ROB510 Robot Kinematics and Dynamics
Homework 1

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Problem I

Use Lemma 2.3 on page 28 of MLS to prove Rodrigues' formula, Equation (2.14).

Hint: We can prove the following two lemmas first using mathematical induction, where
Lemma 0.1: $\theta^{2n-1}\hat{\omega}^{2n-1} = (-1)^{n-1}\theta^{2n-1}\hat{\omega}$ for $n = 1, \ldots, \infty$.
Lemma 0.2: $\theta^{2n}\hat{\omega}^{2n} = (-1)^{n+1}\theta^{2n}\hat{\omega}^2$ for $n = 1, \ldots, \infty$.
Using the results of the lemmas and Taylor expansion of $\sin\theta$ and $\cos\theta$, we can get the Rodrigues' formula.

Proof:

On page 28 of MLS, we have:
Lemma 2.3. *Given $\widehat{a} \in \mathrm{so}(3)$, the following relations hold:*

$$\widehat{a}^2 = aa^T - \|a\|^2 I \tag{2.12}$$

$$\widehat{a}^3 = -\|a\|^2 \widehat{a} \tag{2.13}$$

and higher powers of $\widehat{a}$ can be calculated recursively.

For **Lemma 0.1:**
Base: when $n = 1$, we have $\theta\hat{\omega} = 1\theta\hat{\omega}$, which is obviously True.
Inductive Step: We suppose **Lemma 0.1** is True for $n = k$, then when $n = k + 1$,

$$\begin{aligned}
\theta^{2k+1}\hat{\omega}^{2k+1} &= \theta^{2k+1}\hat{\omega}^{2k-2}(-\|\hat{\omega}\|^2\widehat{\hat{\omega}}) \text{ from Eqn. (2.13)} \\
&= \theta^{2k+1}\hat{\omega}^{2k-2}(-\widehat{\hat{\omega}}) \\
&= -\theta^{2k+1}\hat{\omega}^{2k-1} \\
&= -\theta^2\theta^{2k-1}\hat{\omega}^{2k-1} \\
&= -\theta^2(-1)^{n-1}\theta^{2k-1}\hat{\omega} \text{ from the base step} \\
&= (-1)^k\theta^{2k+1}\hat{\omega}
\end{aligned}$$

Thus, the statement is True for induction step. Hence, by mathematical induction, the statement is True for $\forall n \in \mathcal{N}^+$.

For **Lemma 0.2:**
Base: when $n = 1$, we have $\theta^2\hat{\omega}^2 = \theta^2\hat{\omega}^2$, which is obviously True.
Inductive Step: We suppose **Lemma 0.2** is True for $n = k$, then when $n = k + 1$,

$$\begin{aligned}
\theta^{2k+2}\hat{\omega}^{2k+2} &= \theta^2\theta^{2k}\hat{\omega}^{2k}(\omega\omega^T - \|\omega\|^2 I) \text{ from Eqn. (2.12)} \\
&= -\theta^2\theta^{2k}\hat{\omega}^{2k} I \text{ since } \omega\omega^T = 0 \\
&= -\theta^2(-1)^{k+1}\theta^{2k}\hat{\omega}^2 \text{ by the base step} \\
&= (-1)^{k+2}\theta^{2k+2}\hat{\omega}^2
\end{aligned}$$

Thus, the statement is True for induction step. Hence, by mathematical induction, the statement is True for $\forall n \in \mathcal{N}^+$.

