NA 568 - Winter 2024

Robot Motion and Uncertainty Propagation

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Objectives

1. Study robot motion described by $\mathrm{SE}(2)$ and $\mathrm{SE}(3)$ groups;
2. Focus on fully actuated models and without nonholonomic constraints commonly used in state estimation;
3. Introduce robot motion model and integration technique for rigid body motion;
4. A method for covariance propagation in the Lie algebra.

For convenience, we define the following isomorphism

$$(\cdot)^\wedge : \mathbb{R}^n \to \mathfrak{g},$$

that maps an element in the vector space $\mathbb{R}^n$ to the tangent space of the matrix Lie group at the identity.

We also define the inverse of $(\cdot)^\wedge$ map as

$$(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^n.$$

Exp and Log Maps

For any $\phi \in \mathbb{R}^n$, we can define the Lie exponential map as

$$\exp(\cdot) : \mathbb{R}^n \to \mathcal{G}, \quad \exp(\phi) = \exp_{\mathrm{m}}(\phi^{\wedge}),$$

where $\exp_{\mathrm{m}}(\cdot)$ is the exponential of square matrices.

We also define the Lie logarithmic map as the inverse of the Lie exponential map

$$\log(\cdot) : \mathcal{G} \to \mathbb{R}^n \quad \text{and} \quad \log_m(\cdot) : \mathcal{G} \to \mathfrak{g}.$$

- $\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \mid RR^{\mathsf{T}} = R^{\mathsf{T}} R = I, \det R = 1 \right\}$
- Its Lie algebra
$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A = -A^{\mathsf{T}}\}$$
is the space of 3×3 skew-symmetric matrices.
- $\mathfrak{so}(3)$ elements are three-dimensional vectors that describe angular velocities.

- Let $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3$ be a vector of angular velocity defined using the $\mathbb{R}^3$ standard basis as $\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$.

- We define the *wedge* (also called hat) notation:

$$\omega^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} =: S \in \mathfrak{so}(3).$$

- $S^\vee = (\omega^\wedge)^\vee = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$

- We can also write $\omega^\wedge = \omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3$
- $G_1 = e_1^\wedge$, $G_2 = e_2^\wedge$, and $G_3 = e_3^\wedge$
- $$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- The matrices $G1$,$G2$, and $G3$ give a basis for vector space $\mathfrak{so}(3)$ and called generators.

- $\mathrm{SO}(2) = \left\{R \in \mathbb{R}^{2\times 2} \mid RR^\mathsf{T} = R^\mathsf{T} R = I, \det R = 1\right\}$
- Its Lie algebra

$$\mathfrak{so}(2) = \{A \in \mathbb{R}^{2\times 2} \mid A = -A^\mathsf{T}\}$$

  is the space of 2×2 skew-symmetric matrices.
- Let $\omega \in \mathbb{R}$, then $\omega^\wedge = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}_{2\times 2} \in \mathfrak{so}(2)$

- $\mathrm{SE}(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}_{4\times 4} \mid R \in \mathrm{SO}(3),\ p \in \mathbb{R}^3 \right\}$
- Its Lie algebra is

$$\mathfrak{se}(3) = \left\{ \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \mid \omega^\wedge \in \mathfrak{so}(3),\ v \in \mathbb{R}^3 \right\}$$

- We call $\xi := \mathrm{vec}(\omega, v) \in \mathbb{R}^6$ twist where $\omega \in \mathfrak{so}(3)$ is the angular velocity and $v \in \mathbb{R}^3$ is the linear velocity.

- $X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$ and $\xi^\wedge = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \in \mathfrak{se}(3)$
- For $\xi^\wedge \in \mathfrak{se}(3)$ we have

$$\xi^\wedge = \omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3 + v_1 G_4 + v_2 G_5 + v_3 G_6$$

where

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$ and $\xi^\wedge = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \in \mathfrak{se}(3)$
- From the reconstruction equation, we have

$$X^{-1}\dot{X} = \xi^\wedge \in \mathfrak{se}(3)$$

$$\begin{bmatrix} R^\mathsf{T} & -R^\mathsf{T} p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^\mathsf{T}\dot{R} & R^\mathsf{T}\dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}$$

- or

$$\dot{X}X^{-1} = \xi^\wedge \in \mathfrak{se}(3)$$

$$\begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\mathsf{T} & -R^\mathsf{T} p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^\mathsf{T} & -\dot{R}R^\mathsf{T} p + \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}$$

we need to make sense of these (body vs. spatial velocities)!

