

18.02: Multivariable Calculus

Problem Set 2

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Introduction

“Begin at the beginning,” the King said, gravely, “and go on till you come to an end; then stop.”

– Lewis Carroll, *Alice in Wonderland*

Course Content

This course covers aspects of calculus involving:

Calculus of several variables. Vector algebra in 3-space, determinants, matrices. Vector-valued functions of one variable, space motion. Scalar functions of several variables: partial differentiation, gradient, optimization techniques. Double integrals and line integrals in the plane; exact differentials and conservative fields; Green’s theorem and applications, triple integrals, line and surface integrals in space, Divergence theorem, Stokes’ theorem; applications.¹

Structure of Homework

Problems will be arranged as they are on the assignment page. Problems broken into parts one and two, as per assignment page. Within part one, problems divided first by Lecture number, then by book section.

Distribution

All relevant notes and information are made available on github². This includes:

- Plaintext notes from lecture and recitation
- The raw .tex and .md source for these PDF’s
- These PDF files

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¹<http://student.mit.edu/catalog/m18a.html#18.02>

²<https://github.com/JacksonKearl/MIT>

³<http://creativecommons.org/licenses/by-sa/4.0/>

Part I

1

Lecture 10

1.1 Section 10.9

1.1.1 Problem 2

1. Define Functions:

$$\begin{array}{ll}\text{Function:} & f(x, y) = x + y \\ \text{Constraint:} & x^2 + 4y^2 = 1\end{array}$$

2. Rearrange such that the condition is satisfied when $g(x, y) = 0$:

$$g(x, y) = x^2 + 4y^2 - 1$$

3. Plug into Lagrange Multiplier equation, $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$:

$$L(x, y, \lambda) = x + y - \lambda(x^2 + 4y^2 - 1)$$

4. Take partials with respect to x , y , and λ , and set to 0:

$$\frac{\partial L}{\partial x} = 1 - 2\lambda x = 0 \implies x = \frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial y} = 1 - 8\lambda y = 0 \implies y = \frac{1}{8\lambda}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + 4y^2 - 1 = 0$$

5. Solve as a system of equations:

$$\frac{1}{2\lambda^2} + 4\frac{1}{8\lambda^2} - 1 = 0$$

$$\frac{1}{4\lambda^2} + 4\frac{1}{64\lambda^2} - 1 = 0$$

$$\frac{4}{16\lambda^2} + \frac{1}{16\lambda^2} - 1 = 0$$

$$\frac{5}{16\lambda^2} = 1$$

$$\lambda = \pm \frac{\sqrt{5}}{4}$$

6. Substitute back to find x and y :

$$x = \frac{\pm 4}{2\sqrt{5}} = \frac{\pm 2}{\sqrt{5}}$$

$$y = \frac{\pm 4}{8\sqrt{5}} = \frac{\pm 1}{2\sqrt{5}}$$

7. Empirically determine minimum and maximum from original $f(x, y)$:

$$\begin{array}{ll} \text{Minimum:} & f\left(\frac{-2}{\sqrt{5}}, \frac{-1}{2\sqrt{5}}\right) = \frac{-2}{\sqrt{5}} + \frac{-1}{2\sqrt{5}} = \frac{-5}{2\sqrt{5}} \\ \text{Maximum:} & f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{5}{2\sqrt{5}} \end{array}$$

1.1.2 Problem 8

1. Define Functions:

$$\begin{array}{ll} \text{Function:} & f(x, y, z) = 3x + 2y + z \\ \text{Constraint:} & x^2 + y^2 + z^2 = 1 \end{array}$$

2. Rearrange such that the condition is satisfied when $g(x, y, z) = 0$:

$$g(x, y, z) = x^2 + y^2 + z^2 - 1$$

3. Plug into Lagrange Multiplier equation, $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$

$$L(x, y, z, \lambda) = 3x + 2y + z - \lambda(x^2 + y^2 + z^2 - 1)$$

4. Take partials with respect to x , y , z , and λ , and set to 0:

$$\frac{\partial L}{\partial x} = 3 - 2\lambda x = 0 \implies x = \frac{3}{2\lambda}$$

$$\frac{\partial L}{\partial y} = 2 - 2\lambda y = 0 \implies y = \frac{1}{\lambda}$$

$$\frac{\partial L}{\partial z} = 1 - 2\lambda z = 0 \implies z = \frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

5. Solve as a system of equations:

$$\begin{aligned}
\frac{3^2}{2\lambda} + \frac{1^2}{\lambda} + \frac{1^2}{2\lambda} - 1 &= 0 \\
\frac{9}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} - 1 &= 0 \\
\frac{9}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 &= 0 \\
\frac{14}{4\lambda^2} - 1 &= 0 \\
\frac{7}{2\lambda^2} &= 1 \\
\lambda &= \pm \frac{\sqrt{14}}{2}
\end{aligned}$$