Since the Taylor Expansion of matrix exponential is:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$$

Then,

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^3\theta^3}{3!} + \frac{\hat{\omega}^4\theta^4}{4!} + \cdots \\ &= I + (\hat{\omega}\theta + \frac{\hat{\omega}^3\theta^3}{3!} + \cdots) + (\frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^4\theta^4}{4!} + \cdots) \\ &= I + (\hat{\omega}\theta - \frac{\hat{\omega}\theta^3}{3!} + \cdots) + (\frac{\hat{\omega}^2\theta^2}{2!} - \frac{\hat{\omega}^4\theta^4}{4!} + \cdots) \end{aligned}$$

Since

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots ,$$

and

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots ,$$

Thus,

$$e^{\hat{\omega}\theta} = I + \sin(\theta)\hat{\omega} + (1 - \cos(\theta))\hat{\omega}^2$$

Q.E.D

Problem II

Equation (2.18) on page 30 of MLS shows one way to extract ω from a 3×3 rotation matrix R. This mapping breaks down for $\theta = 0$ and $\theta = \pi$, however, which can be found from (2.17). This is to be expected at $\theta = 0$, because this is a singularity (the axis of rotation ω is undefined when $\theta = 0$). When $\theta = \pi$, there is not a singularity; one should be able to find ω. Use Rodrigues' formula to show that when $R = I, \theta = 0$ and ω is arbitrary.

Proof:

From Rodrigues's formula, we have

$$e^{\hat{\omega}\theta} = I + \sin(\theta)\hat{\omega} + (1 - \cos(\theta))\hat{\omega}^2,$$

when $R = I$,

$$e^{\hat{\omega}\theta} = R = I = I + \sin(\theta)\hat{\omega} + (1 - \cos(\theta))\hat{\omega}^2.$$

Thus,

$$\sin(\theta)\hat{\omega} = (\cos(\theta) - 1)\hat{\omega}^2$$

We now have a matrix equals to another matrix, indicating each corresponding element of them should be the same. However for $\hat{\omega} \in so(3)$, suppose

$$\hat{\omega} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix},$$

we have

$$\hat{\omega}^2 = \begin{bmatrix} -a^2 - b^2 & 0 & 0 \\ 0 & -a^2 - c^2 & 0 \\ 0 & 0 & -b^2 - c^2 \end{bmatrix},$$

which are impossible to be equal to each other. Thus, $\sin(\theta) = 1 - \cos(\theta) = 0$, indicating

$$\theta = 2k\pi, k \in \mathbf{Z}.$$

Thus, $\hat{\omega}$ can be of any value, which indicates it's indeed arbitrary.

Q.E.D

Problem III

Equation (2.18) is just one way of finding ω from the nine terms in R-you have nine equations you can use and only three unknowns. From (2.16), find an alternative to equation (2.18) that will give ω for the particular case that $\theta = \pi$.

Note: when we see methods to solve the inverse kinematics of a multi-link manipulator in a few weeks, our knowledge of the structure of the manipulator will give us ω, and we will use a more robust method of finding θ from the known ω.

From (2.16), we have

$$e^{\widehat{\omega}\theta} = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1-\cos\theta).$$

Plug in $\theta = \pi$,

$$\begin{aligned} e^{\widehat{\omega}\theta} &= \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix} \\ &= \begin{bmatrix} 2\omega_1^2 - 1 & 2\omega_1\omega_2 & 2\omega_1\omega_3 \\ 2\omega_1\omega_2 & 2\omega_2^2 - 1 & 2\omega_2\omega_3 \\ 2\omega_1\omega_3 & 2\omega_2\omega_3 & 2\omega_3^2 - 1 \end{bmatrix}. \end{aligned}$$

Thus, we get

$$\begin{cases} 2\omega_1^2 - 1 = R_{11} \\ 2\omega_2^2 - 1 = R_{22} \\ 2\omega_3^2 - 1 = R_{33}, \end{cases}$$

which yields

$$\begin{cases} \omega_1 = \pm\sqrt{R_{11}+1} \\ \omega_2 = \pm\sqrt{R_{22}+1} \\ \omega_3 = \pm\sqrt{R_{33}+1}. \end{cases}$$

Problem IV

Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is, R satisfies $R = e^{\hat{\omega}\theta}$.

Hint: Note that the eigenvalues of R are the exponentials of the eigenvalues of $\hat{\omega}$ times θ. You don't need to find the eigenvectors to do this problem.

(a) Show that the eigenvalues of $\hat{\omega}$ are $0, i$, and $-i$, where $i = \sqrt{-1}$. Show that the eigenvector associated with $\lambda = 0$ is ω.

