ROB510 Robot Kinematics and Dynamics
Homework 3

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Problem I

Use Paden-Kahan subproblems 1&2 to find θ_1 and θ_2 necessary to rotate an initial point p = $\begin{bmatrix} 0 \\ l \\ l \end{bmatrix}$ to a final position q = $\begin{bmatrix} -l/\sqrt{2} \\ 0 \\ l+l/\sqrt{2} \end{bmatrix}$. See MLS Pgs. 99-103.

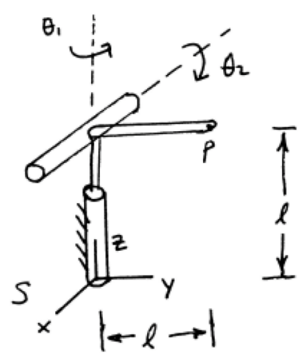

Figure 1: Problem 1

Follow the procedure from MLS pgs. 99-103, for Subproblem 1:

Let r be a point on the axis of ξ, where $r = \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}$. Define $u = (p-r) = \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}$ to be the vector between r and p, and $v = (q-r) = \begin{bmatrix} -l/\sqrt{2} \\ 0 \\ l \end{bmatrix}$ the vector between r and q. Now we define u' and v' to be the projections of u and v onto the plane perpendicular to the axis of ξ. If $\omega \in \mathbb{R}^3$ is a unit vector in the direction of the axis of ξ, then

$$u' = u - \omega\omega^T u \quad \text{and} \quad v' = v - \omega\omega^T v.$$

The problem has a solution only if the projections of u and v onto the ω-axis and onto the plane perpendicular to ω have equal lengths. Then we can find

$$\theta = \operatorname{atan2}\left(\omega^T\left(u' \times v'\right), u'^T v'\right).$$

Now we need to find ω. We have $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ Applying Law of Cosine, we have

$$a = \frac{\omega_1 \cdot \omega_2 \times \omega_2 \cdot u - \omega_1 \cdot v}{(\omega_1 \cdot \omega_2)^2 - 1} = \frac{l}{\sqrt{2}}.$$

$$b = \frac{(\omega_1 \cdot \omega_2)(\omega_1 \cdot v) - \omega_2 \cdot u}{(\omega_1 \cdot \omega_2)^2 - 1} = 0.$$

$$c=\sqrt{\frac{\|u\|^2-a^2-b^2-2ab\times\omega_1\cdot\omega_2}{\|\omega_1\times\omega_2\|^2}}=\sqrt{\frac{l^2}{2}}=\pm\frac{l}{\sqrt{2}}.$$

Now we can calculate u_1, v_1, and the corresponding u_1' and v_1': Let $e^1p=e^{-1}q=s$, where $s=a\omega_1+b\omega_2+c\left(\omega_1\times\omega_2\right)+r=\begin{bmatrix}0\\ \pm\frac{l}{\sqrt{2}}\\ \frac{l}{\sqrt{2}}\end{bmatrix}+\begin{bmatrix}0\\0\\l\end{bmatrix}=\begin{bmatrix}0\\ \pm\frac{l}{\sqrt{2}}\\ l+\frac{l}{\sqrt{2}}\end{bmatrix}$.

$$u_1=s-r=\begin{bmatrix}0\\ \pm l/\sqrt{2}\\ l/\sqrt{2}\end{bmatrix}$$

$$v_1=q-r=\begin{bmatrix}-l/\sqrt{2}\\ 0\\ l/\sqrt{2}\end{bmatrix}$$

$$u_1'=u_1-\omega_1\omega_1^Tu_1=\begin{bmatrix}0\\ \pm l/\sqrt{2}\\ 0\end{bmatrix}$$

$$v_1'=v_1-\omega_1\omega_1^\top v_1=\begin{bmatrix}-\frac{l}{\sqrt{2}}\\ 0\\ 0\end{bmatrix}$$

Plug everything into the equation for θ_1:

$$\theta=\operatorname{atan}2\left(\omega^T\left(u'\times v'\right),u'^Tv'\right)$$

we get,

$$\theta_1=\operatorname{atan}2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\left(\begin{bmatrix}0\\ \pm l/\sqrt{2}\\ 0\end{bmatrix}\times\begin{bmatrix}-\frac{l}{\sqrt{2}}\\ 0\\ 0\end{bmatrix}\right),0\right)=\operatorname{atan}2\left(\pm\frac{l^2}{\sqrt{2}},0\right)=\pm\frac{\pi}{2}.$$

