18.02: Multivariable Calculus

Massachusetts Institute of Technology
As taught by Prof. John Bush

Jackson Kearl

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Contents

Introduction

"Begin at the beginning," the King said, gravely, "and go on till you come to an end; then stop."

– Lewis Carroll, *Alice in Wonderland*

Course Content

This course covers aspects of calculus involving:

> Calculus of several variables. Vector algebra in 3-space, determinants, matrices. Vector-valued functions of one variable, space motion. Scalar functions of several variables: partial differentiation, gradient, optimization techniques. Double integrals and line integrals in the plane; exact differentials and conservative fields; Green's theorem and applications, triple integrals, line and surface integrals in space, Divergence theorem, Stokes' theorem; applications.[1]

Structure of Notes

Each chapter will focus on some broad concept of the course (along the lines of one chapter per test). Within a chapter, each section approximately corresponds to one lecture. Each section contains an overview of the lecture's concepts, followed by example problems taken from the lecture, recitations, or from homework.

Distribution

All relevant notes and information are made available on github[2]. This includes:

- Plaintext notes from lecture and recitation
- The raw .tex source for these PDF's
- These PDF files

[1] `http://student.mit.edu/catalog/m18a.html#18.02`
[2] `https://github.com/JacksonKearl/MIT`
[3] `http://creativecommons.org/licenses/by-sa/4.0/`

1

Matrices, Vectors, and their Basic Operations

17 39

– Professor J. W. Bush

1.1 Vectors

1.1.1 Scalars vs. Vectors

Before entering into vectors, it is necessary to define the concept of a scalar. Most every value dealt with in day to day life is a scalar: from weights of an apple, to the time it takes to fall on your head.
However, things that also have a direction, such as traveling 25 km *northwest* to get to Welesley, are vectors.
Put simply: a vector is a scalar **with a direction**.

Examples

The following are various ways to represent a vector going up and to the left equal amounts, with magnitude 1:

$$
\begin{aligned}
\vec{v} &= \vec{AB} && \text{Where } A = (0,0), \text{ and } B = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) && (1.1)\\
&= \left\langle \frac{\sqrt{2}}{2}\hat{\imath}, \frac{\sqrt{2}}{2}\hat{\jmath} \right\rangle && \text{Where } \hat{\imath} \text{ and } \hat{\jmath} \text{ are unit vectors along the axies} && (1.2)\\
&= \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle && \text{Similar to above, but with implied } \hat{\imath} \text{ and } \hat{\jmath} && (1.3)\\
&= \left(1, \frac{\pi}{4}\right) && \text{In polar coordinates} && (1.4)
\end{aligned}
$$

1.1.2 Vector Magnitude

As previously stated, a vector is a scalar with direction. But what about times when you want only the scalar? This is called the *magnitude* of a vector, and can be represented as

$|\vec{v}|$ for a vector $\vec{v}$. It is computed by taking the square root of the sum of the squares of each component of a given vector. If this sounds similar to the Pythagorean Theorem, that's because they're actually one in the same: the Pythagorean Theorem is a special case where the vectors are of order 2. In equation form, the magnitude of a vector with order n is:

$$|\vec{v}| = \sqrt{\sum_{i=1}^{n} \vec{v}_i^2} \tag{1.5}$$

Example

$$\vec{v} = \langle 2, 4, 9 \rangle \tag{1.6}$$
$$\tag{1.7}$$
$$|\vec{v}| = \sqrt{2^2 + 4^2 + 9^2} \tag{1.8}$$
$$= \sqrt{4 + 16 + 81} \tag{1.9}$$
$$= \sqrt{101} \tag{1.10}$$

