FEM for 2D Stress Analysis

An Example

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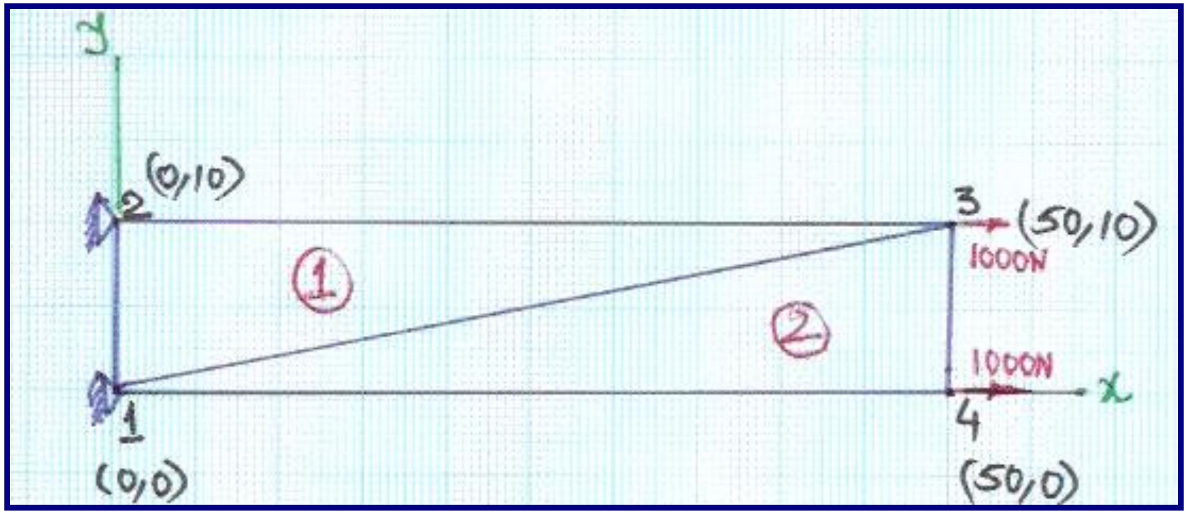
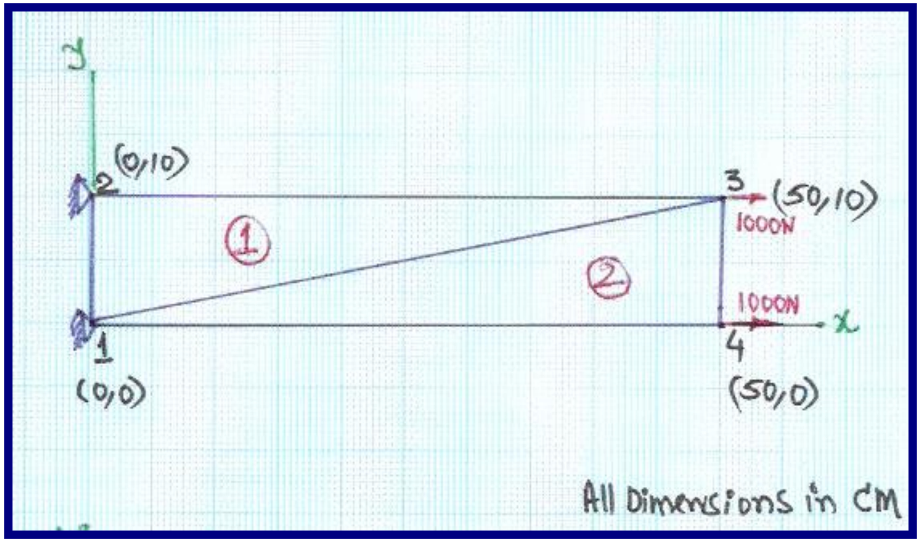
EXAMPLE #1

The plate of 50 CmX10 Cm is fixed at the left end and pulled by the two concentrated loads at the right as shown in the figure below. Considering a uniform thickness of 1 Cm, neglecting all body forces and initial strains and residual stresses, solve for the elemental stress components, using the Finite Element approach. Plane stress conditions prevail. Consider E as 210 GPa and v as 0.3



