Course: Mech 567: Robot Kinematics and Dynamics
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Solution 1

Problem 1

We will first prove the following 2 lemmas.
Lemma 0.1: $\theta^{2n-1}\hat{\omega}^{2n-1} = (-1)^{n-1}\theta^{2n-1}\hat{\omega}$ for $n = 1, \ldots, \infty$.
Proof: We will prove by induction. The base case for $n = 1$ is trivial since $\theta\hat{\omega} = (-1)^0\theta\hat{\omega}$. We will next show the inductive case for $n + 1$.

$$\begin{aligned}
\theta^{2(n+1)-1}\hat{\omega}^{2(n+1)-1} &= \theta^{2n+1}\hat{\omega}^{2n-2}(-\hat{\omega}) \text{ using Lemma 2.3, eq 2.13, from text} \\
&= -\theta^2\theta^{2n-1}\hat{\omega}^{2n-1} \\
&= -\theta^2(-1)^{n-1}(\theta^{2n-1}\hat{\omega}) \text{ by the inductive hypothesis} \\
&= (-1)^{(n+1)-1}\theta^{2(n+1)-1}\hat{\omega} \ \blacksquare
\end{aligned}$$

Lemma 0.2: $\theta^{2n}\hat{\omega}^{2n} = (-1)^{n+1}\theta^{2n}\hat{\omega}^2$ for $n = 1, \ldots, \infty$.
Proof: We will prove by induction. The base case for $n = 1$ is trivial since $\theta^2\hat{\omega}^2 = (-1)^2\theta^2\hat{\omega}^2$. We will next show the inductive case for $n + 1$.

$$\begin{aligned}
\theta^{2(n+1)}\hat{\omega}^{2(n+1)} &= \theta^{2n+2}\hat{\omega}^{2n}(\omega\omega^T - ||\omega||^2 I) \text{ using Lemma 2.3, eq 2.12, from text} \\
&= \theta^{2n+2}(\hat{\omega}^{2n-1}\hat{\omega}\omega\omega^T - \hat{\omega}^{2n}I) \text{ where we know } ||\omega|| = 1 \\
&= -\theta^{2n+2}\hat{\omega}^{2n} \text{ by } \hat{\omega}\omega = 0 \text{ since cross product of } \omega \text{ with itself is 0} \\
&= -\theta^2((-1)^{n+1}\theta^{2n}\hat{\omega}^2) \text{ by the inductive hypothesis} \\
&= (-1)^{(n+1)+1}\theta^{2(n+1)}\hat{\omega}^2 \ \blacksquare
\end{aligned}$$

The exponential $\hat{\omega}$ can be written in series form:

$$e^{\hat{\omega}\theta} = I + \theta\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \ldots$$

Reorganizing,

$$e^{\hat{\omega}\theta} = I + (\theta\hat{\omega} + \frac{\theta^3}{3!}\hat{\omega}^3 + \ldots) + (\frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^4}{4!}\hat{\omega}^4 + \ldots).$$

Using the results of the lemmas proven above for the even and odd powers of the series, we get

$$e^{\hat{\omega}\theta} = I + (\theta\hat{\omega} - \frac{\theta^3}{3!}\hat{\omega} + \ldots) + (\frac{\theta^2}{2!}\hat{\omega}^2 - \frac{\theta^4}{4!}\hat{\omega}^2 + \ldots).$$

Factoring,

$$e^{\hat{\omega}\theta} = I + \hat{\omega}(\theta - \frac{\theta^3}{3!} + \ldots) + \hat{\omega}^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \ldots).$$

The series in the first and second parentheses match the series for $\sin\theta$ and $1-\cos\theta$, respectively. Hence,

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta). \blacksquare$$

Problem 2

Rodrigues' formula is

$$R = e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

When $R = I$,

$$I = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$
$$0 = \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$
$$\hat{\omega}\sin\theta = -\hat{\omega}^2(1-\cos\theta)$$

Since $\hat{\omega}$ is skew-symetric, its diogonals are 0. However, this will not hold for $\hat{\omega}^2$. Thus,

$$\theta \neq 0 \qquad \Rightarrow \qquad e^{\hat{\omega}\theta} \neq I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Therefore,

$$R = I \qquad \Rightarrow \qquad \theta = 0$$

When $\theta = 0$,

$$\sin\theta = 0$$
$$1 - \cos\theta = 0$$

Thus, Rodrigues' formula simplified to

$$R = e^{\hat{\omega}\theta} = I$$

This is always true regardless of ω, so ω is arbitrary.