1.1.3 The Dot Product

The dot product of two vectors can be thought of as a scalar measure of the extent to which they are in the same direction. From this description, along with some rudimentary trigonometric knowledge, a definition for the dot product arises:

$$\vec{m} \cdot \vec{v} = |\vec{m}|\,|\vec{v}| \cos(\theta) \tag{1.11}$$

Additionally, the dot product can be computed by multiplying corresponding terms of vectors of the same order together, then summing the results. In a more mathematical notation, the dot product of two order-n vectors, $\vec{m}$ and $\vec{v}$, is:

$$\vec{m} \cdot \vec{v} = \sum_{i=1}^{n} \vec{m}_i \vec{v}_i \tag{1.12}$$

It can be useful to note that when taking the dot product of a vector with itself, the $\cos(\theta)$ of Equation 1.11 goes to 1, leaving

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2 \tag{1.13}$$

Important to note: for any unit vector (represented with a hat, '$\hat{a}$'), its cross with itself will always be its magnitude squared, or 1.

Usage and Examples

The dot product sees many uses in physics, where it can be important to know how much a force for instance is acting in a given direction. Additionally, through Equation 1.11, the dot product can be used to calculate angles between arbitrary vectors. Some sample problems follow:

1. Find the cosine of the angle between the main diagonals of a cube.

- First, find vectors representing the two diagonals. For instance:

$$\vec{v} = \langle 1, 1, 1 \rangle \qquad \vec{m} = \langle -1, 1, 1 \rangle$$

- Combining Equations 1.11 and 1.12, determine an equation for θ.

$$\begin{aligned}
|\vec{m}|\,|\vec{v}|\cos(\theta) &= \sum_{i=1}^{n} \vec{m}_i \vec{v}_i \\
\sqrt{3}\sqrt{3}\cos(\theta) &= -1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 \\
\cos(\theta) &= \frac{1}{3}
\end{aligned}$$

2. Find the component of $\vec{v} = \langle 1, -2, 4 \rangle$ along a $\vec{m} = \langle 4, 2, -3 \rangle$ direction.

- First, find the amount to which they are in the same direction as each other, using a dot product.

$$\langle 1, -2, 4 \rangle \cdot \langle 4, 2, -3 \rangle = -12$$

- This however is proportional to $\vec{m}$, so in order to remove the relation, one must simply divide by the magnitude of $\vec{m}$.

$$\frac{-12}{|\vec{m}|} = \frac{-12}{\sqrt{29}}$$

- Hence, $\vec{v}$ is $\frac{12}{\sqrt{29}}$ units along $\vec{m}$, in a direction opposite to that of $\vec{m}$
- This concept is called a ***projection***, and is commonly used in math and physics.

1.1.4 The Cross Product

While the dot product measures how much two vectors are in the same direction, the cross product can be thought of as a measure of the extent to which two vectors are in different directions. Once more, this concept, along with some trigonometry, gives a definition for the cross product:

$$\vec{m} \times \vec{v} = |\vec{m}|\,|\vec{v}|\sin(\theta) \qquad \textit{In a direction } \textbf{\textit{perpendicular}} \textit{ to the two vectors.} \tag{1.14}$$

This direction component is actually very important in some scenarios. For example, in physics, cross products are uses for concepts such as torque, where the direction of $\vec{r} \times \vec{F}$ yields a torsion normal to the plane of a given circle.

Additionally, the cross product can be computed as the *determinant* of the two vectors, represented as a matrix. Deriving a determinant will be covered in more detail in Section 2, but for now a simple formulas for crossing order-2 vectors is as follows:

$$\begin{aligned}
\vec{m} &= \langle m_i, m_j \rangle & \vec{v} &= \langle v_i, v_j \rangle \\
\vec{r} &= \langle m_i, m_j, m_k \rangle & \vec{s} &= \langle v_i, v_j, v_k \rangle
\end{aligned}$$

$$\vec{m} \times \vec{v} = m_i v_j - m_j v_i \tag{1.15}$$

$$\vec{r} \times \vec{s} = (r_j s_k - r_k s_j)\,\hat{\imath} - (r_i s_k - r_k s_i)\,\hat{\jmath} + (r_i s_j - r_j s_i)\,\hat{k} \tag{1.16}$$

Do not be overly concerned with memorizing this formula. Just know that it is the determinant of the matrix:

$$\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_i & r_j & r_k \\ s_i & s_j & s_k \end{pmatrix}$$

It can be useful to note that when taking the dot product of a vector with itself, the $\sin(\theta)$ of Equation 1.14 goes to 0, leaving:

$$\vec{v} \times \vec{v} = 0 \tag{1.17}$$

Usage and Examples

As previously mentioned, the cross product sees many uses in physics, especially with things involving torque, E&M, or any other right hand rule oriented field.

