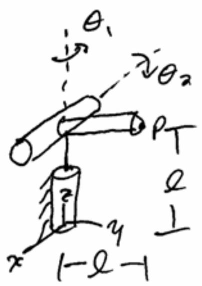
Course: Mech 567: Robot Kinematics and Dynamics
Instructor: Daniel Bruder, PhD

Solution 3

Problem 1



$$p = \begin{bmatrix} 0 \\ \ell \\ \ell \end{bmatrix} \qquad q = \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ \ell + \ell/\sqrt{2} \end{bmatrix}$$

SP2: $e^{1} e^{2} p = q$

$$r = \begin{bmatrix} 0 \\ 0 \\ \ell \end{bmatrix}$$

$$u = p - r = \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ \ell/\sqrt{2} \end{bmatrix}$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha = \frac{\cancelto{0}{(\omega_1 \cdot \omega_2)}(\omega_2 \cdot u) - \omega_1 \cdot v}{\cancelto{0}{(\omega_1 \cdot \omega_2)^2} - 1}$$

$$= +\ell/\sqrt{2}$$

$$\beta = \frac{\cancelto{0}{(\omega_1 \cdot \omega_2)}(\omega_1 \cdot v) - \cancelto{0}{\omega_2 \cdot u}}{\cancel{(\omega_1 \cdot \omega_2)^2} - 1}$$

$$= 0$$

$$\gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\cancelto{0}{\omega_1 \cdot \omega_2}}{\cancelto{1}{\|\omega_1 \times \omega_2\|}}$$

$$= \ell^2 - \ell^2/2$$

$$= \ell^2/2$$

$$\gamma = \pm \ell/\sqrt{2} \qquad z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2)$$

$$z = \begin{bmatrix} 0 \\ 0 \\ \ell/\sqrt{2} \end{bmatrix} + \pm \ell/\sqrt{2} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix}$$

$$c = z + r = \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell + \ell/\sqrt{2} \end{bmatrix}$$

Use SP1 to solve $e^{2} p = c$ and $e^{-1} q = c$

$$\theta_1 : \quad e' c = q \qquad r = \begin{bmatrix} 0 \\ 0 \\ \ell \end{bmatrix} \qquad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_1 = c - r = \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix}$$

$$v_1 = q - r = \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ \ell/\sqrt{2} \end{bmatrix}$$

$$u_1' = u_1 - \omega_1 \omega_1^T u_1$$
$$= \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix} \right)$$
$$= \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ 0 \end{bmatrix}$$

$$v_1' = v_1 - \omega_1 \omega_1^T v_1$$
$$= \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ \ell/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ \ell/\sqrt{2} \end{bmatrix} \right)$$
$$= \begin{bmatrix} -\ell/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$\theta_1 = \operatorname{atan2}\left(\omega_1 \cdot (u_1' \times v_1'),\ \overbrace{u_1' \cdot v_1'}^{0}\right)$$
$$= \operatorname{atan2}\left(\pm \ell^2/2,\ 0\right)$$
$$= \pm \pi/2$$

$\theta_2: \quad e^{2} p = c \qquad r = \begin{bmatrix} 0 \\ 0 \\ \ell \end{bmatrix} \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$u_2 = p - r = \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix}$

$$u_2' = u_2 - \omega_2(\omega_2 \cdot u_2)$$

$$= \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ \ell \\ 0 \end{bmatrix}$$

$$v_2' = v_2 - \omega_2(\omega_2 \cdot v_2)$$

$$= \begin{bmatrix} 0 \\ \pm \ell/\sqrt{2} \\ \ell/\sqrt{2} \end{bmatrix}$$

$$\theta_2 = \operatorname{atan2}\left(\omega_2 \cdot (u_2' \times v_2'), u_2' \cdot v_2' \right)$$

$$= \operatorname{atan2}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \ell^2/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \pm \ell^2/\sqrt{2} \right)$$

$$= \operatorname{atan2}\left(-\ell^2/\sqrt{2}, \pm \ell^2/\sqrt{2} \right)$$

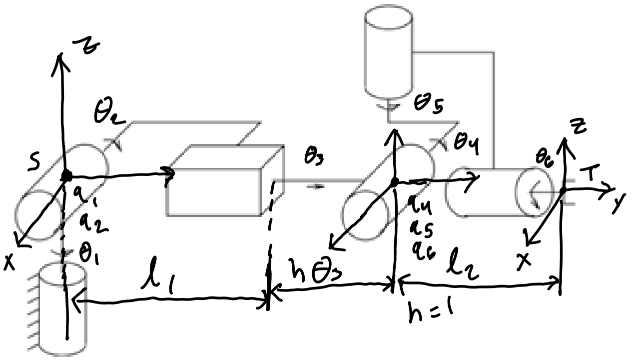
$$= \quad 3\pi/4 \quad \& \quad -\pi/4$$

Total soln:

