Course: Mech 567: Robot Kinematics and Dynamics
Instructor: Robert Gregg, PhD

Solution 4

Problem 1

ZYZ

$$\xi_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0\\0\\0\\0\\1\\0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix}$$

$$e^1 = \begin{bmatrix} c\alpha & -s\alpha & 0 & 0 \\ s\alpha & c\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad e^2 = \begin{bmatrix} c\beta & 0 & s\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s\beta & 0 & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} 0\\0\\0\\-s\alpha\\c\alpha\\0 \end{bmatrix} \qquad \xi_3' = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s\beta \\ 0 \\ c\beta \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\s\beta c\alpha\\s\alpha s\beta\\c\beta \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix}$$

Singularity when $\beta = n\pi$ (columns 1, 3 degenerate)

ZXY:

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c\alpha & -s\alpha c\beta \\ 0 & -s\alpha & c\alpha c\beta \\ 1 & 0 & -s\beta \end{bmatrix}$$

Singularity when $\beta = \pi/2 + n\pi$
(columns 1, 3 degenerate)

Problem 2

$K = \frac{1}{2}m\dot{x}^2$: Kinetic Energy

$p = m\dot{x} = \frac{dK}{d\dot{x}}$: Momentum

For a mechanical system with generalized coordinates $q_1, \ldots, q_n$.

generalized momentum: $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial K}{\partial \dot{q}_k} - \cancelto{0}{\frac{\partial V}{\partial \dot{q}_k}}$

Kinetic Energy: $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$

Lagrangian: $L = K - V$

$$\sum_{k=1}^{n} \dot{q}_k p_k = \dot{q}^T p = \dot{q}^T \frac{\partial}{\partial \dot{q}}[\frac{1}{2}\dot{q}^T D(q)\dot{q}] = \dot{q}^T D(q)\dot{q} = 2K$$

Problem 3

(a)

$$
\begin{aligned}
&H = \sum_{k=1}^{n} \dot{q}_k p_k - L = \dot{q}^T p - L = \dot{q}^T \frac{\partial L}{\partial \dot{q}} - L \qquad\qquad p = \frac{\partial L}{\partial \dot{q}}\\
&L = K - V = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q)\\
&\Downarrow\\
&H = \dot{q}^T \Big(\frac{\partial}{\partial \dot{q}}\big(\frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q)\big)\Big) - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q)\\
&H = \dot{q}^T \left(\frac{1}{2}\Big(\big(\dot{q}^T D(q)\big)^T + D(q)\dot{q}\Big)\right) - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q)\\
&H = \dot{q}^T D(q)\dot{q} - \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q) = \frac{1}{2}\dot{q}^T D(q)\dot{q} + V(q) = K + V \qquad \text{Total Energy}
\end{aligned}
$$

An alternative way of doing this: from Problem 2 we know

$$\sum_{k=1}^{n} \dot{q}_k p_k = 2K \ ,$$

and $L = K - V$. Therefore,

$$H = 2K - K + V = K + V.$$

(b)

E-L Equations

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} L - \frac{\partial L}{\partial q_k} = \tau_k \qquad\qquad k = 1, \ldots, n\\
&H(q,p) = \sum_{e=1}^{n} \dot{q}_e p_e - L = \Big(\dot{q}(q,p)\Big)^T p - L(q, \dot{q}(q,p))
\end{aligned}
$$

From E-L

$$\cdot \quad \frac{\partial L}{\partial q_k} = \frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot{q}_k}L}_{p_k} - \tau_k = \frac{d}{dt}p_k - \tau_k$$

$$\cdot \quad \frac{\partial L\big(q, \dot{q}(q,p)\big)}{\partial q_k} = \frac{\partial L}{\partial q_k} + \cancel{\left(\frac{\partial L}{\partial \dot{q}}\right)^T \frac{\partial \dot{q}}{\partial q_k}} = \cancel{\left(\frac{\partial \dot{q}}{\partial q_k}\right)^T p} - \frac{\partial H}{\partial q_k} \Rightarrow \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k}$$

$$\Downarrow$$

$$-\frac{\partial H}{\partial q_k} = \dot{p}_k - \tau_k \Rightarrow \boxed{\dot{p}_k = -\frac{\partial H}{\partial q_k} + \tau_k}$$