Problem 3

Here we derive an alternative method to compute ω for the particular case $\theta = \pi$.
The rotation matrix R has the form

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

We consider the equation 2.16 simplified for the case $\theta = \pi$.

$$R = e^{\hat{\omega}} = \begin{bmatrix} 2\,\omega_1^2 - 1 & 2\,\omega_1\,\omega_2 & 2\,\omega_1\,\omega_3 \\ 2\,\omega_1\omega_2 & 2\,\omega_2^2 - 1 & 2\,\omega_2\,\omega_3 \\ 2\,\omega_1\,\omega_3 & 2\,\omega_2\,\omega_3 & 2\,\omega_3^2 - 1 \end{bmatrix}.$$

Using the diagonals of R, we solve for the components of ω.

$$r_{11} = 2\,\omega_1^2 - 1 \qquad \Rightarrow \qquad \omega_1 = \sqrt{\frac{r_{11}+1}{2}}$$

$$r_{22} = 2\,\omega_2^2 - 1 \qquad \Rightarrow \qquad \omega_2 = \sqrt{\frac{r_{22}+1}{2}}$$

$$r_{33} = 2\,\omega_3^2 - 1 \qquad \Rightarrow \qquad \omega_3 = \sqrt{\frac{r_{33}+1}{2}}$$

The signs of the ω components must be compatible with the signs of the off-diagonal elements $r_{12} = 2\omega_1\omega_2, r_{13} = 2\omega_1\omega_3$, and $r_{23} = 2\omega_2\omega_3$. However, there will always be multiple solutions because you can multiply ω by -1 and still get the same rotation for $\theta = \pi$.

Problem 4

(a) The eigenvalues of $\hat{\omega}$ are computed using characteristic equation of $\hat{\omega}$.

$$|\hat{\omega} - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -\omega_3 & \omega_2 \\ \omega_3 & -\lambda & -\omega_1 \\ -\omega_2 & \omega_1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 - \omega_1\,\omega_2\,\omega_3 + \omega_1\,\omega_2\,\omega_3 - \lambda\,\omega_1^2 - \lambda\,\omega_2^2 - \lambda\,\omega_3^2 = 0$$

$$-\lambda^3 - \lambda\,\omega_1^2 - \lambda\,\omega_2^2 - \lambda\,\omega_3^2 = 0$$

$$-\lambda\,(\lambda^2 + \omega_1^2 + \omega_2^2 + \omega_3^2) = 0$$

$$-\lambda\,(\lambda^2 + 1) = 0$$

$$\lambda\,(\lambda^2 + 1) = 0$$

Therefore, $\lambda = 0$, $\lambda = i$, and $\lambda = -i$ are the eigenvalues of $\hat{\omega}$.

The eigenvector associated with $\lambda = 0$ is inside the nullspace of $\hat{\omega} - \lambda I$

$$\hat{\omega} v = \lambda v$$

$$\lambda = 0 \quad \rightarrow \quad \hat{\omega} v = 0$$

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies

$$-\omega_3\,v_2 + \omega_2\,v_3 = 0$$

$$\omega_3\,v_1 - \omega_1\,v_3 = 0$$

$$-\omega_2\,v_1 + \omega_1\,v_2 = 0$$

Which implies

$$v_1 = \frac{\omega_1\, v_2}{\omega_2}$$
$$v_3 = \frac{\omega_3\, v_1}{\omega_1} = \frac{\omega_3\, v_2}{\omega_2}$$
$$v_2 = arbitrary$$

Since $||\omega|| = 1$, we normalize v by setting $v_2 = \omega_2$ to obtain $v = [\omega_1,\, \omega_2,\, \omega_3] = \omega$.

(b) Using the Taylor expansion of R, we compute Rv, where v is an eigenvector of $\hat{\omega}$.

$$R\, v = e^{\hat{\omega}\,\theta}\, v$$
$$R\, v = (\sum_{k=0}^{\infty} \frac{(\theta\, \hat{\omega})^k}{k!})\, v$$
$$R\, v = (\sum_{k=0}^{\infty} \frac{\theta^k\, \hat{\omega}^k}{k!})\, v$$
$$R\, v = \sum_{k=0}^{\infty} \frac{\theta^k\, \hat{\omega}^k v}{k!}$$

Since v is an eigenvector of $\hat{\omega}$

$$R\, v = \sum_{k=0}^{\infty} \frac{\theta^k\, \lambda^k v}{k!}$$
$$R\, v = (\sum_{k=0}^{\infty} \frac{\theta^k\, \lambda^k}{k!})\, v$$
$$R\, v = e^{\lambda\,\theta}\, v$$

Which is what the hint is saying.
Thus, v is also an eigenvectors of R. Since $\lambda = 0$, $\lambda = i$, or $\lambda = -i$, the eigenvectors of R have eigenvalues 1, $e^{i\,\theta}$, and $e^{-i\,\theta}$.

Problem 5

(a)

$$SO(n) = \{R \in \mathbb{R}^{n\times n} \mid R\, R^T = I,\, |R| = 1\}$$
$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$
$$R\, R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}$$

$$R\,R^T = \begin{bmatrix} r_{11}^2 + r_{12}^2 & r_{11}\,r_{21} + r_{12}\,r_{22} \\ r_{21}\,r_{11} + r_{22}\,r_{21} & r_{21}^2 + r_{22}^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11}^2 + r_{12}^2 & r_{11}\,r_{21} + r_{12}\,r_{22} \\ r_{21}\,r_{11} + r_{22}\,r_{21} & r_{21}^2 + r_{22}^2 \end{bmatrix}$$

