NA 568 - Winter 2023

Matrix Lie Groups for Robotics

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Objectives

1. Build a geometric understanding of matrix Lie groups to complement our algebraic knowledge;
2. Understanding the concept of tangent space and Lie algebra for working with velocities and change of frame;
3. Use the general framework to move beyond $\mathrm{SO}(3)$ to work with $\mathrm{SE}(3)$ and $\mathrm{SE}_K(3)$;
4. Set the foundation for studying robot motion and sensing, uncertainty propagation in 2D and 3D, and optimization on manifold for robot perception.

$$\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \mid RR^\mathsf{T} = R^\mathsf{T} R = I \det R = 1 \right\}$$

1. $RR^\mathsf{T} = R^\mathsf{T} R = I$ enforces rigid motion, but $\det R = \pm 1$;
2. $\det R = 1$ ensures we have rotation, not reflection (right-hand rule).

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}, \mathrm{vec}(R) = \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{33} \end{bmatrix}_{9\times 1} \in \mathbb{R}^9$$

if $R(t)$ is a function of time then each $r_{ij}(t)$ is a function of time $r_{ij} : \mathbb{R} \to \mathbb{R} \quad , t \in \mathbb{R}$.

Previously on $\mathrm{SO}(3)$

Paths in $\mathrm{SO}(3)$: $\gamma(t) = R(t) = \begin{bmatrix} r_{11}(t) & r_{12}(t) & r_{13}(t) \\ r_{21}(t) & r_{22}(t) & r_{23}(t) \\ r_{31}(t) & r_{32}(t) & r_{33}(t) \end{bmatrix}$

If $R(t)$ is a function of time we can take its derivatives

$$\frac{d}{dt}R(t) = \dot{R} = \begin{bmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{bmatrix} =?$$

$$RR^\mathsf{T} = I_3, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{d}{dt}RR^\mathsf{T} = \dot{R}R^\mathsf{T} + R\dot{R}^\mathsf{T} = 0_{3\times 3}$$

$$\dot{R}R^\mathsf{T} + \left(\dot{R}R^\mathsf{T}\right)^\mathsf{T} = 0 \Rightarrow \dot{R}R^\mathsf{T} = -\left(\dot{R}R^\mathsf{T}\right)^\mathsf{T}$$

Define $A := \dot{R}R^\mathsf{T}$, then $A = -A^\mathsf{T}$ and A must be skew-symmetric, $A \in \mathrm{skew}(3)$.

$$\dot{R}R^\mathsf{T} = A \text{ or } \dot{R}R^\mathsf{T}R = AR \Rightarrow \boxed{\dot{R} = AR} \quad (I)$$

Next, we differentiate $R^\mathsf{T}R = I$:

$$\frac{d}{dt}R^\mathsf{T}R = \dot{R}^\mathsf{T}R + R^\mathsf{T}\dot{R} = \left(R^\mathsf{T}\dot{R}\right)^\mathsf{T} + R^\mathsf{T}\dot{R} = 0$$

Define $B := R^\mathsf{T}\dot{R}$ then $B = -B^\mathsf{T}$ and $B \in \mathrm{skew}(3)$.

$$R^\mathsf{T}\dot{R} = B \text{ or } \boxed{\dot{R} = RB} \quad (II)$$

Equations (I) and (II) are known as reconstruction equations. Given constant A or B, by solving them, we can reconstruct rotations.

$$\dot{R} = AR = RB \Rightarrow A = RBR^{\mathsf{T}}$$

This is called conjugation. For matrices, it is called matrix similarity. We will learn to derive a similar relation for general rigid transformation (rotation and translation simultaneously).

A group is a nonempty set $\mathcal{G}$ together with a binary group operation $\cdot$, e.g., $g \cdot h$ where $g,h \in \mathcal{G}$, that satisfies the following properties:

1. **Closure:** if $g,h \in \mathcal{G}$ then also $g \cdot h \in \mathcal{G}$;
2. **Associativity:** for all $g,h,l \in \mathcal{G}$, $(g \cdot h) \cdot l = g \cdot (h \cdot l)$;
3. **Identity:** there exist a unique identity element $e \in \mathcal{G}$ such that $e \cdot g = g \cdot e = g$ for all $g \in \mathcal{G}$;
4. **Inverse:** if $g \in \mathcal{G}$ there exists an element $g^{-1} \in \mathcal{G}$ such that $g^{-1} \cdot g = g \cdot g^{-1} = e$.

