Course: MECH 567: Robot Kinematics and Dynamics

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Solution 5

Problem 1

Define the positive x-axis pointing towards right, positive y-axis pointing upwards, and positive z-axis pointing out of the page. Euler-Lagrange equations can be found using these two equivalent formula below and should give the exactly same result if the calculation is correct:

where, $\mathcal{L}$ is the Lagrangian function, q_k and $\dot{q_k}$ are the dynamic system states for each joint.

$$D(q)\ddot{q} + C(q, \dot{q})\,\dot{q} + g(p) = \tau$$

Where, $D(q)$ is the manipulator mass/inertia matrix, $C(q, \dot{q})$ is the Coriolis matrix, and $g(q)$ is the conservative/potential forces vector. You can use both methods to verify your results for E-L equations but it's optional regarding the grading.

Solution V1

In[1183]:=
```
(**********************(i) two-link RR manipulator***********************)
ClearAll["Global`*"];
Needs["Screws`", "C:\\Mathematica\\Screws.m"]
Needs["RobotLinks`", "C:\\Mathematica\\RobotLinks.m"]
a = 1;
b = 0.2;
c = 0;
m = 2;
Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {a / 2, 0, 0, 1}];
MatrixForm[P20 = {a + a / 2, 0, 0, 1}];
w1 = {0, 0, 1};
w2 = {0, 0, 1};
q1 = {0, 0, 0};
q2 = {a, 0, 0};

xi1 = Flatten[Append[-Cross[w1, q1], w1]];
xi2 = Flatten[Append[-Cross[w2, q2], w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1193]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1194]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1205]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,\text{Cos}[\text{th1}[t]] \\ \frac{1}{2}\,\text{Sin}[\text{th1}[t]] \\ 0 \\ 1 \end{pmatrix}$$

Out[1206]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\text{th1}[t]] + \frac{1}{2}\,\text{Cos}[\text{th1}[t] + \text{th2}[t]] \\ \text{Sin}[\text{th1}[t]] + \frac{1}{2}\,\text{Sin}[\text{th1}[t] + \text{th2}[t]] \\ 0 \\ 1 \end{pmatrix}$$

In[1207]:=

```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a + a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1214]//MatrixForm=

$$\begin{pmatrix} 3.34667 + 2\,\text{Cos}[\text{th2}[t]] & 0.673333 + \text{Cos}[\text{th2}[t]] \\ 0.673333 + \text{Cos}[\text{th2}[t]] & 0.673333 \end{pmatrix}$$

In[1215]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1215]= $\{(1.67333 + \mathrm{Cos}[\mathrm{th2}[t]])\, \mathrm{th1}'[t]^2 + (0.673333 + \mathrm{Cos}[\mathrm{th2}[t]])\, \mathrm{th1}'[t]\, \mathrm{th2}'[t] + 0.336667\, \mathrm{th2}'[t]^2\}$

Out[1216]= $2.g.\left(\frac{1}{2}\mathrm{Sin}[\mathrm{th1}[t]]\right) + 2.g.\left(\mathrm{Sin}[\mathrm{th1}[t]] + \frac{1}{2}\mathrm{Sin}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right)$

Out[1217]= $\{-2.g.\left(\frac{1}{2}\mathrm{Sin}[\mathrm{th1}[t]]\right) - 2.g.\left(\mathrm{Sin}[\mathrm{th1}[t]] + \frac{1}{2}\mathrm{Sin}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) + (1.67333 + \mathrm{Cos}[\mathrm{th2}[t]])\, \mathrm{th1}'[t]^2 + (0.673333 + \mathrm{Cos}[\mathrm{th2}[t]])\, \mathrm{th1}'[t]\, \mathrm{th2}'[t] + 0.336667\, \mathrm{th2}'[t]^2\}$

Out[1226]//MatrixForm=

$$\left(\begin{array}{l} 2.g.\left(\frac{1}{2}\mathrm{Cos}[\mathrm{th1}[t]]\right) + 2.g.\left(\mathrm{Cos}[\mathrm{th1}[t]] + \frac{1}{2}\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) - 2\,\mathrm{Sin}[\mathrm{th2}[t]]\, \mathrm{th1}'[t]\, \mathrm{th2}'[t] - \mathrm{Sin}[\mathrm{th2}[t]]\, \mathrm{th2}'[t]^2 + 2\,(1.67333 \\ 2.g.\left(\frac{1}{2}\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) + \mathrm{Sin}[\mathrm{th2}[t]]\, \mathrm{th1}'[t]^2 + (0.673333 + \mathrm{Cos}[\mathrm{th2}[t]])\, \mathrm{th1}''[t] + 0 \end{array}\right.$$

