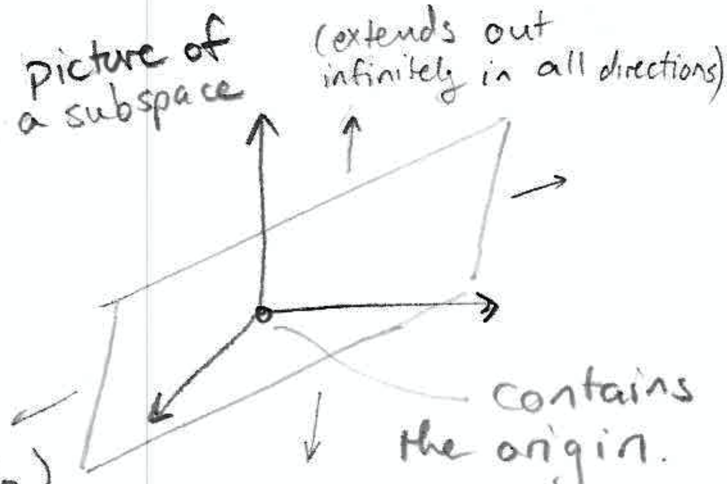
LINEAR ALGEBRA REVIEW (PART 1)

D. CALDERONE

9/6/13

CAUTION: THESE NOTES ERR ON THE SIDE OF BEING INFORMAL



- SUBSPACE:
  "a vector space contained inside another vector space"
  (a subspace always contains the origin)
- for a set of vectors $\{v_1, \dots, v_n\}$

  $\text{span}\{v_1, \dots, v_n\} :=$ the subspace containing all $v_1, \dots, v_n$

- a set of vectors is <u>linearly</u> <u>independent</u> if
  v_i does not lie in the subspace spanned by $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ for all i
- v_n is linear dependent on $\{v_1, \dots, v_{n-1}\}$ if $\exists$ scalars $\{\alpha_1, \dots, \alpha_{n-1}\}$ such that

$$v_n = \sum_{i=1}^{n-1} \alpha_i v_i$$

- a set of n vectors $\{V_1, \dots, V_n\}$, $V_i \in \mathbb{R}^n$ for all i span $\mathbb{R}^n$ iff they are all lin. independent.
- a basis for a vector space is a linear independent set of vectors that span that vector space.

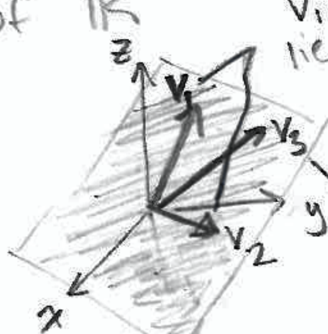
ex. the x, y, and z axes span $\mathbb{R}^3$

→ if a vector space has dimension, n a basis for that space always contains n vectors.

Ex.

2 vectors that span a subspace of $\mathbb{R}^2$

V_1, V_2 vectors lie in the shaded plane



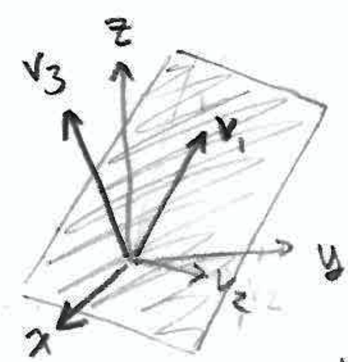
V_3 lies in the shaded plane

V_3 is lin dep on $\{V_1, V_2\}$

plane = span$\{V_1, V_2\}$

the span of 2 vectors $\{V_1, V_2\}$ is the set of all vectors that are lin dep on $\{V_1, V_2\}$

Ex.



