A Very Brief Linear Algebra Review

ME599-004: Data-Driven Methods for Control Systems, Winter 2024

1 Vector Space

A vector space (V, F) is a set of vectors V and a field of scalars F, along with two operations: vector addition $(+)$ and scalar multiplication $(\cdot)$; such that
Addition $(+) : V \times V \to V : (x, y) \mapsto x + y$

(i) associative $(x + y) + z = x + (y + z) \quad \forall x, y, z \in V$.

(ii) commutative $x + y = y + x$.

(iii) $\exists$ additive identity $0 \in V$ such that $x + 0 = 0 + x = x$.

(iv) $\exists$ additive inverse, i.e., $\forall x \in V, \exists(-x)$ such that $x + (-x) = 0$

Scalar Multiplication $(\cdot) : F \times V \to V : (\alpha, x) \mapsto \alpha x$

(v) (v) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x) \quad \forall x \in V, \quad \forall \alpha, \beta \in F$.

(v) $1 \cdot x = x$, where 1 is the multiplicative identity for the field F.

(v) $0 \cdot x = 0$, where 0 is the additive identity for the field F.

(v) distributive (1) $\forall x \in V, \forall \alpha, \beta \in F (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

(v) distributive (2) $\forall x, y \in V, \forall \alpha \in F \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.

Let (V, F) be a linear space (vector space) and $W \subset V$. Then, (W, F) is called a subspace of (V, F) if (W, F) itself is a vector space (with the same inherited operations).

2 Linear Independence

Suppose (V, F) is a linear space. The set of vectors $\{v_1, v_2, \ldots, v_p\}$ is said to be linearly independent iff $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$. Conversely, the set of vectors is said to linearly dependent iff $\exists$ scalars $\alpha_1, \alpha_2, \cdots \alpha_p$ not all zero, such that, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0$.

3 Basis

Suppose (V, F) is a linear space. Then a set of vectors $B = \{b_1, b_2, \cdots b_n\}$ is called a basis if $\{b_1, b_2, \cdots b_n\}$ spans V; and $\{b_1, b_2, \cdots b_n\}$ is a linearly independent set.

4 Linear Map/Operator

Let (V, F) and (W, F) be linear spaces over the same field F. Let $\mathscr{A}$ be a map from V to W, i.e. $\mathscr{A} : V \to W$ such that $\mathscr{A}(v) = w$. $\mathscr{A}$ is said to be a linear map (or a linear operator) iff $\mathscr{A}\left(\alpha_1 v_1 + \alpha_2 v_2\right) = \alpha_1 \mathscr{A}\left(v_1\right) + \alpha_2 \mathscr{A}\left(v_2\right) \quad \forall v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in F$

4.1 Matrices as Linear Operators

Consider a matrix $A \in \mathbb{R}^{n \times m}$ and a vector x in $\mathbb{R}^m$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^n \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We can interpret matrix multiplication in many ways.

1. $\mathbf{y} = A\mathrm{x}$ is the linear combination of the colums of A, i.e. $\mathbf{a}_i$, with weights corresponding to x_i, i.e. $\mathbf{y} = \sum_{i=1}^m x_i \mathbf{a}_i$.
2. $\mathbf{y} = A\mathbf{x}$ can also be thought of as a stack of inner products of the rows of A, namely $\mathbf{r}_j^T$, with $\mathbf{x}$. Therefore $y_j = \mathbf{r}_j^T \mathbf{x}$.
3. A is a linear map from one finite (m) dimensional vector space to another finite (n) dimensional vector space.

5 Range Space and Null Space

Given a linear operator $\mathscr{A} : U \to V$, define the range space (or image) of $\mathscr{A}$ to be the subspace

$$\mathscr{R}(\mathscr{A}) := \{v \mid v = \mathscr{A}(u), u \in U\}$$

We define the null space (or kernel) of $\mathscr{A}$ to be the subspace

$$\mathscr{N}(\mathscr{A}) := \{u \mid \mathscr{A}(u) = 0_V, u \in U\}$$

6 Norms and Normed Vector Spaces

Let F be a field (either $\mathbb{R}$ or $\mathbb{C}$). A normed vector space is a pair $(V, \|\cdot\|)$ where V is a vector space over F and $\|\cdot\| : V \to \mathbb{R}$ is a function such that

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$ in V (positive definiteness)
2. $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in V$ and all $\lambda \in F$ (homogeneity)
3. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ (the triangle inequality)

The function $\|\cdot\|$ is called a norm on V.

7 Eigenspace

Given a square matrix $A \in \mathbb{R}^{n \times n}$, an eigenvalue $\lambda_i \in \mathbb{C}$ and the corresponding eigenvector(s) v_i are such that

$$Av_i = \lambda_i v_i$$

The eigenvalues of a matrix can be determined by solving

$$\det\left(\lambda I_n - A\right) = 0$$

The polynomial $[\det\left(\lambda I_n - A\right)]$ is called the characteristic polynomial. The equation $\det\left(\lambda I_n - A\right) = 0$ is called the characteristic equation.

Eigenspace: Once the eigenvalue(s) λ_i have been found, we find eigenvectors v_i such that $\left(\lambda_i I - A\right) v_i = 0$. Thus, the eigenvectors are elements of the subspace $\mathscr{N}\left(\lambda_i I - A\right)$; this is called the eigenspace corresponding to λ_i.