In[1227]:=
```
(*Coriolic matrix*)
(*C11=1/2. D[Mth⟦1,1⟧,th1[t]].th1'[t]+1/2. D[Mth⟦1,1⟧,th2[t]].th2'[t];
C12=1/2. (D[Mth⟦1,2⟧,th1[t]]+D[Mth⟦1,1⟧,th2[t]]-D[Mth⟦1,2⟧,th1[t]]).th1'[t]+
  1/2. (D[Mth⟦1,2⟧,th2[t]]+D[Mth⟦1,2⟧,th2[t]]-D[Mth⟦2,2⟧,th1[t]]).th2'[t];
C21=1/2. (D[Mth⟦2,1⟧,th1[t]]+D[Mth⟦2,1⟧,th1[t]]-D[Mth⟦1,1⟧,th2[t]]).th1'[t]+
  1/2. (D[Mth⟦2,1⟧,th2[t]]+D[Mth⟦2,2⟧,th1[t]]-D[Mth⟦2,1⟧,th2[t]]).th2'[t];
C22=1/2. D[Mth⟦2,2⟧,th1[t]].th1'[t];*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1231]= {{Null}, {Null}}

Out[1232]//MatrixForm=

$$\begin{pmatrix} -1.\,\mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th2}'[t] & -1.\,\mathrm{Sin}[\mathrm{th2}[t]]\,(\mathrm{th1}'[t]+\mathrm{th2}'[t]) \\ \mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th1}'[t] & 0 \end{pmatrix}$$

In[1233]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1236]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\text{Cos}[\text{th1}[t]]\right) + 2.g.\left(\text{Cos}[\text{th1}[t]] + \frac{1}{2}\text{Cos}[\text{th1}[t] + \text{th2}[t]]\right) \\ 2.g.\left(\frac{1}{2}\text{Cos}[\text{th1}[t] + \text{th2}[t]]\right) \end{pmatrix}$$

Out[1237]//MatrixForm=

$$\begin{pmatrix} 0. + 0.\,i & \text{Sin}[\text{th2}[t]]\,(2.\,\text{th1}'[t] + \text{th2}'[t]) \\ -2.\,\text{Sin}[\text{th2}[t]]\,(\text{th1}'[t] + 0.5\,\text{th2}'[t]) & 0. \end{pmatrix}$$

Out[1238]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\text{Cos}[\text{th1}[t]]\right) + 2.g.\left(\text{Cos}[\text{th1}[t]] + \frac{1}{2}\text{Cos}[\text{th1}[t] + \text{th2}[t]]\right) - 2.\,\text{Sin}[\text{th2}[t]]\,\text{th1}'[t]\,\text{th2}'[t] - 1.\,\text{Sin}[\text{th2}[t]]\,\text{th2}'[t]^2 + 3.34667 \\ 2.g.\left(\frac{1}{2}\text{Cos}[\text{th1}[t] + \text{th2}[t]]\right) + \text{Sin}[\text{th2}[t]]\,\text{th1}'[t]^2 + 0.673333\,\text{th1}''[t] + \text{Cos}[\text{th2}[t]]\,\text{t} \end{pmatrix}$$

In[1239]:=
```
(**********************(ii) two-link PP manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {0, 0, 0, 1}];
MatrixForm[P20 = {a / 2, 0, 0, 1}];
w1 = {0, 0, 0};
w2 = {0, 0, 0};
q1 = {0, 0, 0};
q2 = {0, 0, 0};

xi1 = Flatten[Append[{1, 0, 0}, w1]];
xi2 = Flatten[Append[{0, 1, 0}, w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1249]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1250]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1261]//MatrixForm=

$$\begin{pmatrix} \text{th1}[t] \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Out[1262]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} + \text{th1}[t] \\ \text{th2}[t] \\ 0 \\ 1 \end{pmatrix}$$

In[1263]:=

```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a / 2}, {0 , 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1270]//MatrixForm=

$$\begin{pmatrix} 4. & 0. \\ 0. & 2. \end{pmatrix}$$

In[1271]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1271]= $\{0. + 2.\,\text{th1}'[t]^2 + \text{th2}'[t]^2\}$

Out[1272]= $2.g.0 + 2.g.\text{th2}[t]$

Out[1273]= $\{0. - 2.g.0 - 2.g.\text{th2}[t] + 2.\,\text{th1}'[t]^2 + \text{th2}'[t]^2\}$

Out[1282]//MatrixForm=

$$\begin{pmatrix} 4.\,\text{th1}''[t] \\ 2.g.1 + 2\,\text{th2}''[t] \end{pmatrix}$$

In[1283]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1287]= {{Null}, {Null}}

Out[1288]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In[1289]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1292]//MatrixForm=

$$\begin{pmatrix} 0 \\ 2.g.1 \end{pmatrix}$$

Out[1293]//MatrixForm=

$$\begin{pmatrix} 0. & 0. \\ 0. & 0. \end{pmatrix}$$

Out[1294]//MatrixForm=

$$\begin{pmatrix} 0. + 4.\,th1''[t] \\ 0. + 2.g.1 + 2.\,th2''[t] \end{pmatrix}$$

In[1295]:=
```
(***********************(iii) two-link PR manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {0, 0, 0, 1}];
MatrixForm[P20 = {a, 0, 0, 1}];
w1 = {0, 0, 0};
w2 = {0, 0, 1};
q1 = {0, 0, 0};
q2 = {a / 2, 0, 0};

xi1 = Flatten[Append[{1, 0, 0}, w1]];
xi2 = Flatten[Append[-Cross[w2, q2], w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1305]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1306]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1317]//MatrixForm=

$$\begin{pmatrix} \text{th1}[t] \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Out[1318]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(1 + \text{Cos}[\text{th2}[t]] + 2\,\text{th1}[t]\right) \\ \frac{1}{2}\,\text{Sin}[\text{th2}[t]] \\ 0 \\ 1 \end{pmatrix}$$

In[1319]:=
```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
xi2b = xi2;(*Body velocity*)
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2b, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1327]//MatrixForm=

$$\begin{pmatrix} 4. & -1.\,\text{Sin}[\text{th2}[t]] \\ -1.\,\text{Sin}[\text{th2}[t]] & 0.673333 \end{pmatrix}$$

```
In[1328]:= K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1328]= $\{2\, \text{th1}'[t]^2 - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th1}'[t]\, \text{th2}'[t] + 0.336667\, \text{th2}'[t]^2\}$

Out[1329]= $2.g.0 + 2.g.\left(\frac{1}{2}\, \text{Sin}[\text{th2}[t]]\right)$

Out[1330]= $\left\{-1.\, \left(2.g.0 + 2.g.\left(\frac{1}{2}\, \text{Sin}[\text{th2}[t]]\right) - 2.\, \text{th1}'[t]^2 + \text{Sin}[\text{th2}[t]]\, \text{th1}'[t]\, \text{th2}'[t] - 0.336667\, \text{th2}'[t]^2\right)\right\}$

