NA 565 - Fall 2024

State Space Models & Linear Systems

September 9, 2024

UNIVERSITY OF MICHIGAN

- The equation type of

$$\frac{d}{dt}x = \dot{x} = f(x)$$

  that involves derivatives of the dependent variable is called an Ordinary Differential Equation (ODE).

- here x depends on time t, i.e., $x(t)$;

- In mathematics, physics, and engineering, ODEs are extremely useful for modeling both physical and non-physical processes.

Let the state be $s := [x, y, \psi, v]^\mathsf{T}$ where (x, y) is the position, ψ is the yaw angle, and v is the velocity. The vehicle is controlled by the steering angle of the front wheel δ and the acceleration a.

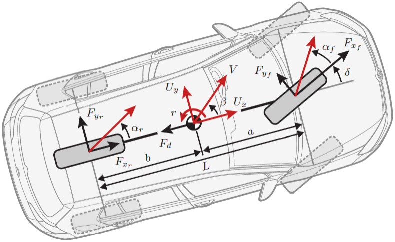
The equations of motion of the vehicle are given by

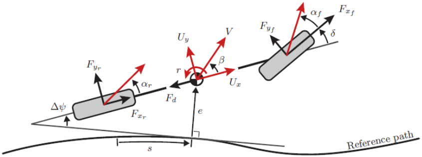
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \\ v \end{bmatrix} = \begin{bmatrix} v\cos(\psi+\beta) \\ v\sin(\psi+\beta) \\ \frac{v}{L_r}\sin\beta \\ a \end{bmatrix}, \text{with } \beta := \arctan\left(\frac{L_r}{L_r+L_f}\arctan\delta\right),$$

where L_r and L_f are the distance from the rear or front axes to the center of the vehicle (given data).

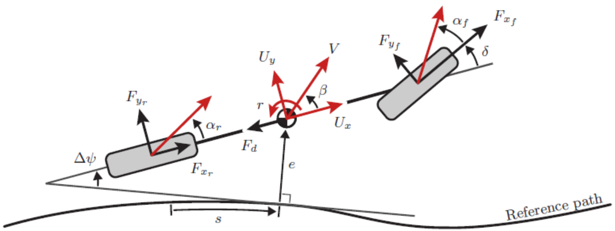
Example: Dynamic Single-Track Vehicle Model

- Longitudinal speed U_x
- Lateral speed U_y
- Yaw rate r
- Total velocity V
- Sideslip angle β
- Longitudinal force on the front and rear tire F_{x_f}, F_{x_r}
- Lateral axle forces F_{y_f}, F_{y_r}
- Steering angle δ
- Disturbance force F_d
- Mass m, inertial I_{zz}
- distance along a reference path s
- Lateral distance to the path e
- Heading error $\Delta\psi$





Example: Dynamic Single-Track Vehicle Model



$$\dot{U}_x = \frac{-F_{y_f}\sin\delta + F_{x_f}\cos\delta + F_{x_r} - F_d}{m} + rU_y$$

$$\dot{U}_y = \frac{F_{y_f}\cos\delta + F_{x_f}\sin\delta + F_{y_r}}{m} - rU_x$$

$$\dot{r} = \frac{a(F_{y_f}\cos\delta + F_{x_f}\sin\delta) - bF_{y_r}}{I_{zz}}$$

$$\dot{s} = \frac{U_x\cos\Delta\psi - U_y\sin\Delta\psi}{1-\kappa e}$$

$$\dot{e} = U_x\sin\Delta\psi + U_y\cos\Delta\psi, \quad \Delta\dot{\psi} = r - \kappa\dot{s}$$

- The state of a system is a collection of variables that summarize the past of a system for the purpose of predicting the future.
- The state variables are gathered in a vector $x \in \mathbb{R}^n$ called the *state vector*.
- The control variables are represented in a vector $u \in \mathbb{R}^p$;
- measured signal by the vector $y \in \mathbb{R}^q$

A system can be represented by the differential equation

$$\frac{d}{dt}x = f(x,u)$$

$$y = h(x,u)$$

$$f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \quad \text{and} \quad h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$$

We call a model of this form a state space model.

