Physical System Representation

PDE

$\downarrow$ Transform, Graph, approximation

ODE's

Nested Integrals

Laplace Transforms

State Space

$G(s) = C(sI-A)^{-1}B + D$

$G(s)$ Transfer function

$Y(s) = G(s)\,U(s)$

first-order system representation

$\dot{x} = Ax + Bu$

$y = Cx + Du$

inverse Laplace Transform

Impulse Response

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$$

$$= g(t) * u(t)$$

NOTE: Note the following

(1) SS → TF (on MATLAB): ss2tf
TF → SS (on MATLAB): tf2ss

(2) ODE's & TF's are unique for a given system

(3) SS Representations are not unique

(4) We may go from transfer function to ODE

E.g: $G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^2+4s+5} \Rightarrow (s^2+4s+5)Y(s) = 10U(s)$

with all zero initial conditions in time, we have

$$\ddot{y}(t) + 4\dot{y}(t) + 5y = 10u$$

(5) Discrete-Time System Representation

Given

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

← Linear Time Invariant (LTI) System

Suppose during interval Δt, $u(t)$ is constant (zero-order hold)

$$u_k \equiv u(t) \text{ for } t \in [k\Delta t, (k+1)\Delta t)$$

then,

$$x_{k+1} = A_d x_k + B_d u_k$$

$$y_k = C_d x_k + D_d u_k$$

where $x_k = x(k\Delta t)$

$$A_d = e^{A\Delta t}$$

$$B_d = \int_0^{\Delta t} e^{A\tau} B \, d\tau$$

$$C_d = C$$

$$D_d = D.$$

Infact,

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \Delta t}$$

STABILITY OF LTI SYSTEMS

$u \rightarrow$ [System] $\rightarrow y$

$G(s)$, SS

Bounded output $|y(t)| < \infty \ \forall t$
~~if $|u(t)| < \infty \ \forall t$~~

① $G(s)$ given $\rightarrow$ find poles of $G(s)$ and confirm that they lie in the open LHP

② State Space given: $\dot{x}(t) = Ax(t) + Bu(t)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$= e^{At}x(0) + e^{At}B * u(t)$$

Assuming x is a scalar, $A = a$, $B = b$. For small changes in u to cause only small changes in x, e^{at} must always remain bounded. Hence the system is stable iff e^{at} decays to zero or $a < 0$

For vector x, e^{At} must also decay to zero: that happens iff all eig(A) lie in the Open Left Hand Plane

NOTE Poles of $G(s)$ $\equiv$ Eigenvalues of A

Hint:

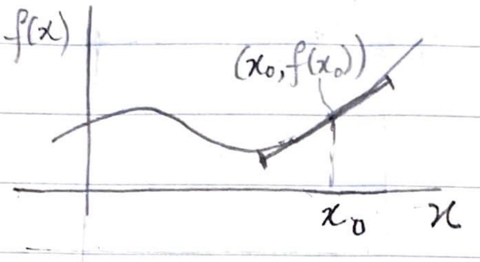
$$(sI - A)^{-1} = \mathcal{L}(e^{At})$$

$\downarrow$

Poles: values for which $|sI - A| = 0$

Eigenvalue: characteristic (Equation) roots $|\lambda I - A| = 0$

LINEARIZATION



For a nonlinear function $f(x)$, given x_0, we can approximate $f(x)$ in the neigborhood of x_0 by a linear function of the form:

$$\delta f(x) = a\,\delta x$$

$$f(x) - f(x_0) = a(x - x_0)$$

$$a = \left.\frac{df}{dx}\right|_{x = x_0}$$

In general, for multivariate functions

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad f(x) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \in \mathbb{R}^q$$

$$\delta f(x) = \underbrace{\nabla_x f}_{\text{Jacobian } (q \times n \text{ matrix})} \delta x$$

$$\nabla_x f = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_2 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & & & \vdots \\ \partial f_q/\partial x_1 & \cdots & \cdots & \partial f_n/\partial x_n \end{bmatrix}$$

Hence, given $\dot{x} = f(x,u)$ $\quad x \in \mathbb{R}^n$ state

$y = h(x,u)$ $\quad u \in \mathbb{R}^m$ inputs

$y \in \mathbb{R}^p$ output

$$f = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix} \qquad h = \begin{bmatrix} h_1(x,u) \\ \vdots \\ h_p(x,u) \end{bmatrix}$$

