) \qquad (1.115)$$

$$\frac{1}{M}\sum_{i=1}^{M} x_i \qquad (arithmetic\ mean)$$

$$\sqrt[M]{\prod_{i=1}^{M} x_i} \qquad (geometric\ mean)$$

Proof)

명제, "the arithmetic mean of a set of real numbers is never less than their geometrical mean."는 아래와 같은 식으로 나타낼 수 있다.

$$\frac{1}{M}\sum_{i=1}^{M} x_i \ge \sqrt[M]{\prod_{i=1}^{M} x_i}$$

위 부등식의 양변에 자연로그(ln)을 취하면 아래와 같다.

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge ln\left(\sqrt[M]{\prod_{i=1}^{M} x_i}\right)$$

그리고 위 부등식을 변형하면, 아래와 같이 변형할 수 있다.

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge ln\left(\left(\prod_{i=1}^{M} x_i\right)^{\frac{1}{M}}\right)$$

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge \frac{1}{M} \times ln\left(\prod_{i=1}^{M} x_i\right)$$

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge \frac{1}{M} \times ln(x_1 x_2 x_3 \cdots x_m)$$

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge \frac{1}{M} \times (ln\, x_1 + ln\, x_2 + ln\, x_3 + \cdots + ln\, x_M)$$

$$ln\left(\frac{1}{M}\sum_{i=1}^{M} x_i\right) \ge \frac{1}{M}\sum_{i=1}^{M} (ln\, x_i)$$

λ_i의 정의에 의해서 $\sum_i \lambda_i = 1$이므로, 위 식을 아래와 같이 변형할 수 있다.

$$ln\left(\frac{1}{M}\sum_{i=1}^{M}\lambda_i x_i\right) \geq \frac{1}{M}\sum_{i=1}^{M}\lambda_i(ln\, x_i)$$

$$\boldsymbol{ln\left(\sum_{i=1}^{M}\frac{\lambda_i x_i}{M}\right) \geq \sum_{i=1}^{M}\frac{\lambda_i(ln\, x_i)}{M}}$$

$f(x) = ln\, x$가 concave function이므로, Jensen's inequality에 의해 위 부등식은 참이다.

(증명 끝)

1.41. Using the sum and product rules of probability, show that the mutual information $I(x,y)$ satisfies the relation (1.121).

$$\mathbf{I}[\boldsymbol{x},\boldsymbol{y}] = \mathbf{H}[\boldsymbol{x}] - \mathbf{H}[\boldsymbol{x} \mid \boldsymbol{y}] = \mathbf{H}[\boldsymbol{y}] - \mathbf{H}[\boldsymbol{y} \mid \boldsymbol{x}] \qquad (\mathbf{1.121})$$

Proof)

Mutual information, $I[x,y]$는 아래와 같이 정의된다.

$$I[x,y] = -\iint p(x,y)ln\left(\frac{p(x)p(y)}{p(x,y)}\right)dx\, dy \qquad \cdots \qquad 식 ①$$

Continuous variable의 product rule of probability, $p(x,y) = p(y \mid x)p(x) = p(x \mid y)p(y) = p(y,x)$ 이용하면, 아래와 같이 식을 전개할 수 있다.

$$\begin{aligned}
(준\ 식) &= -\iint p(x,y)ln\left(\frac{p(x)p(y)}{p(y \mid x)p(x)}\right)dx\, dy \\
&= -\iint p(x,y)ln\left(\frac{p(y)}{p(y \mid x)}\right)dx\, dy \\
&= -\iint p(x,y)\, ln\, p(y) - p(x,y)\, ln\, p(y \mid x)\, dx\, dy \\
&= -\iint p(x,y)\, ln\, p(y)\, dx\, dy + \iint p(x,y)\, ln\, p(y \mid x)\, dx\, dy \\
&= -\int ln\, p(y)\left\{\int p(x,y)\, dx\right\}dy + \iint p(x,y)\, ln\, p(y \mid x)\, dx\, dy \\
&= -\int ln\, p(y)\left\{\int p(y,x)\, dx\right\}dy + \iint p(x,y)\, ln\, p(y \mid x)\, dx\, dy
\end{aligned}$$

Continuous variable의 sum rule of probability, $\int p(y,x)\, dx = p(y)$에 의해 위 식을 변형하면, 아래와 같다.

