Problem 1:

(a) Assuming that the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ where $\omega = \frac{\pi}{T}$ spans the space of all square integrable functions $f(t)$ defined on the domain $t \in [-T, T)$, show that the set $\{1, \cos k\omega t, \sin k\omega t\}_{k=1}^{\infty}$ also forms a basis for any $f(t) \in L^2([-T, T))$.

(b) Show that the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

can be written as the complex series expansion

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}.$$

where c_k's for $k \in \mathbb{Z}$ are constants that may be expressed in terms of a_k, b_k for $k = 0, \ldots, \infty$.

(c) Show that the Discrete Fourier Transform $\hat{\mathbf{f}}$ of a signal vector $\mathbf{f}$ preserves its energy, that is, $\|\hat{\mathbf{f}}\|^2 = \|\mathbf{f}\|^2$. [1]

Problem 2:

(a) Active noise cancellation (used in headphones) typically work by generating an anti-noise signals. The file `hwk1_p2a.mat` (attached with this assignment) is a noisy audio signal. (After downloading the file, load the variables `piano_noisy` and the sample rate `Fs`. Listen to the audio using the MATLAB command `sound(piano_noisy,Fs)`). Generate an anti-noise signal, which when added to the original signal, eliminates the noise. Listen to verify that the anti-noise signal does indeed eliminate the noise. Submit the anti-noise signal.

(b) 2D convolution of an image with a Gaussian (or Gaussian-like) kernel is often used for blurring images. The variable `Xblurred`, in the file `hwk1_p2b.mat`, is the image of a dog (in floating points; use `imshow(uint8(Xblurred))` to display image) that was blurred using the filter

$$\frac{1}{100}\begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 2 & 4 & 6 & 4 & 2 \\ 4 & 6 & 8 & 6 & 4 \\ 2 & 4 & 6 & 4 & 2 \\ 0 & 2 & 4 & 2 & 0 \end{bmatrix}.$$

Recover the original image. Submit a side-by-side image of the blurred and recovered image. (For example, assuming `X` is the recovered image, in floating points, you may use the command `imshow([uint8(Xblurred),uint8(X)])` to display both images).

[1] It is also okay to show that $\|\hat{\mathbf{f}}\|^2 = C\|\mathbf{f}\|^2$ where C is some constant.