Check if $(\mathbb{R},+)$ and $(\mathbb{R}\backslash\{0\},\cdot)$ are groups.

1. **Closure:**

2. **Associativity:**

3. **Identity:**

4. **Inverse:**

- A matrix group is a group of invertible matrices.

In general we can work with the set of all m by n matrices with entries in $\mathbb{R}$ denoted $\mathrm{M}_{m,n}(\mathbb{R})$.

Definition

The *general linear group* over $\mathbb{R}$ is:

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) : \det(A) \neq 0\}.$$

- The n-dimensional *affine group* over $\mathbb{R}$ is

$$\mathrm{Aff}_n(\mathbb{R}) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : A \in \mathrm{GL}_n(\mathbb{R}),\ t \in \mathbb{R}^n \right\}.$$

- If we identify $x \in \mathbb{R}^n$ with $\begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$, then as a consequence of the formula

$$\begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + t \\ 1 \end{bmatrix}$$

we obtain an action of $\mathrm{Aff}_n(\mathbb{R})$ on $\mathbb{R}^n$.

- The vector space $\mathbb{R}^n$ itself can be viewed as the *translation subgroup* of $\mathrm{Aff}_n(\mathbb{R})$,

$$\mathrm{Trans}_n(\mathbb{R}) = \left\{ \begin{bmatrix} I_n & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R}^n \right\} \subseteq \mathrm{Aff}_n(\mathbb{R}),$$

and this is a closed subgroup.

Definition

The *orthogonal group* over $\mathbb{R}$ is denoted $\mathrm{O}(n)$ and defined as:

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A \cdot A^\mathsf{T} = I_n\},$$

where "$\cdot$" denotes the standard matrix multiplication as the group operation and is dropped hereafter, i.e., AA^T.

- Looking closer into the orthogonal group, we see that $\det(AA^\mathsf{T}) = \det(A)^2 = \det(I_n) = 1$;
- therefore, $\det(A) = \pm 1$. Thus we have $\mathrm{O}(n) = \mathrm{O}(n)^+ \cup \mathrm{O}(n)^-$ where

$$\mathrm{O}(n)^+ = \{A \in \mathrm{O}(n) : \det(A) = 1\},$$

$$\mathrm{O}(n)^- = \{A \in \mathrm{O}(n) : \det(A) = -1\}.$$

- Notice that $\mathrm{O}(n)^+ \cap \mathrm{O}(n)^- = \varnothing$, so $\mathrm{O}(n)$ is the *disjoint union* of the subsets $\mathrm{O}(n)^+$ and $\mathrm{O}(n)^-$.

- The important subgroup $\mathrm{SO}(n) = \mathrm{O}(n)^+ \leq \mathrm{O}(n)$ is the $n \times n$ *special orthogonal group.*

- One of the main reasons for the study of the orthogonal groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ is their relationships with *isometries*, where an isometry of $\mathbb{R}^n$ is a distance-preserving bijection $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e.,

$$\|f(x) - f(y)\| = \|x - y\| \qquad x,y \in \mathbb{R}^n$$

- If such an isometry fixes the origin, 0, then it is a *linear transformation*, often referred to as *linear isometry*, and so with respect with the standard basis it corresponds to a matrix $A \in \mathrm{GL}_n(\mathbb{R})$

Remark

The special orthogonal group $\mathrm{SO}(n)$ is the simultaneous rotation of n perpendicular planes!

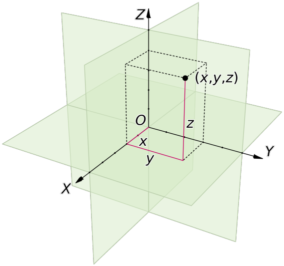


Figure: For example, $\mathrm{SO}(3)$ is the rotation group of $\mathbb{R}^3$ and defines the simultaneous rotation of three perpendicular planes which construct the three-dimensional (3D) Euclidean space.

