NA 568 - Winter 2024

Invariant Kalman Filtering II

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Objectives

1. Study the extended pose groups $\mathrm{SE}_K(3)$ for navigation;
2. Modeling the Inertial measurement Unit using $\mathrm{SE}_2(3)$;
3. IMU-GPS Left-Invariant EKF Example;
4. Switching Between Left and Right-Invariant Errors.

- A process dynamics evolving on the Lie group, for state $X_t \in \mathcal{G}$, is

$$\frac{d}{dt} X_t = f_{u_t}(X_t).$$

- $\bar{X}_t$ denotes an estimate of the state.

- The state estimation error is defined using right or left multiplication of X_t^{-1}.

Definition (Left and Right Invariant Error)

The right- and left-invariant errors between two trajectories X_t and $\bar{X}_t$ are:

$$\eta_t^r = \bar{X}_t X_t^{-1} = (\bar{X}_t L)(X_t L)^{-1} \quad \text{(Right-Invariant)}$$
$$\eta_t^l = X_t^{-1} \bar{X}_t = (L\bar{X}_t)^{-1}(LX_t), \quad \text{(Left-Invariant)}$$

where $L \in \mathcal{G}$ is an arbitrary element of the group.

Theorem (Autonomous Error Dynamics)

A system is group affine if the dynamics, $f_{u_t}(\cdot)$, satisfies:

$$f_{u_t}(X_1 X_2) = f_{u_t}(X_1) X_2 + X_1 f_{u_t}(X_2) - X_1 f_{u_t}(I) X_2$$

for all $t > 0$ and $X_1, X_2 \in \mathcal{G}$. Furthermore, if this condition is satisfied, the right- and left-invariant error dynamics are trajectory independent and satisfy:

$$\frac{d}{dt}\eta_t^r = g_{u_t}(\eta_t^r) \quad \textit{where} \quad g_{u_t}(\eta^r) = f_{u_t}(\eta^r) - \eta^r f_{u_t}(I)$$
$$\frac{d}{dt}\eta_t^l = g_{u_t}(\eta_t^l) \quad \textit{where} \quad g_{u_t}(\eta^l) = f_{u_t}(\eta^l) - f_{u_t}(I)\eta^l$$

Define A_t to be a $\dim\mathfrak{g} \times \dim\mathfrak{g}$ matrix satisfying

$$g_{u_t}(\exp(\xi)) := (A_t\xi)^\wedge + \mathcal{O}(||\xi||^2).$$

For all $t \geq 0$, let $\xi_t \in \mathbb{R}^{\dim\mathfrak{g}}$ be the solution of the linear differential equation $\frac{d}{dt}\xi_t = A_t\xi_t$.

Theorem (Log-Linear Property of the Error)

Consider the right-invariant error, η_t, between two trajectories (possibly far apart). For arbitrary initial error $\xi_0 \in \mathbb{R}^{\dim\mathfrak{g}}$, if $\eta_0 = \exp(\xi_0)$, then for all $t \geq 0$,

$$\eta_t = \exp(\xi_t);$$

that is, the nonlinear estimation error η_t can be exactly recovered from the time-varying linear differential equation.

- A noisy process dynamics evolving on the Lie group, take the following form:

$$\frac{d}{dt}X_t = f_{u_t}(X_t) + X_t w_t^\wedge$$

- $w_t^\wedge \in \mathfrak{g}$ is a continuous white noise whose covariance matrix is denoted by Q_t.

- $\frac{d}{dt}\eta_t^r = g_{u_t}(\eta_t^r) - (\bar{X}_t w_t^\wedge \bar{X}_t^{-1})\eta_t^r = g_{u_t}(\eta_t^r) - (\mathrm{Ad}_{\bar{X}_t} w_t)^\wedge \eta_t^r$

- $\frac{d}{dt}\eta_t^l = g_{u_t}(\eta_t^l) - w_t^\wedge \eta_t^l$

- If observations take a particular form, then the linearized observation model and the innovation will also be autonomous.