Interestingly, the cross product of n order-n vectors can be used to determine the area/volume/etc. of the parallelogram type object formed by joining all those vectors together at one point. To illustrate, sample problems follow:

1. Derive a formula for the area of a cube with side length s.
   - First, find vectors representing the three sides. For instance:
     $$\vec{v_1} = \langle 0, 0, s\rangle \qquad \vec{v_2} = \langle 0, s, 0\rangle \qquad \vec{v_3} = \langle s, 0, 0\rangle$$
   - Next, cross two of the vectors to determine the area of one face of the solid.
     $$\begin{aligned} \vec{v_1} \times \vec{v_2} &= (v_{1j}v_{2k} - v_{1k}v_{2j})\,\hat{\imath} - (v_{1i}v_{2k} - v_{1k}v_{2i})\,\hat{\jmath} + (v_{1i}v_{2j} - v_{1j}v_{2i})\,\hat{k} \\ &= \left(s^2\right)\hat{\imath} - 0\hat{\jmath} + 0\hat{k} \qquad = \left(s^2\right)\hat{\imath} \end{aligned}$$
   - Now, with the area of one face equal to s^2, we must go back to the dot product to determine the extent to which this vector normal to this face lies in direction with the third vector coming up out of the face
     $$\begin{aligned} V &= (s^2)\hat{\imath} \cdot \vec{v_3} \\ &= (s^2)\hat{\imath} * (s)\hat{\imath} + 0 + 0 \\ &= s^3 \end{aligned}$$
   - Thus, we show the volume of the cube to be s^3. Not earth shattering, but a worthwhile proof of concept.
2. Find a unit vector perpendicular to both $\vec{v} = \langle 1, -2, 4\rangle$ and $\vec{m} = \langle 4, 2, -3\rangle$.
   - First, find cross product, which will be in a direction perpendicular to both $\vec{v}$ and $\vec{m}$ by definition.
     $$\langle 1, -2, 4\rangle \times \langle 4, 2, -3\rangle = 2\hat{\imath} + 19\hat{\jmath} + 10\hat{k}$$
   - While this vector is in the proper direction, it is not of proper magnitude, namely 1, so we must then devide this vectors components by its magnitude.
     $$\begin{aligned} \hat{v} &= \frac{2\hat{\imath} + 19\hat{\jmath} + 10\hat{k}}{|\vec{v}|} \\ &= \frac{-2\hat{\imath}}{\sqrt{465}} + \frac{19\hat{\jmath}}{\sqrt{465}} + \frac{10\hat{k}}{\sqrt{465}} \end{aligned}$$

1.2 Matricies

Where a vector is a one dimensional list of values, a matrix is a two dimensional list, or an *array*.

Table 1.1: Sample table

S. No.	Column#1	Column#2	Column#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

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1.2.1 Matrix Addition

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1.2.2 Matrix Multiplication

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1.2.3 Determinants

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1.2.4 Inverting a Matrix

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1.2.5 Linear Systems

A small set of linear equations is trivial to solve such a system using algebraic techniques such as distribution. However, as the matrices get more complicated, Linear Systems provide a more systematic approach.

The method relies on the following equation:

$$\vec{A}\vec{x} = \vec{B}$$

Where $\vec{A}$ is the matrix of coefficients for a given system, $\vec{x}$ is the unknown array for a solution to the given system,and $\vec{B}$ is the right hand side of a given system. In example: given

$$\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}$$

then,

$$\vec{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} m \\ n \end{pmatrix}$$