- The *special Euclidean group* is the isometry group that requires A to be a valid right-handed rotation matrix:

$$\mathrm{SE}(n) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : A \in \mathrm{SO}(n),\ t \in \mathbb{R}^n \right\}.$$

- This is the group of valid rigid body transformations of $\mathbb{R}^n$.

Manifolds are higher-dimensional analogs of smooth curves and surfaces.
We can "chop" up manifold M into pieces that each look like $\mathbb{R}^n$.

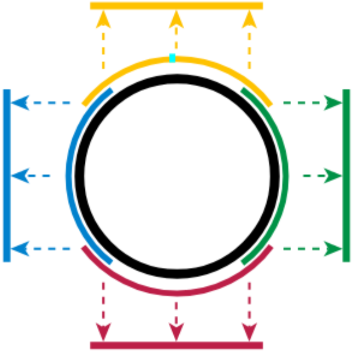
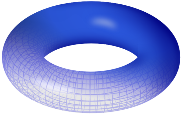


Figure: https://en.wikipedia.org/wiki/Manifold

Many common objects are manifolds.

1. Every Euclidean space, $\mathbb{R}^n$.
2. The 2-sphere, S^2

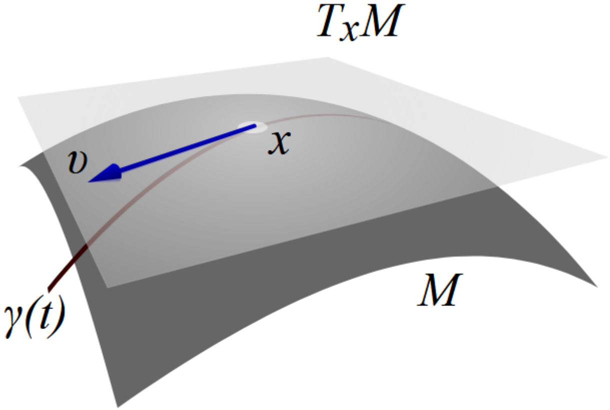
3. The Torus T^2



https://www.jpl.nasa.gov/edu/teach/activity/ocean-world-earth-globe-toss-game/

https://en.wikipedia.org/wiki/Torus

To study the geometry of a manifold, we need the notion of a tangent space. Let γ be some curve in some manifold M, then its derivative $\dot{\gamma}$ is a *tangent vector*.



- If $x \in M$ is a point in the manifold, then the space of *all possible* tangent vectors is called the *tangent space* and is denoted by T_xM.

- It is important to point out that T_xM is a vector space and

$$\dim T_xM = \dim M.$$

- A matrix group is an algebraic object. However, it can also be seen as a geometric object since it is a subset of a Euclidean space:

$$\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

- In addition, looking at a matrix group as a subset of a Euclidean space means we can discuss its tangent space.

Definition (Tangent space)

Let $\mathcal{G} \subset \mathbb{R}^m$ be a subset, and let $g \in \mathcal{G}$. The *tangent space* to $\mathcal{G}$ at g is:

$$T_g\mathcal{G} = \{\gamma'(0) : \gamma : (-\epsilon, \epsilon) \to \mathcal{G} \text{ is differentiable with } \gamma(0) = g\}.$$

$T_g\mathcal{G}$ means the set of initial velocity vectors of differentiable paths through g in $\mathcal{G}$. The term *differentiable* means that, when we consider γ as a path in $\mathbb{R}^m$, the m components of γ are differentiable functions from $(-\epsilon, \epsilon)$ to $\mathbb{R}$.

Definition (Lie algebra)

The *Lie algebra* of a matrix group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R})$ is the tangent space to $\mathcal{G}$ at the identity e. It is denoted $\mathfrak{g} = \mathfrak{g}(\mathcal{G}) = T_e\mathcal{G}$.

- Note that the choice of identity is due to the fact that all groups contain at least the identity element.

The set $\mathfrak{so}(n) = \{A \in \mathrm{M}_n(\mathbb{R}) : A + A^{\mathsf{T}} = 0\}$ is denoted $\mathfrak{so}(n)$ and called *skew-symmetric* matrices.