Out[1339]//MatrixForm=

$$\begin{pmatrix} -1.\, \text{Cos}[\text{th2}[t]]\, \text{th2}'[t]^2 + 4\, \text{th1}''[t] - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th2}''[t] \\ 0. + 2.g.\left(\frac{1}{2}\, \text{Cos}[\text{th2}[t]]\right) - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th1}''[t] + 0.673333\, \text{th2}''[t] \end{pmatrix}$$

In[1340]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1344]= {{Null}, {Null}}

Out[1345]//MatrixForm=

$$\begin{pmatrix} 0. & -1.\,\mathrm{Cos}[\mathrm{th2}[t]]\,\mathrm{th2}'[t] \\ 0. & 0. \end{pmatrix}$$

In[1346]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1349]//MatrixForm=

$$\begin{pmatrix} 0 \\ 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th2}[t]]\right) \end{pmatrix}$$

Out[1350]//MatrixForm=

$$\begin{pmatrix} 0. & \mathrm{Cos}[\mathrm{th2}[t]]\,\mathrm{th2}'[t] \\ -1.\,\mathrm{Cos}[\mathrm{th2}[t]]\,\mathrm{th2}'[t] & 0. \end{pmatrix}$$

Out[1351]//MatrixForm=

$$\begin{pmatrix} -1.\,\left(\mathrm{Cos}[\mathrm{th2}[t]]\,\mathrm{th2}'[t]^2 - 4.\,\mathrm{th1}''[t] + \mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th2}''[t]\right) \\ 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th2}[t]]\right) - 1.\,\mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th1}''[t] + 0.673333\,\mathrm{th2}''[t] \end{pmatrix}$$

In[1352]:=
```
(**********************(iv) two-link RP manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

P10 = {a / 2, 0, 0, 1};
P20 = {a, 0, 0, 1};
w1 = {0, 0, 1};
w2 = {0, 0, 0};
q1 = {0, 0, 0};
q2 = {0, 0, 0};

xi1 = Flatten[Append[-Cross[w1, q1], w1]];
xi2 = Flatten[Append[{0, -1, 0}, w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1362]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1363]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1374]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,\text{Cos}[\text{th1}[t]] \\ \frac{1}{2}\,\text{Sin}[\text{th1}[t]] \\ 0 \\ 1 \end{pmatrix}$$

Out[1375]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\text{th1}[t]] + \text{Sin}[\text{th1}[t]]\ \text{th2}[t] \\ \text{Sin}[\text{th1}[t]] - \text{Cos}[\text{th1}[t]]\ \text{th2}[t] \\ 0 \\ 1 \end{pmatrix}$$

In[1376]:=
```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
(*Mth=Transpose[Jb1].M.Jb1+Transpose[Jb2].M.Jb2;*)
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify
```

Out[1383]//MatrixForm=

$$\begin{pmatrix} 2.84667 + 2.\ \text{th2}[t]^2 & -2. \\ -2. & 2. \end{pmatrix}$$

In[1384]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1384]= $\left\{\left(1.42333+\text{th2}[t]^2\right)\text{th1}'[t]^2-2.\,\text{th1}'[t]\,\text{th2}'[t]+\text{th2}'[t]^2\right\}$

Out[1385]= $2.g.\left(\frac{1}{2}\text{Sin}[\text{th1}[t]]\right)+2.g.(\text{Sin}[\text{th1}[t]]-\text{Cos}[\text{th1}[t]]\,\text{th2}[t])$

Out[1386]= $\left\{-2.g.\left(\frac{1}{2}\text{Sin}[\text{th1}[t]]\right)-2.g.(\text{Sin}[\text{th1}[t]]-\text{Cos}[\text{th1}[t]]\,\text{th2}[t])+\left(1.42333+\text{th2}[t]^2\right)\text{th1}'[t]^2-2.\,\text{th1}'[t]\,\text{th2}'[t]+\text{th2}'[t]^2\right\}$

Out[1395]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\text{Cos}[\text{th1}[t]]\right)+2.g.(\text{Cos}[\text{th1}[t]]+\text{Sin}[\text{th1}[t]]\,\text{th2}[t])+4\,\text{th2}[t]\,\text{th1}'[t]\,\text{th2}'[t]+2\left(1.42333+\text{th2}[t]^2\right)\text{th1}''[t]-2.\,\text{th2}''[t] \\ 2.g.(-\text{Cos}[\text{th1}[t]])-2\,\text{th2}[t]\,\text{th1}'[t]^2-2.\,\text{th1}''[t]+2\,\text{th2}''[t] \end{pmatrix}$$

In[1396]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1400]= {{Null}, {Null}}

Out[1401]//MatrixForm=

$$\begin{pmatrix} 0. + 2.\ \mathrm{th2}[t]\ \mathrm{th2}'[t] & 0. + 2.\ \mathrm{th2}[t]\ \mathrm{th1}'[t] \\ 0. - 2.\ \mathrm{th2}[t]\ \mathrm{th1}'[t] & 0 \end{pmatrix}$$

In[1402]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1405]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\mathrm{Cos}[\mathrm{th1}[t]]\right) + 2.g.(\mathrm{Cos}[\mathrm{th1}[t]] + \mathrm{Sin}[\mathrm{th1}[t]]\ \mathrm{th2}[t]) \\ 2.g.(-\mathrm{Cos}[\mathrm{th1}[t]]) \end{pmatrix}$$

Out[1406]//MatrixForm=

$$\begin{pmatrix} 0. & 0. - 4.\ \mathrm{th2}[t]\ \mathrm{th1}'[t] \\ 0. + 4.\ \mathrm{th2}[t]\ \mathrm{th1}'[t] & 0. \end{pmatrix}$$

Out[1407]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\mathrm{Cos}[\mathrm{th1}[t]]\right) + 2.g.(\mathrm{Cos}[\mathrm{th1}[t]] + \mathrm{Sin}[\mathrm{th1}[t]]\ \mathrm{th2}[t]) + 4.\ \mathrm{th2}[t]\ \mathrm{th1}'[t]\ \mathrm{th2}'[t] + 2.\ \left(1.42333 + \mathrm{th2}[t]^2\right)\mathrm{th1}''[t] - 2.\ \mathrm{th2}''[t \\ 2.g.(-\mathrm{Cos}[\mathrm{th1}[t]]) - 2.\ \left(\mathrm{th2}[t]\ \mathrm{th1}'[t]^2 + \mathrm{th1}''[t] - 1.\ \mathrm{th2}''[t]\right) \end{pmatrix}$$

Solution V2

The given solutions assume that the a=1 dimension is always in the x-direction. The following solutions assume that the a=1 dimension is always along the long axis of the link as shown in the homework. This will end up changing the inertia tensors, which trickles down.