- This happens when the measurement, Y_{t_k}, can be written as either

$$Y_{t_k} = X_{t_k} b + V_{t_k} \quad \text{(Left-Invariant Observation)} \quad \text{or}$$
$$Y_{t_k} = X_{t_k}^{-1} b + V_{t_k} \quad \text{(Right-Invariant Observation).}$$

- b is a constant vector and V_{t_k} is a vector of Gaussian noise.

Left-Invariant Extended Kalman Filter

1. LI-EKF Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \leq t < t_k,$$
$$\frac{d}{dt}P_t^l = A_t^l P_t^l + P_t^l {A_t^l}^{\mathsf{T}} + Q_t.$$

2. LI-EKF Update:

$$\bar{X}_{t_k}^{+} = \bar{X}_{t_k} \exp\left(L_{t_k}\left(\bar{X}_{t_k}^{-1} Y_{t_k} - b\right)\right),$$
$$P_{t_k}^{l+} = (I - L_{t_k} H) P_{t_k}^l (I - L_{t_k} H)^{\mathsf{T}} + L_{t_k} \bar{N}_k L_{t_k}^{\mathsf{T}},$$

where

$$L_{t_k} = P_{t_k}^l H^{\mathsf{T}} S^{-1}, \quad S = H P_{t_k}^l H^{\mathsf{T}} + \bar{N}_k.$$

Given these equations, once we know A_t^l and H matrices, we can implement the LI-EKF.

Right-Invariant Extended Kalman Filter

1. RI-EKF Propagation:

$$\frac{d}{dt}\bar{X}_t = f_{u_t}(\bar{X}_t), \quad t_{k-1} \leq t < t_k,$$
$$\frac{d}{dt}P_t^r = A_t^r P_t^r + P_t^r {A_t^r}^\mathsf{T} + \mathrm{Ad}_{\bar{X}_t} Q_t \mathrm{Ad}_{\bar{X}_t}^\mathsf{T}.$$

2. RI-EKF Update:

$$\bar{X}_{t_k}^+ = \exp\left(L_{t_k}\left(\bar{X}_{t_k} Y_{t_k} - b\right)\right)\bar{X}_{t_k},$$
$$P_{t_k}^{r+} = (I - L_{t_k} H) P_{t_k}^r (I - L_{t_k} H)^\mathsf{T} + L_{t_k} \bar{N}_k L_{t_k}^\mathsf{T},$$

where

$$L_{t_k} = P_{t_k}^r H^\mathsf{T} S^{-1}, \quad S = H P_{t_k}^r H^\mathsf{T} + \bar{N}_k.$$

Given these equations, once we know A_t^r and H matrices, we can implement the RI-EKF.

A navigation state often includes the following variables.

1. Rotation: $R \in \mathrm{SO}(3)$;
2. Velocity: $v \in \mathbb{R}^3$;
3. Position: $p \in \mathbb{R}^3$;

The state can be constructed as a tuple $X := (R, v, p)$.

3D Extended Pose Matrix Lie Group

- $$\mathrm{SE}_2(3) = \left\{ \begin{bmatrix} R & v & p \\ 0_{1\times 3} & 1 & 0 \\ 0_{1\times 3} & 0 & 1 \end{bmatrix}_{5\times 5} \mid R \in \mathrm{SO}(3),\ v,\, p \in \mathbb{R}^3 \right\}$$
- Its Lie algebra is

$$\mathfrak{se}_2(3) = \left\{ \begin{bmatrix} \xi^{R\wedge} & \xi^v & \xi^p \\ 0_{1\times 3} & 0 & 0 \\ 0_{1\times 3} & 0 & 0 \end{bmatrix}_{5\times 5} \mid \xi^{R\wedge} \in \mathfrak{so}(3),\ \xi^v,\, \xi^p \in \mathbb{R}^3 \right\}$$

- We call $\xi := \mathrm{vec}(\xi^R,\xi^v,\xi^p) \in \mathbb{R}^9$ extended twist (or just twist for convenience) where $\xi^{R\wedge} \in \mathfrak{so}(3)$ is the angular velocity, $\xi^p \in \mathbb{R}^3$ is the linear velocity, and $\xi^v \in \mathbb{R}^3$ can be considered the linear acceleration.

