MODULE 1.1: FOURIER TRANSFORM & ITS APPLICATIONS

INNER PRODUCTS

Consider 2 real finite vectors $\bar{f}, \bar{g} \in \mathbb{R}^n$

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

Their dot product, that is, the projection of one vector on another to get a magnitude is given by

$$\vec{f} \cdot \vec{g} = \vec{f}^{\,\prime} \vec{g} = \sum_{i=1}^{n} f_i g_i \qquad \dots (1)$$

But we can generalize this idea of projecting one vector onto the other to all vector spaces (complex, infinite dimensional), and we refer to this as the inner product. Given a vector space V and a field F ($\mathbb{R}$ or $\mathbb{C}$), the inner product is the mapping $\langle \cdot , \cdot \rangle : V \times V \to F$ such that $\forall\, \alpha, \beta \in F$

and $f, g, h \in V$, the following rules/properties hold

1.) Linearity: $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$

FN : denotes that everything under the line is a footnote

2.) Conjugate Symmetry: $\langle f, g \rangle = \overline{\langle g, f \rangle}$ ← complex conjugation

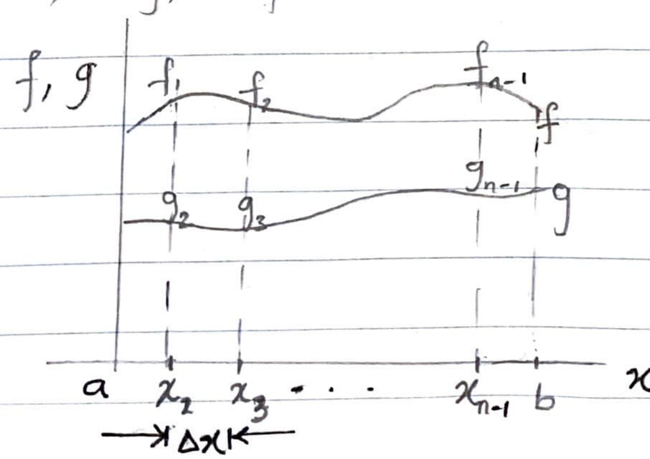
3.) Positive definiteness: $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ iff $f = \vec{0}$

Two interesting points to note here:
First, the inner product induces a norm when we find the inner product of a vector with itself, that is, it turns out that for the vector $f \in V$, the inner product $\langle f, f \rangle$ satisfies the definition of a norm:

$$\langle f, f \rangle = \|f\|^2 \quad \ldots \quad (2)$$

Remark: This is not the 1-norm, ∞-norm or some other p-norm: this is "the norm induced by the inner product". It turns out that this norm is equivalent to the 2-norm (see wikipedia: Polarization Identity for more information)

Second, if f and g are infinite dimensional vectors, or function, say, defined over some interval $[a, b]$



$f, g \in C_{[a,b]}$

We know that

$$\langle (f_1, \cdots f_n), (g_1, \cdots g_n) \rangle = \sum_{i=1}^{n} f(x_i) g(x_i) \cdots (3)$$

We can normalize (3) by noting $\Delta x = \frac{b-a}{n-1}$:

$$\frac{b-a}{n-1} \langle \vec{f}, \vec{g} \rangle = \sum_{i=1}^{n} f(x_i) g(x_i) \Delta x;$$

and taking the limit

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) g(x_i) \Delta x = \int_a^b f(x) g(x) dx$$

assuming the now functions f, g are Reinmann integrable*

$$= \langle f(x), g(x) \rangle$$

If you think about it, we use the inner product $\langle \cdot, \cdot \rangle$ in finite-dimensional vector spaces to find the projection of a vector to some basis**

FN

* The limit exists.

**Recall that a basis is a set of linear independent vectors that span a vector space e.g

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, that is irrespective of the vector $v \in \mathbb{R}^2$ you give me $\exists\, \alpha, \beta$ such that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We can do the same for the inner product of function by projecting them to an orthorgonal (and possibly orthonormal) set of sine and cosine functions with integer periods on the domain $[a, b]$. This is exactly what the Fourier Series is.

NOTE: For complex functions, we can more generally write the inner product as

$$\langle f(x), g(x) \rangle_{[a,b]} = \int_a^b f(x)\,\overline{g}(x)\,dx$$

where $\overline{g}(x)$ is the complex conjugate

FN

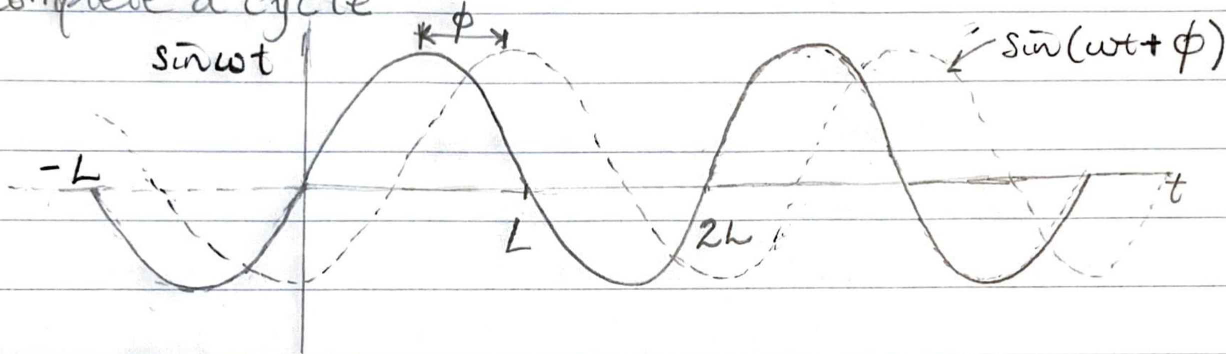
** A basis is orthonormal if each element yields a zero inner product with other element, and has a unit induced norm.
e.g the standard basis for $\mathbb{R}^3$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that ① the dimension of the vector space = # elements in a basis
② The basis for a vector space is not unique