To see this, consider the path $\gamma(t) \in \mathrm{SO}(n)$ that satisfies $\gamma(0) = I$ and $\gamma'(0) = A$. For every matrix in $\mathrm{SO}(n)$, we have $\gamma(t) \cdot \gamma(t)^{\mathsf{T}} = I$. Using the product rule to differentiate both sides of

$$\gamma(t) \cdot \gamma(t)^{\mathsf{T}} = I$$

gives $\gamma'(0) + \gamma'^{\mathsf{T}}(0) = A + A^{\mathsf{T}} = 0$, so A must be skew-symmetric.

Example: Lie Algebras of (Real) Orthogonal Group

Corollary

$\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$.

Proof.

Skew-symmetric matrices have zeros on the diagonal, arbitrary real numbers above, and entries below determined by those above, so $\dim(\mathfrak{so}(n)) = \frac{n(n-1)}{2}$. $\square$

For a general matrix group $\mathcal{G}$, we have

$$T_g\mathcal{G} = g \cdot \mathfrak{g}.$$

This means if we can describe the tangent space $\mathfrak{g}$ at the identity, then $g \cdot \mathfrak{g}$ will describe the tangent space $T_g\mathcal{G}$ at any point $g \in \mathcal{G}$.

Corollary

For $\gamma(t) \in \mathcal{G}$, $\gamma(0) = I$ and $\gamma'(0) = A \in \mathfrak{g}$, via a left translation for a constant $g \in \mathcal{G}$ we have $g \cdot \gamma(t)$.

Differentiating this gives $g \cdot \gamma'(t)\Big|_{t=0} = g \cdot A =: \dot{g} \in T_g\mathcal{G}$.

Hence, we arrive at the following first-order differential equation known as the reconstruction equation.

$$\boxed{\dot{g} = g \cdot A \quad \text{or} \quad g^{-1}\dot{g} = A \in \mathfrak{g}}.$$

Corollary

A similar argument can be made for the right translation for any point $g \in \mathcal{G}$ *to get* $\gamma'(t)\Big|_{t=0} \cdot g = A \cdot g =: \dot{g} \in T_g\mathcal{G}$.

$$\boxed{\dot{g} = A \cdot g \quad \text{or} \quad \dot{g}g^{-1} = A \in \mathfrak{g}}.$$

Example

Let γ be a curve in $\mathrm{SO}(n)$ such that $\gamma(0) = g$. Then we know that $\dot{\gamma}(0) \in T_g\mathrm{SO}(n)$.

$$T_g\mathrm{SO}(n) = \left\{ gA : A^\mathsf{T} = -A \right\} = g \cdot \mathfrak{so}(n).$$

Solution of Reconstruction Equation

See Chapter 4.4.3 of Lecture Notes for Mobile Robotics.

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ represented in the standard basis $e_1, \cdots, e_n$ and an alternative representation of the same points denoted x' and y' in another basis $e'_1, \cdots, e'_n$ such that $e_i = P^{-1}e'_i$ for $i = 1, \cdots, n$, i.e., $x = P^{-1}x'$ and $y = P^{-1}y'$. Let A be a linear map such that $y = Ax$ and T another linear map as $y' = Tx'$.

Then A and T are similar via the following change of basis.

$$\begin{aligned} y &= Ax \\ P^{-1}y' &= AP^{-1}x' \\ y' &= PAP^{-1}x' = Tx', \\ \implies T &= PAP^{-1}. \end{aligned}$$

Let $\mathcal{G}$ be a matrix group with Lie algebra $\mathfrak{g}$. For all $g \in \mathcal{G}$, the *conjugation map* $C_g : \mathcal{G} \to \mathcal{G}$, define as

$$C_g(a) = gag^{-1},$$

is a smooth isomorphism. The derivative $\mathrm{d}(\mathrm{C}_g)_I : \mathfrak{g} \to \mathfrak{g}$ is a vector space isomorphism, which we denote as Ad_g (adjoint):

$$\mathrm{Ad}_g = \mathrm{d}(\mathrm{C}_g)_I$$

To derive a simple formula for $\mathrm{Ad}_g(B)$, notice that any $B \in \mathfrak{g}$ can be represented as $B = b'(0)$, where $b(t)$ is a differentiable path in $\mathcal{G}$ with $b(0) = I$. The product rule gives:

$$\mathrm{Ad}_g(B) = \mathrm{d}(\mathrm{C}_g)_I(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g b(t) g^{-1} = gBg^{-1}.$$