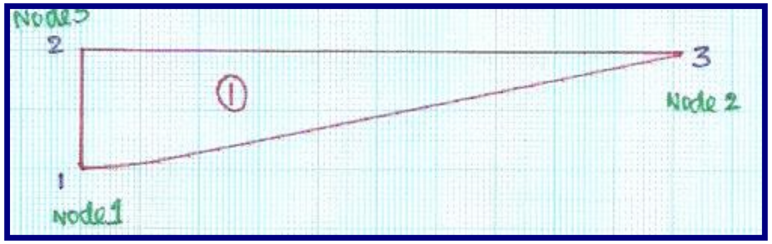
FE Approach



$$[K]\{\delta\}-\{f\}=0$$

$$\begin{Bmatrix} F1x \\ F1y \\ F2x \\ F2y \\ F3x \\ F3y \\ F4x \\ F4y \end{Bmatrix} = \begin{Bmatrix} R1x \\ R1y \\ R2x \\ R2y \\ 1000 \\ 0 \\ 1000 \\ 0 \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Element 1



$$a_1 = x_2 y_3 - x_3 y_2;\ a_2 = x_3 y_1 - x_1 y_3;\ a_3 = x_1 y_2 - x_2 y_1$$

$$b_1 = y_2 - y_3;\ b_2 = y_3 - y_1;\ b_3 = y_1 - y_2$$

$$c_1 = x_3 - x_2;\ c_2 = x_1 - x_3;\ c_3 = x_2 - x_1$$

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

B1 =

$$\begin{bmatrix} 0 & 0 & 0.1000 & 0 & -0.1000 & 0 \\ 0 & -0.5000 & 0 & 0 & 0 & 0.5000 \\ -0.5000 & 0 & 0 & 0.1000 & 0.5000 & -0.1000 \end{bmatrix}$$

$$[D] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

B1 =

$$\begin{bmatrix} 0 & 0 & 0.1000 & 0 & -0.1000 & 0 \\ 0 & -0.5000 & 0 & 0 & 0 & 0.5000 \\ -0.5000 & 0 & 0 & 0.1000 & 0.5000 & -0.1000 \end{bmatrix}$$

$$[D] = \left(\frac{210X10^9}{1-0.3^2}\right) \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

B1TD =

B1T =

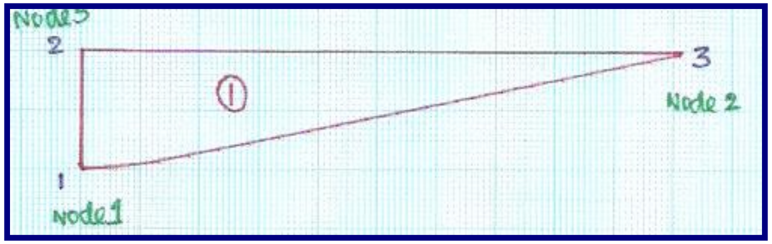
$$\begin{bmatrix} 0 & 0 & -0.5000 \\ 0 & -0.5000 & 0 \\ 0.1000 & 0 & 0 \\ 0 & 0 & 0.1000 \\ -0.1000 & 0 & 0.5000 \\ 0 & 0.5000 & -0.1000 \end{bmatrix}$$

$$\left(\frac{210X10^9}{1-0.3^2}\right) \begin{bmatrix} 0 & 0 & -0.1750 \\ -0.1500 & -0.5000 & 0 \\ 0.1000 & 0.0300 & 0 \\ 0 & 0 & 0.0350 \\ -0.1000 & -0.0300 & 0.1750 \\ 0.1500 & 0.5000 & -0.0350 \end{bmatrix}$$

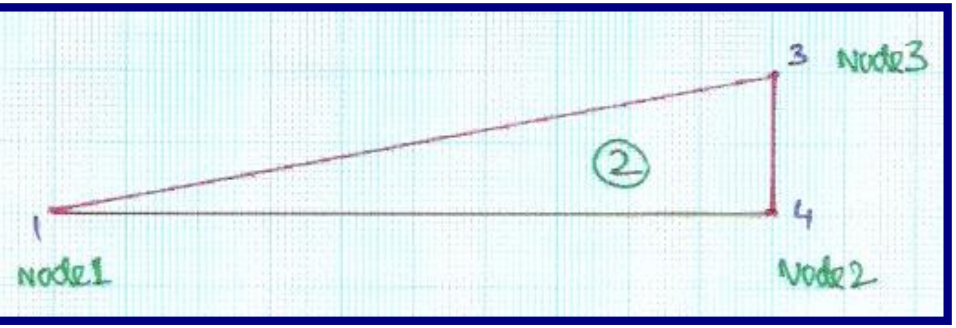
Elemental stiffness matrix1

Ke¹ = B1TDB1 =

$$
\begin{bmatrix}
0.0875 & 0 & 0 & -0.0175 & -0.0875 & 0.0175 \\
0 & 0.2500 & -0.0150 & 0 & 0.0150 & -0.2500 \\
0 & -0.0150 & 0.0100 & 0 & -0.0100 & 0.0150 \\
-0.0175 & 0 & 0 & 0.0035 & 0.0175 & -0.0035 \\
-0.0875 & 0.0150 & -0.0100 & 0.0175 & 0.0975 & -0.0325 \\
0.0175 & -0.2500 & 0.0150 & -0.0035 & -0.0325 & 0.2535
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \\ u_2 \\ v_2 \end{Bmatrix}
$$