$$\cdot \quad \frac{\partial H}{\partial p_k} = \frac{\partial}{\partial p_k}\Big(\sum_{e=1}^{n} \dot{q}_e p_e\Big) - \frac{\partial L}{\partial p_k} = \dot{q}_k + \sum_{e=1}^{n} \frac{\partial \dot{q}_e}{\partial p_k} p_e - \Big(\frac{\partial K}{\partial p_k} - \cancelto{0}{\frac{\partial V}{\partial p_k}}\Big)$$

only k-th component is $\neq 0$

$$\cdot \quad \frac{\partial H}{\partial p_k} = \dot{q}_k + \sum_{e=1}^{n} \frac{\partial \dot{q}_e}{\partial p_k} p_e - \sum_{e=1}^{n} \frac{\partial K}{\partial \dot{q}_e}\frac{\partial \dot{q}_e}{\partial p_k} = \dot{q}_k + \sum_{e=1}^{n} \frac{\partial \dot{q}_e}{\partial p_k} p_e - \sum_{e=1}^{n} p_e \frac{\partial \dot{q}_e}{\partial p_k} = \dot{q}_k$$

$$\Rightarrow \boxed{\dot{q}_k = \frac{\partial H}{\partial p_k}}$$

Problem 4

From the textbook (SHV) pages 186-188.

$$K = \frac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}^T D(q_1,q_2)\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}p_1\\ p_2\end{bmatrix}^T D^{-1}(q_1,q_2)\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

where $D(q_1,q_2) = \begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}$.

$$H = K + V$$

$$\frac{\partial H}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} = p = \frac{\partial K}{\partial \dot{q}} = D(q)\dot{q} \Rightarrow \dot{q} = D^{-1}(q)p$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\frac{\partial H}{\partial p_1}\\ \frac{\partial H}{\partial p_2}\end{bmatrix} = D^{-1}(q)\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = \begin{bmatrix}-\frac{\partial H}{\partial q_1} + \tau_1\\ -\frac{\partial H}{\partial q_2} + \tau_2\end{bmatrix} = -\begin{bmatrix}\frac{\partial K(q,p)}{\partial q_1} + \frac{\partial V}{\partial q_1} - \tau_1\\ \frac{\partial K(q,p)}{\partial q_2} + \frac{\partial V}{\partial q_2} - \tau_2\end{bmatrix}$$

See the the next page for the extra (optional) step of calculating out these equations based on the robot's dynamics terms.

In[4]:=
```
ClearAll["Global`*"];
Needs["Screws`", "C://Mathematica//Screws.m"]
Needs["RobotLinks`", "C://Mathematica//RobotLinks.m"]
```

In[7]:=
```
q = {{q1[t]}, {q2[t]}};
p = {{p1[t]}, {p2[t]}};
d11 = m1 * lc1^2 + m2 * (l1^2 + lc2^2 + 2 * l1 * lc2 * Cos[q2[t]]) + I1 + I2;
d12 = m2 * (lc2^2 + l1 * lc2 * Cos[q2[t]]) + I2;
d22 = m2 * lc2^2 + I2;
MatrixForm[Dq = {{d11, d12}, {d12, d22}}]
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \mathrm{I1} + \mathrm{I2} + \mathrm{lc1}^2\,\mathrm{m1} + \mathrm{m2}\left(\mathrm{l1}^2 + \mathrm{lc2}^2 + 2\,\mathrm{l1}\,\mathrm{lc2}\,\mathrm{Cos}[\mathrm{q2}[t]]\right) & \mathrm{I2} + \mathrm{m2}\left(\mathrm{lc2}^2 + \mathrm{l1}\,\mathrm{lc2}\,\mathrm{Cos}[\mathrm{q2}[t]]\right) \\ \mathrm{I2} + \mathrm{m2}\left(\mathrm{lc2}^2 + \mathrm{l1}\,\mathrm{lc2}\,\mathrm{Cos}[\mathrm{q2}[t]]\right) & \mathrm{I2} + \mathrm{lc2}^2\,\mathrm{m2} \end{pmatrix}$$

In[13]:=
```
MatrixForm[DqInverse = Inverse[Dq] // Simplify]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} \frac{\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2}}{(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})-\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[\mathrm{q2}[t]]^2} & -\frac{\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2}+\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]}{(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})-\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[\mathrm{q2}[t]]^2} \\ -\frac{\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2}+\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]}{(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})-\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[\mathrm{q2}[t]]^2} & \frac{\mathrm{I1}+\mathrm{I2}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2}+\mathrm{lc2}^2\,\mathrm{m2}+2\,\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]}{(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})-\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[\mathrm{q2}[t]]^2} \end{pmatrix}$$