Now we solve Subproblem 2 following the same pattern:

$$u_2=p-r=\begin{bmatrix}0\\ l\\ 0\end{bmatrix}$$

$$v_2=\begin{bmatrix}0\\ \pm\frac{l}{\sqrt{2}}\\ \frac{l}{\sqrt{2}}\end{bmatrix}$$

$$u_2'=u_2-w_2\left(\omega_2\cdot u_2\right)=\begin{bmatrix}0\\ l\\ 0\end{bmatrix}$$

$$v_2'=v_2-\omega_2\left(\omega_2\cdot v_2\right)=\begin{bmatrix}0\\ \pm\frac{l}{\sqrt{2}}\\ \frac{l}{\sqrt{2}}\end{bmatrix}$$

Plug in for θ_2:

$$\theta_2 = \operatorname{atan2}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix} \cdot \left(\begin{bmatrix}0\\l\\0\end{bmatrix} \times \begin{bmatrix}0\\\pm\frac{l}{\sqrt{2}}\\\frac{l}{\sqrt{2}}\end{bmatrix}\right), \pm\frac{l^2}{\sqrt{2}}\right) = \operatorname{atan2}\left(-\frac{l^2}{\sqrt{2}}, \pm\frac{l^2}{\sqrt{2}}\right) = \pm\frac{3\pi}{4}.$$

Thus, we have

$$\theta_1 = \frac{\pi}{2}, \theta_2 = -\frac{\pi}{4}$$

and

$$\theta_1 = -\frac{\pi}{2}, \theta_2 = -\frac{3\pi}{4}$$

Problem II

Show how you would solve for the inverse kinematics of the manipulator shown in Figure 3.24 (iii) of MLS book, given a desired g_d. Show which subproblems you would solve and which points you would use to solve them (or how you would use geometric reasoning instead of one or more of the subproblems). How many solutions are possible?

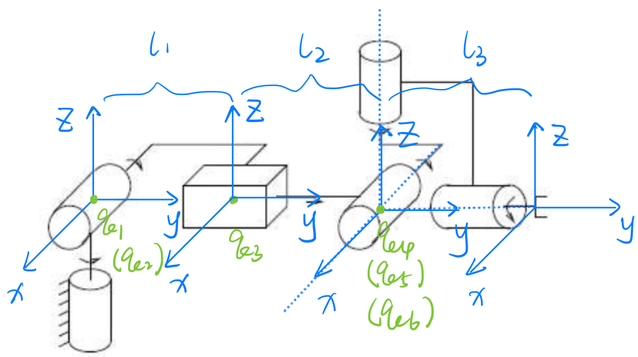

Figure 2: Standford Manipulator

To decide which Subproblem to use and how to solve the systematic problem step by step, we need to understand what every Subproblem is capable of. As illustrated in MLS book,

1. Subproblem 1 is to solve rotation about a single axis.
2. Subproblem 2 is to solve rotation about two subsequent axes.
3. Subproblem 3 is to solve rotation to a given distance.

Define the problem to be

$$e^1 e^2 e^3 e^4 e^5 e^6 g_{st}(0) = g_d \iff g_{st} g_d^{-1} = e^{-1} e^{-2} \dots e^{-6}$$

We can see that the 4 to 6 rotation is offset by the previous prismatic joint, which could be solved using Subproblem 3, i.e., to solve rotation to a given distance. We need to construct a setup where we can use Subproblem 3 to solve for θ_4.
Let p_5 be on ω_5 and ω_6, which should be at the location of q_4. Let p_1 be on ω_1, ω_2, and ω_3, which should be at the location of q_1. Then we have

$$g_{st}(0) g_d^{-1} p_1 - p_5 = e^{-6} e^{-5} e^{-4} p_1 - p_5.$$

Since p_5 on ω_5 and ω_6, we have

$$\|g_{st}(0) g_d^{-1} p_1 - p_5\|^2 = \|e^{-4} p_1 - p_5\|^2.$$

Thus, we could use Subproblem 3 to solve for θ_4,

$$\theta = \theta_0 \pm \cos^{-1}\left(\frac{\|u'\|^2 + \|v'\|^2 - \delta'^2}{2\,\|u'\|\,\|v'\|}\right),$$

$$\theta_0 = \operatorname{atan2}\left(\omega^T \left(u' \times v'\right), u'^T v'\right),$$

which will yield 2 solutions.

Now, given that we already have θ_4, we can write

$$g_{st}(0) g_d^{-1} p_1 = e^{-6} e^{-5} e^{-4} p_1 = e^{-6} e^{-5} p_4,$$

where p_4 is on ω_4, but not on ω_5 nor ω_6. This equation actually satisfies

$$e^{\widehat{\xi}_1 \theta_1} e^{\widehat{\xi}_2 \theta_2} p = q,$$

where ξ_1 and ξ_2 are two zero-pitch, unit magnitude twists with intersecting axes and $p, q \in \mathbb{R}^3$ two points. We could solve for θ_5 and θ_6 using Subproblem 2, wich will eventually go down to Subproblem 1, yielding another 2 different solutions.