```
In[757]:= ClearAll["Global`*"]
Needs["Screws`", "C:\\Mathematica\\Screws.m"]
Needs["RobotLinks`", "C:\\Mathematica\\RobotLinks.m"]
Needs["VariationalMethods`"]
```

Problem 1: Two-link RR manipulator

```
In[761]:= m = 2;
a = 1;
b = 0.2;
c = 0;

θ = {θ1[t], θ2[t]};
θdot = D[θ, t];

xi1 = {0, 0, 0, 0, 0, 1};
xi2 = {0, -a, 0, 0, 0, 1};

gst0b1 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst0b2 = {{1, 0, 0, a + a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

e1 = TwistExp[xi1, θ[[1]]];
e2 = TwistExp[xi2, θ[[2]]];

gstb1 = TwistExp[xi1, θ[[1]]].gst0b1;
gstb2 = TwistExp[xi1, θ[[1]]].TwistExp[xi2, θ[[2]]].gst0b2 // FullSimplify;

Labeled[MatrixForm[COM1 = RigidPosition[gstb1]], "COM1"]
Labeled[MatrixForm[COM2 = RigidPosition[gstb2]], "COM2"]
Labeled[
 MatrixForm[I1 = {{m / 12 * (b^2 + c^2), 0, 0}, {0, m / 12 * (a^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor]
I2 = I1;

GenM1 = {{m * IdentityMatrix[3], 0}, {0, I1}} // ArrayFlatten;
```

```
GenM2 = {{m * IdentityMatrix[3], 0}, {0, I2}} // ArrayFlatten;

xi1cross = Inverse[RigidAdjoint[e1.gst0b1]].xi1 // Simplify;
Jbsl1 = {xi1cross, {0, 0, 0, 0, 0, 0}} // Transpose;

xi1cross = Inverse[RigidAdjoint[e1.e2.gst0b2]].xi1 // Simplify;
xi2cross = Inverse[RigidAdjoint[e2.gst0b2]].xi2 // Simplify;
Jbsl2 = {xi1cross, xi2cross} // Transpose;

M = Transpose[Jbsl1].GenM1.Jbsl1 + Transpose[Jbsl2].GenM2.Jbsl2 // Simplify;

P = m * g * COM1[[2]] + m * g * COM2[[2]];
G = {0, 0};
G[[1]] = D[P, θ[[1]]];
G[[2]] = D[P, θ[[2]]];
christoffel = ConstantArray[0, {2, 2, 2}];

For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  For[k = 1, k ≤ 2, k++,
   christoffel[[i, j, k]] = 1 / 2 * (D[M[[i, j]], θ[[k]]] + D[M[[i, k]], θ[[j]]] - D[M[[k, j]], θ[[i]]]);
  ]
 ]
]

MatrixForm[christoffel];

Coriolis = ConstantArray[0, {2, 2}];
For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  Coriolis[[i, j]] = christoffel[[i, j, All]].θdot;
 ]
]
```

```
K = 1 / 2 * Transpose[θdot].M.θdot;
L = K - P // FullSimplify;

EL = {0, 0};
For [i = 1, i ≤ 2, i++,
 EL⟦i⟧ = D[D[L, θdot⟦i⟧], t] - D[L, θ⟦i⟧];
]

Labeled[MatrixForm[EL // FullSimplify], "Euler-Lagrange Equations"]

Labeled[MatrixForm[M], "M(θ)"]
Labeled[MatrixForm[Coriolis // FullSimplify], "C(θ,θ̇)"]
MatrixForm[InertiaToCoriolis[M, θ, θdot] // FullSimplify];
Labeled[MatrixForm[G // FullSimplify], "G(θ)"]