Element 2



B2 =

$$\begin{bmatrix} -0.1000 & 0 & 0.1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5000 & 0 & 0.5000 \\ 0 & -0.1000 & -0.5000 & 0.1000 & 0.5000 & 0 \end{bmatrix}$$

B2T =

$$\begin{bmatrix} -0.1000 & 0 & 0 \\ 0 & 0 & -0.1000 \\ 0.1000 & 0 & -0.5000 \\ 0 & -0.5000 & 0.1000 \\ 0 & 0 & 0.5000 \\ 0 & 0.5000 & 0 \end{bmatrix}$$

$$[D]=\left(\frac{210X10^9}{1-0.3^2}\right)\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \dfrac{1-0.3}{2} \end{bmatrix}$$

B2TD =

$$\left(\frac{210X10^9}{1-0.3^2}\right)\begin{bmatrix} -0.1000 & -0.0300 & 0 \\ 0 & 0 & -0.0350 \\ 0.1000 & 0.0300 & -0.1750 \\ -0.1500 & -0.5000 & 0.0350 \\ 0 & 0 & 0.1750 \\ 0.1500 & 0.5000 & 0 \end{bmatrix}$$

Elemental stiffness matrix 2

Ke^2 = B2TDB2 =

$$\begin{bmatrix} 0.0100 & 0 & -0.0100 & 0.0150 & 0 & -0.0150 \\ 0 & 0.0035 & 0.0175 & -0.0035 & -0.0175 & 0 \\ -0.0100 & 0.0175 & 0.0975 & -0.0325 & -0.0875 & 0.0150 \\ 0.0150 & -0.0035 & -0.0325 & 0.2535 & 0.0175 & -0.2500 \\ 0 & -0.0175 & -0.0875 & 0.0175 & 0.0875 & 0 \\ -0.0150 & 0 & 0.0150 & -0.2500 & 0 & 0.2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix}$$

Stiffness matrices $[K^e]$

Contribution from Element 1

$$\left(0.01X\,0.025X\,\frac{1}{2X\,0.025}X\,\frac{210X10^9}{1-0.3^2}X\,\frac{1}{2X\,0.025}\right)\begin{bmatrix} 0.0875 & 0.0000 & -0.0875 & 0.0175 & 0.0000 & -0.0175 & 0.0000 & 0.0000 \\ 0.0000 & 0.2500 & 0.1500 & -0.2500 & -0.0150 & 0.0000 & 0.0000 & 0.0000 \\ -0.0875 & 0.0150 & 0.0975 & -0.0325 & -0.0100 & 0.0175 & 0.0000 & 0.0000 \\ 0.0175 & -0.2500 & -0.0325 & 0.2535 & 0.0150 & -0.0035 & 0.0000 & 0.0000 \\ 0.0000 & -0.0150 & -0.0100 & 0.0150 & 0.0100 & 0.0000 & 0.0000 & 0.0000 \\ -0.0175 & 0.0000 & 0.0175 & -0.0035 & 0.0000 & 0.0035 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.00000 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\left(23.1X10^9\right)\begin{bmatrix} 0.0875 & 0.0000 & -0.0875 & 0.0175 & 0.0000 & -0.0175 & 0.0000 & 0.0000 \\ 0.0000 & 0.2500 & 0.1500 & -0.2500 & -0.0150 & 0.0000 & 0.0000 & 0.0000 \\ -0.0875 & 0.0150 & 0.0975 & -0.0325 & -0.0100 & 0.0175 & 0.0000 & 0.0000 \\ 0.0175 & -0.2500 & -0.0325 & 0.2535 & 0.0150 & -0.0035 & 0.0000 & 0.0000 \\ 0.0000 & -0.0150 & -0.0100 & 0.0150 & 0.0100 & 0.0000 & 0.0000 & 0.0000 \\ -0.0175 & 0.0000 & 0.0175 & -0.0035 & 0.0000 & 0.0035 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.00000 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Contribution from Element 2

$$
\left(23.1X10^{9}\right)\begin{bmatrix}
0.0100 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0150 & -0.0100 & 0.0150 \\
0.0000 & 0.0035 & 0.0000 & 0.0000 & -0.0175 & 0.0000 & 0.0175 & -0.0035 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0175 & 0.0000 & 0.0000 & 0.0875 & 0.0000 & -0.0875 & 0.0175 \\
-0.0150 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2500 & 0.0150 & -0.2500 \\
-0.0100 & 0.0175 & 0.0000 & 0.0000 & -0.0875 & 0.0150 & 0.0975 & -0.0325 \\
0.0150 & -0.0035 & 0.0000 & 0.0000 & 0.0175 & -0.2500 & -0.0325 & 0.2535
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
$$