Proof:

Let $\hat{\omega} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$, where $a^2 + b^2 + c^2 = 1$ since $\|\omega\| = 1$.
Solve for eigenvalues of $\hat{\omega}$:

$$det(\lambda I - \hat{\omega}) = 0,$$

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda^3 + (a^2 + b^2 + c^2)\lambda = \lambda^3 + \lambda = 0,$$

$$\lambda = \pm i \text{ or } 0.$$

Now, for eigenvalue $\lambda = 0$, we have

$$\lambda v = \hat{\omega} v = 0,$$

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -cv_2 + bv_3 \\ cv_1 - av_3 \\ -bv_1 + av_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which yields

$$v_2 = \frac{b}{c} v_3, \text{ and } v_1 = \frac{a}{c} v_3,$$

indicating that $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \omega.$

Q.E.D

(b) Verify that the eigenvalues of R are $1, e^{i\theta}$, and $e^{-i\theta}$ and that the eigenvectors of R are the same as $\hat{\omega}$.

Proof:

We know from the **Hint** that eigenvalues of R is given by $\lambda_R = e^{\lambda\theta}$. By Cayley-Hamilton Theory, we have $R = e^{\hat{\omega}\theta} = I + a\hat{\omega}\theta + b(\hat{\omega}\theta)^2$. Then, since v is the eigenvector of $\hat{\omega}$ with eigenvalue λ, we have

$$Rv = Iv + a\hat{\omega}\theta v + b\hat{\omega}^2 v\theta^2 = Iv + a\lambda\theta v + b\lambda^2 v\theta^2 = I + a\lambda\theta + b(\lambda\theta)^2)v = e^{\lambda\theta} v.$$

Thus, we proved that the statement from the **Hint** is indeed True.
Now, plug in the λ values, we get

$$\lambda_R = e^{\pm i\theta} \text{ and } 1.$$

Q.E.D

Problem V

Let $SO(2)$ be the set of all 2×2 orthogonal matrices with determinant equal to $+1$.

(a) Show that $SO(2)$ can be identified with the $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$.

Hint: Where you are on a circle can be identified by a single parameter θ with periodicity 2π. Show, using the definition of the properties of $SO(n)$ in section 2.1, that any member of $SO(2)$ can also be written as a function of θ.

Proof:

Let $A \in SO(2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since $A \in SO(2)$, we have

$$\begin{cases} ac + bd = 0 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \end{cases}$$

With these properties, we can let

$$\begin{cases} a = \sin(\alpha) \\ b = \cos(\alpha) \\ c = \sin(\alpha) \\ d = \cos(\beta), \end{cases}$$

with which we have

$$\sin(\alpha)\sin(\alpha) + \cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) = 0.$$

Thus, we can let $\beta = \alpha + \frac{\pi}{2} + k\pi$, $k \in \mathbf{Z}$ which yields

$$A_1 = \begin{bmatrix} \sin(\alpha) & \cos(\alpha) \\ \cos(\alpha) & -\sin(\alpha) \end{bmatrix},$$

or

$$A_2 = \begin{bmatrix} \sin(\alpha) & \cos(\alpha) \\ -\cos(\alpha) & \sin(\alpha) \end{bmatrix}.$$

Since $det(A) = 1$, we have

$$A = A_2 = \begin{bmatrix} \sin(\alpha) & \cos(\alpha) \\ -\cos(\alpha) & \sin(\alpha) \end{bmatrix}.$$

Thus, the mapping: $\alpha \rightarrowtail \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is injective for $\alpha \in [0, 2\pi)$.
Thus, $SO(2)$ can be identified with the $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$.

Q.E.D

(b) Let $\omega \in \mathbb{R}$ be a real number and define $\hat{\omega} \in so(2)$ as the skew-symmetric matrix

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

Show that

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \sin\omega\theta & \cos\omega\theta \end{bmatrix}.$$

Is the exponential map $\exp : so(2) \to SO(2)$ surjective? injective?

