Course: MECH 567: Robot Kinematics and Dynamics
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Solution 5

Problem 1

$$\dot{x}_1 = x_1 - x_1 x_2 = f_1(x_1, x_2)$$
$$\dot{x}_2 = 2x_1^2 - 2x_2 = f_2(x_1, x_2)$$

Equilibrium Points

$$x_1 - x_1 x_2 = 0 \rightarrow x_1(1 - x_2) = 0 \qquad (1)$$
$$2x_1^2 - 2x_2 = 0 \rightarrow x_1^2 - x_2 = 0 \qquad (2)$$

From (1)

$$x_1 = 0, x_2 = 1$$

Plugging into (2)

$$x_2 = 0, x_1 = \pm 1$$

Equilibrium points are: (0,0), (-1,1), (1,1)

Linearize about equilibrium points (stability determined by eigen values)

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - x_2 & -x_1 \\ 4x_1 & -2 \end{bmatrix}$$

$$A|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \lambda_1 = 1, \lambda_2 = -2 \text{ (unstable)}$$

$$A|_{(-1,1)} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \rightarrow \lambda_1 = -1 + 1.73i, \lambda_2 = -1 - 1.73i \text{ (stable)}$$

$$A|_{(1,1)} = \begin{bmatrix} 0 & -1 \\ 4 & -2 \end{bmatrix} \rightarrow \lambda_1 = -1 + 1.73i, \lambda_2 = -1 - 1.73i \text{ (stable)}$$

Problem 2

$$\dot{x}_1 = -x_1 - x_1\,x_2^2$$
$$\dot{x}_2 = -x_2 - x_2\,x_1^2$$

at equilibrium point we have

$$-x_1 - x_1\,x_2^2 = 0$$
$$-x_2 - x_2\,x_1^2 = 0$$

by factoring

$$-x_1\,(x_2^2 + 1) = 0 \qquad (1)$$
$$-x_2\,(x_1^2 + 1) = 0 \qquad (2)$$

By solving only (1), we have $x_1 = 0$ or $x_2 = \pm i$. If $x_1 = 0$, from (2) we can conclude $x_2 = 0$. If $x_2 = \pm i$, from (2) we can conclude $x_1 = \pm i$. Therefore, the system has 5 equilibrium points $(0, 0)$, $(-i, -i)$, $(-i, i)$,(i,-i), (i, i). Thus, $(0, 0)$ is the unique real equilibrium point of the system.

To investigate local stability, we find jacobian of the system.

$$J(x_1,\, x_2) = \frac{\partial f}{\partial x} = \begin{bmatrix} -1 - x_2^2 & -2x_1x_2 \\ -2x_1x_2 & -1 - x_2^2 \end{bmatrix}$$

For $(0, 0)$, we have

$$J(0,\, 0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of the Jacobian at point $(0, 0)$ are $\lambda_1 = -1$ and $\lambda_2 = -1$. Both eigenvalues are negative. Therefore, system is locally stable.

To investigate global stability we use the Lyapanov function candidate

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

$$\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = (\,-x_1 - x_1\,x_2^2\,)\,x_1 + (\,-x_2 - x_2\,x_1^2\,)\,x_2 = -x_1^2 - x_2^2 - 2x_1^2x_2^2$$

$$\begin{cases} \dot{V} = 0 & \text{when } (x_1, x_2) = (0, 0) \\ \dot{V} < 0 & \text{when } (x_1, x_2) \neq (0, 0) \end{cases}$$

Thus, the system is asymptotically stable.

$$\text{if } x_1 \to \infty \text{ or } x_2 \to \infty \qquad \Rightarrow \qquad V \to \infty$$

Therefore, the system is globally asymptotically stable.

Problem 3

Computing $\dot{V}$ from (9.20) using the skew-symmetry property and (9.18) with the gravity term $g(q_1) = 0$, we obtain

$$
\begin{aligned}
\dot{V} =& \dot{q}_1^T D \ddot{q}_1^T + \frac{1}{2} \dot{q}_1^T \dot{D} \dot{q}_1^T + \dot{q}_2^T J \ddot{q}_2^T + (q_1 - q_2)^T K (\dot{q}_1 - \dot{q}_2) + \tilde{q}^T K_p \dot{\tilde{q}}^T \\
& \ddot{q}_1 = D^{-1}[-K(q_1 - q_2) - C\dot{q}_1], \ddot{q}_2 = J^{-1}[u - K(q_2 - q_1)] \\
\dot{V} =& \dot{q}_1^T D[D^{-1}[-C\dot{q}_1 - \cancelto{0}{G(q_1)} - K(q_1 - q_2)]] + \frac{1}{2} \dot{q}_1^T \dot{D} \dot{q}_1 + \\
& \dot{q}_2^T J[J^{-1}[-K_p \tilde{q}_2 - K_d \dot{\tilde{q}}_2 - K(q_2 - q_1)]] + (q_1 - q_2)^T K (\dot{q}_1 - \dot{q}_2) + \tilde{q}^T K_p \dot{\tilde{q}}^T \\
\dot{V} =& \frac{1}{2} \dot{q}_1^T \cancelto{0}{(\dot{D} - 2C)} \dot{q}_1^T - (\dot{q}_1 - \dot{q}_2)^T K (q_1 - q_2) - \dot{q}_2^T K \tilde{q}_2 \\
& - \dot{q}_2^T K_d \dot{\tilde{q}}_2 + (q_1 - q_2)^T K (\dot{q}_1 - \dot{q}_2) + \tilde{q}^T K_p \dot{\tilde{q}}^T \\
& \textit{Cancel terms} \\
\dot{V} =& -\dot{q}_2^T K_D \dot{q}_2
\end{aligned}
$$

Thus $\dot{V} < 0$ as long as $\dot{q}_2 \neq 0$. If $\dot{q}_2 \equiv 0$, then the second equation in (9.18) implies $K(q_2 - q_1) = -K_P \tilde{q}_2$. By taking derivatives on both sides, since $\dot{q}_2$ is constant, we have $\dot{q}_1 \equiv 0$, $\ddot{q} \equiv 0$. Therefore from (9.18) we have $q_1 \equiv q_2$ and, hence, $\tilde{q}_2 = 0$. Asymptotic stability follows from Lasalle's Theorem.