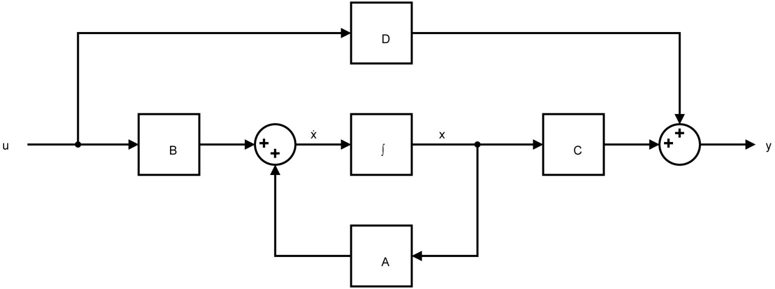
A *linear* and time-invariant system, or LTI, can be represented by the differential equation

$$\frac{d}{dt}x = Ax + Bu$$
$$y = Cx + Du$$

A (dynamics or system matrix), B (control matrix), C (sensor matrix), and D (direct sum) are constant matrices.

Often, $D = 0$; indicating that the control signal u does not influence the output directly.

Block Diagram Representation



Remark (Existence and uniqueness theorem)

Picard–Lindelöf theorem.

Power Series of $\exp(x)$

For real values $x \in \mathbb{R}$:

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

- When the power series of the function $f(x) = \exp(x)$ is applied to a real $n \times n$ matrix A, the result is called *matrix exponentiation*:

$$\exp(A) = e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots .$$

- This series converges for all real $n \times n$ matrices (`https://math.stackexchange.com/questions/131013/matrix-exponential-convergence`).

Theorem

The convolution equation gives the solution to the linear differential equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

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Proof.

Using the Leibniz integral rule, we have

$$\frac{d}{dt}x = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) = Ax + Bu.$$

$\square$

Assuming a constant timestep T and a zero-order hold on the input $u(t)$ during T, we have

$$
\begin{aligned}
x_{t_{k+1}} :=\ & x\left((k+1)T\right) = \\
&= e^{AT}x(kT) + \left(\int_0^T e^{A\nu}d\nu\right)Bu(kT) \\
&=: e^{AT}x_{t_k} + \left(\int_0^T e^{A\nu}d\nu\right)Bu_{t_k} \\
&=: A_d x_{t_k} + B_d u_{t_k}.
\end{aligned}
$$

$$
\boxed{A_d = e^{AT}} \quad \textbf{and} \quad \boxed{B_d = \left(\int_0^T e^{A\nu}d\nu\right)B}.
$$

Common notations are (all equivalent and acceptable):

$$x_{t_{k+1}} = A_d x_{t_k} + B_d u_{t_k}$$
$$x_{k+1} = A_d x_k + B_d u_k$$
$$x_{t+1} = A_d x_t + B_d u_t$$

$$t_k, k, t = 0, 1, 2, \ldots$$

The simplest discretization of an ODE is known as the Euler method.

- Euler method: At timestep k, use a forward-difference approximation $\dot{x} \approx \frac{x_{k+1} - x_k}{T}$,

$$x_{k+1} = (I + TA)x_k + TBu_k.$$

- Works for nonlinear ODEs as well,

$$x_{k+1} = x_k + Tf(x_k, u_k).$$

Notation $h = T$ as timestep (or step size) is also very common ($x_{k+1} = x_k + hf(x_k, u_k)$).

We can linearize nonlinear models around an operating point to obtain a linear model. Note that this model is only locally valid.

Linearization of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ around point x_0 is

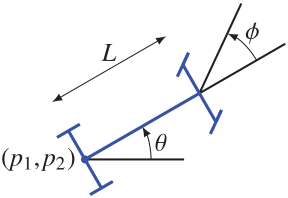
$$\begin{aligned} f(x) &\approx f(x_0) + \frac{\partial f}{\partial x}\bigg|_{x=x_0}(x - x_0) = f(x_0) + \underbrace{J(x_0)}_{\text{Jacobian}}(x - x_0) \\ &= (f(x_0) - J(x_0)x_0) + J(x_0)x \\ &=: a + Fx, \end{aligned}$$

which is an affine function.

