Problem 1:

(a) Assuming that the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ where $\omega = \frac{\pi}{T}$ spans the space of all square integrable functions $f(t)$ defined on the domain $t \in [-T, T)$, show that the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ also forms a basis for any $f(t) \in L^2([-T, T))$.

Solution (2 points): A set $B = \{b_1, b_2, \cdots b_n\}$ is called a basis of the space V if $\{b_1, b_2, \cdots b_n\}$ *spans* V; and $\{b_1, b_2, \cdots b_n\}$ is a *linearly independent* set.

The question already assumes that the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ spans $L^2([-T, T))$. Hence we only need to show that the set is linearly independent. The set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ is linearly independent if for constants $\{\alpha_0, \alpha_l, \beta_l\}_{l=1}^{\infty}$, $\alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t) = 0$ implies $\alpha_0 = \alpha_l = \beta_l = 0 \ \forall l$

We show this by noting that each element of the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ is orthogonal to every other element over the domain $t \in [-T, T)$ for $\omega = \frac{\pi}{T}$. For any integers m, n:

- $\int_{-T}^{T} 1 \cdot \cos\left(m\frac{\pi}{T}t\right) dt = 0$
- $\int_{-T}^{T} 1 \cdot \sin\left(m\frac{\pi}{T}t\right) dt = 0$
- $\int_{-T}^{T} \cos\left(m\frac{\pi}{T}t\right) \sin\left(n\frac{\pi}{T}t\right) dt = 0$
- for $m \neq n$, $\int_{-T}^{T} \cos\left(m\frac{\pi}{T}t\right) \cos\left(n\frac{\pi}{T}t\right) dt = 0$
- for $m \neq n$, $\int_{-T}^{T} \sin\left(m\frac{\pi}{T}t\right) \sin\left(n\frac{\pi}{T}t\right) dt = 0$

With the above results, given the constants $\{\alpha_0, \alpha_l, \beta_l\}_{l=1}^{\infty}$,
$f(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos\left(k\frac{\pi}{T}t\right) + \beta_k \sin\left(k\frac{\pi}{T}t\right) = 0$ implies that $\int_{-T}^{T} f(t) dt = 2L\alpha_0 = 0$, $\int_{-T}^{T} f(t) \cos\left(l\frac{\pi}{T}t\right) dt = T\alpha_l = 0$, and $\int_{-T}^{T} f(t) \sin\left(l\frac{\pi}{T}t\right) dt = T\beta_l = 0$. Thus the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ is linearly independent.

(b) Show that the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

can be written as the complex series expansion

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}.$$

where c_k's for $k \in \mathbb{Z}$ are constants that may be expressed in terms of a_k, b_k for $k = 0, \dots, \infty$.

Solution (1.5 points): From Euler's formula,

$$\begin{cases} \mathrm{e}^{i\theta} = \cos\theta + i\sin\theta \\ \mathrm{e}^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \qquad \begin{cases} \cos\theta = \frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2} \\ \sin\theta = \frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i} = -i\frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2} \end{cases},$$

we have that

$$\begin{aligned} a_k\cos kx + b_k\sin kx &= a_k\frac{\mathrm{e}^{ikx}+\mathrm{e}^{-ikx}}{2} - ib_k\frac{\mathrm{e}^{ikx}-\mathrm{e}^{-ikx}}{2} \\ &= \frac{a_k - ib_k}{2}\mathrm{e}^{ikx} + \frac{a_k + ib_k}{2}\mathrm{e}^{-ikx} \\ &= c_k\mathrm{e}^{ikx} + c_{-k}\mathrm{e}^{-ikx} \end{aligned}$$

where $c_k := \dfrac{a_k - ib_k}{2}$ and $c_{-k} := \dfrac{a_k + ib_k}{2}$; in addition, $c_0 := \dfrac{a_0}{2}$. Hence,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + b_k\sin kx = c_0 + \sum_{k=1}^{\infty} c_k\mathrm{e}^{ikx} + \sum_{k=1}^{\infty} c_{-k}\mathrm{e}^{-ikx} = \sum_{k\in\mathbb{Z}} c_k e^{ikx}.$$

(c) Show that the Discrete Fourier Transform $\hat{\mathbf{f}}$ of a signal vector $\mathbf{f}$ preserves its energy, that is, $\|\hat{\mathbf{f}}\|^2 = \|\mathbf{f}\|^2$. [1]

Solution (1.5 points): Recall that for a signal vector $\mathbf{f} \in \mathbb{R}^N$, the DFT $\hat{\mathbf{f}} \in \mathbb{R}^N$ can be written as:

$$\hat{\mathbf{f}} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^1 & \omega_N^2 & & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix}}_{=\Omega} \mathbf{f}$$

where $\omega_N := e^{-i\frac{2\pi}{N}}$.