6. Substitute back to find x and y :

$$\begin{aligned}
x &= \pm \frac{3}{\sqrt{14}} \\
y &= \pm \frac{2}{\sqrt{14}} \\
z &= \pm \frac{1}{\sqrt{14}}
\end{aligned}$$

7. Empirically determine min and max from original $f(x, y)$:

$$\begin{aligned}
\text{Min:} \quad f\left(\frac{-3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right) &= \frac{-3}{\sqrt{14}} + \frac{-2}{\sqrt{14}} + \frac{-1}{\sqrt{14}} = \frac{-6}{\sqrt{14}} \\
\text{Max:} \quad f\left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) &= \frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{1}{\sqrt{14}} = \frac{6}{\sqrt{14}}
\end{aligned}$$

1.1.3 Problem 38

1. Define Functions:

$$\begin{aligned}
\text{Function:} \quad & f(x, y) = x^2 + y^2 \\
\text{Constraint:} \quad & x^2 + xy + y^2 = 3
\end{aligned}$$

2. Rearrange such that the condition is satisfied when $g(x, y) = 0$:

$$g(x, y) = x^2 + xy + y^2 - 3$$

3. Plug into Lagrange Multiplier equation, $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$$

4. Take partials with respect to x , y , and λ , and set to 0:

$$\frac{\partial L}{\partial x} = 2x - 2\lambda x - \lambda y = 0 \implies x = \frac{\lambda y}{2 - 2\lambda}$$

$$\frac{\partial L}{\partial y} = 2y - 2\lambda y - \lambda x = 0 \implies y = \frac{\lambda x}{2 - 2\lambda}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + xy + y^2 - 3 = 0$$

5. Solve as a system of equations:

$$x = \frac{\lambda y}{2 - 2\lambda}$$

$$\frac{x}{y} = \frac{\lambda}{2 - 2\lambda}$$

$$y = \frac{\lambda x}{2 - 2\lambda}$$

$$\frac{y}{x} = \frac{\lambda}{2 - 2\lambda}$$

$$\frac{x}{y} = \frac{y}{x}$$

$$x^2 = y^2$$

$$y = \pm x$$

$$g(x, y) = 3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = y = \pm 1$$

$$g(x, y) = x^2 - 3 = 0$$

$$x^2 = 3$$

$$x^2 = 3$$

$$x = -y = \pm\sqrt{3}$$

6. Substitute back to find x and y :

$$(x, y) = (1, 1), (-1, -1), (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$$

7. Empirically determine min and max from original $f(x, y)$:

$$\text{Distance from Origin:} \quad D(x, y) = \sqrt{x^2 + y^2}$$

$$\text{Minimums @ } (1, 1), (-1, -1) : \quad D(1, 1) = D(-1, -1) = \sqrt{2}$$

$$\text{Maximums @ } (-\sqrt{3}, \sqrt{3}), (\sqrt{3}, -\sqrt{3}) : \quad D(-\sqrt{3}, \sqrt{3}) = \sqrt{6}$$

1.1.4 Problem 52

1. Define Functions:

$$\text{Function:} \quad f(x, y) = (x - 3)^2 + (y - 2)^2$$

$$\text{Constraint:} \quad 4x^2 + 9y^2 = 36$$

2. Rearrange such that the condition is satisfied when $g(x, y) = 0$:

$$g(x, y) = 4x^2 + 9y^2 - 36$$

3. Plug into Lagrange Multiplier equation, $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$L(x, y, \lambda) = (x - 3)^2 + (y - 2)^2 - \lambda(4x^2 + 9y^2 - 36)$$

4. Take partials with respect to x , y , and λ , and set to 0:

$$\frac{\partial L}{\partial x} = 2x - 6 - 8\lambda x = 0 \implies x = \frac{3}{1 - 4\lambda}$$

$$\frac{\partial L}{\partial y} = 2y - 4 - 18\lambda y = 0 \implies y = \frac{2}{1 - 9\lambda}$$

$$\frac{\partial L}{\partial \lambda} = 4x^2 + 9y^2 - 36 = 0$$

5. Solve as a system of equations, using CAS software:

$$\begin{aligned} 4x^2 + 9y^2 - 36 &= 0 \\ 4\left(\frac{3}{1 - 4\lambda}\right)^2 + 9\left(\frac{2}{1 - 9\lambda}\right)^2 - 36 &= 0 \\ \lambda &= [0.5103, -0.0684] \end{aligned}$$