So we learn that (notice the similarity transformation):

$$\mathrm{Ad}_g(B) = gBg^{-1}.$$

Definition (Lie bracket)

The Lie bracket of two vectors A and B in $\mathfrak{g}$ is:

$$[A, B] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{a(t)}(B),$$

where $a(t)$ is any differentiable path in $\mathcal{G}$ with $a(0) = I$ and $a'(0) = A$.

Proposition

For all $A, B \in \mathfrak{g}$, $[A, B] = AB - BA$.

Proof.

Left as exercise. $\square$

Example: Lie Bracket on $\mathfrak{so}(3)$ and Cross Product

See `so3_cross_example.m` for numerical examples and details.

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[G_1, G_2] = G_3,\ [G_2, G_3] = G_1,\ [G_3, G_1] = G_2$$

$$e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2$$

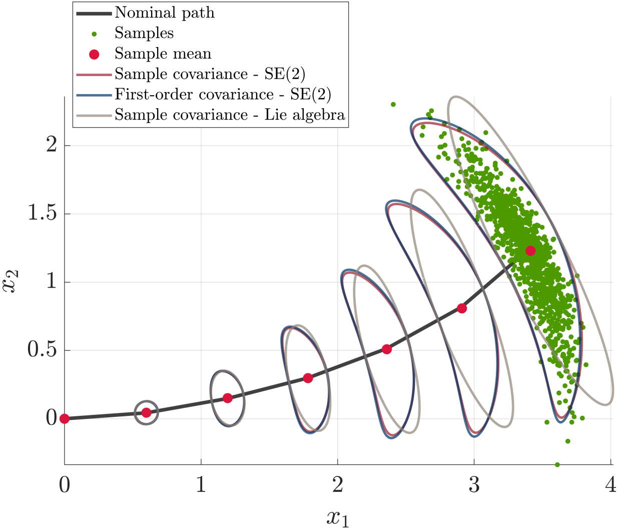
For $X, Y, Z \in \mathfrak{g}$ with sufficiently small norm, the equation $\exp(X)\exp(Y) = \exp(Z)$ has a power series solution for Z in terms of repeated Lie bracket of X and Y. The beginning of the series is:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots .$$

- Group of 3D rotation matrices, $\mathrm{SO}(3)$; it can model rotations without any singularities or ambiguities.
- Group of direct spatial isometries (3D Rigid Body Transformations), $\mathrm{SE}(3)$.
- Group of K direct isometries, $\mathrm{SE}_K(3)$; for example, it is used for modeling IMU sensors and robot pose plus landmarks and/or contact points.
- Group of 3D similarity transformations, $\mathrm{Sim}(3)$; it is more general than $\mathrm{SE}(3)$ and includes a scale factor and used in monocular vision where the scale is not known.

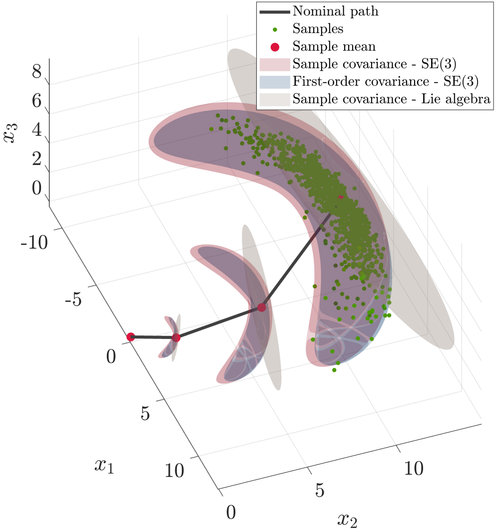
Example: Uncertainty Propagation on $\mathrm{SE}(2)$

See `odometry_propagation_se2.m` or `.py` for code.



Example: Uncertainty Propagation on $\mathrm{SE}(3)$

See `odometry_propagation_se3.m` or `.py` for code.



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