Overall stiffness matrix

K =

$$
23.1X10^{9}
\begin{matrix}
0.0975 & 0.000 & -0.0875 & 0.0175 & 0.000 & -0.0325 & -0.0100 & 0.0150 \\
0.000 & 0.2535 & 0.1500 & -0.2500 & -0.0325 & 0.000 & 0.0175 & -0.003 \\
-0.0875 & 0.0150 & 0.0975 & -0.0325 & -0.0100 & 0.0175 & 0.000 & 0.000 \\
0.0175 & -0.2500 & -0.0325 & 0.2535 & 0.0150 & -0.0035 & 0.000 & 0.000 \\
0.000 & -0.0325 & -0.0100 & 0.0150 & 0.0975 & 0.000 & -0.0875 & 0.0175 \\
-0.0325 & 0.000 & 0.0175 & -0.0035 & 0.000 & 0.2535 & 0.0150 & -0.2500 \\
-0.0100 & 0.0175 & 0.000 & 0.000 & -0.0875 & 0.0150 & 0.0975 & -0.0325 \\
0.0150 & -0.0035 & 0.000 & 0.000 & 0.0175 & -0.2500 & -0.0325 & 0.2535
\end{matrix}
$$

Unknown Displacements

The first four rows and columns can be eliminated from these equations as u1, v1 and u2 and v2 are all equal to zero. This means

$$\left(23.1X10^{9}\right)\begin{bmatrix} 0.0975 & 0.0000 & -0.0875 & 0.0175 \\ 0.0000 & 0.2535 & 0.0150 & -0.2500 \\ -0.0875 & 0.0150 & 0.0975 & -0.0325 \\ 0.0175 & -0.2500 & -0.0325 & 0.2535 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}=\begin{Bmatrix} 1000 \\ 0 \\ 1000 \\ 0 \end{Bmatrix}$$

Upon inversion,

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}=\left(1.0X10^{-5}\right)\begin{Bmatrix} 0.4490 \\ 0.0092 \\ 0.4582 \\ 0.0368 \end{Bmatrix}$$

Reactions at nodes 1 and 2

$$\boxed{F = [K]\{\delta\}}$$

$$\left\{\begin{array}{l} R1x \\ R1y \\ R2x \\ R2y \\ F3x \\ F3y \\ F4x \\ F4y \end{array}\right\} = (1.0e+003)\left\{\begin{array}{r} -1.000 \\ -1.5483 \\ -1.0000 \\ 1.5483 \\ 1.0000 \\ 0.0012 \\ 1.0000 \\ -0.0012 \end{array}\right\}$$

Elemental stress components

Element 1

$$\sigma = [D][B1]\{d1\}$$

$$\sigma = \frac{210X10^{9}\,X10^{-5}}{2X\,0.025\left(1-0.3^{2}\right)}\begin{bmatrix} 1.0 & 0.3 & 0 \\ 0.3 & 1.0 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \\ -0.5 & 0 & 0 & 0.1 & 0.5 & -0.1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.4490 \\ 0.0092 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma = \begin{Bmatrix} \sigma x \\ \sigma y \\ \tau xy \end{Bmatrix} = 1.0e+006\,X\begin{Bmatrix} 2.0723 \\ 0.6217 \\ 0.0149 \end{Bmatrix}$$

Elemental stress components

Element 2

$$\sigma = [D][B2]\{d2\}$$

$$\sigma = \frac{210X10^{9}X10^{-5}}{2X0.025\left(1-0.3^{2}\right)}\begin{bmatrix} 1.0 & 0.3 & 0 \\ 0.3 & 1.0 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -0.1000 & 0 & 0.1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5000 & 0 & 0.5000 \\ 0 & -0.1000 & -0.5000 & 0.1000 & 0.5000 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.4582 \\ 0.0368 \\ 0.4490 \\ 0.0092 \end{Bmatrix}$$

$$\sigma = \begin{Bmatrix} \sigma x \\ \sigma y \\ \tau xy \end{Bmatrix} = 1.0e+006\,X \begin{Bmatrix} 1.9237 \\ -0.0025 \\ -0.0149 \end{Bmatrix}$$