PDE to ODE: Using Graphs

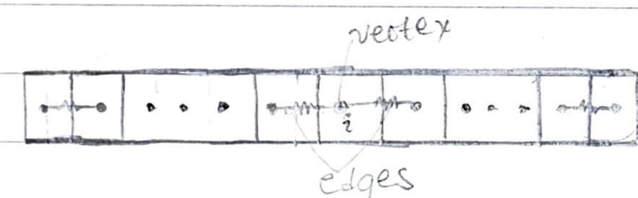
We can also just discretize the domain so that our PDE becomes a system of ODE's. Assume our rod is discretized into n units or volumes. We can write the ODE for each volume approximately as

$$\dot{u}_i = \sum_{j \in \mathcal{N}_i} \alpha_{ij} (u_j - u_i) \quad (*)$$

where $\mathcal{N}_i$ are the neighbouring nodes

We may think of this as approximating the 2nd order derivative with a unit step

$$\dot{u}_i = \frac{\alpha_{ij_1} \dfrac{u_{j_1} - u_i}{\Delta x} - \alpha_{ij_2} \dfrac{u_i - u_{j_2}}{\Delta x}}{\Delta x}$$



We can write $(*)$ as

$$\dot{u}_i = -\left(u_i \sum_j A_{ij} - \sum_j A_{ij} u_j \right)$$

$$= -\sum_j \Big(\delta_{ij} \underbrace{\sum_j A_{ij}}_{} - A_{ij} \Big) u_j$$

→ degree of vertex i
i.e. total edge weights incident on vertex i

$$= -\sum_j L_{ij} u_j$$

In matrix-vector notation, we have

$\dot{u} = -Lu$ for $u \in \mathbb{R}^n$

discrete Laplacian / Laplacian matrix

where $L = D - A$

$A_{i,j} = \begin{cases} \alpha_{ij} & \text{if } \exists \text{ edge } (i,j) \\ 0 \end{cases}$ is the adjacency matrix

$D_{i,j} = \sum_{j}^{n} A_{ij}$ is the Degree matrix

Thus $L_{ij} = \begin{cases} \deg(v_i) & \text{if } i=j \\ -\alpha_{ij} & \text{if } i \neq j \text{ and edge } (i,j) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$

where $\deg(v_i)$ is the degree of vertex $i = \sum_{j=1}^{n} A_{i,j}$

The Laplacian for an undirected graph can also be written as

$L = B \Lambda B^T$

where Λ is a diagonal matrix containing all the weighting of all edges,

$B_{i,k} = \begin{cases} -1 & \text{edge } k \text{ starts @ vertex } i \\ 1 & \text{link } k \text{ ends @ vertex } i \\ 0 & \text{otherwise} \end{cases}$

$B \in \mathbb{R}^{n \times n_l}$ where n_l are the total number of graph links and is called the incidence matrix. Though the above incidence matrix suggests a directed graph, the directions can be arbitrary and would still result in the same laplacian matrix.

NOTE: The laplacian matrix, by construction, has zero column sum

$$L \vec{1} = \vec{0}$$

Moreover, if the graph is undirected and connected, or directed and strongly connected, then L is irreducible and e^{-L} is row stochastic.

Higher Order ODE's to 1st Order ODE (State Space)

Consider the n^{th} order ODE

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 z = b_m \frac{d^m u}{dt^m} + \dots + b_0 u$$

We can convert this to a set of n coupled 1st order ODE's (or system of ODE's):

$$\dot{x}_1 = f_1(x_1, x_2, \dots x_n, u)$$
$$x_2 = f_2(x_1, x_2, \dots x_n, u)$$
$$\vdots \qquad\qquad \vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \dots x_n, u)$$

where $x_1, x_2, \dots, x_n$ are intermediate variable. For example, for the ODE

$$\ddot{y} + \dot{y} + 2y = u$$

let $x_1 = y \quad \Rightarrow \quad \dot{x}_1 = x_2 \quad \Leftarrow f_1(x_1, x_2, u)$