- $X = \begin{bmatrix} R & v & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SE}_2(3)$ and $\xi^\wedge = \begin{bmatrix} \xi^{R\wedge} & \xi^v & \xi^p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{se}_2(3)$
- From the reconstruction equation, we have

$$X^{-1}\dot{X} = \xi_b^\wedge \in \mathfrak{se}_2(3)$$

$$\begin{bmatrix} R^\mathsf{T} & -R^\mathsf{T}v & -R^\mathsf{T}p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{v} & \dot{p} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R^\mathsf{T}\dot{R} & R^\mathsf{T}\dot{v} & R^\mathsf{T}\dot{p} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \xi_b^\wedge$$

-

$$\begin{aligned} \dot{R} &= R\xi^{R\wedge} = R\omega^\wedge, \quad \omega := \xi^{R\wedge} \\ \dot{v} &= R\xi^v = Ra, \quad a := \xi^v \\ \dot{p} &= R\xi^p = v \end{aligned}$$

An accelerometer sensor measures the body frame's linear acceleration. Assuming R is given, we have:

$$\dot{v} = Ra + g$$
$$\dot{p} = v$$

where g is the gravity vector.

Remark

So, the $\mathrm{SE}_2(3)$ reconstruction equation is very close to the accelerometer model, but it does not include the gravity term. Do we lose the group affine property?

- The inertial Measurement Units (IMUs) are ubiquitous and are available in most of the modern robotic systems. For an example of an IMU sensor see: `https://www.vectornav.com/products/vn-100`
- The IMU measurements, angular velocity $\tilde{\omega}_t$ and linear acceleration $\tilde{a}_t$ in the body frame.
- They are modeled as $\tilde{\omega}_t = \omega_t + w_t^g$, $w_t^g \sim \mathcal{GP}\left(0_{3,1}, \Sigma^g\, \delta(t - t')\right)$ and $\tilde{a}_t = a_t + w_t^a$, $w_t^a \sim \mathcal{GP}\left(0_{3,1}, \Sigma^a\, \delta(t - t')\right)$, where $\mathcal{GP}$ denotes a Gaussian process and $\delta(t - t')$ denotes the Dirac delta function.

- The IMU dynamics can be written as:

$$
\begin{aligned}
\dot{R}_t &= R_t(\tilde{\omega}_t - w_t^g)^\wedge \\
\dot{v}_t &= R_t(\tilde{a}_t - w_t^a) + g \\
\dot{p}_t &= v_t,
\end{aligned}
$$

  where g is the gravity vector.

- This model in deterministic form satisfies the group affine property (post your proof on Piazza to increase your friendship score with me!).

IMU Discrete Dynamics

- Assuming a zero-order hold on the incoming IMU measurements between times t_k and t_{k+1}, we have:

$$
\begin{aligned}
R_{k+1} &= R_k\,\Gamma_0(\bar{\omega}_k \Delta t) = R_k \exp(\bar{\omega}_k \Delta t) \\
v_{k+1} &= v_k + R_k \Gamma_1(\bar{\omega}_k \Delta t) \bar{a}_k \Delta t + g \Delta t \\
p_{k+1} &= p_k + v_k \Delta t + R_k \Gamma_2(\bar{\omega}_k \Delta t) \bar{a}_k \Delta t^2 + \frac{1}{2} g \Delta t^2,
\end{aligned}
$$

  where $\bar{\omega}_t := \tilde{\omega}_t - \bar{b}_t^g$ and $\bar{a}_t := \tilde{a}_t - \bar{b}_t^a$ are the "bias-corrected" inputs.

- These discrete dynamics are an exact integration of the continuous-time system under the assumption that the IMU measurements are constant over Δt.