In[19]:=
```
(*Kinetic energy*)
K = First[First[1/2 * Transpose[p].DqInverse.p // Simplify]]
```

Out[19]=

$$\Big(\left(\mathrm{I2} + \mathrm{lc2}^2\,\mathrm{m2}\right) \mathrm{p1}[t]^2 - 2\left(\mathrm{I2} + \mathrm{lc2}^2\,\mathrm{m2} + \mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]\right) \mathrm{p1}[t] \times \mathrm{p2}[t] + \left(\mathrm{I1} + \mathrm{I2} + \mathrm{lc1}^2\,\mathrm{m1} + \mathrm{l1}^2\,\mathrm{m2} + \mathrm{lc2}^2\,\mathrm{m2} + 2\,\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]\right) \mathrm{p2}[t]^2\Big) \Big/ \left(2\left(\mathrm{I1} + \mathrm{lc1}^2\,\mathrm{m1} + \mathrm{l1}^2\,\mathrm{m2}\right)\left(\mathrm{I2} + \mathrm{lc2}^2\,\mathrm{m2}\right) - 2\,\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[\mathrm{q2}[t]]^2\right)$$

In[20]:=
```
MatrixForm[DKDq  =  Transpose[{D[K, Transpose[q]]}] // FullSimplify]
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 \\ -\frac{4\,\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\left(-(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})\,(\mathrm{p1}[t]-\mathrm{p2}[t])+\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]\,\mathrm{p2}[t]\right)\left((\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,\mathrm{p2}[t]+\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]\,(-\mathrm{p1}[t]+\mathrm{p2}[t])\right)}{\left(2\,\mathrm{I2}\,(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1})+2\,(\mathrm{I2}\,\mathrm{l1}^2+\mathrm{lc2}^2\,(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}))\,\mathrm{m2}+\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2-\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2\,\mathrm{Cos}[2\,\mathrm{q2}[t]]\right)^2} \end{pmatrix}$$

In[21]:=
```
PE = m1 * g * lc1 * Sin[q1[t]] + m2 * g * (l1 * Sin[q1[t]] + lc2 * Sin[q1[t] + q2[t]]);
MatrixForm[DPEDq = Transpose[{D[PE, Transpose[q]]}] // FullSimplify]
```

Out[22]//MatrixForm=

$$\begin{pmatrix} g\left(\left(\mathrm{lc1}\,\mathrm{m1} + \mathrm{l1}\,\mathrm{m2}\right)\mathrm{Cos}[\mathrm{q1}[t]] + \mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q1}[t] + \mathrm{q2}[t]]\right) \\ g\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q1}[t] + \mathrm{q2}[t]] \end{pmatrix}$$

In[23]:=
```
MatrixForm[DKDq + DPEDq]
```

Out[23]//MatrixForm=

$$\begin{pmatrix} g\left(\left(\mathrm{lc1}\,\mathrm{m1} + \mathrm{l1}\,\mathrm{m2}\right)\mathrm{Cos}[\mathrm{q1}[t]] + \mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q1}[t] + \mathrm{q2}[t]]\right) \\ g\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q1}[t] + \mathrm{q2}[t]] - \frac{4\,\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\left(-(\mathrm{I2}+\mathrm{lc2}^2\,\mathrm{m2})\,(\mathrm{p1}[t]-\mathrm{p2}[t])+\mathrm{l1}\,\mathrm{lc2}\,\mathrm{m2}\,\mathrm{Cos}[\mathrm{q2}[t]]\,\mathrm{p2}[t]\right)\left((\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}+\mathrm{l1}^2\,\mathrm{m2})\,\mathrm{p2}[t\right.}{\left(2\,\mathrm{I2}\,(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1})+2\,(\mathrm{I2}\,\mathrm{l1}^2+\mathrm{lc2}^2\,(\mathrm{I1}+\mathrm{lc1}^2\,\mathrm{m1}))\,\mathrm{m2}+\mathrm{l1}^2\,\mathrm{lc2}^2\,\mathrm{m2}^2-\mathrm{l1}^2\right.} \end{pmatrix}$$

Problems 5-6

Define the positive x-axis pointing towards right, positive y-axis pointing upwards, and positive z-axis pointing out of the page. Euler-Lagrange equations can be found using these two equivalent formula below and should give the exactly same result if the calculation is correct:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k; k = 1, \ldots, n \tag{1}$$

where, $\mathcal{L}$ is the Lagrangian function, q_k and $\dot{q}_k$ are the dynamic system states for each joint.

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{2}$$

where, $D(q)$ is the manipulator mass/inertia matrix, $C(q, \dot{q})$ is the Coriolis matrix, and $g(q)$ is the conservative/potential forces vector. You can use both methods to verify your results for E-L equations but it's optional regarding the grading.

In[1183]:=
```
(**********************(i) two-link RR manipulator************************)
ClearAll["Global`*"];
Needs["Screws`", "C:\\Mathematica\\Screws.m"]
Needs["RobotLinks`", "C:\\Mathematica\\RobotLinks.m"]
a = 1;
b = 0.2;
c = 0;
m = 2;
Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {a / 2, 0, 0, 1}];
MatrixForm[P20 = {a + a / 2, 0, 0, 1}];
w1 = {0, 0, 1};
w2 = {0, 0, 1};
q1 = {0, 0, 0};
q2 = {a, 0, 0};