Labeled[MatrixForm[D[M, t] - 2 * Coriolis // Simplify], "(Ṁ - 2C)"]
MatrixForm[D[M, t] - 2 * Coriolis // Simplify] == MatrixForm[-1 * Transpose[D[M, t] - 2 * Coriolis // Simplify]]
```

Out[775]=
$$\begin{pmatrix} \frac{1}{2}\,\text{Cos}[\theta 1[t]] \\ \frac{1}{2}\,\text{Sin}[\theta 1[t]] \\ 0 \end{pmatrix}$$
COM1

Out[776]=
$$\begin{pmatrix} \text{Cos}[\theta 1[t]] + \frac{1}{2}\,\text{Cos}[\theta 1[t] + \theta 2[t]] \\ \text{Sin}[\theta 1[t]] + \frac{1}{2}\,\text{Sin}[\theta 1[t] + \theta 2[t]] \\ 0 \end{pmatrix}$$
COM2

Out[777]=
$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$
Inertia Tensor

Out[800]=

$$\begin{pmatrix} g\,(3.\,\text{Cos}[\theta 1[t]] + \text{Cos}[\theta 1[t] + \theta 2[t]]) - \text{Sin}[\theta 2[t]]\,\theta 2'[t]\,(2\,\theta 1'[t] + \theta 2'[t]) + 2.\,(1.673 + \text{Cos}[\theta 2[t]])\,\theta 1''[t] + (0.673 + \text{Cos}[\theta 2[t]])\theta 2''[t] \\ g\,\text{Cos}[\theta 1[t] + \theta 2[t]] + \text{Sin}[\theta 2[t]]\,\theta 1'[t]^2 + (0.673333 + \text{Cos}[\theta 2[t]])\,\theta 1''[t] + 0.673333\,\theta 2''[t] \end{pmatrix}$$

Euler−Lagrange Equations

Out[801]=

$$\begin{pmatrix} 2.\,(1.67333 + \text{Cos}[\theta 2[t]]) & 0.673333 + \text{Cos}[\theta 2[t]] \\ 0.673333 + \text{Cos}[\theta 2[t]] & 0.673333 \end{pmatrix}$$

$M(\theta)$

Out[802]=

$$\begin{pmatrix} -1.\,\text{Sin}[\theta 2[t]]\,\theta 2'[t] & -1.\,\text{Sin}[\theta 2[t]]\,(\theta 1'[t] + \theta 2'[t]) \\ \text{Sin}[\theta 2[t]]\,\theta 1'[t] & 0 \end{pmatrix}$$

$C(\theta,\dot{\theta})$

Out[804]=

$$\begin{pmatrix} g\,(3\,\text{Cos}[\theta 1[t]] + \text{Cos}[\theta 1[t] + \theta 2[t]]) \\ g\,\text{Cos}[\theta 1[t] + \theta 2[t]] \end{pmatrix}$$

$G(\theta)$

Out[805]=

$$\begin{pmatrix} 0. & \text{Sin}[\theta 2[t]]\,(2.\,\theta 1'[t] + \theta 2'[t]) \\ -\text{Sin}[\theta 2[t]]\,(2\,\theta 1'[t] + \theta 2'[t]) & 0 \end{pmatrix}$$

$(\dot{M} - 2C)$

Out[806]= True

Problem 2: Two-link PP manipulator

In[807]:=

```
m = 2;
a = 1;
b = 0.2;
c = 0;

θ = {θ1[t], θ2[t]};
θdot = D[θ, t];

xi1 = {1, 0, 0, 0, 0, 0};
xi2 = {0, 1, 0, 0, 0, 0};

gst0b1 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
```

```
gst0b2 = {{0, -1, 0, a / 2}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

e1 = TwistExp[xi1, θ[[1]]];
e2 = TwistExp[xi2, θ[[2]]];

gstb1 = TwistExp[xi1, θ[[1]]].gst0b1;
gstb2 = TwistExp[xi1, θ[[1]]].TwistExp[xi2, θ[[2]]].gst0b2 // FullSimplify;

Labeled[MatrixForm[COM1 = RigidPosition[gstb1]], "COM1"]
Labeled[MatrixForm[COM2 = RigidPosition[gstb2]], "COM2"]
Labeled[
 MatrixForm[I1 = {{m / 12 * (b^2 + c^2), 0, 0}, {0, m / 12 * (a^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor1]
Labeled[
 MatrixForm[I2 = {{m / 12 * (a^2 + c^2), 0, 0}, {0, m / 12 * (b^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor2]

GenM1 = {{m * IdentityMatrix[3], 0}, {0, I1}} // ArrayFlatten;
GenM2 = {{m * IdentityMatrix[3], 0}, {0, I2}} // ArrayFlatten;

xi1cross = Inverse[RigidAdjoint[e1.gst0b1]].xi1 // Simplify;
Jbsl1 = {xi1cross, {0, 0, 0, 0, 0, 0}} // Transpose;

xi1cross = Inverse[RigidAdjoint[e1.e2.gst0b2]].xi1 // Simplify;
xi2cross = Inverse[RigidAdjoint[e2.gst0b2]].xi2 // Simplify;
Jbsl2 = {xi1cross, xi2cross} // Transpose;

M = Transpose[Jbsl1].GenM1.Jbsl1 + Transpose[Jbsl2].GenM2.Jbsl2 // Simplify;

P = m * g * COM1[[2]] + m * g * COM2[[2]];
G = {0, 0};
G[[1]] = D[P, θ[[1]]];
G[[2]] = D[P, θ[[2]]];
christoffel = ConstantArray[0, {2, 2, 2}];

For[i = 1, i ≤ 2, i++,
```

```
 For [j = 1, j ≤ 2, j++,
  For[k = 1, k ≤ 2, k++,
   christoffel⟦i, j, k⟧ = 1 / 2 * (D[M⟦i, j⟧, θ⟦k⟧] + D[M⟦i, k⟧, θ⟦j⟧] - D[M⟦k, j⟧, θ⟦i⟧]);
  ]
 ]
]

MatrixForm[christoffel];

Coriolis = ConstantArray[0, {2, 2}];
For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  Coriolis⟦i, j⟧ = christoffel⟦i, j, All⟧.θdot;
 ]
]

K = 1 / 2 * Transpose[θdot].M.θdot;
L = K - P // FullSimplify;

EL = {0, 0};
For [i = 1, i ≤ 2, i++,
 EL⟦i⟧ = D[D[L, θdot⟦i⟧], t] - D[L, θ⟦i⟧];
]

Labeled[MatrixForm[EL // FullSimplify], "Euler-Lagrange Equations"]

Labeled[MatrixForm[M], "M(θ)"]
Labeled[MatrixForm[Coriolis // FullSimplify], "C(θ,θ̇)"]
MatrixForm[InertiaToCoriolis[M, θ, θdot] // FullSimplify];
Labeled[MatrixForm[G // FullSimplify], "G(θ)"]