V_3 does not lie in the shaded plane, it is linear independent from $\{V_1, V_2\}$

- we say a basis is orthogonal if all the vectors in the basis are orthogonal

  i.e. for basis $\{w_1, \dots, w_n\}$

$$w_i^T w_j = 0 \quad \text{if } i \neq j$$

- we say a basis is orthonormal if all the vectors are orthogonal and have unit norm.

  i.e.

$$w_i^T w_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Ex. the standard basis for $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis

COORDINATES:

- a set of coefficients $\{\alpha_1, \dots, \alpha_n\}$ such that a vector v is given by

$$v = \sum_{i=1}^{n} \alpha_i w_i \quad \text{for some basis } \{w_1, \dots, w_n\}$$

is called a set of coordinates for v with respect to the basis $\{w_1, \dots, w_n\}$

we usually represent these coordinates in a vector $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

and thus $v = [w_1 \cdots w_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

We can do this w/ a <u>coordinate</u> <u>transformation</u> <u>matrix</u>

$$V = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

ea. w_i has a coordinate representation in the basis $\{u_1 \dots u_n\}$,

i.e.

$$w_i = \sum_{k=1}^{n} a_i^k u_k = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \underbrace{\begin{bmatrix} a_i^1 \\ \vdots \\ a_i^n \end{bmatrix}}_{\text{coordinates of } w_i \text{ w.r.t. } \{u_1 \dots u_n\}}$$

$$V = \begin{bmatrix} \begin{bmatrix} u_1 \cdots u_n \end{bmatrix} \begin{bmatrix} a_1^1 \\ \vdots \\ a_1^n \end{bmatrix} & \cdots & \begin{bmatrix} u_1 \cdots u_n \end{bmatrix} \begin{bmatrix} a_n^1 \\ \vdots \\ a_n^n \end{bmatrix} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \underbrace{\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}$$

$$= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

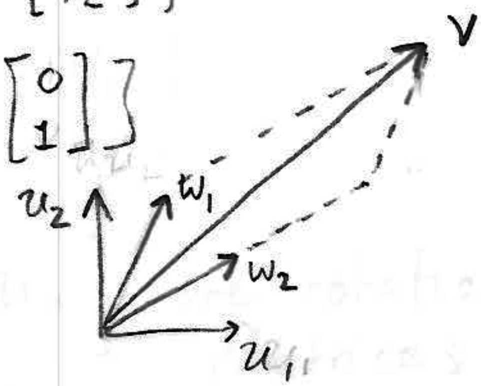
where

$$\underbrace{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}_{\text{coordinates w.r.t. } \{u_1 \dots u_n\}} = \underbrace{\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{bmatrix}}_{\text{coordinate transform. mat. } A} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\text{coordinates w.r.t } \{w_1 \dots w_n\}}$$

this method for obtaining a coord. transform mat. always works!

Ex. let $\{w_1, w_2\} = \left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}\right\}$

$\{u_1, u_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$



the coordinates of v w.r.t. $\{w_1, w_2\}$ are $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

the coordinate representation of w_1 w.r.t. $\{u_1, u_2\}$ is $\begin{bmatrix} a_1^1 \\ a_1^2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

i.e. $w_1 = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a_1^1 \\ a_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

the coord. rep. of w_2 w.r.t. $\{u_1, u_2\}$ is $\begin{bmatrix} a_2^1 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

i.e. $w_2 = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a_2^1 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

the coordinates of v w.r.t $\{u_1, u_2\}$, $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ is given by

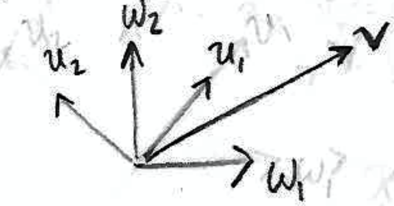
$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ 2 \end{bmatrix}$$

which is the coordinates of v w.r.t. the standard basis as we would expect.

