Lecture 09 : Matrix Lie Groups

objectives:

1) Derive Kinematic EOM on Lie groups (Reconstruction equations)
2) Solve it
3) A deep dive into SO(3)

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; RR^T = R^T R = I, \det R = 1 \right\}$$

I $RR^T = R^T R = I$ enforces rigid motion (orthonormality)

II $\det R = 1$ " fixed scale

w/o I the shape can deform (shear)

w/o II " " " be scaled

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad r_{ij} \in \mathbb{R}$$

$i,j = 1, 2, 3$

$$\mathrm{Vec}(R) = \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{33} \end{bmatrix}_{9\times 1} \in \mathbb{R}^9$$

if $R(t)$ is a function of time

then each $r_{ij}(t)$ is a function of time

$r_{ij} : \mathbb{R} \to \mathbb{R}, \quad t \in \mathbb{R}$

paths in SO(3)

$$\gamma(t) = R(t) = \begin{bmatrix} r_{11}(t) & r_{12}(t) & r_{13}(t) \\ r_{21}(t) & r_{22}(t) & r_{23}(t) \\ r_{31}(t) & r_{32}(t) & r_{33}(t) \end{bmatrix}$$

If $R(t)$ is a function of time
we can take its derivatives

$$\frac{d}{dt} R(t) = \dot{R} = ?$$

$$\dot{R} = \begin{bmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{bmatrix}$$

Side idea 1) $R(q(t))$, $\dot{R} = \frac{\partial R}{\partial q} \cdot \dot{q}$

〃 〃 2) Discrete R_k, R_{k+1} ...

Q: How to compute angular vel?

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1) $RR^T = I_3$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\frac{d}{dt} RR^T = \dot{R}R^T + R\dot{R}^T = 0_{3\times 3}$$

$$\dot{R}R^T + (\dot{R}R^T)^T = 0 \Rightarrow \dot{R}R^T = -(\dot{R}R^T)^T$$

Define $A := \dot{R}R^T$ then $A = -A^T$

A must be skew-symmetric, $A \in skew(3)$

$$\boxed{\dot{R}R^T = A \quad \text{or} \quad \dot{R}R^T R = AR \Rightarrow \dot{R} = AR} \quad (1)$$

2) $R^T R = I$

$$\frac{d}{dt} R^T R = \dot{R}^T R + R^T \dot{R} = (R^T \dot{R})^T + R^T \dot{R} = 0$$

$$R^T \dot{R} = -(R^T \dot{R})^T$$

$B := R^T \dot{R}$ then $B = -B^T \Rightarrow B \in skew(3)$

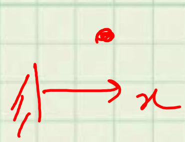
$$\boxed{R^T \dot{R} = B \quad \text{or} \quad \dot{R} = RB} \quad (2)$$

Equations (1) and (2) are reconstruction equations.

$$\dot{R} = AR = RB \Rightarrow \boxed{A = RBR^T}$$

This is called conjugation and for matrices is called matrix similarity.

Example:



x position

$\dot{x}$ velocity

$$\left.\begin{aligned} f &= ma = m\ddot{x} \\ y_1 &= x \\ y_2 &= \dot{x} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= F/m \end{aligned}$$

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \dot{y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ F/m \end{bmatrix}, \quad y(t=0) = y(0) = y_0$$

if $F=0$, $\underset{2\times1}{\dot{y}} = \underset{:=A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \underset{2\times1}{y}$, $y(0)=y_0$

for constant A, $\dot{y}=Ay$

Solution is $y(t) = \exp(At) y_0 = e^{At} y_0$

Matrix exp

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

How to solve $\dot{R} = RB$? (constant B)

$\dot{R} = RB$, $R(0) = R_0$

$\ddot{R} = \dot{R}B = RB^2$

$\vdots$

$R^{(k)} = RB^k$

$R^{(k+1)} = RB^{k+1}$

$R^{(n)} = RB^n$

$$R(t) = \sum_{n=0}^{\infty} \frac{1}{n!} R^{(n)}(0) t^n = \sum_{n=0}^{\infty} \frac{1}{n!} R_0 B^n t^n$$

$$= R_0 \exp(Bt)$$

Solution to $\dot{R} = RB$ and $\dot{R} = AR$

$$R(t) = R_0 \exp(Bt)$$

$$R(t) = \exp(At) R_0$$

for SE(3)

$T \in SE(3)$

$\dot{T} = TC$

$$T^{-1}\dot{T} = \begin{bmatrix} A & v \\ 0 & 0 \end{bmatrix}_{4\times4}$$