Problem 1:

Consider the dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Show that for a zero-order hold on the input, that is, $u(t) \equiv u_k$ for $t \in [k\Delta t, (k+1)\Delta t)$, $k = 1, 2, \dots$, the discrete-time system is

$$x_{k+1} = A_d x_k + B_d u_k$$

where $x_k = x(k\Delta t)$,

$$A_d = e^{A\Delta t},$$
$$B_d = \int_{\tau=0}^{\Delta t} e^{A\tau} B d\tau.$$

BONUS: Show that, in fact,

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\Delta t}.$$

Problem 2:

Consider the system shown below where a DC motor is used to drive a slender spotlight (which may be thought of as a pendulum).

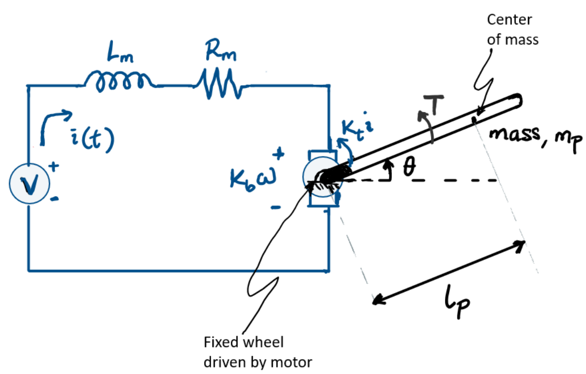

Figure 1: Motor driving slender spotlight (pendulum) of length l_p and mass m_p.

Our goal is to design a feedback controller that is able to drive the spotlight back-and-forth from the horizontal to the vertical, that is, from $\theta = 0$ *rad* to $\theta = \pi/2$ *rad*. Our measurable output is angle θ and the input to the system is voltage V. The parameters of the system are given in attached file `hwk4_p2.m`. Assume that:

- the torque $T(t)$ is counter-clockwise positive when a positive current $i(t)$ flows through the motor;

- angles are measured counter-clockwise positive from the positive $x-$axis, as shown in the figure;
- a viscous friction torque $T_f(t) = -B_m\dot{\theta}(t)$ opposes motion; and
- the motor electric circuit has negligible inductance.

(a) Find a state space set of equations for this system. (You may refresh modeling of DC motors circuits and find other related excellent resources at the CTMS website).

(b) Find the equilibrium points and comment on the stability of the points. Perturb the (original nonlinear) system about the equilibrium points and simulate what happens to the angular displacement and velocity over $10s$. How does the system behavior change if the electric circuit is open? Compare plots of the angular displacement vs time and velocity vs time for both closed and open circuits.

(c) What is the impulse response function of the system at the equilibrium points?

(d) Now, design a feedback controller that will stabilize the system about $\theta = pi/4\ rad$ with an overshoot of less than 15% and a settling time less than 0.2 s.

(e) Verify that your controller will enable the spotlight track the following angular displacement reference

$$r(t) = \begin{cases} \pi/2, & 0 \leq t < 5s, \\ 0, & 5 \leq t < 10s, \\ \pi/2, & 10 \leq t < 15s, \\ 0, & 15 \leq t < 20s, \\ \pi/2, & 20 \leq k < 25s; \end{cases}$$

with a maximum input voltage of $9V$ available. Using a `subplot` or a *double y-axes* figure, contrast a plot of the controlled input voltage against time with a plot of the angular displacement against time. Comment on your results.

Problem 3:

Consider the model of an active suspension control system on the following MATLAB webpage:

https://www.mathworks.com/help/robust/gs/active-suspension-control-design.html

The suspension system in discrete time can be expressed as

$$x_{k+1} = Ax_k + Bu_k + Fd_k,$$

where $x \in \mathbb{R}^4$ is the state of the system as described in the webpage above. The control input u is the actuator force f_s in the website, while the external input d is the road disturbance r in the website. Our goal is to design an optimal control input for this system to reject a known disturbance profile (e.g., a speed bump). Specifically, we wish to minimize the objective

$$J = \frac{1}{2}\sum_{k=0}^{N} x_k^T Q x_k + \rho u_k^2,$$

where $N = 300$. Assume that the system is initially at rest, $x_0 = 0$. The parameters A, B, F, Q, and ρ are provided in the script `hwk4_p3.m` attached to this assignment.

(a) Calculate the optimal input and state trajectories if the road disturbance is

$$d(k) = \begin{cases} 0, & 0 \leq k < 100, \\ 0.1, & 100 \leq k \leq 299. \end{cases}$$

This represents the vehicle going up a step. In your solution, use the `subplot` command to plot (i) the optimal state (displacement/travel) trajectories (both wheel and body displacements on one subplot, distinguished by a legend), (ii) the optimal state (velocity) trajectories (both wheel and body velocities on one subplot, distinguished by a legend), and (iii) the corresponding optimal control input. In addition, separately plot a comparison of the body acceleration when you use your optimal input and when the active suspension is turned off ($u \equiv 0$). Properly label your figures.

(b) Comment on the causality of your optimal control with respect to the disturbance based on (a). Does your controller anticipate the step?

NOTE: Your report should explain your approach for solving this problem.