Ex. Let $\{w_1, w_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Let $\{u_1, u_2\} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\}$ $\mathbb{R}^2$



you can check these both are orthonormal bases for $\mathbb{R}^2$.

the coordinates of v w.r.t. $\{w_1, w_2\}$ are $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

i.e. $v = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

the coordinates of v w.r.t. $\{u_1, u_2\}$ are $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

(we applied a similar coordinate change when we derived the Rodrigues formula in discussion)

PARTITIONING OF MATRICES:

Suppose we have matrices $M_1 \in \mathbb{R}^{m \times n}$ & $M_2 \in \mathbb{R}^{n \times p}$ that can be decomposed into blocks

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} M_1 \end{bmatrix} \end{array} = \begin{array}{c|cc} & l & n-l \\ k & A_1 & B_1 \\ m-k & C_1 & D_1 \end{array} \qquad \begin{array}{cc} & p \\ n & \begin{bmatrix} M_2 \end{bmatrix} \end{array} = \begin{array}{c|cc} & r & p-r \\ l & A_2 & B_2 \\ n-l & C_2 & D_2 \end{array}$$

we can multiply $M_1 M_2$ blockwise

$$M_1 M_2 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{array}{c|cc} & r & p-r \\ k & A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ m-k & C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{array}$$

you can check that the dimensions work out.

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2 SPECIAL (IMPORTANT) CASES:

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \qquad W = \begin{bmatrix} - & w_1^T & - \\ - & w_n^T & - \end{bmatrix}$$

v_i are col vectors
w_i^T are row vectors

Case 1: $VW = \sum_{i=1}^{n} \underbrace{v_i w_i^T}_{\text{these are called dyads}} \longrightarrow \begin{bmatrix} v_i \end{bmatrix} \begin{bmatrix} w_i^T \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

colvec × rowvec = full matrix

Case 2: $WV = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} = \begin{bmatrix} w_1^T v_1 & \cdots & w_1^T v_n \\ \vdots & & \\ w_n^T v_1 & \cdots & w_n^T v_n \end{bmatrix}$ $\quad [WV]_{ij} = w_i^T v_j$

$[\quad][\quad] = [\,]$

row vec × col vec = scalar

LINEAR ALGEBRA REVEW (PART 2) D. CALDERONE

9/7/13

OUTLINE:

- MATRICES AS LINEAR MAPS
- DECOMPOSITION OF THE DOMAIN & CODOMAIN OF LINEAR MAPS

$$\text{DOMAIN} = \underbrace{N(A)}_{\substack{\text{nullspace} \\ \text{of } A}} \oplus \underbrace{R(A^T)}_{\substack{\text{rangespace} \\ \text{of } A^T}} \qquad \text{CODOMAIN} = \underbrace{N(A^T)}_{\substack{\text{nullspace} \\ \text{of } A^T}} \oplus \underbrace{R(A)}_{\substack{\text{range space} \\ \text{of } A}}$$

- SQUARE MATRICES
  - ↳ EIGENVALUE PROBLEM
  - ↳ SIMILARITY TRANSFORMS (transforming a linear map into new coordinates)
  - ↳ INVERTIBILITY OF A MATRIX (a square matrix is invertible iff it does not collapse a dimension of the domain.)

