NA 568 - Winter 2022

Rigid Body Motion

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Rigid Body Motion

- A rigid motion of an object is a motion which preserves distance between points.
- The study of robot kinematics, dynamics, and control has at its heart the study of the motion of rigid objects.

- A rigid body is a collection of particles such that the distance between any two particles remains fixed, regardless of any motions of the body or forces exerted on the body.

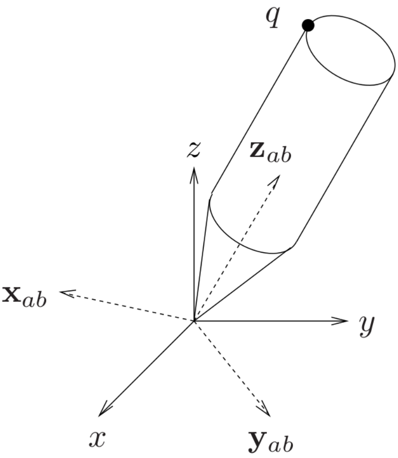
- A rigid motion of an object is a continuous movement of the particles in the object such that the distance between any two particles remains fixed at all times.
- The net movement of a rigid body from one location to another via a rigid motion is called a rigid displacement.
- In general, a rigid displacement may consist of both translation and rotation of the object.

- Relative orientation between a coordinate frame attached to the body and a fixed or inertial coordinate frame.
- All coordinate frames will be right-handed unless stated otherwise.
- Let A be the **inertial frame** and B the **body frame**.

- Let $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \in \mathbb{R}^3$ be the coordinates of the principal axes of B relative to A.

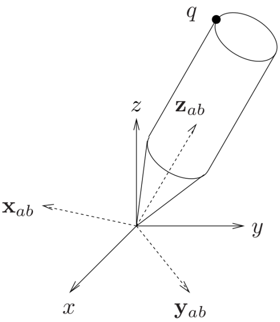
- Define the 3×3 matrix $\mathbf{R}_{ab} = \begin{bmatrix} \mathbf{x}_{ab} & \mathbf{y}_{ab} & \mathbf{z}_{ab} \end{bmatrix}$.

- $\mathbf{R}_{ab}$ is a **rotation matrix**.



The familiar rotation matrices about each axis are:

- $\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.
- $\mathbf{R}_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$.
- $\mathbf{R}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- Yaw-Pitch-Roll; $\mathbf{R}_{zyx}(\theta,\beta,\alpha) = \mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)$.



- Yaw-Pitch-Roll; $\mathbf{R}_{zyx}(\theta,\beta,\alpha) = \mathbf{R}_z(\theta)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)$.
- When $\beta = \frac{\pi}{2}$,
  $$\mathbf{R}_{zyx}(\theta,\tfrac{\pi}{2},\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) & 0 \\ -\cos(\theta+\alpha) & \sin(\theta+\alpha) & 0 \end{bmatrix}.$$
- θ and α correspond to the same rotation.
- This is a topological constraint! (Can't fix it using coordinates)

Remark

For a rotation by an angle $\theta = \pi$ about some axis e, both $R_e(\pi)$ and $R_{-e}(\pi)$ correspond to the same rotation.

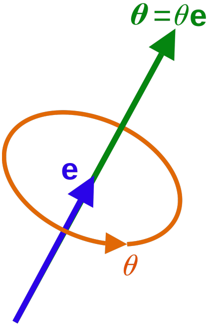


Figure: Image credit: https://en.wikipedia.org/wiki/Axis%E2%80%93angle_representation

- Let $\mathbf{R} \in \mathrm{GL}_3(\mathbb{R})$ (3×3 invertible real matrix) be a rotation matrix.
- Then $\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathbf{I}$;
- and $\det(\mathbf{R}) = 1$.

- $\mathrm{SO}(3) = \{\mathbf{R} \in \mathrm{GL}_n(\mathbb{R}) : \mathbf{R}\mathbf{R}^\mathsf{T} = \mathbf{I} \text{ and } \det(\mathbf{R}) = 1\}$
- We refer to $\mathrm{SO}(3)$ as the rotation group of $\mathbb{R}^3$.
- Every configuration of a rigid body that is free to rotate relative to a fixed frame can be identified with a unique $\mathbf{R} \in \mathrm{SO}(3)$.

Composition Rule for Rotations

- Rotation matrices can be combined using matrix multiplication.
- $\mathbf{R}_{bc}$: orientation of a frame C relative to another frame B.
- $\mathbf{R}_{ab}$: orientation of the frame B relative to another frame A.
- Then $\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$ is the orientation of frame C relative to A.

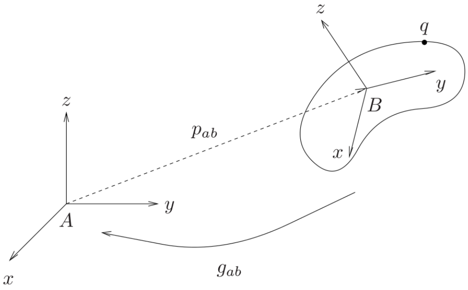
Composition Rule for Rotations

- Rotations about the same axis commute.
- Let $\mathbf{R}_1 = \mathbf{R}_z(\theta_1)$ and $\mathbf{R}_2 = \mathbf{R}_z(\theta_2)$. Then $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_z(\theta_1 + \theta_2) = \mathbf{R}_2\mathbf{R}_1$. This is similar to rotation in the 2D plane.
- In general, the order of composition matters because matrix multiplication is noncommutative.
- For arbitrary $\mathbf{R}_1$ and $\mathbf{R}_2$, $\mathbf{R}_1 + \mathbf{R}_2$ is not a valid rotation matrix. Mathematically, $\mathrm{SO}(3)$ is not closed under the addition.