Now we could do the same thing by selecting a point on ω_1 but not on ω_2 nor ω_3 and use the same method to solve for θ_2 and θ_3, which is another 2 solutions. This leaves us with

$$e^{-1} = e^2 e^3 e^4 e^5 e^6 g_{st}(0) g_d^{-1}.$$

This can be solved using Subproblem 1 by choosing a point on ω_1, which will yield one only solution for θ_1.

Thus, the total number of solutions

$$n = 2 \times 2 \times 2 \times 1 = 8.$$

Problem III

(a) A body B is initially located with origin $(1, 2, 0)$ relative to a fixed frame A, and coordinate axes parallel to A. What is the spatial velocity V_{ab}^s if the body rotates at $\frac{\pi}{2}$rad/s about its z axis?

By utilizing the twist ξ, we can have rotation axix $\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, point $q = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, angular velocity $\dot{\theta} = \frac{\pi}{2}$, and pitcc $h = 0$. Thus, we have

$$V_{ab}^s = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix} \frac{\pi}{2} = \begin{bmatrix} \pi \\ -\frac{\pi}{2} \\ 0 \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix}.$$

(b) What is the initial velocity, relative to the A frame (i.e., v_{q_a}), of a point with coordinates $q_b = (0, 1, 0)$ in the body frame? Show your math.

$$v_{q_a} = \hat{V}_{ab}^s q_a = \begin{bmatrix} \hat{\omega}_{ab}^s & v_{ab}^s \\ 0 & 0 \end{bmatrix} q_a = \begin{bmatrix} 0 & -\frac{\pi}{2} & 0 & \pi \\ \frac{\pi}{2} & 0 & 0 & -\frac{\pi}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that

$$q_a = g_{ab} q_b = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Problem IV

(a) Continued with Problem 3, what is the location of the point $\boldsymbol{q}$ after 1 second, in A coordinates? Show your math.

$$q_a(\frac{\pi}{2}) = e^{\hat{\xi}\frac{\pi}{2}} q_a(0) = \begin{bmatrix} e^{\hat{z}\frac{\pi}{2}} & \left(I - e^{\hat{z}\frac{\pi}{2}}\right)\omega \times v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Note that

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Thus, we have

$$q_a(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Suppose that B is translating with a velocity of 2π units per second along the z axis, in addition to the rotation described in Problem 3(a). Now what is V^s_{ab} ? What would the location of the point q be after 1 second, in A coordinates? Again, show your math.

Note that pitch $h = \frac{2\pi}{\frac{\pi}{2}} = 4$ now.

$$V^s_{ab} = \xi\dot{\theta} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta} = \begin{bmatrix} \pi \\ -\frac{\pi}{2} \\ 2\pi \\ 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix}$$

$$q_a(t) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2\pi \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2\pi \\ 1 \end{bmatrix}.$$

Problem V

Figure 2.17 shows a two degree of freedom manipulator. Let l_0, l_1, l_2 be the link length parameters and θ_1, θ_2 the joint angle variables of link 1 and link 2 , respectively. The coordinate axes for this problem have z pointing up, y pointing to the right, and x pointing out of the page. Use the RightHand-Rule (RHR) to determine the sign of the rotation. For simplicity, assume $C_1, C_2,$ &C3 are on the same plane. **Use Mathematica to solve the following problems:**

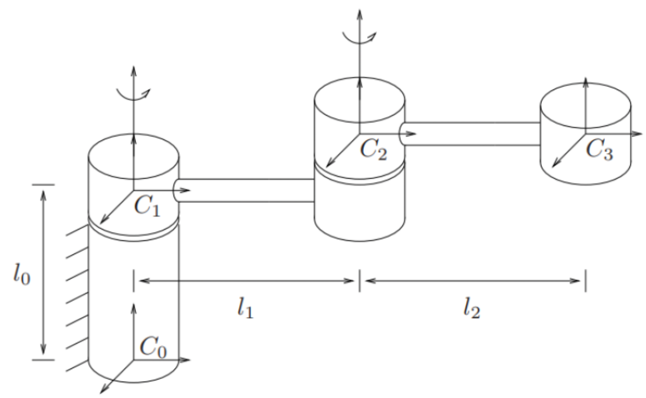

Figure 3: A two degree of freedom manipulator

(a) Express the position and orientation of frame C3 relative to frame C0 in terms of the joint angle variables and the link parameters.

(b) Compute the spatial velocity of C3 relative to C0 as functions of the joint angles and the joint rates.

(c) Compute the body velocity of C3 relative to C0 as functions of the joint angles and the joint rates.