FOURIER SERIES

Assuming I have a sine wave oscillating at angular velocity $\omega = \frac{2\pi}{2L}$, that is, it takes $2L$ units of time to complete a cycle



Assume I have another wave, say $\sin(\omega t + \phi)$, and I want to find what the projection of $\sin(\omega t + \phi)$ is on $\sin \omega t$, that is, the Hilbert space* inner product.

$$\int_{-L}^{L} \sin(\omega t + \phi) \sin \omega t \, dt$$

$$= L \cos \phi .$$

The inner product is a function of ϕ, and for each multiple of $\pi/2$, it is zero; that is, $\sin(\omega t)$ and $\sin(\omega t + \pi/2)$ [or $\cos \omega t$] are orthogonal. Any sinusoid oscillating at ω can be written in terms of $\sin \omega t$ and $\cos \omega t$. (In fact, these can act as a basis: $\{\sin \omega t, \cos \omega t\}$)

How about a sine wave oscillating at higher frequencies?

$$\langle \sin k\omega t, \sin \omega t \rangle = \int_{-L}^{L} \sin k\omega t \sin \omega t \, dt \quad \text{for } k = 0, 2, 3, \ldots, \infty$$

FN

* complete inner product space

$$= \frac{-2L}{\pi(\kappa^2-1)} \sin \kappa\pi = 0 \quad \text{for } \kappa = 0, 2, \dots, \infty$$

This means that $\{\sin \omega t, \sin 2\omega t, \dots, \cos \omega t, \cos 2\omega t, \dots\}$ or

$\{1, \sin \kappa\omega t, \cos \kappa\omega t\}_{\kappa=1}^{\infty}$ forms a set of orthorgonal functions.

But moreso, just like we write vectors in terms of standard, orthorgonal vectors e.g in 3D euclidean space, for $f \in \mathbb{R}^3$

$$\bar{f} = \frac{\langle f, e_x \rangle}{\langle e_x, e_x \rangle} e_x + \frac{\langle f, e_y \rangle}{\langle e_y, e_y \rangle} e_y + \frac{\langle f, e_z \rangle}{\langle e_z, e_z \rangle} e_z,$$

we can write

$$f(t) = \sum_{\kappa=0}^{\infty} \frac{\langle f, \cos \kappa\omega t \rangle}{\|\cos \kappa\omega t\|^2} \cos \kappa\omega t + \sum_{\kappa=1}^{\infty} \frac{\langle f, \sin \kappa\omega t \rangle}{\|\sin \kappa\omega t\|^2} \sin \kappa\omega t$$

for all square integrable functions $L_2([-L, L])$

$$\text{Let } a_\kappa = \frac{\langle f, \cos \kappa\omega t \rangle}{\|\cos \kappa\omega t\|^2} = \frac{1}{\|\cos \kappa\omega t\|^2} \int_{-L}^{L} f(t) \cos \kappa\omega t \, dt$$

$$= \begin{cases} \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(t) \, dt & \text{if } \kappa = 0 \\ \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(t) \cos \kappa\omega t \, dt & \text{else} \end{cases}$$

Similarly, let $b_k = \dfrac{\langle f, \sin k\omega t \rangle}{\| \sin k\omega t \rangle}$

for $k = 1, \ldots, \infty$

$$= \frac{1}{L} \int_{-L}^{L} f(t) \sin k\omega t \quad \text{else}$$

Hence

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega t + b_k \sin k\omega t$$

FOURIER TRANSFORM

Based on Euler's formula, $e^{ik\omega t} = \cos k\omega t + i \sin k\omega t$,

we can write the fourier series as

$$f(t) = \sum_{k=0}^{\infty} c_k e^{ik\pi t/L}$$

where $c_k = \dfrac{1}{2L} \langle f(t), \psi_k \rangle = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(t) e^{-ik\pi t/L}$

where $\psi_k = e^{ik\pi t/L}$

Let us assume $f(t)$ is not just a periodic function defined in, the domain $[-L, L)$, and then repeating itself outside the domain Instead let $L \to \infty$, and thus π/L which we now

denote with $\Delta\omega$ tend to 0 $\left(\Delta\omega = \pi/L \to 0 \right)$

$$f(t) = \lim_{\Delta\omega \to 0} \sum \left[\frac{\Delta\omega}{2\pi} \underbrace{\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-ik\Delta\omega\xi} \, d\xi}_{\langle f(t), \psi_k(t) \rangle} \right] e^{k\Delta\omega t}$$

In the limit, $\langle f(t), \psi_k(t) \rangle$ becomes the Fourier transform,

and the entire summation can be written as the Riemann integral:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \triangleq \mathcal{F}(f(t))$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \triangleq \mathcal{F}^{-1}(\hat{f}(\omega))$$

These are regarded as the Fourier transform pair