Labeled[MatrixForm[D[M, t] - 2 * Coriolis // Simplify], "(Ṁ - 2C)"]
MatrixForm[D[M, t] - 2 * Coriolis // Simplify] == MatrixForm[-1 * Transpose[D[M, t] - 2 * Coriolis // Simplify]]
```

Out[821]= $\begin{pmatrix} \theta 1[t] \\ 0 \\ 0 \end{pmatrix}$

COM1

Out[822]= $\begin{pmatrix} \frac{1}{2} + \theta 1[t] \\ \theta 2[t] \\ 0 \end{pmatrix}$

COM2

Out[823]= $\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$

Inertia Tensor1

Out[824]= $\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & 0.00666667 & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$

Inertia Tensor2

Out[846]= $\begin{pmatrix} 4.\ \theta 1''[t] \\ 2\,(g + \theta 2''[t]) \end{pmatrix}$

Euler−Lagrange Equations

Out[847]= $\begin{pmatrix} 4. & 0. \\ 0. & 2. \end{pmatrix}$

$M(\theta)$

Out[848]= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$C(\theta,\dot{\theta})$

Out[850]= $\begin{pmatrix} 0 \\ 2 g \end{pmatrix}$

$G(\theta)$

Out[851]=
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$(\dot{M} - 2C)$

Out[852]= True

Problem 3: Two-link PR manipulator

In[853]:=
```
m = 2;
a = 1;
b = 0.2;
c = 0;

θ = {θ1[t], θ2[t]};
θdot = D[θ, t];

xi1 = {1, 0, 0, 0, 0, 0};
xi2 = {0, a / 2, 0, 0, 0, 1};

gst0b1 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst0b2 = {{1, 0, 0, a}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

e1 = TwistExp[xi1, θ[[1]]];
e2 = TwistExp[xi2, θ[[2]]];

gstb1 = TwistExp[xi1, θ[[1]]].gst0b1;
gstb2 = TwistExp[xi1, θ[[1]]].TwistExp[xi2, θ[[2]]].gst0b2 // FullSimplify;

Labeled[MatrixForm[COM1 = RigidPosition[gstb1]], "COM1"]
Labeled[MatrixForm[COM2 = RigidPosition[gstb2]], "COM2"]
Labeled[
 MatrixForm[I1 = {{m / 12 * (b^2 + c^2), 0, 0}, {0, m / 12 * (a^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor]
I2 = I1;
```

```
GenM1 = {{m * IdentityMatrix[3], 0}, {0, I1}} // ArrayFlatten;
GenM2 = {{m * IdentityMatrix[3], 0}, {0, I2}} // ArrayFlatten;

xi1cross = Inverse[RigidAdjoint[e1.gst0b1]].xi1 // Simplify;
Jbsl1 = {xi1cross, {0, 0, 0, 0, 0, 0}} // Transpose;

xi1cross = Inverse[RigidAdjoint[e1.e2.gst0b2]].xi1 // Simplify;
xi2cross = Inverse[RigidAdjoint[e2.gst0b2]].xi2 // Simplify;
Jbsl2 = {xi1cross, xi2cross} // Transpose;

M = Transpose[Jbsl1].GenM1.Jbsl1 + Transpose[Jbsl2].GenM2.Jbsl2 // Simplify;

P = m * g * COM1[[2]] + m * g * COM2[[2]];
G = {0, 0};
G[[1]] = D[P, θ[[1]]];
G[[2]] = D[P, θ[[2]]];
christoffel = ConstantArray[0, {2, 2, 2}];

For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  For[k = 1, k ≤ 2, k++,
   christoffel[[i, j, k]] = 1 / 2 * (D[M[[i, j]], θ[[k]]] + D[M[[i, k]], θ[[j]]] - D[M[[k, j]], θ[[i]]]);
  ]
 ]
]

MatrixForm[christoffel];

Coriolis = ConstantArray[0, {2, 2}];
For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  Coriolis[[i, j]] = christoffel[[i, j, All]].θdot;
 ]
]
```

```
K = 1 / 2 * Transpose[θdot].M.θdot;
L = K - P // FullSimplify;

EL = {0, 0};
For [i = 1, i ≤ 2, i++,
 EL[[i]] = D[D[L, θdot[[i]]], t] - D[L, θ[[i]]];
]

Labeled[MatrixForm[EL // FullSimplify], "Euler-Lagrange Equations"]

Labeled[MatrixForm[M], "M(θ)"]
Labeled[MatrixForm[Coriolis // FullSimplify], "C(θ,θ̇)"]
MatrixForm[InertiaToCoriolis[M, θ, θdot] // FullSimplify];
Labeled[MatrixForm[G // FullSimplify], "G(θ)"]