LINEAR ALGEBRA REVIEW (PART 2) D. CALDERONE

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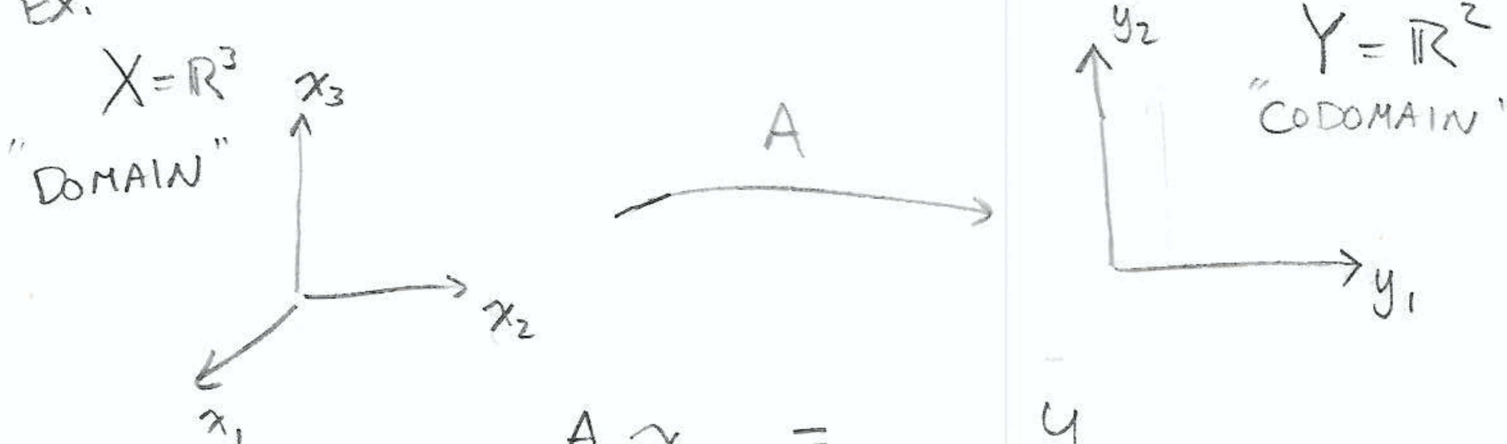
LINEARITY:

A map $f: X \to Y$ between two vector spaces X & Y is <u>linear</u> if

$$f(ax + by) = a f(x) + b f(y) \qquad \begin{array}{c} x \in X \\ y \in Y \\ a, b \text{ scalars} \end{array}$$

MATRICES ARE LINEAR MAPS BETWEEN VECTOR SPACES

Ex.



$$A x = y$$

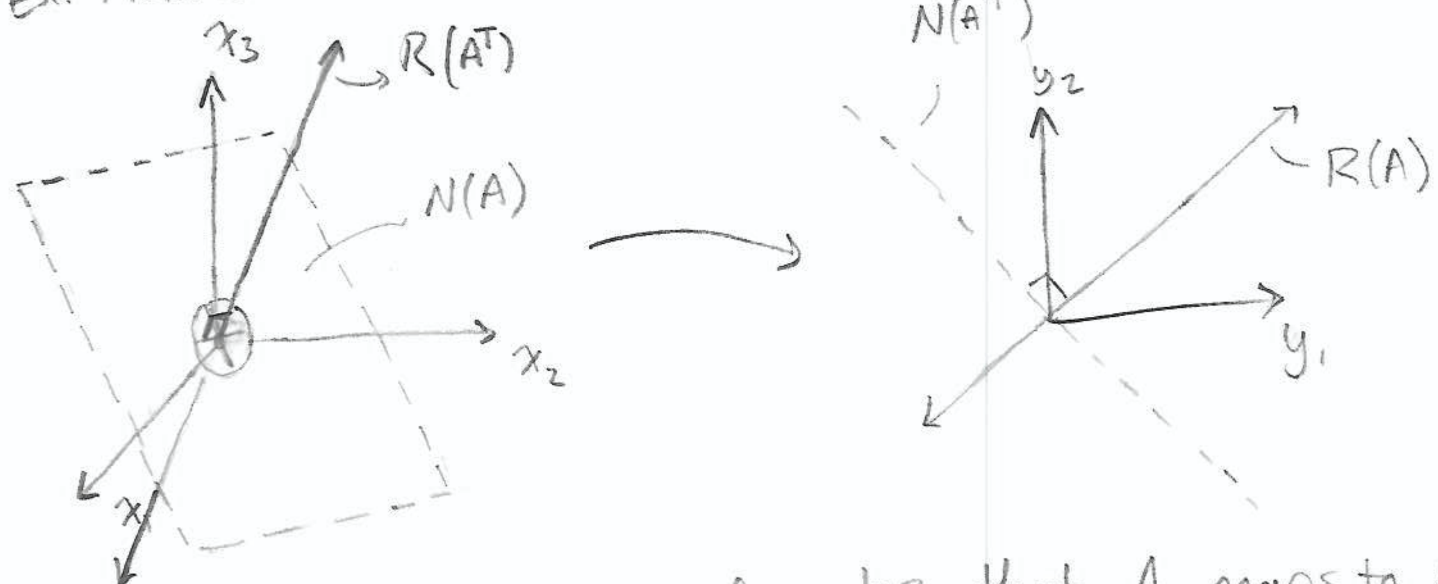
$$\underset{2 \times 3}{\begin{bmatrix} & A & \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} x \end{bmatrix}} = \underset{2 \times 1}{\begin{bmatrix} y \end{bmatrix}}$$

