1

BACKGROUND AND CURRENT UNDERSTANDING

Is there a beautiful fundamental law, a theory of everything, governing the rule of how our nature operates? To my best knowledge, we did find two promising theories—gravity theory and the Standard Model, which have explained and predicted most phenomena in our universe. They demonstrate that there are four fundamental interactions (forces) in nature: electromagnetic interaction, weak interaction, strong interaction, and gravity.

However, they are not yet beautiful enough. The Standard Model constructed from quantum field theory unifies only three of the four fundamental forces[1–4], except for gravity which is described by Einstein's theory of General Relativity. Due to its long-range interaction of gravity and its substantial dependence on the masses of the objects involved, gravity plays a dominant role at large length scales, such as the scales of planets and all the way to the scale of the entire universe. At small length scales—the scale of subatomic particles, the other three fundamental forces play a much more important role than gravity and are described by the Standard Model. However, at an even smaller scale—the Planck scale, gravity becomes as important as the other fundamental forces. Neglecting any of them may lead to incomplete or incorrect descriptions of physical phenomena. Unfortunately, the current formulation of quantum field theory breaks down when including gravity.

For gravity theory, it states that spacetime is curved by matter following the general relativity theory proposed by Albert Einstein[5, 6]. While on the other hand, quantum field theory is formulated on a fixed background, typically the Minkowski spacetime, and is based on the principles of quantum mechanics and special relativity. In other words, quantum field theory has not fully taken into account the effects of general relativity. No self-consistent approach has been found to extend quantum field theory to a dynamic background to date. Suppose one naively applies a similar process to gravity as we do, for example, for spin-1 massless fields in quantum field theory. The results turn out to be non-renormalizable, which deviates from a physically realistic outcome. Besides, although general relativity predicts the existence of black holes, the singularity at the center of black holes leads to infinite curvature and thus the breakdown of general relativity theory.

Overall, our understanding of the natural world remains incomplete due to the limitations of current theories. This gives rise to a pivotal goal for theoretical physicists around the world: the unification of the Standard Model and gravity theory, or more ambitiously, the development of a comprehensive Theory of Everything.

Currently, a promising candidate to unify gravity theory and quantum mechanics is string theory. Instead of treating 0-dimensional particles as elementary objects, string theory regards 1-dimensional strings as elementary objects in nature. Particles, including photons, quarks, gravitons, and others, are all represented by different vibrational modes of strings. Although there is to date no direct experimental evidence to verify it (largely due to our inability to achieve super high-energy experiments), string theory does offer a framework that potentially unifies all known fundamental forces, and it generates numerous insights and developments both in physics and mathematics.

2

DOUBLE COPY AND RESEARCH FINDINGS

One of the intriguing topics emerging from string theory is the double copy. It can be traced back to the discovery of the relationships between the amplitudes of closed and open strings in 1986, now known as the KLT relations[7], which shows that any closed string tree amplitude can be expressed as a sum of the products, consisting of corresponding open string tree amplitudes. As we know, in the low-energy limit, closed string theory gives rise to massless particles called gravitons, which mediate gravitational interactions; while open string theory gives rise to massless particles associated with gauge fields, which mediate the fundamental forces in Standard Model. Therefore, inspired by KLT relations, a new relationship between gravity and gauge tree amplitudes was noticed about two decades later—BCJ duality[8–10]. It demonstrates that tree-level gravity amplitudes can be obtained using double copies of gauge-theory diagram numerators. Shortly after, this finding was extended to the loop level, and at that point, the "double copy" concept was formally proposed.

Let's take the case of tree-level as an example. The general massless m-point tree-level gauge theory amplitude can be represented by

$$\mathcal{A}_m^{(tree)} = g^{m-2} \sum_i \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{1}$$

where the sum is over all distinct diagrams with cubic vertices, the product runs over all propagators $1/p^2_{\alpha_i}$ of each diagram, and α_i refers to the α_i-th internal line in the i-th diagram. The numerators n_i are called the kinematic numerators, which are dependent on the particle momenta and polarisation vectors and can be deformed by a so-called gauge transformation. The c_i are the color factors obtained by dressing each triple vertex with structure constants. Interestingly, the BCJ duality states that as long as there exist a set of kinematic numerators n_i such that they obey the same Jacobi-like identities as the color factors c_i, one is able to write directly the tree-level gravity amplitudes by replacing the color factors with another kinematic numerator $\tilde{n}_i$ from the tree-level gauge theory amplitudes Eq.(1),

$$\mathcal{M}_m^{(tree)} = (\frac{\kappa}{2})^{m-2} \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}, \tag{2}$$

where the coupling constant g is replaced with the gravitational counterpart κ.

In recent years, a significant amount of research has been dedicated to understanding this duality relation [11–39]. In the meantime, due to its potential implications in gravitational wave astronomy, cosmology, and particle physics, much attention has been devoted to investigating exact classical solutions related to the double copy[40–70]. Two main types of classical double copy prescriptions have been extensively studied. One is the Kerr-Schild double copy prescription, which is based on a special metric known as Kerr-Schild spacetime. This metric allows the Einstein equation to be linearized, leading to a double copy relation between solutions of linearized gravity equations and associated gauge field solutions. The other one is the Weyl double copy prescription, which demonstrates a gravity and gauge duality in a wider range of 4-dimensional spacetimes, such as non-twisting vacuum Petrov type N spacetime and Petrov type D spacetime. We will provide a more detailed introduction to these two types of classical double copy in the next part, then we will pay full attention to the research on Weyl double copy throughout this thesis.