Now $\|\hat{\mathbf{f}}\|^2 = \hat{\mathbf{f}}^H\hat{\mathbf{f}} = \mathbf{f}^H\Omega^H\Omega\mathbf{f} = \mathbf{f}^H\mathbf{f} = \|\mathbf{f}\|^2$ (since $\Omega^H\Omega = I$; let $W = \Omega^H\Omega$, you can show this by noting that $W_{m,n} = \sum_{k=0}^{N-1}\omega_N^{(n-m)k} = \delta_{m,n}$ where $\delta_{m,n}$ is the kronecker delta function). This is known as Parseval's theorem.

[1] It is also okay to show that $\|\hat{\mathbf{f}}\|^2 = C\|\mathbf{f}\|^2$ where C is some constant.

Problem 2:

(a) Active noise cancellation (used in headphones) typically work by generating an anti-noise signal. The file `hwk1_p2a.mat` (attached with this assignment) is a noisy audio signal. (After downloading the file, load the variables `piano_noisy` and the sample rate `Fs`. Listen to the audio using the MATLAB command `sound(piano_noisy,Fs)`). Generate an anti-noise signal, which when added to the original signal, eliminates the noise. Listen to verify that the anti-noise signal does indeed eliminate the noise. Submit the anti-noise signal.

Solution(2.5 points): Denote the noisy signal `piano_noisy` by $\tilde{\mathbf{f}}$. We assume that $\tilde{\mathbf{f}}$ is the sum of a pure signal $\mathbf{f}$ and noise signal $\mathbf{s}$. Hence,

$$\mathcal{F}\left(\tilde{\mathbf{f}}\right) = \mathcal{F}(\mathbf{f}+\mathbf{s}) = \mathcal{F}(\mathbf{f}) + \mathcal{F}(\mathbf{s})$$

Figure 1 shows the one-sided power spectrum of $\tilde{\mathbf{f}}$ $\left(\mathcal{F}(\tilde{\mathbf{f}}) \cdot \overline{\mathcal{F}(\tilde{\mathbf{f}})}\right)$.The plot suggest two distinct bandwidths: one below $5kHz$ and the other centered between $5kHz$ and $10kHz$. We assume the higher frequency range constitutes noise, that is, in fact, the spectrum above $5kHz$ represents the (high-frequency) noise component added by $\mathcal{F}(\mathbf{s})$. We find the inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\mathbf{s}))$ and the anti-noise signal becomes $-\mathbf{s}$. We verify that this assumption is reasonable by adding $-\mathbf{s}$ to $\tilde{\mathbf{f}}$ and listening to the sound to see if a pure signal $\mathbf{f}$ is played. The audio confirms the noise is gone and, hence, that our assumption is tenable.