xi1 = Flatten[Append[-Cross[w1, q1], w1]];
xi2 = Flatten[Append[-Cross[w2, q2], w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1193]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1194]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1205]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t]] \\ \frac{1}{2}\,\mathrm{Sin}[\mathrm{th1}[t]] \\ 0 \\ 1 \end{pmatrix}$$

Out[1206]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\mathrm{th1}[t]] + \frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]] \\ \mathrm{Sin}[\mathrm{th1}[t]] + \frac{1}{2}\,\mathrm{Sin}[\mathrm{th1}[t] + \mathrm{th2}[t]] \\ 0 \\ 1 \end{pmatrix}$$

In[1207]:=

```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a + a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1214]//MatrixForm=

$$\begin{pmatrix} 3.34667 + 2\,\mathrm{Cos}[\mathrm{th2}[t]] & 0.673333 + \mathrm{Cos}[\mathrm{th2}[t]] \\ 0.673333 + \mathrm{Cos}[\mathrm{th2}[t]] & 0.673333 \end{pmatrix}$$

In[1215]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1215]= $\{(1.67333 + \mathrm{Cos}[th2[t]])\ th1'[t]^2 + (0.673333 + \mathrm{Cos}[th2[t]])\ th1'[t]\ th2'[t] + 0.336667\ th2'[t]^2\}$

Out[1216]= $2.g.\left(\frac{1}{2}\mathrm{Sin}[th1[t]]\right) + 2.g.\left(\mathrm{Sin}[th1[t]] + \frac{1}{2}\mathrm{Sin}[th1[t] + th2[t]]\right)$

Out[1217]= $\{-2.g.\left(\frac{1}{2}\mathrm{Sin}[th1[t]]\right) - 2.g.\left(\mathrm{Sin}[th1[t]] + \frac{1}{2}\mathrm{Sin}[th1[t] + th2[t]]\right) + (1.67333 + \mathrm{Cos}[th2[t]])\ th1'[t]^2 + (0.673333 + \mathrm{Cos}[th2[t]])\ th1'[t]\ th2'[t] + 0.336667\ th2'[t]^2\}$

Out[1226]//MatrixForm=

$$\left(\begin{array}{l} 2.g.\left(\frac{1}{2}\mathrm{Cos}[th1[t]]\right) + 2.g.\left(\mathrm{Cos}[th1[t]] + \frac{1}{2}\mathrm{Cos}[th1[t] + th2[t]]\right) - 2\,\mathrm{Sin}[th2[t]]\ th1'[t]\ th2'[t] - \mathrm{Sin}[th2[t]]\ th2'[t]^2 + 2\ (1.67333 \\ 2.g.\left(\frac{1}{2}\mathrm{Cos}[th1[t] + th2[t]]\right) + \mathrm{Sin}[th2[t]]\ th1'[t]^2 + (0.673333 + \mathrm{Cos}[th2[t]])\ th1''[t] + 0 \end{array}\right.$$

In[1227]:=
```
(*Coriolic matrix*)
(*C11=1/2. D[Mth⟦1,1⟧,th1[t]].th1'[t]+1/2. D[Mth⟦1,1⟧,th2[t]].th2'[t];
C12=1/2. (D[Mth⟦1,2⟧,th1[t]]+D[Mth⟦1,1⟧,th2[t]]-D[Mth⟦1,2⟧,th1[t]]).th1'[t]+
  1/2. (D[Mth⟦1,2⟧,th2[t]]+D[Mth⟦1,2⟧,th2[t]]-D[Mth⟦2,2⟧,th1[t]]).th2'[t];
C21=1/2. (D[Mth⟦2,1⟧,th1[t]]+D[Mth⟦2,1⟧,th1[t]]-D[Mth⟦1,1⟧,th2[t]]).th1'[t]+
  1/2. (D[Mth⟦2,1⟧,th2[t]]+D[Mth⟦2,2⟧,th1[t]]-D[Mth⟦2,1⟧,th2[t]]).th2'[t];
C22=1/2. D[Mth⟦2,2⟧,th1[t]].th1'[t];*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1231]= {{Null}, {Null}}

