Book of Proof I: Fundamentals

April 4, 2018

Sets: A mathematical structure

$$\{1, 2, 3\}$$
$$\{a, b, c, d\}$$
$$\{cat, dog, pig\}$$
$$\{2, 4, 6, 8, \ldots\}$$
$$\emptyset = \{\}$$
$$\emptyset \neq \{\emptyset\}$$

Note: $\{1, 2, 3\}$ is not the same as $1, 2, 3$ or $(1, 2, 3)$ or *etc.*

Sets have no order or duplicates

$$\begin{aligned}\{1,2,3\} &= \{2,3,1\} \\ &= \{2,1,3\} \\ &= \{1,1,2,2,3,3\} \\ &= \{2,3,3,2,1,1,1,1,2,3,2,2,2,3,1\}\end{aligned}$$

Some important sets

The integers, the natural numbers, the nonnegative integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$
$$\mathbb{N}^0 = \{0, 1, 2, 3, 4, \ldots\}$$

We have limited use for the real numbers and the rational numbers

$$\mathbb{R} = \{0, -17, \pi, \sqrt{2}, \ldots\}$$
$$\mathbb{Q} = \left\{\frac{3}{4}, \frac{1}{3}, \frac{9}{3}, \frac{-4}{333}, \ldots\right\}$$

The size of a finite set

$$
\begin{aligned}
3 &= |\{a, b, c\}| \\
5 &= |\{a, b, c, d, e\}| \\
&= |\{a, b, c, d, e, a, d, b\}| \\
0 &= |\emptyset| \\
1 &= |\{\emptyset\}| \\
1 &= |\{\{\emptyset\}\}|
\end{aligned}
$$

Membership and subsets

$$3 \in \{1, 2, 3, 4, 5\}$$
$$3 \notin \{2, 4, 6, 8\}$$
$$cat \in \{cat, dog, pig\}$$
$$3 \in \mathbb{N}^0$$
$$\pi \notin \mathbb{Z}$$

$$\{2, 5, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$\{2, 5, 8\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7\}$$
$$\{3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{3\} \not\subseteq \{2, 4, 6, 8\}$$
$$\mathbb{N}^0 \subseteq \mathbb{Z}$$
$$\mathbb{R} \not\subseteq \mathbb{N}$$

Set builder notation

$$
\begin{aligned}
\{n : n \text{ is odd and } 4 \leq n \leq 16\} &= \{5, 7, 9, 11, 13, 15\} \\
\{2n + 5 : n \in \{3, 6, 7\}\} &= \{11, 17, 19\} \\
\left\{2n : n \in \mathbb{N}^0\right\} &= \{0, 2, 4, 6, 8, \ldots\} \\
\{n \in \mathbb{N} : n < 5\} &= \{1, 2, 3, 4\} \\
\{3n : n \in \mathbb{N} \text{ and } n < 5\} &= \{3, 6, 9, 12\}
\end{aligned}
$$

Ordered pairs, triples, n-tuples (lists, sequences, strings)

$$
\begin{aligned}
(2,4) &\neq (4,2) \\
(2,2) &\neq (2) \\
(1,2,3) &\neq (3,2,1) \\
(1,1,2) &\neq (1,2) \\
(5,3,2,1,6) &\neq (1,2,3,5,6)
\end{aligned}
$$

Cartesian product

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$
$$\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

Higher order Cartesian products

$$
\begin{aligned}
A \times B \times C &= \{(a, b, c) : a \in A, b \in B, c \in C\} \\
A^n &= A \times A \times A \times ... \times A \\
&= \{(x_1, x_2, x_3, ..., x_n) : x_1, x_2, x_3, ..., x_n \in A\}
\end{aligned}
$$