Labeled[MatrixForm[D[M, t] - 2 * Coriolis // Simplify], "(Ṁ - 2C)"]
MatrixForm[D[M, t] - 2 * Coriolis // Simplify] == MatrixForm[-1 * Transpose[D[M, t] - 2 * Coriolis // Simplify]]
```

Out[867]=

$$\begin{pmatrix} \theta 1[t] \\ 0 \\ 0 \end{pmatrix}$$

COM1

Out[868]=

$$\begin{pmatrix} -\frac{1}{2} + \frac{3}{2} \text{Cos}[\theta 2[t]] + \theta 1[t] \\ \frac{3}{2} \text{Sin}[\theta 2[t]] \\ 0 \end{pmatrix}$$

COM2

Out[869]=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Inertia Tensor

Out[892]=
$$\begin{pmatrix} -3.\ (\text{Cos}[\theta 2[t]]\ \theta 2'[t]^2 - 1.33333\ \theta 1''[t] + \text{Sin}[\theta 2[t]]\ \theta 2''[t]) \\ 3.\ g\ \text{Cos}[\theta 2[t]] - 3.\ \text{Sin}[\theta 2[t]]\ \theta 1''[t] + 4.67333\ \theta 2''[t] \end{pmatrix}$$
Euler−Lagrange Equations

Out[893]=
$$\begin{pmatrix} 4. & -3\ \text{Sin}[\theta 2[t]] \\ -3\ \text{Sin}[\theta 2[t]] & 4.67333 \end{pmatrix}$$
$M(\theta)$

Out[894]=
$$\begin{pmatrix} 0 & -3\ \text{Cos}[\theta 2[t]]\ \theta 2'[t] \\ 0 & 0 \end{pmatrix}$$
$C(\theta,\dot{\theta})$

Out[896]=
$$\begin{pmatrix} 0 \\ 3\ g\ \text{Cos}[\theta 2[t]] \end{pmatrix}$$
$G(\theta)$

Out[897]=
$$\begin{pmatrix} 0 & 3\ \text{Cos}[\theta 2[t]]\ \theta 2'[t] \\ -3\ \text{Cos}[\theta 2[t]]\ \theta 2'[t] & 0 \end{pmatrix}$$
$(\dot{M} - 2C)$

Out[898]= True

Problem 4: Two-link RP manipulator

In[899]:=
```
m = 2;
a = 1;
b = 0.2;
c = 0;

θ = {θ1[t], θ2[t]};
θdot = D[θ, t];

xi1 = {0, 0, 0, 0, 0, 1};
xi2 = {0, 1, 0, 0, 0, 0};

gst0b1 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
```

```
gst0b2 = {{0, -1, 0, a}, {1, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

e1 = TwistExp[xi1, θ[[1]]];
e2 = TwistExp[xi2, θ[[2]]];

gstb1 = TwistExp[xi1, θ[[1]]].gst0b1;
gstb2 = TwistExp[xi1, θ[[1]]].TwistExp[xi2, θ[[2]]].gst0b2 // FullSimplify;

Labeled[MatrixForm[COM1 = RigidPosition[gstb1]], "COM1"]
Labeled[MatrixForm[COM2 = RigidPosition[gstb2]], "COM2"]
Labeled[
 MatrixForm[I1 = {{m / 12 * (b^2 + c^2), 0, 0}, {0, m / 12 * (a^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor1]
Labeled[
 MatrixForm[I1 = {{m / 12 * (a^2 + c^2), 0, 0}, {0, m / 12 * (b^2 + c^2), 0}, {0, 0, m / 12 * (a^2 + b^2)}}], Inertia Tensor2]

GenM1 = {{m * IdentityMatrix[3], 0}, {0, I1}} // ArrayFlatten;
GenM2 = {{m * IdentityMatrix[3], 0}, {0, I2}} // ArrayFlatten;

xi1cross = Inverse[RigidAdjoint[e1.gst0b1]].xi1 // Simplify;
Jbsl1 = {xi1cross, {0, 0, 0, 0, 0, 0}} // Transpose;

xi1cross = Inverse[RigidAdjoint[e1.e2.gst0b2]].xi1 // Simplify;
xi2cross = Inverse[RigidAdjoint[e2.gst0b2]].xi2 // Simplify;
Jbsl2 = {xi1cross, xi2cross} // Transpose;

M = Transpose[Jbsl1].GenM1.Jbsl1 + Transpose[Jbsl2].GenM2.Jbsl2 // Simplify;

P = m * g * COM1[[2]] + m * g * COM2[[2]];
G = {0, 0};
G[[1]] = D[P, θ[[1]]];
G[[2]] = D[P, θ[[2]]];
christoffel = ConstantArray[0, {2, 2, 2}];

For[i = 1, i ≤ 2, i++,
```

```
 For [j = 1, j ≤ 2, j++,
  For[k = 1, k ≤ 2, k++,
   christoffel[[i, j, k]] = 1 / 2 * (D[M[[i, j]], θ[[k]]] + D[M[[i, k]], θ[[j]]] - D[M[[k, j]], θ[[i]]]);
  ]
 ]
]

MatrixForm[christoffel];

Coriolis = ConstantArray[0, {2, 2}];
For[i = 1, i ≤ 2, i++,
 For [j = 1, j ≤ 2, j++,
  Coriolis[[i, j]] = christoffel[[i, j, All]].θdot;
 ]
]

K = 1 / 2 * Transpose[θdot].M.θdot;
L = K - P // FullSimplify;

EL = {0, 0};
For [i = 1, i ≤ 2, i++,
 EL[[i]] = D[D[L, θdot[[i]]], t] - D[L, θ[[i]]];
]

Labeled[MatrixForm[EL // FullSimplify], "Euler-Lagrange Equations"]

Labeled[MatrixForm[M], "M(θ)"]
Labeled[MatrixForm[Coriolis // FullSimplify], "C(θ,θ̇)"]
MatrixForm[InertiaToCoriolis[M, θ, θdot] // FullSimplify];
Labeled[MatrixForm[G // FullSimplify], "G(θ)"]