Out[1232]//MatrixForm=

$$\begin{pmatrix} -1.\ \mathrm{Sin}[th2[t]]\ th2'[t] & -1.\ \mathrm{Sin}[th2[t]]\ (th1'[t] + th2'[t]) \\ \mathrm{Sin}[th2[t]]\ th1'[t] & 0 \end{pmatrix}$$

In[1233]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1236]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t]]\right) + 2.g.\left(\mathrm{Cos}[\mathrm{th1}[t]] + \frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) \\ 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) \end{pmatrix}$$

Out[1237]//MatrixForm=

$$\begin{pmatrix} 0. + 0.\,i & \mathrm{Sin}[\mathrm{th2}[t]]\,(2.\,\mathrm{th1}'[t] + \mathrm{th2}'[t]) \\ -2.\,\mathrm{Sin}[\mathrm{th2}[t]]\,(\mathrm{th1}'[t] + 0.5\,\mathrm{th2}'[t]) & 0. \end{pmatrix}$$

Out[1238]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t]]\right) + 2.g.\left(\mathrm{Cos}[\mathrm{th1}[t]] + \frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) - 2.\,\mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th1}'[t]\,\mathrm{th2}'[t] - 1.\,\mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th2}'[t]^2 + 3.34667 \\ 2.g.\left(\frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t] + \mathrm{th2}[t]]\right) + \mathrm{Sin}[\mathrm{th2}[t]]\,\mathrm{th1}'[t]^2 + 0.673333\,\mathrm{th1}''[t] + \mathrm{Cos}[\mathrm{th2}[t]] \end{pmatrix}$$

In[1239]:=
```
(***********************(ii) two-link PP manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {0, 0, 0, 1}];
MatrixForm[P20 = {a / 2, 0, 0, 1}];
w1 = {0, 0, 0};
w2 = {0, 0, 0};
q1 = {0, 0, 0};
q2 = {0, 0, 0};

xi1 = Flatten[Append[{1, 0, 0}, w1]];
xi2 = Flatten[Append[{0, 1, 0}, w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1249]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1250]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1261]//MatrixForm=

$$\begin{pmatrix} th1[t] \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Out[1262]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} + th1[t] \\ th2[t] \\ 0 \\ 1 \end{pmatrix}$$

In[1263]:=
```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a / 2}, {0 , 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1270]//MatrixForm=

$$\begin{pmatrix} 4. & 0. \\ 0. & 2. \end{pmatrix}$$

In[1271]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1271]= $\{0. + 2.\,\text{th1}'[t]^2 + \text{th2}'[t]^2\}$

Out[1272]= $2.g.0 + 2.g.\text{th2}[t]$

Out[1273]= $\{0. - 2.g.0 - 2.g.\text{th2}[t] + 2.\,\text{th1}'[t]^2 + \text{th2}'[t]^2\}$

Out[1282]//MatrixForm=

$$\begin{pmatrix} 4.\,\text{th1}''[t] \\ 2.g.1 + 2\,\text{th2}''[t] \end{pmatrix}$$

In[1283]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1287]= {{Null}, {Null}}