1 $\quad \theta_1 = -\pi/2 \; ; \; \theta_2 = -3\pi/4$

2 $\quad \theta_1 = \pi/2 \; ; \; \theta_2 = -\pi/4$

Problem 2



(iii) Stanford manipulator

$e^1 e^2 e^3 e^4 e^5 e^6 g_{st}(0) = g_d$ where $e^1 = e^{\hat{\xi}_1 \theta}$

Solve for Prismatic Joint first! θ_3

$e^1 e^2 e^3 e^4 e^5 e^6 = g_d g_{st}^{-1}(0) = g_1$ ← This can be calculated

then choose point where joints 4, 5, & 6 and multiply each side

so,

$$e^1 e^2 e^3 e^4 e^5 e^6 p_{456} = g p_{456}$$

We can then remove $e^4 e^5 e^6$ because they have no impact on p_{456}:

$$e^1 e^2 e^3 p_{456} = g_1 p_{456}$$

Now subtract point p_{123} located at intersections of $\hat{\xi}_1, \hat{\xi}_2$

$$e^1 e^2 e^3 p_{456} - p_{123} = g_1 p_{456} - p_{123}$$

$$e^1 e^2 e^3 p_{456} - e^1 e^2 p_{123} = g_1 p_{456} - p_{123}$$

$$e^1 e^2 (e^3 p_{456} - p_{123}) = g_1 p_{456} - p_{123}$$

Now take the norm of both sides

$$\| e^1 e^2 (e^3 p_{456} - p_{123}) \| = \| g_1 p_{456} - p_{123} \|$$

RBT preserve distance so:

$$\| e^3 p_{456} - p_{123} \| = \| g_1 p_{456} - p_{123} \|$$

← Important: Not SP3

We know $\| e^3 p_{456} - p_{123} \| = h\theta_3 + l_1$ & $h = 1$ cause prismatic

$$\theta_3 + l_1 = \| g_1 p_{456} - p_{123} \|$$

$$\theta_3 = \| g_1 p_{456} - p_{123} \| - l_1 \longrightarrow 1 \text{ solution}$$

Next solve for θ_1 & θ_2

Since we know θ_3,

$$e^1 e^2 (e^3 p_{456}) = g_1 p_{456}$$

Let $p = e^3 p_{456}$ & $q = g_1 p_{456}$, then apply SP2 to get values for θ_1, θ_2 → 2 solutions

Next solve for θ_4 & θ_5 by choosing a point p_T on $\hat{\xi}_6$ but not $\hat{\xi}_5$ or $\hat{\xi}_4$

$$e^4 e^5 e^6 p_T = e^{-3} e^{-2} e^{-1} g_1 p_T$$

$$\downarrow$$

$$e^5 e^6 p_T = g_2 p_T$$

let $p = p_T$ & $q = g_2 p_T$, then apply SP2 to find $\theta_4, \theta_5 \rightarrow$ 2 solutions

Finally solve for θ_6

$$e^1 e^2 e^3 e^4 e^5 e^6 = g_1$$

$$e^6 = e^{-5} e^{-4} e^{-3} e^{-2} e^{-1} g_1$$

Apply point p not on $\hat{\xi}_6$ to both sides

$$e^6 p = e^{-5} e^{-4} e^{-3} e^{-2} g_1 p = q$$

then solve for θ_6 using SP1 $\rightarrow$ 1 solution

Max # of Solutions	Total
• θ_3 prismatic : 1 sol	1
• θ_1, θ_2 SP2 : 2 sol	2
• θ_4, θ_5 SP2 : 2 sol	4
• θ_6 SP1 : 1 sol	4

4 solutions ~~total~~

Problem 3

(a)

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab}\, R_{ab}^\top p_{ab} + \dot{p}_{ab} \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\hat{\omega}_{ab}^s\, p_{ab} + \dot{p}_{ab} \\ \omega_{ab}^s \end{bmatrix}$$

$$V_{ab}^s = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \pi \\ -\frac{\pi}{2} \\ 0 \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix}$$

Or

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \dot{\theta} = \frac{\pi}{2} \qquad h = 0$$

$$V_{ab}^s = \xi\dot{\theta} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix} \dot{\theta} = \begin{bmatrix} \pi \\ -\frac{\pi}{2} \\ 0 \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix}$$

(b)

$$q_a(0) = g_{ab}(0)\, q_b = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$v_{q_a} = \hat{V}_{ab}^s\, q_a = \begin{bmatrix} \hat{\omega}_{ab}^s & v_{ab}^s \\ 0 & 0 \end{bmatrix} q_a$$

$$v_{q_a}(0) = \begin{bmatrix} 0 & -\frac{\pi}{2} & 0 & \pi \\ \frac{\pi}{2} & 0 & 0 & -\frac{\pi}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4

(a)

$$\dot{\theta} = \frac{\pi}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{2} \quad \text{at t=1}$$

Note that q_b is constant; point is fixed in body coordinates.