Let $r_{11} = \cos\theta$ and $r_{12} = \sin\theta$ where $\theta \in \mathbb{S}^1$. This satisfies the condition that $r_{11}^2 + r_{12}^2$.

$$r_{11}\,r_{21} + r_{12}\,r_{22} = 0$$

$$(\cos\theta)\,r_{21} + (\sin\theta)\,r_{22} = 0$$

This condition is satisfied if $r_{21} = -\sin\theta$ and $r_{22} = \cos\theta$ where $\theta \in \mathbb{S}^1$. R is 2π periodic.

$$\cos(\theta + 2\pi n) = \cos\theta$$

$$\sin(\theta + 2\pi n) = \sin\theta$$

Therefore

$$\mathbb{S}^1 \to SO(2)\ :\ \theta \to \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thus, $SO(2)$ can be identified with $\mathbb{S}^1$, which is the unit circle in $\mathbb{R}^2$.

(b)

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \omega\,J$$

$$\hat{\omega}^2 = \omega^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\omega^2\,I$$

$$\hat{\omega}^3 = -\omega^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\omega^3\,J$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^3\theta^3}{3!} + \frac{\hat{\omega}^4\theta^4}{4!} + \frac{\hat{\omega}^5\theta^5}{5!} + ...$$

Group according to I and J

$$e^{\hat{\omega}\theta} = (1 - \frac{\omega^2\theta^2}{2!} + \frac{\omega^4\theta^4}{4!} + ...)\,I + (\omega\theta - \frac{\omega^3\theta^3}{3!} + \frac{\omega^5\theta^5}{5!} + ...)\,J$$

$$e^{\hat{\omega}\theta} = \cos\omega\theta\,I + \sin\omega\theta\;J$$

$$e^{\hat{\omega}\theta} = \cos\omega\theta\, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\omega\theta\, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\omega\theta & 0 \\ 0 & \cos\omega\theta \end{bmatrix} + \begin{bmatrix} 0 & -\sin\omega\theta \\ \sin\omega\theta & 0 \end{bmatrix}$$

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \sin\omega\theta & \cos\omega\theta \end{bmatrix}$$

$e^{\hat{\omega}\theta}$ is surjective but not injective.

$$e^{\hat{\omega}(\theta+2\pi n)} = \begin{bmatrix} \cos(\omega\theta + 2\pi n) & -\sin(\omega\theta + 2\pi n) \\ \sin(\omega\theta + 2\pi n) & \cos(\omega\theta + 2\pi n) \end{bmatrix} = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \sin\omega\theta & \cos\omega\theta \end{bmatrix} = e^{\hat{\omega}\theta}$$

Problem 6

(a) Let $R \in SO(2)$ and $\hat{\omega} \in so(2)$

$$R\hat{\omega}R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}$$

$$R\hat{\omega}R^T = \begin{bmatrix} r_{12}\omega & -r_{11}\omega \\ r_{22}\omega & -r_{21}\omega \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}$$

$$R\hat{\omega}R^T = \begin{bmatrix} r_{11}r_{12}\,\omega - r_{11}r_{12}\,\omega & r_{12}r_{21}\,\omega - r_{11}r_{22}\,\omega \\ r_{11}r_{22}\,\omega - r_{21}r_{12}\,\omega & r_{22}r_{21}\,\omega - r_{21}r_{22}\,\omega \end{bmatrix}$$

$$R\hat{\omega}R^T = \begin{bmatrix} 0 & r_{12}r_{21}\,\omega - r_{11}r_{22}\,\omega \\ r_{11}r_{22}\,\omega - r_{21}r_{12}\,\omega & 0 \end{bmatrix}$$

$$R\hat{\omega}R^T = \begin{bmatrix} 0 & r_{12}r_{21} - r_{11}r_{22} \\ r_{11}r_{22} - r_{21}r_{12} & 0 \end{bmatrix} \omega$$

$$|R| = 1 \quad \Rightarrow \quad r_{11}r_{22} - r_{21}r_{12} = 1$$

$$R\hat{\omega}R^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \omega$$

$$R\hat{\omega}R^T = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

$$R\hat{\omega}R^T = \hat{\omega}$$

(b)

$$\begin{aligned} Re^{\hat{\omega}\theta}R^T &= R(I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \cdots)R^T \\ &= (I + R\hat{\omega}R^T\theta + \frac{(R\hat{\omega}R^T)^2\theta^2}{2!} + \cdots) = e^{R\hat{\omega}R^T\theta} = e^{\hat{\omega}\theta}. \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}e^{\hat{\omega}\theta} &= \frac{d}{dt}(I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \cdots) \\ &= (\hat{\omega}\dot{\theta} + \frac{\hat{\omega}^2}{2!} \cdot 2\theta\dot{\theta} + \frac{\hat{\omega}^3}{3!} \cdot 3\theta^2\dot{\theta} + \cdots) \text{ (Product Rule is used)} \\ &= \hat{\omega}\dot{\theta}(I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \cdots) = \hat{\omega}\dot{\theta} \cdot e^{\hat{\omega}\theta} = e^{\hat{\omega}\theta} \cdot \hat{\omega}\dot{\theta}. \end{aligned}$$