Labeled[MatrixForm[D[M, t] - 2 * Coriolis // Simplify], "(Ṁ - 2C)"]
MatrixForm[D[M, t] - 2 * Coriolis // Simplify] == MatrixForm[-1 * Transpose[D[M, t] - 2 * Coriolis // Simplify]]
```

Out[913]=
$$\begin{pmatrix} \frac{1}{2}\,\mathrm{Cos}[\theta 1[t]] \\ \frac{1}{2}\,\mathrm{Sin}[\theta 1[t]] \\ 0 \end{pmatrix}$$
COM1

Out[914]=
$$\begin{pmatrix} \mathrm{Cos}[\theta 1[t]] - \mathrm{Sin}[\theta 1[t]]\,\theta 2[t] \\ \mathrm{Sin}[\theta 1[t]] + \mathrm{Cos}[\theta 1[t]]\,\theta 2[t] \\ 0 \end{pmatrix}$$
COM2

Out[915]=
$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$
Inertia Tensor1

Out[916]=
$$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & 0.00666667 & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$
Inertia Tensor2

Out[938]=
$$\begin{pmatrix} 3.\,g\,\mathrm{Cos}[\theta 1[t]] + 2.84667\,\theta 1''[t] + \theta 2[t]\,(-2.\,g\,\mathrm{Sin}[\theta 1[t]] + 4\,\theta 1'[t]\,\theta 2'[t] + 2\,\theta 2[t]\,\theta 1''[t]) + 2.\,\theta 2''[t] \\ 2.\,\left(g\,\mathrm{Cos}[\theta 1[t]] - 1.\,\theta 2[t]\,\theta 1'[t]^2 + \theta 1''[t] + \theta 2''[t]\right) \end{pmatrix}$$
Euler−Lagrange Equations

Out[939]=
$$\begin{pmatrix} 2.84667 + 2\,\theta 2[t]^2 & 2. \\ 2. & 2. \end{pmatrix}$$
$M(\theta)$

Out[940]=
$$\begin{pmatrix} 2\,\theta 2[t]\,\theta 2'[t] & 2\,\theta 2[t]\,\theta 1'[t] \\ -2\,\theta 2[t]\,\theta 1'[t] & 0 \end{pmatrix}$$
$C(\theta,\dot{\theta})$

Out[942]=
$$\begin{pmatrix} g\,(3\,\mathrm{Cos}[\theta 1[t]] - 2\,\mathrm{Sin}[\theta 1[t]]\,\theta 2[t]) \\ 2\,g\,\mathrm{Cos}[\theta 1[t]] \end{pmatrix}$$
$G(\theta)$

Out[943]= $\begin{pmatrix} 0 & -4\,\theta 2[t]\,\theta 1'[t] \\ 4\,\theta 2[t]\,\theta 1'[t] & 0 \end{pmatrix}$

$(\dot{M} - 2C)$

Out[944]= True

In[945]:=

Problem 2

$$\dot{x}_1 = x_1 - x_1 x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = 2x_1^2 - 2x_2 = f_2(x_1, x_2)$$

<u>Equilibrium Points</u>

$$x_1 - x_1 x_2 = 0 \to x_1(1 - x_2) = 0 \tag{1}$$

$$2x_1^2 - 2x_2 = 0 \to x_1^2 - x_2 = 0 \tag{2}$$

From (1)

$$x_1(1 - x_2) = 0 \to x_1 = 0 \ \& \ (1 - x_2) = 0$$

$$x_1 = 0, \ x_2 = 1$$

Plugging into (2)

$$x_1 = 0 \to x_2 = 0$$

$$x_2 = 1 \to x_1 = \pm 1$$

Equilibrium points are: (0,0), (-1,1), (1,1)

<u>Linearize about equilibrium points (stability determined by eigen values)</u>

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - x_2 & x_1 \\ 4x_1 & -2 \end{bmatrix}$$

$$A|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \to \lambda_1 = 1, \lambda_2 = -2 \text{ (unstable)}$$

$$A|_{(-1,1)} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \to \lambda_1 = -1 + 1.73i, \lambda_2 = -1 - 1.73i \text{ (stable)}$$

$$A|_{(1,1)} = \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} \to \cancel{\lambda_1 = -1 + 1.73i, \lambda_2 = -1 - 1.73i \text{ (stable)}}$$

$\lambda_1 = 1.2361 \qquad \lambda_2 = -3.2361$ (unstable)

Problem 3

$$\dot{x}_1 = -x_1 - x_1 x_2^2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -x_2 - x_2 x_1^2 = f_2(x_1, x_2)$$

<u>Equilibrium Points</u>

$$-x_1 - x_1 x_2^2 = 0 \rightarrow\ -x_1(1 + x_2^2) = 0 \quad (1)$$

$$-x_2 - x_2 x_1^2 = 0 \rightarrow\ -x_2(1 + x_1^2) = 0 \quad (2)$$

By solving only (1), we have $x_1 =\ 0$ or $x_2 = \pm i$. If $x_1 =\ 0$, from (2) we can conclude $x_2\ =\ 0$. If $x_2 = \pm i$, from (2) we can conclude $x_1 =\ \pm i$. Therefore, the system has 5 equilibrium points (0, 0), $(-i, -i)$, $(-i, i)$, $(i, -i)$, (i, i). Thus, (0, 0) is the unique real equilibrium point of the system.

<u>Stability</u>

To investigate local stability, we find jacobian of the system.

$$J(x_1, x_2) =\ \frac{\partial f}{\partial x} = \begin{bmatrix} -1 - x_2^2 & -2x_1 x_2 \\ -2x_1 x_2 & -1 - x_2^2 \end{bmatrix}$$

For (0,0), we have

$$J(0{,}0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of the Jacobian at point (0, 0) are $\lambda_1 = -1$ and $\lambda_2 = -1$. Both eigenvalues are negative. Therefore, system is locally stable.

To investigate global stability we use the Lyapanov function candidate.

$$V =\ \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$$

$$\dot{V} =\ \dot{x}_1 x_1 + \dot{x}_2 x_2 = (-x_1 - x_1 x_2^2) x_1 + (-x_2 - x_2 x_1^2) x_2 = -x_1^2 - x_2^2 - 2x_1^2 x_2^2$$

$$\begin{cases} \dot{V} = 0 & \text{when } (x_1, x_2) = (0{,}0) \\ \dot{V} < 0 & \text{when } (x_1, x_2) \neq (0{,}0) \end{cases}$$

Thus, the system is asymptotically stable.

$$x_1 \rightarrow \infty \text{ or } x_1 \rightarrow \infty \qquad \Rightarrow \qquad V \rightarrow \infty$$

Therefore, the system is globally asymptotically stable.