Problem 4

(a) The state space is four dimensional.
(b) Choose state and control variables

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}; \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

Then

$$
\dot{x} = \begin{bmatrix} x_2 \\ -3x_1 x_3 - x_3^2 \\ x_4 \\ -x_4 \cos x_1 - 3(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & x_3 \\ 0 & 0 \\ -3x_3 \cos^2 x_1 & 1 \end{bmatrix} u
$$

(c) The system dynamic is

$$
\begin{gathered}
\ddot{y}_1 + 3y_1 y_2 + y_2^2 = u_1 + y_2 u_2 \\
\ddot{y}_2 + \cos y_1 \, \dot{y}_2 + 3(y_1 - y_2) = u_2 - 3(\cos y_1)^2 y_2 u_1
\end{gathered}
$$

We can rewrite it as

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix}+\begin{bmatrix}3y_1y_2+y_2^2\\ \cos y_1\,\dot{y}_2+3(y_1-y_2)\end{bmatrix}=\begin{bmatrix}1 & y_2\\ -3(\cos y_1)^2\,y_2 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}v_1\\ v_2\end{bmatrix},$$

where we define

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}1 & y_2\\ -3(\cos y_1)^2\,y_2 & 1\end{bmatrix}^{-1}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

The system dynamic is then

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix}=\begin{bmatrix}-3y_1y_2-y_2^2+v_1\\ -\cos y_1\,\dot{y}_2-3(y_1-y_2)+v_2\end{bmatrix}\tag{1}$$

The desired system has $\xi=0.5$ and $\omega_n=10$.

$$\ddot{e}+2\xi\omega_n\dot{e}+\omega_n^2e=0$$

$$\begin{aligned}(\ddot{y}_i-\ddot{y}_i^d)+2\xi\omega_n(\dot{y}_i-\dot{y}_i^d)+\omega_n^2(y_i-y_i^d)&=0\qquad for\ i=1,2\\ \ddot{y}_i=\ddot{y}_i^d+2\xi\omega_n(\dot{y}_i^d-\dot{y}_i)+\omega_n^2(y_i^d-y_i)&=0\qquad for\ i=1,2\\ \ddot{y}_i=\ddot{y}_i^d+10(\dot{y}_i^d-\dot{y}_i)+100(y_i^d-y_i)&=0\qquad for\ i=1,2\end{aligned}$$

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix}=\begin{bmatrix}\ddot{y}_1^d+10(\dot{y}_1^d-\dot{y}_1)+100(y_1^d-y_1)\\ \ddot{y}_2^d+10(\dot{y}_2^d-\dot{y}_2)+100(y_i^d-y_2)\end{bmatrix}\tag{2}$$

We want (1) and (2) be equal.

$$\begin{bmatrix}-3y_1y_2-y_2^2+v_1\\ -\cos y_1\,\dot{y}_2-3(y_1-y_2)+v_2\end{bmatrix}=\begin{bmatrix}\ddot{y}_1^d+10(\dot{y}_1^d-\dot{y}_1)+100(y_1^d-y_1)\\ \ddot{y}_2^d+10(\dot{y}_2^d-\dot{y}_2)+100(y_i^d-y_2)\end{bmatrix}$$

Therefore,

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}\ddot{y}_1^d+10(\dot{y}_1^d-\dot{y}_1)+100(y_1^d-y_1)+3y_1y_2+y_2^2\\ \ddot{y}_2^d+10(\dot{y}_2^d-\dot{y}_2)+100(y_i^d-y_2)+\cos y_1\,\dot{y}_2+3(y_1-y_2)\end{bmatrix}$$

Finally, the inverse dynamics control law is

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}1 & y_2\\ -3(\cos y_1)^2\,y_2 & 1\end{bmatrix}^{-1}\begin{bmatrix}\ddot{y}_1^d+10(\dot{y}_1^d-\dot{y}_1)+100(y_1^d-y_1)+3y_1y_2+y_2^2\\ \ddot{y}_2^d+10(\dot{y}_2^d-\dot{y}_2)+100(y_i^d-y_2)+\cos y_1\,\dot{y}_2+3(y_1-y_2)\end{bmatrix}$$

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\frac{1}{1+3(\cos y_1)^2\,y_2^2}\begin{bmatrix}1 & -y_2\\ 3(\cos y_1)^2\,y_2 & 1\end{bmatrix}\begin{bmatrix}\ddot{y}_1^d+10(\dot{y}_1^d-\dot{y}_1)+100(y_1^d-y_1)+3y_1y_2+y_2^2\\ \ddot{y}_2^d+10(\dot{y}_2^d-\dot{y}_2)+100(y_i^d-y_2)+\cos y_1\,\dot{y}_2+3(y_1-y_2)\end{bmatrix}$$