Out[1288]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In[1289]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1292]//MatrixForm=

$$\begin{pmatrix} 0 \\ 2.g.1 \end{pmatrix}$$

Out[1293]//MatrixForm=

$$\begin{pmatrix} 0. & 0. \\ 0. & 0. \end{pmatrix}$$

Out[1294]//MatrixForm=

$$\begin{pmatrix} 0. + 4.\, th1''[t] \\ 0. + 2.g.1 + 2.\, th2''[t] \end{pmatrix}$$

In[1295]:=
```
(***********************(iii) two-link PR manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

MatrixForm[P10 = {0, 0, 0, 1}];
MatrixForm[P20 = {a, 0, 0, 1}];
w1 = {0, 0, 0};
w2 = {0, 0, 1};
q1 = {0, 0, 0};
q2 = {a / 2, 0, 0};

xi1 = Flatten[Append[{1, 0, 0}, w1]];
xi2 = Flatten[Append[-Cross[w2, q2], w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1305]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1306]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1317]//MatrixForm=

$$\begin{pmatrix} \text{th1}[t] \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Out[1318]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,(1 + \text{Cos}[\text{th2}[t]] + 2\,\text{th1}[t]) \\ \frac{1}{2}\,\text{Sin}[\text{th2}[t]] \\ 0 \\ 1 \end{pmatrix}$$

In[1319]:=

```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
xi2b = xi2;(*Body velocity*)
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2b, th2[t]}, gst20];
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify(*Inertia matrix*)
```

Out[1327]//MatrixForm=

$$\begin{pmatrix} 4. & -1.\,\text{Sin}[\text{th2}[t]] \\ -1.\,\text{Sin}[\text{th2}[t]] & 0.673333 \end{pmatrix}$$

In[1328]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1328]= $\{2\, \text{th1}'[t]^2 - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th1}'[t]\, \text{th2}'[t] + 0.336667\, \text{th2}'[t]^2\}$

Out[1329]= $2.g.0 + 2.g.\left(\frac{1}{2}\, \text{Sin}[\text{th2}[t]]\right)$

Out[1330]= $\left\{-1.\left(2.g.0 + 2.g.\left(\frac{1}{2}\, \text{Sin}[\text{th2}[t]]\right) - 2.\, \text{th1}'[t]^2 + \text{Sin}[\text{th2}[t]]\, \text{th1}'[t]\, \text{th2}'[t] - 0.336667\, \text{th2}'[t]^2\right)\right\}$

Out[1339]//MatrixForm=

$$\begin{pmatrix} -1.\, \text{Cos}[\text{th2}[t]]\, \text{th2}'[t]^2 + 4\, \text{th1}''[t] - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th2}''[t] \\ 0. + 2.g.\left(\frac{1}{2}\, \text{Cos}[\text{th2}[t]]\right) - 1.\, \text{Sin}[\text{th2}[t]]\, \text{th1}''[t] + 0.673333\, \text{th2}''[t] \end{pmatrix}$$

In[1340]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth[[i, j]], th[[k]]] + D[Mth[[i, k]], th[[j]]] - D[Mth[[k, j]], th[[i]]]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1344]= {{Null}, {Null}}

Out[1345]//MatrixForm=

$$\begin{pmatrix} 0. & -1.\,\text{Cos}[\text{th2}[t]]\,\text{th2}'[t] \\ 0. & 0. \end{pmatrix}$$

In[1346]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1349]//MatrixForm=

$$\begin{pmatrix} 0 \\ 2.g.\left(\frac{1}{2}\,\text{Cos}[\text{th2}[t]]\right) \end{pmatrix}$$

Out[1350]//MatrixForm=

$$\begin{pmatrix} 0. & \text{Cos}[\text{th2}[t]]\,\text{th2}'[t] \\ -1.\,\text{Cos}[\text{th2}[t]]\,\text{th2}'[t] & 0. \end{pmatrix}$$

Out[1351]//MatrixForm=

$$\begin{pmatrix} -1.\,\left(\text{Cos}[\text{th2}[t]]\,\text{th2}'[t]^2 - 4.\,\text{th1}''[t] + \text{Sin}[\text{th2}[t]]\,\text{th2}''[t]\right) \\ 2.g.\left(\frac{1}{2}\,\text{Cos}[\text{th2}[t]]\right) - 1.\,\text{Sin}[\text{th2}[t]]\,\text{th1}''[t] + 0.673333\,\text{th2}''[t] \end{pmatrix}$$

In[1352]:=
```
(**********************(iv) two-link RP manipulator************************)
ClearAll["Global`*"];
Needs["Screws`"]
Needs["RobotLinks`"]
a = 1;
b = 0.2;
c = 0;
m = 2;