- $\mathrm{SE}(2) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}_{3\times 3} \mid R \in \mathrm{SO}(2),\ p \in \mathbb{R}^2 \right\}$
- Its Lie algebra is

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}_{3\times 3} \mid \omega^\wedge \in \mathfrak{so}(2),\ v \in \mathbb{R}^2 \right\}$$

- We call $\xi := \mathrm{vec}(\omega, v) \in \mathbb{R}^3$ twist where $\omega \in \mathfrak{so}(2)$ is the angular velocity and $v \in \mathbb{R}^2$ is the linear velocity.

Let S and B represent Spatial (World/fixed/inertial) and Body (moving, attached to the robot or sensor) frames.

- The coordinates of a point in frame B, p_b, is related to its coordinates in frame S, p_s, using the action of relative rotation R_{sb}, as
$$p_s = R_{sb} p_b.$$
- We can also define a vector using the difference of two points as $v_b := p_b - q_b$ and show that
$$v_s = R_{sb} v_b = R_{sb} p_b - R_{sb} q_b = p_s - q_s.$$
- We can relate the angular velocity in the body frame B, ω_b, to the angular velocity in the spatial frame S, ω_s, via
$$\omega_s = R_{sb}\omega_b \quad \text{or} \quad \omega_b = R_{sb}^{\mathsf{T}}\omega_s$$

Recall the reconstruction equation $\dot{R} = R\omega_b^\wedge$, and consider ω_b to be in the body frame as the rotation is multiplied in the left.

The consistency with $\dot{R} = \omega_s^\wedge R$ can be proved as follows.

$$\dot{R} = \omega_s^\wedge R = RR^\mathsf{T}\omega_s^\wedge R = R\left(R^\mathsf{T}\omega_s^\wedge R\right) = R\left(R^\mathsf{T}\omega_s\right)^\wedge = R\omega_b^\wedge.$$

$$\implies \omega_s^\wedge = R\omega_b^\wedge R^\mathsf{T} \quad \text{or} \quad \omega_s = R\omega_b$$

$$\omega_b^\wedge = R^\mathsf{T}\omega_s^\wedge R \quad \text{or} \quad \omega_b = R^\mathsf{T}\omega_s$$

Lemma

For any $R \in \mathrm{SO}(3)$ and $\omega^\wedge \in \mathfrak{so}(3)$, we have

$$R\omega^\wedge R^\mathsf{T} = (R\omega)^\wedge.$$

Adjoint

The adjoint map formalizes the previous derivations for all matrix Lie groups.

The definition of the adjoint is given as

$$\mathrm{Ad}_g(\xi^\wedge) = g\xi^\wedge g^{-1},$$

where $\xi^\wedge \in \mathfrak{g}$ and $g \in \mathcal{G}$

Remark

We're after a matrix transformation called adjoint that takes a group element and maps an element of the Lie algebra to another element of the Lie algebra (change of basis for twist).

Let $\omega^\wedge \in \mathfrak{so}(3)$ and $R \in \mathrm{SO}(3)$. We have

$$\mathrm{Ad}_R(\omega^\wedge) = R\omega^\wedge R^{-1} = R\omega^\wedge R^\mathsf{T} = (R\omega)^\wedge.$$

So we learn that

$$(\mathrm{Ad}_R\omega)^\wedge = (R\omega)^\wedge \implies \boxed{\mathrm{Ad}_R = R}.$$

Remark

Since we seek a matrix that acts on $\omega \in \mathbb{R}^3$, the leftmost term must be of this form $(\mathrm{Ad}_R\omega)^\wedge$. Notice that we're solving for matrix Ad_R and $\mathrm{Ad}_R(\cdot)$ is a function in the conjugation form.