- $\Gamma_0(\phi) = I + \frac{\sin(\|\phi\|)}{\|\phi\|}(\phi^\wedge) + \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}(\phi^\wedge)^2$
- $\Gamma_1(\phi) = I + \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}(\phi^\wedge) + \frac{\|\phi\|-\sin(\|\phi\|)}{\|\phi\|^3}(\phi^\wedge)^2$
- $\Gamma_2(\phi) = \frac{1}{2}I + \frac{\|\phi\|-\sin(\|\phi\|)}{\|\phi\|^3}(\phi^\wedge) + \frac{\|\phi\|^2+2\cos(\|\phi\|)-2}{2\|\phi\|^4}(\phi^\wedge)^2$
- $\Gamma_m(\phi) := \left(\sum_{n=0}^{\infty} \frac{1}{(n+m)!}(\phi^\wedge)^n\right)$

Remark

$\Gamma_0(\phi)$ *is simply the exponential map of* $\mathrm{SO}(3)$*.* $\Gamma_1(\phi)$ *is also known as the left Jacobian of* $\mathrm{SO}(3)$*.*

Table: Summary of World-centric State Estimator

State Definition	Deterministic Nonlinear Dynamics
$X_t := \begin{bmatrix} R_{WB} & {}_W v_B & {}_W p_{WB} \\ 0_{1,3} & 1 & 0 \\ 0_{1,3} & 0 & 1 \end{bmatrix}$	$f_{u_t}(\bar{X}_t) = \begin{bmatrix} \bar{R}_t \tilde{\omega}_t^\wedge & \bar{R}_t \tilde{a}_t + g & \bar{v}_t \\ 0_{1,3} & 0 & 0 \\ 0_{1,3} & 0 & 0 \end{bmatrix}$
Log-Linear Right-Invariant Dynamics	**Log-Linear Left-Invariant Dynamics**
$A_t^r = \begin{bmatrix} 0 & 0 & 0 \\ g^\wedge & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$ $\bar{Q}_t^r = \mathrm{Ad}_{\bar{X}_t} \mathrm{Cov}[w_t] \mathrm{Ad}_{\bar{X}_t}^\mathsf{T}$	$A_t^l = \begin{bmatrix} -\tilde{\omega}_t^\wedge & 0 & 0 \\ -\tilde{a}_t^\wedge & -\tilde{\omega}_t^\wedge & 0 \\ 0 & I & -\tilde{\omega}_t^\wedge \end{bmatrix}$ $\bar{Q}_t^l = \mathrm{Cov}[w_t]$

A robot is equipped with an IMU and the Global Positioning System (GPS) sensors. We use a Left-Invariant EKF to estimate its pose (R_k, p_k), and velocity, v_k, in the world frame. The state is modeled using $\mathrm{SE}_2(3)$ such that

$$X_k = \begin{bmatrix} R_k & v_k & p_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SE}_2(3).$$

For the prediction step, we use the discretized IMU model, and we update the predictions using GPS measurements.

Example: IMU-GPS Left-Invariant EKF

- The (world-centric) left-invariant error dynamics matrix only depends on the IMU inputs that are assumed to be constant between times t_k and t_{k+1}. The state transition matrix can be simply computed from the matrix exponential.

$$\Phi^l(t_{k+1}, t_k) = \exp(A_t^l \Delta t).$$

- This state transition matrix also has an analytical solution of the form:

$$\Phi^l(t_{k+1}, t_k) = \begin{bmatrix} \Phi_{11}^l & 0 & 0 \\ \Phi_{21}^l & \Phi_{22}^l & 0 \\ \Phi_{31}^l & \Phi_{32}^l & \Phi_{33}^l \end{bmatrix}.$$

- The individual terms are

$$
\begin{aligned}
\Phi^l_{11} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t) \\
\Phi^l_{21} &= -\Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)(\Gamma_1(\bar{\omega}_t \Delta t)\bar{a}_t)^\wedge \Delta t \\
\Phi^l_{31} &= -\Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)(\Gamma_2(\bar{\omega}_t \Delta t)\bar{a}_t)^\wedge \Delta t^2 \\
\Phi^l_{22} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t) \\
\Phi^l_{32} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)\Delta t \\
\Phi^l_{33} &= \Gamma_0^{\mathsf{T}}(\bar{\omega}_t \Delta t)
\end{aligned}
$$