Ixx = m / 12 * (b^2 + c^2);
Iyy = m / 12 * (a^2 + c^2);
Izz = m / 12 * (a^2 + b^2);
MatrixForm[It1 = {{Ixx, 0, 0}, {0, Iyy, 0}, {0, 0, Izz}}]
MatrixForm[It2 = It1]

P10 = {a / 2, 0, 0, 1};
P20 = {a, 0, 0, 1};
w1 = {0, 0, 1};
w2 = {0, 0, 0};
q1 = {0, 0, 0};
q2 = {0, 0, 0};

xi1 = Flatten[Append[-Cross[w1, q1], w1]];
xi2 = Flatten[Append[{0, -1, 0}, w2]];
MatrixForm[e1 = TwistExp[xi1, th1[t]]];
MatrixForm[e2 = TwistExp[xi2, th2[t]]];
MatrixForm[gs1 = e1.P10 // Simplify]
MatrixForm[gs2 = e1.e2.P20 // Simplify]
```

Out[1362]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1363]//MatrixForm=

$$\begin{pmatrix} 0.00666667 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0.173333 \end{pmatrix}$$

Out[1374]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,\mathrm{Cos}[\mathrm{th1}[t]] \\ \frac{1}{2}\,\mathrm{Sin}[\mathrm{th1}[t]] \\ 0 \\ 1 \end{pmatrix}$$

Out[1375]//MatrixForm=

$$\begin{pmatrix} \mathrm{Cos}[\mathrm{th1}[t]] + \mathrm{Sin}[\mathrm{th1}[t]]\,\mathrm{th2}[t] \\ \mathrm{Sin}[\mathrm{th1}[t]] - \mathrm{Cos}[\mathrm{th1}[t]]\,\mathrm{th2}[t] \\ 0 \\ 1 \end{pmatrix}$$

In[1376]:=

