ROB510 Robot Kinematics and Dynamics
Homework 5

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R	O	B	O	T
T	E	N	E	T
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Problem I

In this problem, use Mathematica with the screw theory package. Copy and paste your code in the homework.

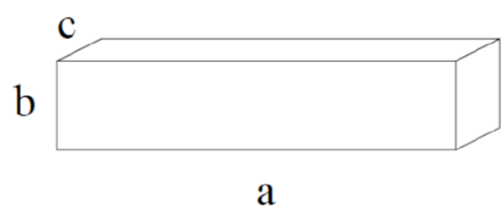

Figure 1: Block for Problem1

Consider a uniform rectangular solid of mass m and dimensions $a \times b \times c$ as shown in Figure 2 . The inertia tensor with respect to a coordinate frame located at the geometric center of the solid can be computed in terms of a, b, c and m.

For each of the two-link planar robots shown on the next page, assume that the links are uniform rectangular solids as above. Let $a = 1, b = 0.2$, and $m = 2$ in each case. The value of c does not matter since the motion is planar. You may take $c = 0$ for simplicity. Assume that the gravity vector is in the direction of the negative y-axis. **Define the positive x-axis pointing towards right, positive y-axis pointing upwards, and positive z-axis pointing out of the page.** Robots (i), (iii), and (iv) are shown with angles to help understand where the links are. For the zero configuration of each of these robots assume that the links that are shown at 45 degrees are only along the x-direction. **This means that the two links for (i) and (iii) are both along** x **only.**

(a) Compute the inertia tensor and center of mass vector for each link.

(b) Compute the Euler-Lagrange dynamic equations using the Lagrangian function L.

(c) Output the inertia matrix $D(q)$, the Coriolis/centrifugal matrix $C(q, \dot{q})$, and the gravitational torques/forces vector $g(q)$ for each robot.

(d) Compute the expression $\dot{D}(q) - 2C(q, \dot{q})$ for each robot and verify that it is a skew symmetric matrix.

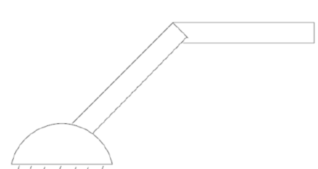

Figure 2: RR Manipulator

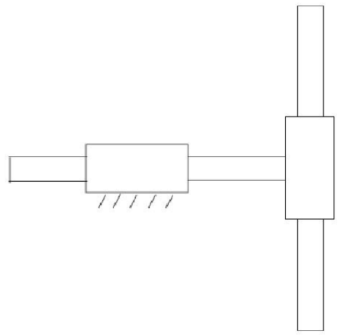

Figure 3: PP Manipulator

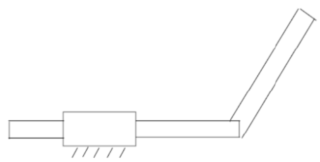

Figure 4: PR Manipulator

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(* Mathematica Code *)
ROB510   Homework5
Needs["Screws`", \"C:\Users\Alex\OneDrive\Desktop\OneDrive\ROB510\Homework\Mathematica\Screws.m"]
Needs["RobotLinks`", \"C:\Users\Alex\OneDrive\Desktop\OneDrive\ROB510\Homework\Mathematica\Robotlinks.m"]
```

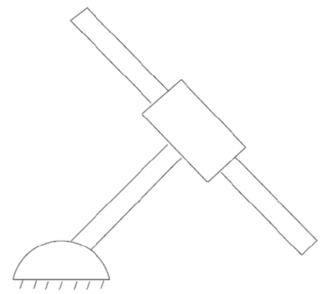

Figure 5: RP Manipulator

Problem II

Consider the nonlinear system:

$$\dot{x}_1 = x_1 - x_1 x_2$$
$$\dot{x}_2 = 2x_1^2 - 2x_2$$

Find the equilibrium points and investigate local stability around each equilibrium point.

Equilibrium points can be calculated by setting both equations equal to 0:

$$\dot{x}_1 = x_1 - x_1 x_2 = 0$$
$$\dot{x}_2 = 2x_1^2 - 2x_2 = 0$$

This yields that

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \text{or} \begin{cases} x_2 = 1 \\ x_1 = \pm 1 \end{cases}$$

To analyze the stability, we compute the Jacobian matrix of the system (linearize the system about the equilibrium points:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - x_2 & -x_1 \\ 4x_1 & -2 \end{bmatrix}.$$

At $(0, 0)$

$$J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

where $\lambda_1 = 1$ and $\lambda_2 = -2$, indicating that $(0, 0)$ is a saddle point and thus unstable.

At $(1, 1)$ **and** $(-1, 1)$

$$J(1, 1) = J(-1, 1) = \begin{bmatrix} 0 & \pm 1 \\ \mp 4 & -2 \end{bmatrix}.$$

The eigenvalues are obtained by solving:

$$\det(J - \lambda I) = \lambda^2 + 2\lambda + 4 = 0,$$

which yields $\lambda = -1 \pm \sqrt{5}i$. The real part of it is negative, indicating that both $(1, 1)$ and $(-1, 1)$ are spiral sink, which are stable.

Problem III

Consider the nonlinear system:

$$\dot{x}_1 = -x_1 - x_1 x_2^2$$
$$\dot{x}_2 = -x_2 - x_2 x_1^2$$

Show that $(0,0)$ is the unique equilibrium point and investigate local stability. Investigate global stability using the Lyapunov function candidate.

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

Similarly, equilibrium points can be calculated by setting both equations equal to 0:

$$\dot{x}_1 = -x_1 - x_1 x_2^2 = 0$$
$$\dot{x}_2 = -x_2 - x_2 x_1^2 = 0$$

This yields that

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \text{or} \begin{cases} x_2 = \pm i \\ x_1 = \pm i, \end{cases}$$

where $(0,0)$ is the only real equilibrium point.

Local Stability at $(0,0)$
Calculate the Jacobian matrix of the system:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 - x_2^2 & -2x_1 x_2 \\ -2x_2 x_1 & -1 - x_1^2 \end{bmatrix},$$

and evaluate at $(0,0)$:

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\lambda_1 = -1$ and $\lambda_2 = -1$ indicates that $(0,0)$ is a node sink, which is locally asymptotically stable.

Global Stability Using Lyapunov Function
Consider the Lyapunov function candidate:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

Compute its derivative along the trajectories of the system:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 - x_1x_2^2) + x_2(-x_2 - x_2x_1^2) = -x_1^2(1+x_2^2) - x_2^2(1+x_1^2) = -x_1^2 - x_2^2 - 2x_1^2x_2^2.$$

This indicates that $\dot{V} \leq 0 \; \forall \; x_1, x_2 \in \mathit{domain}$. Also, $V \sim \infty$ when x_1 or $x_2 \sim \infty$. Thus, the nonlinear system is globally asymptotically stable.