We can pick the point $\begin{Bmatrix} x_e \\ u_e \end{Bmatrix}$, and define the deviation variable

$$\delta x = x - x_e$$
$$\delta u = u - u_e$$

The linearized state space can then be written as

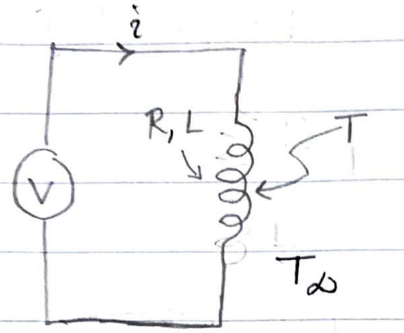
$$\frac{d}{dt}(\delta x) = \nabla_x f \Big|_{\substack{x=x_e \\ u=u_e}} \delta x + \nabla_u f \Big|_{\substack{x=x_e \\ u=u_e}} \delta u$$

$$\delta y = \nabla_x h \Big|_{\substack{x=x_e \\ u=u_e}} \delta x + \nabla_u h \Big|_{\substack{x=x_e \\ u=u_e}} \delta u$$

for short: $$\frac{d}{dt}(\delta x) = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

Ex 1: Electric Heater



$$C\dot{T} = i^2 R + K(T_\infty - T)$$

$$L\frac{di}{dt} + iR = V \leftarrow \text{input}$$

$T \leftarrow$ Output

Linearize about $V = V_0$, $T = T_0$, $i = i_0$, $T_\infty \equiv 0$

Step 1: Write ODE's in standard state-space form

$x_1 = T$, $x_2 = i$, $u = V$

$$\dot{x}_1 = \frac{R}{C}x_2^2 + \frac{K}{C}T_\infty - \frac{K}{C}x_1$$

$$\dot{x}_2 = -\frac{R}{L}x_2 + \frac{u}{L}$$

$$y = x_1$$

$$f = \begin{bmatrix} -\frac{K}{C}x_1 + \frac{R}{C}x_2^2 \\ -\frac{R}{L}x_2 + \frac{u}{L} \end{bmatrix} \qquad h = x_1$$

Step 2: Linearize

$$\nabla_x f = \begin{bmatrix} -K/C & 2\frac{R}{C}x_2 \\ 0 & -R/L \end{bmatrix} \qquad \nabla_u f = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

$$\nabla_x h = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \nabla_u h = 0$$

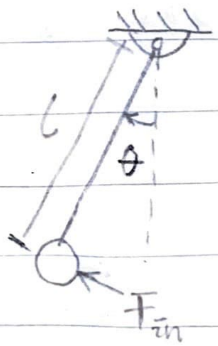
$$A = \nabla_x f \Big|_{\substack{x = x_0 \\ u = u_0}} = \begin{bmatrix} -K/C & 2R/C\, i_0 \\ 0 & -R/L \end{bmatrix}$$

$$B = \nabla_u f \Big|_{\substack{x = x_0 \\ u = u_0}} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

$$C = \nabla_x h \Big|_{\substack{x = x_0 \\ u = u_0}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = \nabla_u h \Big|_{\substack{x = x_0 \\ u = u_0}} = 0$$

Ex2: Simple Pendulum



mass   moment of inertia

$$l F_{in} - mgl \sin\theta = I\ddot{\theta}$$

$y = \theta$ output

$u = F_{in}$ input

Linearize about equilibrium point

Let $x = \theta$, $x_2 = \dot{\theta}$, $u = F_{in}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{\theta} = -\frac{mgl\sin\theta}{I} + \frac{l}{I}u = -\frac{mgl\sin x_1}{I} + \frac{l}{I}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} x_2 \\ -\frac{mgl}{I}\sin(x_1) + \frac{L}{I}u \end{bmatrix}}^{f(x,u)}$$

$$y = u \leftarrow h(x,u)$$

An equilibrium point x_e is a state about which the dynamics are zero

x_e is such that $f(x_e, 0) = 0$

$$f(x_e, 0) = \begin{bmatrix} x_{e,2} \\ -\frac{mgl}{I}\sin(x_{e,1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_{e,2} = 0 \\ \sin(x_{e,1}) = 0 \end{matrix}$$

$$x_{e,1} = k\pi$$

stable - even k

& unstable - odd k

equilibrium

$$x_e = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \qquad k \in \mathbb{Z}$$

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}\right|_{x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{mgl}{I}\cos(x_1) & 0 \end{bmatrix}\right|_{x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{mgL}{I} & 0 \end{bmatrix} \qquad B = \left.\begin{bmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ L/I \end{bmatrix}$$

②

$$C = \left[\frac{\partial h}{\partial x_1} \quad \frac{\partial h}{\partial x_2} \right] \Bigg|_{\substack{x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ u = 0}} = \begin{bmatrix} 1 & 0 \end{bmatrix} ; \; D = 0$$

NOTE: $eig(A) = \pm i \sqrt{\frac{mgL}{I}}$

If we used an odd k, say k = 1; $eig(A) = \pm \sqrt{\frac{mgl}{I}}$