```
(*Kinetic energy*)
M = DiagonalMatrix[Join[{m, m, m}, Diagonal[It1]]];
gst10 = {{1, 0, 0, a / 2}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
gst20 = {{1, 0, 0, a}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
Jb11 = BodyJacobian[{xi1, th1[t]}, gst10];
Jb12 = {0, 0, 0, 0, 0, 0};
Jb1 = MapThread[Append, {Jb11, Jb12}];
Jb2 = BodyJacobian[{xi1, th1[t]}, {xi2, th2[t]}, gst20];
(*Mth=Transpose[Jb1].M.Jb1+Transpose[Jb2].M.Jb2;*)
MatrixForm[Mth = Transpose[Jb1].M.Jb1 + Transpose[Jb2].M.Jb2] // Simplify
```

Out[1383]//MatrixForm=

$$\begin{pmatrix} 2.84667 + 2.\,\mathrm{th2}[t]^2 & -2. \\ -2. & 2. \end{pmatrix}$$

In[1384]:=
```
K = 1 / 2. {th1'[t], th2'[t]}.Mth.{{th1'[t]}, {th2'[t]}} // Simplify
(*Potential energy*)
P = m.g.{0, 1, 0, 0}.gs1 + m.g.{0, 1, 0, 0}.gs2 // Simplify
(*Lagrange equation*)
Simplify[L = K - P]
(*Euler-Lagrange equation*)
dLdqdot1 = D[L, th1'[t]];
dLdqdot1dt = D[dLdqdot1, t];
dLdq1 = D[L, th1[t]];
tau1 = dLdqdot1dt - dLdq1; // Simplify
dLdqdot2 = D[L, th2'[t]];
dLdqdot2dt = D[dLdqdot2, t];
dLdq2 = D[L, th2[t]];
tau2 = dLdqdot2dt - dLdq2 ; // Simplify
EL = {tau1, tau2} // MatrixForm
```

Out[1384]= $\left\{\left(1.42333+\text{th2}[t]^2\right)\text{th1}'[t]^2-2.\,\text{th1}'[t]\,\text{th2}'[t]+\text{th2}'[t]^2\right\}$

Out[1385]= $2.g.\left(\frac{1}{2}\,\text{Sin}[\text{th1}[t]]\right)+2.g.(\text{Sin}[\text{th1}[t]]-\text{Cos}[\text{th1}[t]]\,\text{th2}[t])$

Out[1386]= $\left\{-2.g.\left(\frac{1}{2}\,\text{Sin}[\text{th1}[t]]\right)-2.g.(\text{Sin}[\text{th1}[t]]-\text{Cos}[\text{th1}[t]]\,\text{th2}[t])+\left(1.42333+\text{th2}[t]^2\right)\text{th1}'[t]^2-2.\,\text{th1}'[t]\,\text{th2}'[t]+\text{th2}'[t]^2\right\}$

Out[1395]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\,\text{Cos}[\text{th1}[t]]\right)+2.g.(\text{Cos}[\text{th1}[t]]+\text{Sin}[\text{th1}[t]]\,\text{th2}[t])+4\,\text{th2}[t]\,\text{th1}'[t]\,\text{th2}'[t]+2\left(1.42333+\text{th2}[t]^2\right)\text{th1}''[t]-2.\,\text{th2}''[t] \\ 2.g.(-\text{Cos}[\text{th1}[t]])-2\,\text{th2}[t]\,\text{th1}'[t]^2-2.\,\text{th1}''[t]+2\,\text{th2}''[t] \end{pmatrix}$$

In[1396]:=
```
(*Coriolic matrix*)
th = {{th1[t]}, {th2[t]}};
Tau[i_, j_, k_] := 1 / 2 * (D[Mth⟦i, j⟧, th⟦k⟧] + D[Mth⟦i, k⟧, th⟦j⟧] - D[Mth⟦k, j⟧, th⟦i⟧]);
C11 = Tau[1, 1, 1] * th1'[t] + Tau[1, 1, 2] * th2'[t];
C12 = Tau[1, 2, 1] * th1'[t] + Tau[1, 2, 2] * th2'[t];
C21 = Tau[2, 1, 1] * th1'[t] + Tau[2, 1, 2] * th2'[t];
C22 = Tau[2, 2, 1] * th1'[t] + Tau[2, 2, 2] * th2'[t];
MatrixForm[Cmatrix = {{C11, C12}, {C21, C22}} // Simplify]
```

Out[1400]= {{Null}, {Null}}

Out[1401]//MatrixForm=

$$\begin{pmatrix} 0. + 2.\, th2[t]\, th2'[t] & 0. + 2.\, th2[t]\, th1'[t] \\ 0. - 2.\, th2[t]\, th1'[t] & 0 \end{pmatrix}$$

In[1402]:=
```
(*Gravity vector*)
G1 = D[P, th1[t]];
G2 = D[P, th2[t]];
Gmatrix = {{G1}, {G2}};
MatrixForm[Gmatrix // Simplify]
(*Verify Ddot-2C*)
MatrixForm[D2C = D[Mth, t] - 2. Cmatrix] // Simplify
(*Euler-Lagrange Equations*)
MatrixForm[Simplify[Mth.{th1''[t], th2''[t]} + Cmatrix.{th1'[t], th2'[t]} + Gmatrix]]
```

Out[1405]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\, Cos[th1[t]]\right) + 2.g.\,(Cos[th1[t]] + Sin[th1[t]]\, th2[t]) \\ 2.g.\,(-Cos[th1[t]]) \end{pmatrix}$$

Out[1406]//MatrixForm=

$$\begin{pmatrix} 0. & 0. - 4.\, th2[t]\, th1'[t] \\ 0. + 4.\, th2[t]\, th1'[t] & 0. \end{pmatrix}$$

Out[1407]//MatrixForm=

$$\begin{pmatrix} 2.g.\left(\frac{1}{2}\, Cos[th1[t]]\right) + 2.g.\,(Cos[th1[t]] + Sin[th1[t]]\, th2[t]) + 4.\, th2[t]\, th1'[t]\, th2'[t] + 2.\,\left(1.42333 + th2[t]^2\right) th1''[t] - 2.\, th2''[t \\ 2.g.\,(-Cos[th1[t]]) - 2.\,\left(th2[t]\, th1'[t]^2 + th1''[t] - 1.\, th2''[t]\right) \end{pmatrix}$$