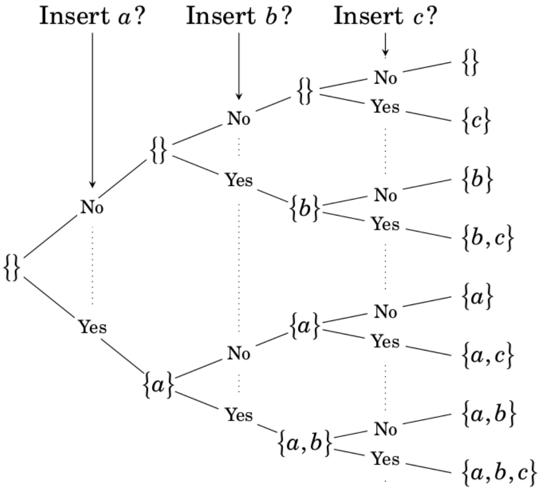
$$
\begin{aligned}
(a, 3, z) &\in \{a, b, c\} \times \{1, 2, 3, 4\} \times \{u, v, w, x, y, z\} \\
(b, 3, z) &\in \{a, b, c\} \times \{1, 2, 3, 4\} \times \{u, v, w, x, y, z\} \\
(a, 1, w) &\in \{a, b, c\} \times \{1, 2, 3, 4\} \times \{u, v, w, x, y, z\} \\
(a, 2, x) &\in \{a, b, c\} \times \{1, 2, 3, 4\} \times \{u, v, w, x, y, z\}
\end{aligned}
$$

Power set: the set of all subsets

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

How many subsets are there?

If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$



Union, Intersection, Difference

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{4, 5, 6, 7, 8\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A \cap B = \{4, 5\}$$
$$A - B = \{1, 2, 3\}$$

Complement

$$\overline{A} = \{x : x \notin A\}$$

$$\overline{\{2, 4, 6, 8, \ldots\}} = \{1, 3, 5, 7, \ldots\}$$

Usually relative to some implied **universal set** or **universe**, in this case, $\mathbb{N}$.

Indexed Sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

Indexed Sets

$$
\begin{aligned}
A_i &= \{ni : n \in \mathbb{N}\} \\
A_1 &= \{1, 2, 3, 4, \ldots\} \\
A_2 &= \{2, 4, 6, 8, \ldots\} \\
A_3 &= \{3, 6, 9, 12, \ldots\} \\
A_4 &= \{4, 8, 12, 16, \ldots\} \\
&\ldots
\end{aligned}
$$

$$\bigcup_{i=2}^{4} A_i = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\}$$

$$\bigcap_{i=2}^{4} A_i = \{12, 24, 36, 48, 72, \ldots\}$$

The Division Algorithm

Given $a, b \in \mathbb{Z}$ with $b > 0$, there exist $q, r \in \mathbb{Z}$ with

$$a = qb + r$$
$$0 \leq r < b$$

Logic

1. Circle X has radius equal to 3.
2. If any circle has radius r, then its area is πr^2.

---

3. Circle X has area 9π.

Statements

NOT Statements:	Statements
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
$\mathbb{Z}$	$42 \in \mathbb{Z}$
42	42 is not a number.
What is the solution of $2x = 84$?	The solution of $2x = 84$ is 42.

We use the letters P, Q, R and S to stand for statements.

Examples

P	:	The function $f(x) = x^2$ is continuous.
$P(x)$	:	If an integer x is a multiple of 6, then x is even.
$Q(x)$	:	The integer x is even.

A sentence whose truth value depends on the value of variables is called an **open sentence**.

And, Or, Not

P	:	The number 4 is even.
Q	:	The number 7 is even.
$P \wedge Q$	:	The number 4 is even **and** the number 7 is even.
$P \vee Q$	:	The number 4 is even **or** the number 7 is even.
$\sim P$	:	The number 4 is **not** even.
$\sim Q$	:	The number 7 is **not** even.

Truth Tables

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\sim P$
T	F
F	T

P	Q	$P \bigoplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

$R(a)$: **If** the integer a is multiple of 6, **then** a is divisible by 2.
$P(a)$: The integer a is multiple of 6.
$Q(a)$: a is divisible by 2.
$R(a)$: If P, then Q.
$R(a)$: $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalent expressions

$$\left.\begin{array}{l}
\text{If } P \text{ then } Q. \\
P \text{ only if } Q. \\
Q, \text{ if } P. \\
Q \text{ whenever } P. \\
Q, \text{ provided that } P. \\
\text{Whenever } P, \text{ then also } Q. \\
P \text{ is sufficient for } Q. \\
Q \text{ is necessary for } P.
\end{array}\right\} P \Rightarrow Q$$

Biconditional or Equivalence Statements

$$\left.\begin{array}{l} P \text{ if and only if } Q. \\ P \text{ iff } Q. \\ P \text{ is necessary and sufficient for } Q. \\ \text{If } P\text{, then } Q\text{, and conversely.} \\ P \text{ is logically equivalent to } Q. \\ (P \Rightarrow Q) \wedge (Q \Rightarrow P) \end{array}\right\} P \iff Q$$

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables for Complex Statements