Adjoint as a Matrix (Linear Map)

$\mathrm{SE}(3)$ Case:

$X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$ and $\xi^\wedge = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix}_{4\times 4} \in \mathfrak{se}(3).$

$$(\mathrm{Ad}_X \xi)^\wedge = X \xi^\wedge X^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\mathsf{T} & -R^\mathsf{T} p \\ 0 & 1 \end{bmatrix},$$

$$\begin{aligned} (\mathrm{Ad}_X \xi)^\wedge &= \begin{bmatrix} R\omega^\wedge R^\mathsf{T} & -R\omega^\wedge R^\mathsf{T} p + Rv \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (R\omega)^\wedge & -(R\omega)^\wedge p + Rv \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$(\mathrm{Ad}_X \xi)^\wedge = \begin{bmatrix} (R\omega)^\wedge & p^\wedge R\omega + Rv \\ 0 & 0 \end{bmatrix}.$$

We can choose to stack the linear velocity $v \in \mathbb{R}^3$ and angular velocity $\omega \in \mathbb{R}^3$ in the following orders.

1. $\xi = \mathrm{vec}(\omega, v) \in \mathbb{R}^6$;

$$\mathrm{Ad}_X \xi = \begin{bmatrix} R & 0 \\ p^\wedge R & R \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}, \quad \text{and} \quad \boxed{\mathrm{Ad}_X = \begin{bmatrix} R & 0 \\ p^\wedge R & R \end{bmatrix}_{6\times 6}}.$$

2. $\xi = \mathrm{vec}(v, \omega) \in \mathbb{R}^6$;

$$\mathrm{Ad}_X \xi = \begin{bmatrix} R & p^\wedge R \\ 0 & R \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}, \quad \text{and} \quad \boxed{\mathrm{Ad}_X = \begin{bmatrix} R & p^\wedge R \\ 0 & R \end{bmatrix}_{6\times 6}}.$$

- The kinematic equation of motion is described by the curve $X(t) \in \mathrm{SE}(2)$ or $\mathrm{SE}(3)$ as

$$\frac{d}{dt}X_t = \dot{X}_t = X_t u_t^{\wedge}, \quad u_t^{\wedge} \in \mathfrak{g}.$$

- The kinematic equation of motion is described by the curve $X(t) \in \mathrm{SE}(2)$ or $\mathrm{SE}(3)$ as

$$\frac{d}{dt}X_t = \dot{X}_t = X_t u_t^{\wedge}, \quad u_t^{\wedge} \in \mathfrak{g}.$$

- The control input $u_t = \mathrm{vec}(\omega_t, v_t) \in \mathbb{R}^3$ or $\mathbb{R}^6$ is a vector of angular velocity, ω_t, and linear velocity, v_t.

- The reconstruction equation is a simple kinematic model that provides a process model to predict the motion of a rigid body.

- The integration of the reconstruction equation also makes it convenient to predict rotation and translation simultaneously.

- To see the separate terms for the rotation and translation, we can write

$$\dot{X}_t = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R\omega^\wedge & Rv \\ 0 & 0 \end{bmatrix}.$$

- Suppose we integrate this process model for a fixed sampling time $\Delta t = t_{k+1} - t_k$ by assuming a zero-order hold on the input.

- Define $u_k := u_{t_k} \cdot \Delta t$. Then we have the following discrete-time motion model.

$$X_{k+1} = X_k \exp(u_k) =: X_k U_k.$$

Noisy Process and Uncertainty Propagation

- The deterministic process model is given by

$$X_{k+1} = f_{u_k}(X_k) := X_k U_k.$$

- Substituting in the noisy process model, we have

$$X_{k+1} = f_{u_k}(X_k) \exp(w_k),$$

where $w_k \sim \mathcal{N}(0, \Sigma_{w_k})$.

Remark

w_k is the zero-mean white Gaussian noise term defined in the Lie algebra. This is convenient as the Lie algebra is a vector space, whereas the group is a nonlinear space.

Definition (Left and Right Invariant Error)

The right- and left-invariant errors between two trajectories $X_t \in \mathcal{G}$ and $\bar{X}_t \in \mathcal{G}$ are:

$$\eta_t^r = \bar{X}_t X_t^{-1} = (\bar{X}_t L)(X_t L)^{-1} \quad \text{(Right-Invariant)}$$
$$\eta_t^l = X_t^{-1} \bar{X}_t = (L\bar{X}_t)^{-1}(LX_t), \quad \text{(Left-Invariant)}$$

where $L \in \mathcal{G}$ is an arbitrary element of the group.