Square Matrices are a special case where $\underline{\dim(X)} = \underline{\dim(Y)}$

Any matrix, $A \in \mathbb{R}^{m \times n}$, decomposes its domain into two significant subspaces and its codomain into two significant subspaces

- DOMAIN = nullspace of A, denoted $N(A)$ $\oplus$ the range space of A^T, denoted $R(A^T)$
- CODOMAIN = nullspace of A^T, denoted $N(A^T)$ $\oplus$ the range space of A, denoted $R(A)$

EX. PICTURE



- Nullspace of A: space of vectors that A maps to 0 i.e. $\{x \mid Ax = 0\}$
- Rangespace of A: space of vectors in the codomain that A maps some vector to i.e. $\{y \mid \exists x \text{ s.t. } Ax = y\}$

(similarly for A^T)

ORTHOGONALITY OF $N(A)$ & $R(A^T)$:

if $x_1 \in N(A)$ & $x_2 \in R(A^T)$ $\Rightarrow$ $x_1^T x_2 = 0$

similarly for $N(A^T)$ & $R(A)$

- the dimension of $R(A)$ & $R(A^T)$ are always the same. $\dim R(A)$ is called the rank of A

Relation of $N(A)$, $R(A)$, $N(A^T)$, $R(A^T)$ to the rows & cols of A:

① $N(A)$ is orthogonal to the rows of A

i.e. $x \in N(A)$ $\quad A = \begin{bmatrix} - A_1^T - \\ \vdots \\ - A_m^T - \end{bmatrix} x = \begin{bmatrix} - A_1^T x - \\ \vdots \\ - A_m^T x - \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

② $R(A)$ is the span of the cols of A

$y \in R(A)$ $\quad y = Ax = \begin{bmatrix} \tilde{A}_1 \cdots \tilde{A}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \tilde{A}_1 x_1 + \cdots + \tilde{A}_n x_n$