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim (P \wedge Q)$	$(P \vee Q) \wedge \sim (P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Quantifiers

Universal quantifier

- For every $n \in \mathbb{Z}$, $2n$ is even.
- $\forall n \in \mathbb{Z}, 2n$ is even.
- $\forall n \in \mathbb{Z}, E(2n)$

Existential quantifier

- There exists a subset X of $\mathbb{N}$ for which $|X| = 5$.
- $\exists X, (X \subseteq \mathbb{N}) \wedge (|X| = 5)$
- $\exists X \subseteq \mathbb{N}, |X| = 5$
- $\exists X \in \mathcal{P}(\mathbb{N}), |X| = 5$

Negating statements

$$\begin{aligned}
\sim (P \Rightarrow Q) &\iff P \wedge (\sim Q) \\
\sim (P \wedge Q) &\iff (\sim P) \vee (\sim Q) \\
\sim (P \vee Q) &\iff (\sim P) \wedge (\sim Q) \\
\sim (\forall x \in S, P(x)) &\iff \exists x \in S, \sim P(x) \\
\sim (\exists x \in S, P(x)) &\iff \forall x \in S, \sim P(x)
\end{aligned}$$

Negating Statements, Example

$$
\begin{array}{rcl}
R & : & \text{The square of every real number is non-negative.} \\
R & : & \forall x \in \mathbb{R}, x^2 \geq 0 \\
\sim R & : & \sim (\forall x \in \mathbb{R}, x^2 \geq 0) \\
\sim R & : & \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \\
\sim R & : & \exists x \in \mathbb{R}, (x^2 \not\geq 0) \\
\sim R & : & \exists x \in \mathbb{R}, x^2 < 0 \\
\sim R & : & \text{There exists a real number whose square is negative.}
\end{array}
$$

Negating Statements, Example

R :
For every real number x there is a real number y for which $y^3 = x$.

$$\begin{aligned} R &= \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x \\ \sim R &= \sim (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x) \\ &= \exists x \in \mathbb{R}, \sim (\exists y \in \mathbb{R}, y^3 = x) \\ &= \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim (y^3 = x) \\ &= \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^3 \neq x \end{aligned}$$

$\sim R$:
There is a real number x for which $y^3 \neq x$ for all real numbers y.

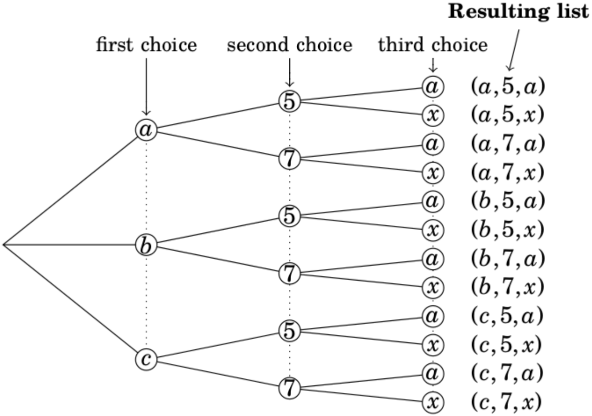
Tuples or Lists or Strings

$$(a, b, c, d, e) \neq (b, a, c, d, e)$$
$$(a, b, c, d, e) \neq (a, a, b, c, d, e)$$

$$SOS = (S, O, S)$$

Counting Tuples: Multiplication Principle

How many different lists of length 3 are there, where
the first entry must be an element of $\{a, b, c\}$,
the second entry must be an element of $\{5, 7\}$,
and the third entry must be an element of $\{a, x\}$?



Some notation: falling factorial powers

$$\begin{aligned} 7^5 &= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\ 7^{\underline{5}} &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \\ 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 7^{\underline{7}} \end{aligned}$$

Lists with repetitions

How many lists of length 4 with selections from $\{A, B, C, D, E, F, G\}$, where repetition is allowed?

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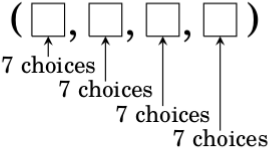

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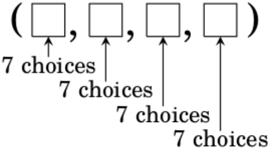

$$7 \cdot 7 \cdot 7 \cdot 7 = 7^4$$

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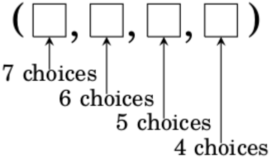

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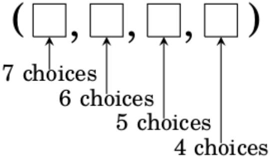

$$7 \cdot 6 \cdot 5 \cdot 4 = 7^{\underline{4}}$$

More complex lists

How many lists of length 4 with selections from $\{A, B, C, D, E, F, G\}$, without repetitions, and the symbol E must appear somewhere in the list?