6. Substitute back to find x and y :

$$(x, y) = \left(\frac{3}{1 - 4\lambda}, \frac{2}{1 - 9\lambda}\right)$$

$$\begin{aligned} (x, y) &= \left(\frac{3}{1 - 4(0.5103)}, \frac{2}{1 - 9(0.5103)}\right) \quad \text{OR} \quad (x, y) = \left(\frac{3}{1 - 4(-0.0684)}, \frac{2}{1 - 9(-0.0684)}\right) \\ (x, y) &= (-2.88, -0.557) \quad \text{OR} \quad (x, y) = (2.36, 1.24) \end{aligned}$$

7. Empirically determine min and max from original $f(x, y)$:

Distance from Origin:	$D(x, y) = \sqrt{(x - 3)^2 + (y - 2)^2}$
Minimum @ (2.36, 1.24) :	$D(2.36, 1.24) = 0.993$
Maximum @ (-2.88, -0.557) :	$D(-2.88, -0.557) = 6.411$

1.2 Section 13.10

1.2.1 Problem 4

1. Calculate partials, set to 0 to find critical points:

$$\begin{aligned}\frac{\partial f}{\partial x} &= y + 3 = 0 \implies y = -3 \\ \frac{\partial f}{\partial y} &= x - 2 = 0 \implies x = 2\end{aligned}$$

2. Evaluate discriminant at critical point, $(2, -3)$:

$$\begin{aligned}\Delta &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 0 \cdot 0 - 1 \\ &= -1\end{aligned}$$

The discriminant at $(2, -3)$ is negative, meaning there is a saddle point here, not a minimum or maximum.

1.2.2 Problem 10

1. Calculate partials, set to 0 to find critical points:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3y - 3x^2 = 0 && \implies y = x^2 \\ \frac{\partial f}{\partial y} &= 3x - 3y^2 = 0 && \implies x = y^2 \\ x &= x^4 && \implies (x, y) = (0, 0) \\ 1 &= x^3 && \implies (x, y) = (1, 1)\end{aligned}$$

2. Evaluate discriminant at critical points $(0, 0)$, and $(1, 1)$:

$$\begin{aligned}\Delta(x, y) &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= -6x \cdot -6y - [3]^2 \\ \Delta(0, 0) &= 0 \cdot 0 - 9 = -9 \\ \Delta(1, 1) &= -6 \cdot -6 - 9 = 27\end{aligned}$$

The discriminant at $(0, 0)$ is negative, meaning it is a saddle point, not a minima or maxima.

However, the discriminant at $(1, 1)$ is positive, meaning it is either a minima or maxima. Because $f_{xx}(1, 1)$ is negative, $(1, 1)$ is a maxima.

2

Lecture 13

2.1 Section 13.7

2.1.1 Problem 2

1. Finding $\frac{dw}{dt}$ via chain rule:

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} \\ &= \frac{2u}{(u^2 + v^2)^2} (-2 \sin(2t)) + \frac{2v}{(u^2 + v^2)^2} (2 \cos(2t)) \\ &= \frac{2 \cos(2t)}{(\cos(2t)^2 + \sin(2t)^2)^2} (-2 \sin(2t)) + \frac{2 \sin(2t)}{(\cos(2t)^2 + \sin(2t)^2)^2} (2 \cos(2t)) \\ &= 2 \cos(2t) (-2 \sin(2t)) + 2 \sin(2t) (2 \cos(2t)) \\ &= 0\end{aligned}$$

2. Finding $\frac{dw}{dt}$ via substitution:

$$\begin{aligned}w(u, v) &= \frac{1}{\cos(2t)^2 + \sin(2t)^2} \\ &= \frac{1}{1} = 1 \\ \frac{dw}{dt} &= 0\end{aligned}$$

2.1.2 Problem 4

1. Finding $\frac{dw}{dt}$ via chain rule:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\
&= \frac{1}{u+v+z} [2 \sin(t) \cos(t) + (-2) \cos(t) \sin(t) + 2t] \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

2. Finding $\frac{dw}{dt}$ via substitution:

$$\begin{aligned}
w(u, v, z) &= \ln(\cos^2 t + \sin^2 t + t^2) \\
&= \ln(1 + t^2) \\
\frac{dw}{dt} &= \frac{2t}{1+t^2}
\end{aligned}$$

2.1.3 Problem 6

1. Finding $\frac{\partial w}{\partial t}$:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial p} \frac{dp}{dt} + \frac{\partial w}{\partial q} \frac{dq}{dt} + \frac{\partial w}{\partial r} \frac{dr}{dt} \\
&= q \sin(r) \cdot 1 + p \sin(r) \cdot -1 + pq \cos(r) \cdot s \\
&= pqs \cos(r) + q \sin(r) - p \sin(r)
\end{aligned}$$