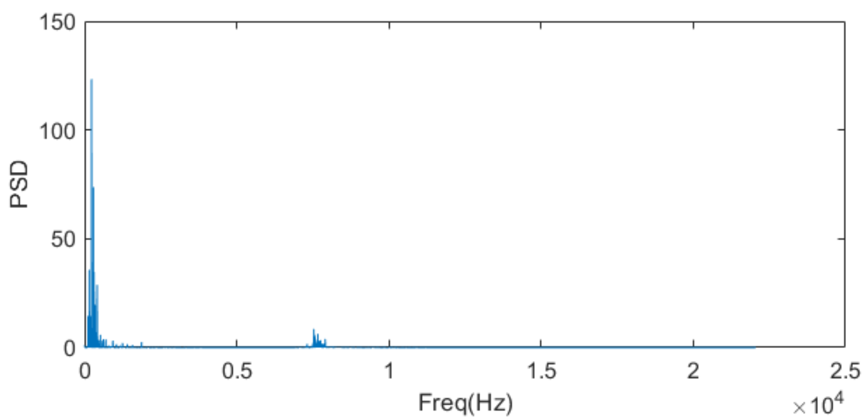

Figure 1: PSD (first half) of given noisy signal.

```
% load data
load('hwk1_p2a.mat');
sound(piano_noisy,Fs);

% Fourier transform the signal and plot PSD
n = length(piano_noisy); %length of signal
fhat = fft(piano_noisy); %FFT of signal
PSD = fhat.*conj(fhat)/n; % Power spectrum (power per freq)
```

```
freq = 1/(n/Fs)*(0:n-1); %frequencies (in Hz, based on sampling rate)
L = 1:floor(n/2); % Only plot the first half of freqs
figure;plot(freq(L),PSD(L)) %plot PSD...
xlabel('Freq(Hz)'); ylabel('PSD'); %...and label

%pick out higher frequencies above str
str = 50000;
fhat_noise = zeros(n,1);
fhat_noise(str+1:end-str) = fhat(str+1:end-str); %This is (assumed to be) the ...
    FFT of the noise signal alone

% Generate, potentially, anti-noise signal and confirm
noise = real(ifft(fhat_noise)); % Inverse FFT for filtered time signal
anti_noise = -noise; %anti_noise signal
sound(piano_noisy+anti_noise,Fs); %Check that noise is eliminated
% ...it is eliminated!
```

(b) 2D convolution of an image with a Gaussian (or Gaussian-like) kernel is often used for blurring images. The variable `Xblurred`, in the file `hwk1_p2b.mat`, is the image of a dog (in floating points; use `imshow(uint8(Xblurred))` to display image) that was blurred using the filter

$$\frac{1}{100}\begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 2 & 4 & 6 & 4 & 2 \\ 4 & 6 & 8 & 6 & 4 \\ 2 & 4 & 6 & 4 & 2 \\ 0 & 2 & 4 & 2 & 0 \end{bmatrix}.$$

Recover the original image. Submit a side-by-side image of the blurred and recovered image. (For example, assuming `X` is the recovered image, in floating points, you may use the command `imshow([uint8(Xblurred),uint8(X)])` to display both images).

Solution(2.5 points): Let $F \in \mathbb{R}^{l\times l}$ be the blurring filter, and suppose that $G \in \mathbb{R}^{m\times m}$ is the original image. The linear 2D convolution is $F * G \in \mathbb{R}^{n\times n}$ where $n = l + m - 1$. This 2D convolution is equivalent to a circular convolution of $\tilde{F} \in \mathbb{R}^{n\times n}$ over $\tilde{G} \in \mathbb{R}^{n\times n}$, that is, $\tilde{F} \circledast \tilde{G}$ where $\tilde{F}$ is F that has been padded with $m-1$ zeros, and $\tilde{G}$ is G that has been padded with $l-1$ zeros. Now according to the Convolutional Theorem,

$$\mathcal{F}(\tilde{F} \circledast \tilde{G}) = \mathcal{F}(\tilde{F}) \cdot \mathcal{F}(\tilde{G})$$

where $\mathcal{F}$ is the DFT. Suppose the given blurred image is $\tilde{F} \circledast \tilde{G}$, the padded original image is therefore $\tilde{G} = \mathcal{F}^{-1}\left(\mathcal{F}(\tilde{F} \circledast \tilde{G})/\mathcal{F}(\tilde{F})\right)$, where '/' denote element-wise division.

```
load ("hwk1_p2b.mat")
b = (1/100)*[0 2 4 2 0 ;
             2 4 6 4 2 ;
             4 6 8 6 4 ;
             2 4 6 4 2 ;
             0 2 4 2 0 ;]; % Filter
nxb = size(Xblurred,1);  nb = size(b,1); % sizes of image and filter

xbhat = fft2(Xblurred); % 2D FFT
bhat = fft2(padarray(b,[(nxb-nb)/2 (nxb-nb)/2],0,'both')); %pad kernel, 2D FFT
```

```

% Get inverse FFT; shift the zero-frequency component to the center
X = fftshift(ifft2(xbhat./bhat));  % padded recovered image
imshow([uint8(Xblurred(3:end-2,3:end-2)),uint8(X(3:end-2,3:end-2))]) %display ...
    blurred and recovered images
```

Figure 2: Blurred (left) and recovered (right) images.