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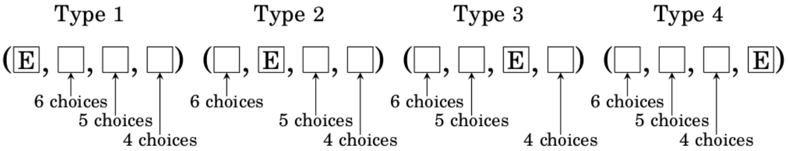

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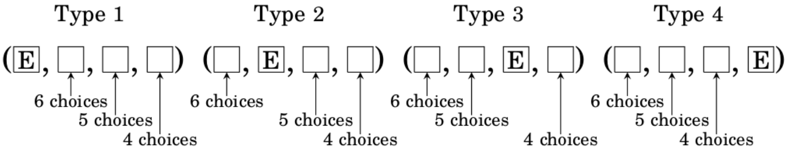

$$4 \cdot 6 \cdot 5 \cdot 4 = 4 \cdot 6^{3}$$

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- There are 6^4 lists that do not contain E.

$$7^4 - 6^4$$

More about falling factorial powers

$$
\begin{aligned}
7^{\underline{3}} &= 7 \cdot 6 \cdot 5 \\
7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
7^{\underline{3}} &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{7!}{4!} \\
&= \frac{7!}{(7-3)!} \\
n^{\underline{k}} &= \frac{n!}{(n-k)!} \\
n! &= n^{\underline{n}}
\end{aligned}
$$

Permutations

How many lists with repetitions of length k can be made from n symbols? $AAAA, BBBB, BABA, EBAB, CABC, \ldots$

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$$n^{\underline{k}}$$

$$\frac{n!}{(n-k)!}$$

Counting subsets

How many subsets of size k can be made by selecting elements from a set of size n?

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$$n^{\underline{k}}$$

But these contain set-equivalent pairs, such as *abc* and *cba*.

Eliminating the duplicates

$$\binom{5}{3}3! = 5^{\underline{3}}$$

$\binom{5}{3}$

$3!$

abc	abd	abe	acd	ace	ade	bcd	bce	bde	cde
acb	adb	aeb	adc	aec	aed	bdc	bec	bed	ced
bac	bad	bae	cad	cae	dae	cbd	cbe	dbe	dce
bca	bda	bea	cda	cea	dea	cdb	ceb	deb	dec
cba	dba	eba	dca	eca	eda	dcb	ecb	edb	edc
cab	dab	eab	dac	eac	ead	dbc	ebc	ebd	ecd

Eliminating the duplicates

$$\binom{5}{3}3! = 5^{\underline{3}}$$

$\binom{5}{3}$

3!

abc	abd	abe	acd	ace	ade	bcd	bce	bde	cde
acb	adb	aeb	adc	aec	aed	bdc	bec	bed	ced
bac	bad	bae	cad	cae	dae	cbd	cbe	dbe	dce
bca	bda	bea	cda	cea	dea	cdb	ceb	deb	dec
cba	dba	eba	dca	eca	eda	dcb	ecb	edb	edc
cab	dab	eab	dac	eac	ead	dbc	ebc	ebd	ecd

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5^{\underline{3}}}{3!} = \frac{5^{\underline{3}}}{3^{\underline{3}}}$$

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} = \frac{n^{\underline{k}}}{k^{\underline{k}}}$$

Easy to prove theorem?

$$\binom{n}{k} = \binom{n}{n-k}$$

Using set theory in counting

$$|A \cup B| = |A| + |B| - |A \cap B|$$

How many 3-card hands are there for which all 3 cards are red, or all three cards are face cards?

$A =$ 3-card hands, all red cards $\quad$ $B =$ 3-card hands, all face cards

$$|A| = \binom{26}{3}$$

$$|B| = \binom{12}{3}$$

$$|A \cap B| = \binom{6}{3}$$

$$|A \cup B| = |A| + |B| - |A \cap B| = \binom{26}{3} + \binom{12}{3} - \binom{6}{3}$$