$x_2 = \dot{y}$

Since $\ddot{y} = -\dot{y} - 2y + u = \dot{x}_2$

$$\dot{x}_2 = -\dot{x} - 2x_1 + u \qquad \Leftarrow f_2(x_1, x_2, u)$$

$$\therefore \begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - x_2 + u \\ y = x_1 \end{bmatrix}$$

State Space form

$$\dot{x}_1 = f_1(x_1, x_2, \cdots, x_n, u)$$
$$x_2 = f_2(x_1, x_2, \cdots, x_n, u)$$
$$\vdots$$
$$x_n = f_n(x_1, x_2, \cdots, x_n, u)$$

$$y = h(x_1, x_2, \cdots, x_n, u)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ State

$y \rightarrow$ Output

$u \rightarrow$ input

$$\dot{\vec{x}} = f(\vec{x}, u)$$
$$y = h(\vec{x}, u)$$

Using Nested Integral Method

Suppose our vibration ODE was

$$\ddot{y} + \dot{y} + 2y = \ddot{u} + u,$$

it is no immediately clear what intermediate steps to pick,
We can re-write this ODE as a nested sum of integrals:

$$\ddot{y} = \ddot{u} - \dot{y} + u - 2y$$

$$\dot{y}(t) = \dot{u}(t) - y(t) + \int_{\tau} \big(u(\tau) - 2y(\tau)\big)\, d\tau$$

$$y(t) = u(t) + \underbrace{\int_{\rho} \Big[-y(\rho) + \underbrace{\int_{\tau} \big(u(\tau) - 2y(\tau)\big)\, d\tau}_{x_2} \Big]\, d\rho}_{x_1}$$

$$y = u + x_1$$

$$x_1(t) = \int_{\rho} \big(-y(\rho) + x_2(\rho)\big)\, d\rho \qquad \Rightarrow \qquad \dot{x}_1(t) = -y(t) + x_2(t) \quad \ldots (1)$$

$$x_2(t) = \int_{\tau} \big(u(\tau) - 2y(\tau)\big)\, d\tau \qquad \dot{x}_2(t) = u(t) - 2y(t) \quad \ldots (2)$$

Plugging in $y = u + x_1$ in (1) and (2)

$$\dot{x}_1 = -x_1 + x_2 - u$$
$$\dot{x}_2 = -2x_1 - u \qquad \Leftarrow \text{SS form}$$
$$y = u + x_1$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

Transfer Functions

We have shown that we can write linear differential equations in (1st order) state space form. For manipulating the differential equations, (e.g getting an explicit expression for control), we can under certain conditions now write our state space equation in algebraic form using Laplace Transform. Recall

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = f(0^-) + s\mathcal{L}\{f(t)\}$$

Thus for zero initial conditions $\mathcal{L}\{df/dt\} = sf(s)$.
Taking the Laplace transform of (*) yields

$$sX(s) = AX(s) + Bu(s) \quad \cdots (1)$$
$$Y(s) = CX(s) + Du(s) \quad \cdots (2)$$

$$(sI - A)X(s) = Bu(s) \Rightarrow X(s) = (sI - A)^{-1}Bu(s)$$

In (2),

$$Y(s) = C(sI - A)^{-1}Bu(s) + Du(s)$$

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)}u(s)$$

Input Output

Transfer function

which is in the linear state space form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \ldots (*)$$
$$y(t) = Cx(t) + Du(t)$$

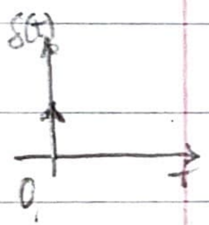
The Inverse Laplace Transform $\mathcal{L}^{-1}(G(s)) = g(t)$ is called the impulse response. Here's why:

For a single-input single-output system:

$$Y(s) = G(s)U(s)$$

$$\downarrow \mathcal{L}^{-1}$$

$$y(t) = g(t) * u(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$$



If $u(t) = \delta(t)$ (impulse), $\mathcal{L}(u(t)) = U(s) = 1$, and $G(s) = Y(s)$, which is the Laplace Transform of the response of the system to an impulse