Module 2: Dynamical Systems & Control

Dynamical System Representation

A dynamical system is one in which the variable of interest changes with time. Typically we express them as:

1. Ordinary Differential Equations (ODE's)
2. Partial Differential Equations (PDE's)
3. A System of Differential Equations (DE's)

1. ODE's: The variable of interest only varies with time

Ex:

linear $\dfrac{dx}{dt} = kx$ [Exponential growth (or decay)]

Nonlinear $\dfrac{dx}{dt} = \dfrac{k}{M}(M-x)x$ [Logistic growth]

Linear Non homogeneous $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = g(t)$ [(Damped) Mechanical Vibration]

Nonlinear $\dfrac{d^2x}{dt^2} - \mu(1-x^2)\dfrac{dx}{dt} + x = 0$ [Van der pol oscillator]
Exhibits limit cycle*

$\dfrac{d}{dt}$ is often denoted with overhead dot

*Limit cycle: isolated closed trajectory

(2) PDE's: Variable of interest also changes wrt other variables, typically, space.

Ex:

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x}$$ [Transport equation (in 1D)]

$$\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$ [Heat equation (in 2D)]

$$\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ [Wave equation (in 2D)]

We use ∇ to denote $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$; $\Delta \triangleq \nabla^2$;

Other notations used:

$$\frac{\partial u}{\partial t} \triangleq u_t, \quad \frac{\partial^2 u}{\partial t^2} \triangleq u_{tt}, \text{ etc}$$

(3) System of Equations: Multiple variables of interest that depend on other variables

Nonlinear set of ODE's

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}\right\}$$ [Lorenz System]

PDE's to ODE's

We shall concern ourselves with ODE's because PDE's can often be rendered as ODE's, and we have good control theory for dealing with ODE's

NOTE: Control methods exist for PDE's (such as backstepping methods) but that is beyond the scope of this class

Consider for example, the heat equation governing 1D thermal conduction in a rod.

$$u_t = \alpha^2 u_{xx}$$

We can take the spatial Fourier transform $\mathcal{F}(u(t,x)) = \hat{u}(t,\omega)$

Recall that

$$\mathcal{F}\left[\frac{du}{dx}\right] = \int_{-\infty}^{\infty} \underbrace{\frac{du}{dx}}_{du} \underbrace{e^{-i\omega x}}_{x} dx$$

$$= u e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\left[-i\omega e^{-\omega dx}\right] dx$$

$$= i\omega \int_{-\infty}^{\infty} u e^{-i\omega x} dx$$

$$= i\omega \mathcal{F}(u(x,t)) = i\omega \hat{u}$$

Infact $\mathcal{F}\left[\frac{d^n u}{dx^n}\right] = i^n \omega^n \hat{u}$

Hence, $u_t = \alpha u_{xx}$

$$\Rightarrow \hat{u}_t = -\alpha \omega^2 \hat{u}$$

which is an ODE for each frequency ω

Notice that this is an exponential decay ODE. The solution is

$$\hat{u}(t,\omega) = e^{-\alpha^2\omega^2 t}\, \hat{u}(0,\omega)$$

where $\hat{u}(0,\omega)$ is the Fourier transform of the initial temperature distribution $u(0,x)$.

To get the solution in time, we can take the Inverse FT using the convolution property $\mathcal{F}(f*g) = \mathcal{F}(f)\,\mathcal{F}(g)$

$$u(t,x) = \mathcal{F}^{-1}(\hat{u}(t,\omega)) = \mathcal{F}^{-1}(e^{-\alpha^2\omega^2 t}) * u(0,x)$$

$$= \frac{1}{2\sqrt{\alpha\pi t}} e^{-x^2/4\alpha t} * u(0,x)$$