③ $N(A^T)$ is orthogonal to the rows of A^T

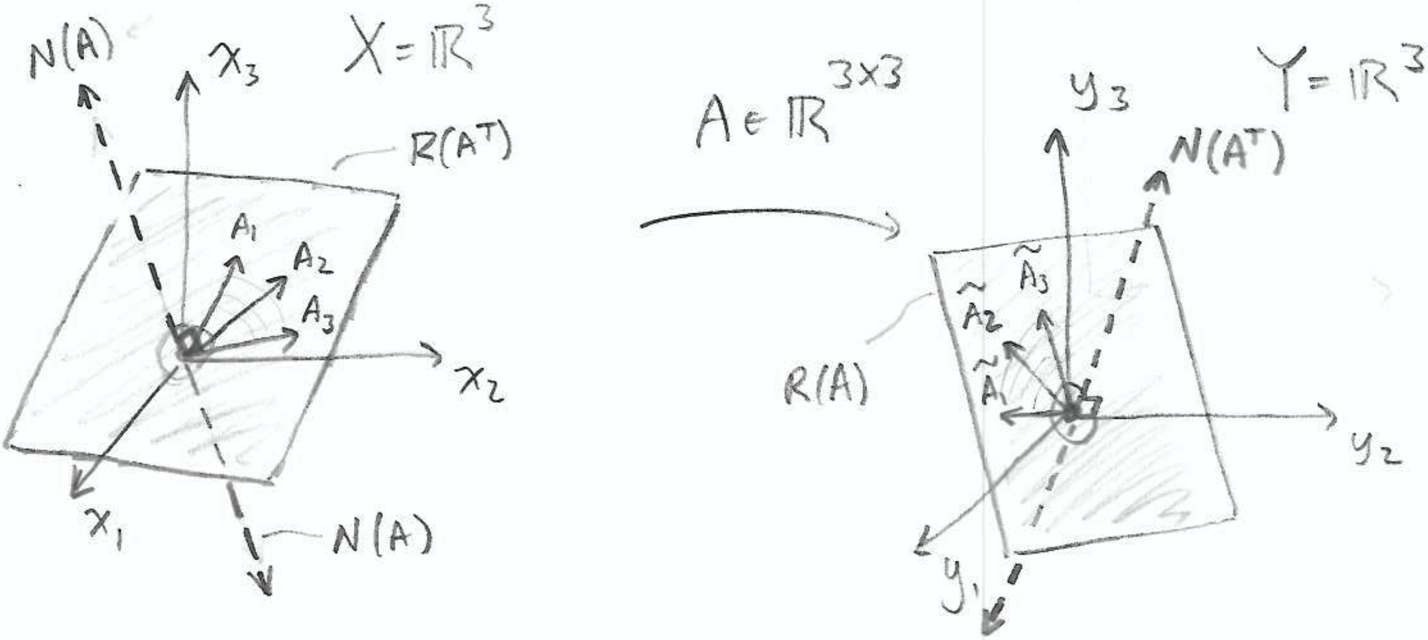
NOTE: that this is equivalent to saying $N(A^T)$ is orthogonal to $R(A)$ and ②

④ $R(A^T)$ is the span of the cols of A^T

NOTE: that this is equivalent to saying $R(A^T)$ is orthogonal to $N(A)$ and ①

Ex. PICTURE: dimension of DOMAIN & CODOMAIN $= 3$.

$\dim N(A) = 1$, $\dim R(A^T) = 2$, $\dim N(A^T) = 1$, $\dim R(A) = 2$



$$A := \begin{bmatrix} - & A_1^T & - \\ - & A_2^T & - \\ - & A_3^T & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 \\ | & | & | \end{bmatrix}$$

A_1, A_2, A_3 all lie in the same plane, $R(A^T)$

$\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ all lie in the same plane, $R(A)$

NOTE: for $A \in \mathbb{R}^{m\times n}$

- if $m > n$ (A is a tall matrix, $\begin{array}{cc} & n \\ m & \begin{bmatrix} A \end{bmatrix} \end{array}$)

  $R(A)$ cannot span all of the codomain: why?

- if $n > m$ (A is a fat matrix, $\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad A \quad \end{bmatrix} \end{array}$)

  $\dim N(A) \geq n - m$ why?

SQUARE MATRICES:

EIGEN VALUE PROBLEM: for a square matrix A, when can we find a vector x & a scalar λ such that

$$Ax = \lambda x$$

this is the eigen value problem, λ is called an eigenvalue and x is called an eigenvector

if $\lambda = 0$, $x \in N(A)$

SIMILARITY TRANSFORMS:

for a linear map $A: X \to X$ where $\dim X = n$ (A is a matrix in $\mathbb{R}^{n \times n}$) if we change coordinates in X, how should A change?

i.e. for a coordinate change $\bar{x} = Tx$

how do we find $\bar{A}$ such that $\bar{A}$ perfoms the same action on vectors in the $\bar{x}$ coordinates as A did to vectors in the x coordinates?