Hints: An injective mapping is one-to-one. Don't use Rodrigues' formula-it is only proven for $so(3)$.

Proof:

The Taylor expansion of a matrix exponential is

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^3\theta^3}{3!} + \frac{\hat{\omega}^4\theta^4}{4!} + \cdots,$$

We have

$$\hat{\omega}^2 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}^2 = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} = -\omega^2 I.$$

Thus,

$$\begin{aligned} e^{\hat{\omega}\theta} &= I(1 - \frac{\omega^2\theta^2}{2!} + \frac{\omega^4\theta^4}{4!} - \cdots) + \hat{\omega}\theta\left(1 - \frac{\omega^2\theta^2}{3!} + \frac{\omega^4\theta^4}{5!} - \cdots\right) \\ &= I\cos(\omega\theta) + \hat{\omega}\sin(\omega\theta) \\ &= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix} \end{aligned}$$

Q.E.D

The mapping is surjective but not injective since for each $\hat{\omega}$, there are infinite number of θ, for example, $\theta + 2k\pi$, for $k \in \mathbf{Z}$ that yield same $e^{\hat{\omega}\theta}$.

Problem VI

Let $R \in SO(2)$ and $\hat{\omega} \in so(2)$.

(a) Show that $R\hat{\omega}R^T = \hat{\omega}$.

Proof:

Let $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and $\omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ we have

$$\begin{aligned} R\hat{\omega}R^T &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} R^T \\ &= \begin{bmatrix} -\sin\theta\omega & -\cos\theta\omega \\ \cos\theta\omega & -\sin\theta\omega \end{bmatrix} R^T \\ &= \begin{bmatrix} -\sin\theta\omega & -\cos\theta\omega \\ \cos\theta\omega & -\sin\theta\omega \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \hat{\omega} \end{aligned}$$

Q.E.D

(b) Verify that $Re^{\hat{\omega}\theta}R^T = e^{\hat{\omega}\theta}$ and $\frac{d}{dt}e^{\hat{\omega}\theta} = (\hat{\omega}\dot{\theta})e^{\hat{\omega}\theta} = e^{\hat{\omega}\theta}(\hat{\omega}\dot{\theta})$.

Hints: Matrix exponential of Λ is defined as: $e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \cdots$.

Proof:

We have $\hat{\omega} \in so(2)$. thus, we have the following property,

$$\hat{\omega}^2 = -\omega^2 I, \quad \hat{\omega}^3 = -\omega^2\hat{\omega}, \quad \hat{\omega}^4 = \hat{\omega}^2\hat{\omega}^2 = \omega^4 I, \cdots$$

Thus, we have

$$\begin{aligned} Re^{\hat{\omega}\theta}R^T &= R(I + \hat{\omega}\theta - \frac{\omega^2\theta^2}{2!}I - \frac{\omega^3\theta^3}{3!}\hat{\omega} + \cdots)R^T \\ &= R(\cos(\omega\theta)I + \sin(\omega\theta)\hat{\omega})R^T \\ &= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix} \\ &= e^{\hat{\omega}\theta} \text{ ,which has been proved in Problem V.} \end{aligned}$$

For the derivative, we have

$$\begin{aligned} \frac{d}{dt}e^{\hat{\omega}\theta} &= \hat{\omega}\dot{\theta} + \hat{\omega}\dot{\theta}\hat{\omega}\theta + \frac{\hat{\omega}\dot{\theta}(\hat{\omega}\theta)^2}{2!} + \cdots \\ &= \hat{\omega}\dot{\theta}\left(I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \cdots\right) \\ &= (\hat{\omega}\dot{\theta})e^{\hat{\omega}\theta} \end{aligned}$$

Since $e^{\hat{\omega}\theta}$ can be written in a series form and θ is a scalar, we can write

$$(\hat{\omega}\dot{\theta})e^{\hat{\omega}\theta} = e^{\hat{\omega}\theta}(\hat{\omega}\dot{\theta}).$$

Q.E.D