Example: IMU-GPS Left-Invariant EKF

The GPS measurement model corresponds to the left-invariant observation form:

$$Y_k = \bar{X}_k b + V_k$$

$$\begin{bmatrix} y_k \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{R}_k & \bar{v}_k & \bar{p}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_k \\ 0 \\ 0 \end{bmatrix}$$

Using the Left-Invariant EKF equations, we proceed to find H.

$$
\begin{aligned}
H\xi_k^l = {\xi_k^l}^\wedge b &= \begin{bmatrix} {\xi_k^\omega}^\wedge & \xi_k^v & \xi_k^p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \xi^p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ 0_{1,3} & 0_{1,3} & 0_{1,3} \\ 0_{1,3} & 0_{1,3} & 0_{1,3} \end{bmatrix} \begin{bmatrix} \xi_k^\omega \\ \xi_k^v \\ \xi_k^p \end{bmatrix},
\end{aligned}
$$

and in its reduced form is

$$
H = \begin{bmatrix} 0 & 0 & I \end{bmatrix}.
$$

Switching Between Left and Right-Invariant Errors

We can switch between the left and right error forms through the use of the adjoint map.

$$\begin{aligned}
\eta_t^r &= \bar{X}_t X_t^{-1} = \bar{X}_t \eta_t^l \bar{X}_t^{-1} \\
\exp(\xi_t^r) &= \bar{X}_t \exp(\xi_t^l) \bar{X}_t^{-1} = \exp(\mathrm{Ad}_{\bar{X}_t} \xi_t^l) \\
\xi_t^r &= \mathrm{Ad}_{\bar{X}_t} \xi_t^l
\end{aligned}$$

Remark

This transformation is exact, which means that we can easily switch between the covariance of the left and right invariant errors using

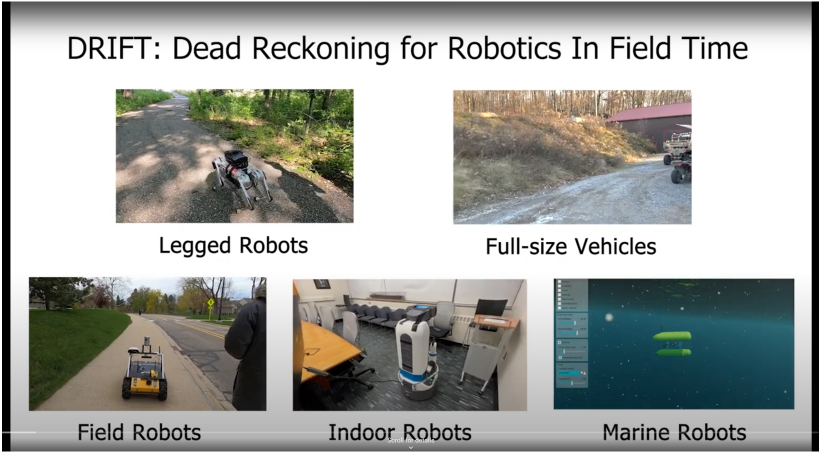
$$P_t^r = \mathrm{Ad}_{\bar{X}_t} \, P_t^l \, \mathrm{Ad}_{\bar{X}_t}^{\mathsf{T}}.$$

Invariant EKF Summary

- Use it if the process dynamics naturally evolves on a Lie group;
- Works well in practice despite the fact by the addition of noise and calibration parameters the theoretical result is lost;
- Excellent consistency and no spurious correlation among state and parameters.
- Highly efficient and suitable for high-frequency state estimation tasks.

Next Time: DRIFT

https://www.youtube.com/watch?v=nqgvFPg-nTY
https://github.com/UMich-CURLY/drift



References and Readings

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