Action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$

- Considered $\mathbf{R}_{ac}$ as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$.
- This map rotates the coordinates of a point from frame C to frame A.

- In general, rigid motions consist of rotation and translation.
- We describe the position and orientation of a coordinate frame B attached to the body relative to an inertial frame A.
- $g_{ab} = (\mathbf{p}_{ab}, \mathbf{R}_{ab})$ where $\mathbf{p}_{ab} \in \mathbb{R}^3$ and $\mathbf{R}_{ab} \in \mathrm{SO}(3)$.



$\mathrm{SE}(3)$: Group of 3D Rigid Body Transformations

The special Euclidean group is the group of rigid body transformations:

- $\mathrm{SE}(3) = \{(\mathbf{p}, \mathbf{R}) : \mathbf{p} \in \mathbb{R}^3$ and $\mathbf{R} \in \mathrm{SO}(3)\}$
- Action of $g \in \mathrm{SE}(3)$ on $\mathbb{R}^3$ is $g(\mathbf{q}) = \mathbf{p} + \mathbf{R}\mathbf{q}$ for $\mathbf{q} \in \mathbb{R}^3$.

$\mathrm{SE}(3)$: Homogeneous Representation

The transformation of points and vectors by rigid transformations has a simple representation in terms of matrices and vectors in $\mathbb{R}^4$:

- We append 1 to the coordinates of a point to yield a vector in $\mathbb{R}^4$, i.e., $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & 1 \end{bmatrix}^\mathsf{T}$;
- These are called the homogeneous coordinates of the point $\mathbf{q}$;
- Vectors, which are the difference of points, then have the form $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 & 0 \end{bmatrix}^\mathsf{T}$.

$\mathrm{SE}(3)$: Homogeneous Representation

A few rules of syntax:

- Sums and differences of vectors are vectors;
- The sum of a vector and a point is a point;
- The difference between two points is a vector;
- The sum of two points is meaningless.

$\mathrm{SE}(3)$: Homogeneous Representation

- The 4×4 matrix $\mathbf{H}$ is called the homogeneous representation of $g \in \mathrm{SE}(3)$;
- If $g = (\mathbf{p}, \mathbf{R}) \in \mathrm{SE}(3)$, then

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

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$$\mathbf{H}_1\mathbf{H}_2 = \begin{bmatrix} \mathbf{R}_1 & \mathbf{p}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{p}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{p}_1 + \mathbf{R}_1\mathbf{p}_2 \\ \mathbf{0} & 1 \end{bmatrix} \in \mathrm{SE}(3)$$

$\mathrm{SE}(3)$: Homogeneous Representation

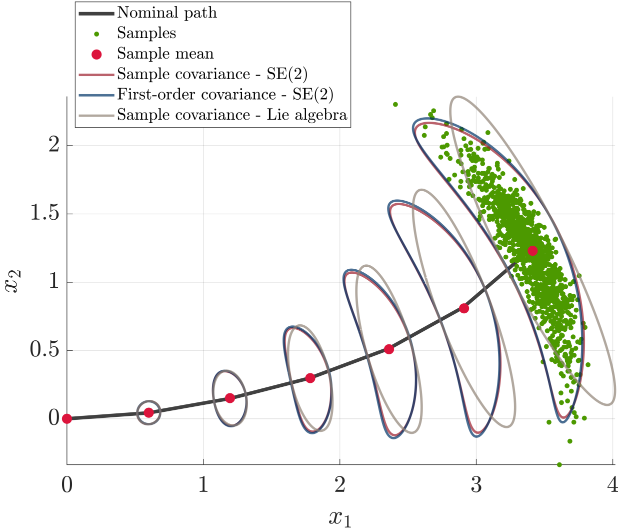
- The 4×4 matrix $\mathbf{H}$ is called the homogeneous representation of $g \in \mathrm{SE}(3)$;

- $$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^\mathsf{T} & -\mathbf{R}^\mathsf{T}\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathrm{SE}(3)$$

- $$\mathbf{H}\mathbf{H}^{-1} = \mathbf{H}^{-1}\mathbf{H} = \mathbf{I}_4 \in \mathrm{SE}(3)$$

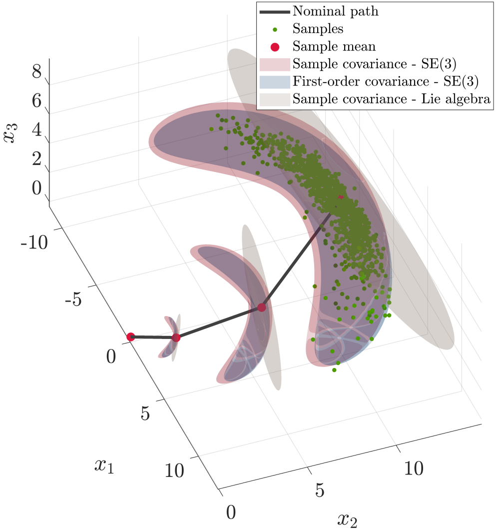
How to Model Uncertainty Propagation on $\mathrm{SE}(2)$

See `odometry_propagation_se2.m` for code.



How to Model Uncertainty Propagation on $\mathrm{SE}(3)$

See `odometry_propagation_se3.m` for code.



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