$$\left(\text{준 식}\right) = -\int p(y)\, ln\, p(y)\ \, dy + \iint p(x,y)\, ln\, p(y \mid x)\, dx\, dy$$

$$= \mathrm{H}[y] + \iint p(x,y)\, ln\, p(y \mid x)\, dx\, dy$$

Continuous variable의 product rule of probability, $p(x,y) = p(y \mid x)p(x) = p(x \mid y)p(y) = p(y,x)$ 이용하면, 아래와 같이 식을 전개할 수 있다.

$$(\text{준 식}) = \mathrm{H}[y] - \left(-\iint p(y,x)\, ln\, p(y \mid x)\, dy\, dx\right)$$

$$= \mathrm{H}[y] - \mathrm{H}[y \mid x]$$

$$\therefore\ \mathbf{I}[\boldsymbol{x},\boldsymbol{y}] = \mathbf{H}[\boldsymbol{y}] - \mathbf{H}[\boldsymbol{y} \mid \boldsymbol{x}]$$

식 ①을 다시 이용하여, $\mathbf{I}[\boldsymbol{x},\boldsymbol{y}] = \mathbf{H}[\boldsymbol{x}] - \mathbf{H}[\boldsymbol{x} \mid \boldsymbol{y}]$를 증명하면, 아래와 같다.

$$\left(\text{식 ①}\right) = -\iint p(x,y) ln\left(\frac{p(x)p(y)}{p(x,y)}\right) dx\, dy$$

Continuous variable의 product rule of probability, $p(x,y) = p(y \mid x)p(x) = p(x \mid y)p(y) = p(y,x)$로부터, $p(x,y) = p(y,x)$라는 결과를 이끌어 낼 수 있으므로, 아래와 같이 식을 변형할 수 있다.

$$\left(\text{준 식}\right) = -\iint p(y,x) ln\left(\frac{p(x)p(y)}{p(y,x)}\right) dx\, dy$$

$$= -\iint p(y,x) ln\left(\frac{p(x)p(y)}{p(x|y)p(y)}\right) dx\, dy$$

$$= -\iint p(y,x) ln\left(\frac{p(x)}{p(x \mid y)}\right) dx\, dy$$

$$= -\iint p(y,x)\, ln\, p(x) - p(y,x)p(x \mid y)\, dx\, dy$$

$$= -\iint p(y,x)\, ln\, p(x)\ \, dx\, dy + \iint p(y,x)p(x \mid y)\, dx\, dy$$

$$= -\iint p(y,x)\, ln\, p(x)\ \, dx\, dy - \left(-\iint p(y,x)p(x \mid y)\, dx\, dy\right)$$

$$= -\iint p(y,x)\, ln\, p(x)\ \, dx\, dy - \mathrm{H}[x \mid y]$$

$$= -\int ln\, p(x)\left\{\int p(y,x) dy\right\} dx\ - \mathrm{H}[x \mid y]$$

$$= -\int ln\, p(x)\left\{\int p(x,y) dy\right\} dx - \mathrm{H}[x \mid y]$$

Continuous variable의 sum rule of probability, $\int p(x,y)\, dy = p(x)$에 의해 위 식을 변형하면, 아래와 같다.

$$\left(\text{준 식}\right) = -\int p(x)\, ln\, p(x) dx\ - \mathrm{H}[x \mid y]$$

$$= \mathrm{H}[x] - \mathrm{H}[x \mid y]$$

$$\therefore \ \mathbf{I}[\boldsymbol{x}, \boldsymbol{y}] = \mathbf{H}[\boldsymbol{x}] - \mathbf{H}[\boldsymbol{x} \mid \boldsymbol{y}]$$

결과적으로, $\mathrm{I}[x, y] = \mathrm{H}[x] - \mathrm{H}[x \mid y]$ 이고, $\mathrm{I}[x, y] = \mathrm{H}[y] - \mathrm{H}[y \mid x]$ 이므로, 명제 $\mathrm{I}[x, y] = \mathrm{H}[x] - \mathrm{H}[x \mid y] = \mathrm{H}[y] - \mathrm{H}[y \mid x]$는 성립한다.

$$\therefore \ \mathbf{I}[\boldsymbol{x}, \boldsymbol{y}] = \mathbf{H}[\boldsymbol{x}] - \mathbf{H}[\boldsymbol{x} \mid \boldsymbol{y}] = \mathbf{H}[\boldsymbol{y}] - \mathbf{H}[\boldsymbol{y} \mid \boldsymbol{x}]$$

(증명 끝)