Jacobian

$$J(x) := \frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Simple model of a car



$$
\begin{aligned}
\frac{dp_1}{dt} &= s(t)\cos\theta(t) \\
\frac{dp_2}{dt} &= s(t)\sin\theta(t) \\
\frac{d\theta}{dt} &= \frac{s(t)}{L}\tan\phi(t)
\end{aligned}
$$

- $s(t)$ is speed of vehicle, $\phi(t)$ is steering angle
- $p(t)$ is position, $\theta(t)$ is orientation

Discretized model

- discretized model (for small time interval h):

$$\begin{aligned} p_1(t+h) &\approx p_1(t) + hs(t)\cos(\theta(t)) \\ p_2(t+h) &\approx p_2(t) + hs(t)\sin(\theta(t)) \\ \theta(t+h) &\approx \theta(t) + h\frac{s(t)}{L}\tan(\phi(t)) \end{aligned}$$

- define input vector $u_k = (s(kh), \phi(kh))$
- define state vector $x_k = (p_1(kh), p_2(kh), \theta(kh))$
- discretized model is $x_{k+1} = f(x_k, u_k)$ with

$$f(x_k, u_k) = \begin{bmatrix} (x_k)_1 + h(u_k)_1\cos((x_k)_3) \\ (x_k)_2 + h(u_k)_1\sin((x_k)_3) \\ (x_k)_3 + h(u_k)_1\tan((u_k)_2)/L \end{bmatrix}$$

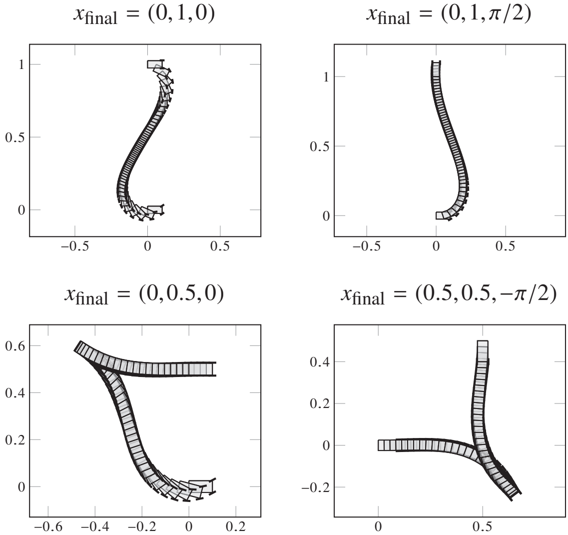
Control problem

- move car from given initial to desired final position and orientation
- using a small and slowly varying input sequence
- this is a constrained nonlinear least squares problem:

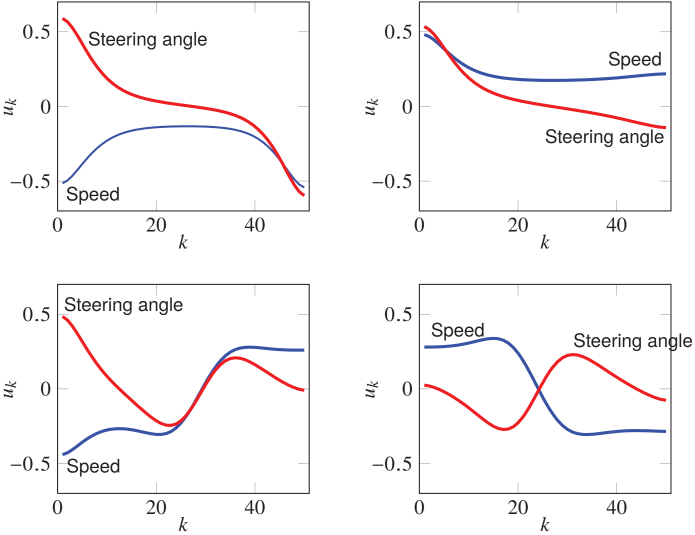
$$
\begin{array}{ll}
\text{minimize} & \sum_{k=1}^{N} \|u_k\|^2 + \gamma \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|^2 \\
\text{subject to} & x_2 = f(0, u_1) \\
& x_{k+1} = f(x_k, u_k), \quad k = 2, \ldots, N-1 \\
& x_{\text{final}} = f(x_N, u_N)
\end{array}
$$

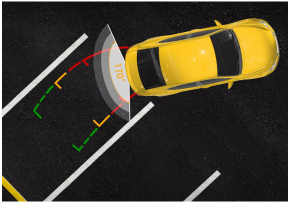
- variables are $u_1, \ldots, u_N, x_2, \ldots, x_N$

Four solution trajectories



Inputs for four trajectories





- See 04_Dynamical_Systems.ipynb at https://umich.instructure.com/courses/689114/files/folder/fall2024
  - study qualitative behavior of linear (or locally linearized) dynamical systems using eigenvalues of the system matrix.
  - learn how to *simulate* linear and nonlinear dynamical systems such as a simple car model.
  - play with built-in ode solvers (integrators; RK4) and your own first-order forward Euler solver.

Next Time

- Constrained & Nonlinear Optimization