$$q_a(\frac{\pi}{2}) = g_{ab}(\frac{\pi}{2})\, q_b = e^{\hat{\xi}\,\frac{\pi}{2}} g_{ab}(0) q_b(0) = e^{\hat{\xi}\,\frac{\pi}{2}} q_a(0)$$

$$e^{\hat{\xi}\,\frac{\pi}{2}} = \begin{bmatrix} e^{\hat{z}\,\frac{\pi}{2}} & (I - e^{\hat{z}\,\frac{\pi}{2}})\omega \times v \\ 0 & 0 \end{bmatrix} \qquad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$q_a(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(b) Like Problem 3 part (a), but now

$$\dot{p}_{ab} = \begin{bmatrix} 0 \\ 0 \\ 2\,\pi \end{bmatrix} \qquad \text{or} \qquad h = \frac{\text{translational velocity parallel to } \omega}{\text{rotational velocity about } \omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$$

$$\xi = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad V_{ab}^s = \begin{bmatrix} \pi \\ -\frac{\pi}{2} \\ 2\pi \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix}$$

$$q_a(t) = g_{ab}(t)\; q_b$$

$$q_a(t) = \begin{bmatrix} cos(\frac{\pi}{2}t) & -sin(\frac{\pi}{2}t) & 0 & 1 \\ sin(\frac{\pi}{2}t) & cos(\frac{\pi}{2}t) & 0 & 2 \\ 0 & 0 & 1 & 2\pi t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

for t=1

$$q_a(1) = \begin{bmatrix} 0 \\ 2 \\ 2\pi \\ 1 \end{bmatrix}$$

Problem 5

```
clearAll["Global`*"];
Needs["Screws`",
  "C:\\Users\\Ross\\Dropbox (University of Michigan)\\Classes\\2023_2024 Spring\\ROB Kinematics,
    Dynamics, and Control\\HelperScripts\\Screws.m"];
```

Part a

(a) Express the position and orientation of frame C3 relative to frame C0 in terms of the joint angle variables and the link parameters.

```
In[200]:= MatrixForm[gst0 = {{1, 0, 0 , 0}, {0, 1, 0, L1 + L2}, {0, 0, 1, L0}, {0, 0 , 0, 1}}];
w1 = {0, 0, 1};
w2 = {0, 0, 1};

q1 = {0, 0, L0};
q2 = {0, L1, L0};

MatrixForm[twist1 = Flatten[Append[-Skew[w1].q1, w1]]];
MatrixForm[twist2 = Flatten[Append[-Skew[w2].q2, w2]]];

MatrixForm[e1 = TwistExp[twist1, th1[t]]];
MatrixForm[e2 = TwistExp[twist2, th2[t]]];

MatrixForm[Simplify[g03[t] = e1.e2.gst0]]
```

Out[209]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\text{th1}[t]+\text{th2}[t]] & -\text{Sin}[\text{th1}[t]+\text{th2}[t]] & 0 & -\text{L1 Sin}[\text{th1}[t]]-\text{L2 Sin}[\text{th1}[t]+\text{th2}[t]] \\ \text{Sin}[\text{th1}[t]+\text{th2}[t]] & \text{Cos}[\text{th1}[t]+\text{th2}[t]] & 0 & \text{L1 Cos}[\text{th1}[t]]+\text{L2 Cos}[\text{th1}[t]+\text{th2}[t]] \\ 0 & 0 & 1 & \text{L0} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Parts b & c

```
In[210]:= MatrixForm[R03 = g03[t][[1 ;; 3, 1 ;; 3]]];

MatrixForm[Simplify[p03 = g03[t][[1 ;; 3, 4]]]];

MatrixForm[R03Dot = D[R03, t]];
MatrixForm[Simplify[p03Dot = D[p03, t]]];
```

(b) Compute the spatial velocity of C3 relative to C0 as functions of the joint angles and the joint rates.

```
In[214]:= MatrixForm[Simplify[vs = -R03Dot.Transpose[R03].p03 + p03Dot]];
MatrixForm[Simplify[ws = UnSkew[R03Dot.Transpose[R03]]]];

MatrixForm[Simplify[Vs = Join[vs, ws]]]
```

Out[216]//MatrixForm=

$$\begin{pmatrix} \text{L1 Cos}[\text{th1}[t]]\ \text{th2}'[t] \\ \text{L1 Sin}[\text{th1}[t]]\ \text{th2}'[t] \\ 0 \\ 0 \\ 0 \\ \text{th1}'[t]+\text{th2}'[t] \end{pmatrix}$$

Calculate Body Velocity of B relative to A

```
In[217]:= MatrixForm[Simplify[vb = Transpose[R03].p03Dot]];
MatrixForm[Simplify[wb = UnSkew[Transpose[R03].R03Dot]]];
MatrixForm[Simplify[Vb = Join[vb, wb]]]
```

Out[219]//MatrixForm=

$$\begin{pmatrix} -((\text{L2}+\text{L1 Cos}[\text{th2}[t]])\ \text{th1}'[t])-\text{L2 th2}'[t] \\ \text{L1 Sin}[\text{th2}[t]]\ \text{th1}'[t] \\ 0 \\ 0 \\ 0 \\ \text{th1}'[t]+\text{th2}'[t] \end{pmatrix}$$