2. Finding $\frac{\partial w}{\partial s}$:

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial p} \frac{dp}{ds} + \frac{\partial w}{\partial q} \frac{dq}{ds} + \frac{\partial w}{\partial r} \frac{dr}{ds} \\
&= q \sin(r) \cdot 2 + p \sin(r) \cdot 1 + pq \cos(r) \cdot t \\
&= pqt \cos(r) + 2q \sin(r) + p \sin(r)
\end{aligned}$$

2.1.4 Problem 16

1. Finding $\frac{\partial p}{\partial x}$:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial p}{\partial u} \frac{\partial u}{\partial x}$$

2. Finding $\frac{\partial p}{\partial y}$:

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial p}{\partial u} \frac{\partial u}{\partial y}$$

3. Finding $\frac{\partial p}{\partial z}$:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial p}{\partial u} \frac{\partial u}{\partial z}$$

4. Finding $\frac{\partial p}{\partial t}$:

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial v} \frac{\partial v}{\partial t} + \frac{\partial p}{\partial u} \frac{\partial u}{\partial t}$$

2.1.5 Problem 20

1. Rearrange to get $z = f(x, y, z) + g(x, y, z) + h(x, y, z)$:

$$\begin{aligned} z &= f(x, y, z) + g(x, y, z) + h(x, y, z) \\ f(x, y, z) &= \frac{x^2}{y} \\ g(x, y, z) &= \frac{y^2}{x} \\ h(x, y, z) &= \frac{z^3}{xy} \end{aligned}$$

2. Use chain rule to find $\frac{\partial z}{\partial x}$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial z}{\partial g} \frac{\partial g}{\partial x} + \frac{\partial z}{\partial h} \frac{\partial h}{\partial x} \\ &= 1 \cdot \frac{2x}{y} + 1 \cdot -\frac{y^2}{x^2} + 1 \cdot -\frac{z^3}{x^2 y} \\ &= \frac{2x}{y} - \frac{y^2}{x^2} - \frac{z^3}{x^2 y} \\ &= \frac{2x^3 - y^3 - z^3}{x^2 y} \end{aligned}$$

3. Use chain rule to find $\frac{\partial z}{\partial y}$:

$$\begin{aligned}
 \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial f} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial g} \frac{\partial g}{\partial y} + \frac{\partial z}{\partial h} \frac{\partial h}{\partial y} \\
 &= 1 \cdot -\frac{x^2}{y^2} + 1 \cdot \frac{2y}{x} + 1 \cdot -\frac{z^3}{xy^2} \\
 &= -\frac{x^2}{y^2} + \frac{2y}{x} - \frac{z^3}{xy^2} \\
 &= \frac{2y^3 - x^3 - z^3}{xy^2}
 \end{aligned}$$

2.1.6 Problem 40

$$\begin{aligned}
 \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 &= \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta}\right)^2 \\
 &= \left(\frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta}\right)^2 \\
 &= \left(\frac{\partial w}{\partial x} \cos(\theta) + \frac{\partial w}{\partial y} \sin(\theta)\right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial x} (-r \sin(\theta)) + \frac{\partial w}{\partial y} r \cos(\theta)\right)^2 \\
 &= \left(\frac{\partial w}{\partial x} \cos(\theta) + \frac{\partial w}{\partial y} \sin(\theta)\right)^2 + \left(\frac{\partial w}{\partial y} (\cos(\theta)) - \frac{\partial w}{\partial x} \sin(\theta)\right)^2 \\
 &= \left(\frac{\partial w}{\partial x}\right)^2 \cos^2(\theta) + 2 \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} \sin(\theta) \cos(\theta) + \left(\frac{\partial w}{\partial y}\right)^2 \sin^2(\theta) + \\
 &\quad \left(\frac{\partial w}{\partial y}\right)^2 \cos^2(\theta) - 2 \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} \sin(\theta) \cos(\theta) + \left(\frac{\partial w}{\partial x}\right)^2 \sin^2(\theta) \\
 &= \left(\frac{\partial w}{\partial x}\right)^2 \cos^2(\theta) + \left(\frac{\partial w}{\partial y}\right)^2 \sin^2(\theta) + \left(\frac{\partial w}{\partial y}\right)^2 \cos^2(\theta) + \left(\frac{\partial w}{\partial x}\right)^2 \sin^2(\theta) \\
 &= \left(\frac{\partial w}{\partial x}\right)^2 (\cos^2(\theta) + \sin^2(\theta)) + \left(\frac{\partial w}{\partial y}\right)^2 (\sin^2(\theta) + \cos^2(\theta)) \\
 &= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2
 \end{aligned}$$