we want: $\bar{A}\bar{x} = TAx$

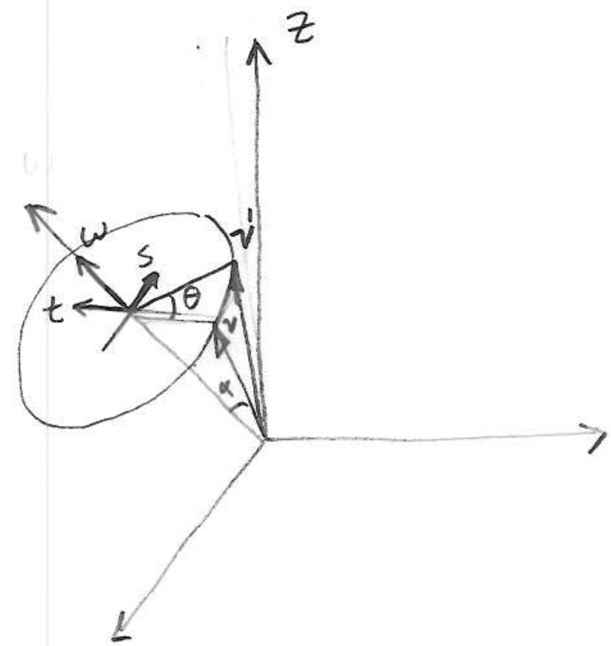
$$\bar{A}\bar{x} = \underbrace{TAT^{-1}}\bar{x}$$

thus we need $\boxed{\bar{A} = TAT^{-1}}$ this is called performing a similarity transform on A

ex. DERIVING THE GENERAL FORM OF A 3D ROTATION: (FROM DIS. 9/4)

In the w, s, t coordinate frame, the rotation is given by

$$R_{wst} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



What is R_{xyz}?

$$R_{xyz} = T R_{wst} T^{-1}$$

where T is a coordinate change s.t. $V = T V_{wst}$

in the xyz frame

$V = T v_{wst}$

$$\Rightarrow T = \left[\omega \;\middle|\; \frac{\hat{\omega} v}{|v| \sin\alpha} \;\middle|\; \frac{\hat{\omega}^2 v}{|v| \sin\alpha} \right]$$

coordinates of ω w.r.t. xyz basis (since xyz is just the standard basis this is just ω)

coordinates of s w.r.t. xyz basis...

coordinates of t w.r.t. the xyz basis...

NOTE SINCE T IS ORTHONORMAL : $T^{-1} = T^T$

$$R_{xyz} = T R_{wst} T^T = \left[\omega \;\middle|\; \frac{\hat{\omega} v}{|v| \sin\alpha} \;\middle|\; \frac{\hat{\omega}^2 v}{|v| \sin\alpha} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} - & \omega^T & - \\ - & \dfrac{v^T \hat{\omega}^T}{|v| \sin\alpha} & - \\ - & \dfrac{v^T \hat{\omega}^{2T}}{|v| \sin\alpha} & - \end{bmatrix}$$

(same as we got in discussion)

INVERTIBILITY OF A MATRIX

We say a square matrix A is invertible if $\exists$ a matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$

A matrix is invertible if there as a one-to-one correspondence between every element of the domain & codomain

(In math lingo, A is invertible if it is a bijection: both injective (one-to-one) and surjective (onto))

One-to-one correspondence between the whole domain & codomain

i.e. x & y come in pairs: $y = Ax$, $x = A^{-1}y$

the following statements are all equivalent: for square A

- A is invertible
- A has full row rank (the only element in $N(A)$ is 0)
- A has full col rank ($R(A)$ spans the whole codomain)
- A has no zero eigenvalues
- $\det(A) \neq 0$

these statements are also all equivalent:

- A is singular (not invertible)
- A has linearly dep. rows – A does not have full row rank (there are nonzero elements in $N(A)$, i.e. $\exists x \neq 0$ s.t. $Ax = 0$)
- A has linearly dep cols – A does not have full col rank ($R(A)$ does not span the whole codomain)
- A has at least one eigenvalue equal to 0
- $\det(A) = 0$

The picture you should have in your head is that a singular matrix collapses one or more dimensions of the domain. (along the directions of vectors in the null space of the matrix)