Using Adjoint Map to Change Order of Multiplication

The adjoint representation of a Lie group is a linear map that captures the non-commutative structure of the group.

$$\begin{aligned} X\exp(\xi)X^{-1} &= \exp\left(\mathrm{Ad}_X\xi\right) \\ X\exp(\xi) &= \exp\left(\mathrm{Ad}_X\xi\right)X \\ \exp(\xi)X &= X\exp\left(\mathrm{Ad}_{X^{-1}}\xi\right). \end{aligned}$$

In the above equations $X \in \mathcal{G}$ and $\xi^\wedge \in \mathfrak{g}$.

Remark

We used the definition of the adjoin and the property of matrix exponential. In particular, $\exp(XYX^{-1}) = X\exp(Y)X^{-1}$ *for any invertible matrix* X*.*

For $X, Y, Z \in \mathfrak{g}$ with the sufficiently small norm, the equation $\exp(X)\exp(Y) = \exp(Z)$ has a power series solution for Z in terms of repeated Lie bracket of X and Y. The beginning of the series is:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots .$$

First-Order Approximation using BCH

- The BCH formula can be used to compound two matrix exponentials.

- If both terms are small, by keeping the first two terms and ignoring the higher order terms, we have:

$$\mathrm{BCH}(\xi_1^\wedge, \xi_2^\wedge) = \xi_1^\wedge + \xi_2^\wedge + \mathrm{HOT},$$
$$\exp(\xi_1)\exp(\xi_2) \approx \exp(\xi_1 + \xi_2).$$

- We use the *left-invariant error* to track the covariance of the spatial error as seen in the body-fixed frame:

$$\eta = \exp(\xi) = X^{-1}\bar{X}.$$

- The state X is of the following form where $\bar{X}$ is the mean or estimated value of the true state:

$$X = \bar{X}\exp(-\xi).$$

- We substitute $X = \bar{X}\exp(-\xi)$ in the noisy process model as follows.

$$\bar{X}_{k+1}\exp(-\xi_{k+1}) = \bar{X}_k \exp(-\xi_k) U_k \exp(w_k),$$

- using the adjoint to shift all noise terms to the right, we get

$$\bar{X}_{k+1}\exp(-\xi_{k+1}) = \bar{X}_k U_k \exp\left(-\mathrm{Ad}_{U_k^{-1}}\xi_k\right)\exp(w_k).$$

This previous equation shows the relation between noise terms at timesteps k and $k+1$. Thus,

$$\exp(-\xi_{k+1}) = \exp\left(-\mathrm{Ad}_{U_k^{-1}}\xi_k\right)\exp(w_k).$$

- Since all noise terms are small, after applying the BCH formula and keeping the first two terms, we arrive at the approximate error dynamics:

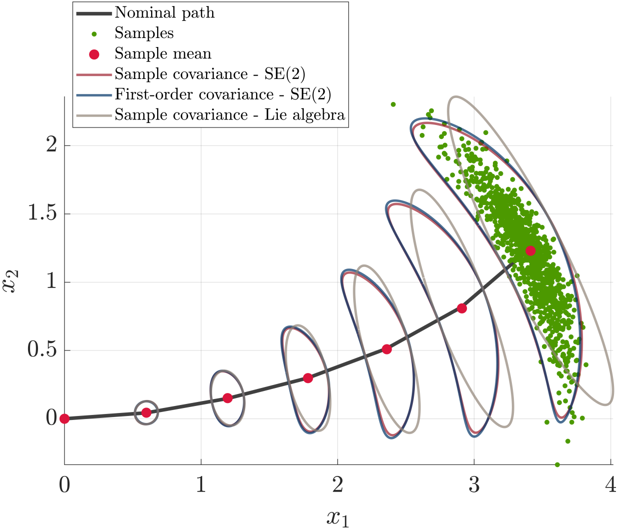
$$\xi_{k+1} = \mathrm{Ad}_{U_k^{-1}}\xi_k - w_k.$$

- Finally, we arrive at

$$\boxed{\Sigma_{k+1} = \mathrm{Ad}_{U_k^{-1}}\Sigma_k\mathrm{Ad}_{U_k^{-1}}^{\mathsf{T}} + \Sigma_{w_k}}.$$

Example: Uncertainty Propagation on $\mathrm{SE}(2)$

See Ch. 5 in Lecture Notes for Mobile Robotics and `odometry_propagation_se2.m` or `.py` for code.



Example: Uncertainty Propagation on $\mathrm{SE}(3)$

See Ch. 5 in Lecture Notes for Mobile Robotics and `odometry_propagation_se3.m` or `.py` for code.