18.02: Multivariable Calculus

Problem Set 4

As taught by Prof. John Bush, MIT

Jackson Kearl

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Contents

Introduction

> *"Begin at the beginning," the King said, gravely, "and go on till you come to an end; then stop."*
>
> – Lewis Carroll, *Alice in Wonderland*

Course Content

This course covers aspects of calculus involving:

> Calculus of several variables. Vector algebra in 3-space, determinants, matrices. Vector-valued functions of one variable, space motion. Scalar functions of several variables: partial differentiation, gradient, optimization techniques. Double integrals and line integrals in the plane; exact differentials and conservative fields; Green's theorem and applications, triple integrals, line and surface integrals in space, Divergence theorem, Stokes' theorem; applications.[1]

Structure of Homework

Problems will be arranged as they are on the assigment page. Problems broken into parts one and two, as per assignment page. Within part one, problems divided first by Lecture number, then by book section.

Distribution

All relevant notes and information are made available on github[2]. This includes:

- Plaintext notes from lecture and recitation
- The raw .tex and .md source for these PDF's
- These PDF files

[1] `http://student.mit.edu/catalog/m18a.html#18.02`
[2] `https://github.com/JacksonKearl/MIT`
[3] `http://creativecommons.org/licenses/by-sa/4.0/`

Part I

Book

1

Lecture 12

Section 13.9

Problem 2

1. Define Functions:

$$\begin{aligned} \text{Function:} &\qquad f(x,y) = x + y \\ \text{Constraint:} &\qquad x^2 + 4y^2 = 1 \end{aligned}$$

2. Rearrange such that the condition is satisfied when $g(x,y) = 0$:

$$g(x,y) = x^2 + 4y^2 - 1$$

3. Plug into Lagrange Multiplier equation, $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$:

$$L(x,y,\lambda) = x + y - \lambda(x^2 + 4y^2 - 1)$$

4. Take partials with respect to x, y, and λ, and set to 0:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 1 - 2\lambda x = 0 \implies x = \frac{1}{2\lambda} \\ \frac{\partial L}{\partial y} &= 1 - 8\lambda y = 0 \implies y = \frac{1}{8\lambda} \\ \frac{\partial L}{\partial \lambda} &= x^2 + 4y^2 - 1 = 0 \end{aligned}$$

5. Solve as a system of equations:

$$\begin{aligned} \frac{1}{2\lambda}^2 + 4\frac{1}{8\lambda}^2 - 1 &= 0 \\ \frac{1}{4\lambda^2} + 4\frac{1}{64\lambda^2} - 1 &= 0 \\ \frac{4}{16\lambda^2} + \frac{1}{16\lambda^2} - 1 &= 0 \\ \frac{5}{16\lambda^2} &= 1 \\ \lambda &= \pm\frac{\sqrt{5}}{4} \end{aligned}$$

6. Substitute back to find x and y:

$$x = \frac{\pm 4}{2\sqrt{5}} = \frac{\pm 2}{\sqrt{5}}$$
$$y = \frac{\pm 4}{8\sqrt{5}} = \frac{\pm 1}{2\sqrt{5}}$$

7. Empirically determine minimum and maximum from original $f(x, y)$:

$$\text{Minimum:} \quad f\left(\frac{-2}{\sqrt{5}}, \frac{-1}{2\sqrt{5}}\right) = \frac{-2}{\sqrt{5}} + \frac{-1}{2\sqrt{5}} = \frac{-5}{2\sqrt{5}}$$
$$\text{Maximum:} \quad f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{5}{2\sqrt{5}}$$

Problem 8

1. Define Functions:

$$\text{Function:} \quad f(x, y, z) = 3x + 2y + z$$
$$\text{Constraint:} \quad x^2 + y^2 + z^2 = 1$$

2. Rearrange such that the condition is satisfied when $g(x, y, z) = 0$:

$$g(x, y) = x^2 + y^2 + z^2 - 1$$

3. Plug into Lagrange Multiplier equation, $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$

$$L(x, y, z, \lambda) = 3x + 2y + z - \lambda(x^2 + y^2 + z^2 - 1)$$

4. Take partials with respect to x, y, z, and λ, and set to 0:

$$\frac{\partial L}{\partial x} = 3 - 2\lambda x = 0 \Longrightarrow x = \frac{3}{2\lambda}$$
$$\frac{\partial L}{\partial y} = 2 - 2\lambda y = 0 \Longrightarrow y = \frac{1}{\lambda}$$
$$\frac{\partial L}{\partial z} = 1 - 2\lambda z = 0 \Longrightarrow z = \frac{1}{2\lambda}$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

5. Solve as a system of equations:

$$\begin{aligned}
\frac{3}{2\lambda}^2 + \frac{1}{\lambda}^2 + \frac{1}{2\lambda}^2 - 1 &= 0 \\
\frac{9}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} - 1 &= 0 \\
\frac{9}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 &= 0 \\
\frac{14}{4\lambda^2} - 1 &= 0 \\
\frac{7}{2\lambda^2} &= 1 \\
\lambda &= \pm\frac{\sqrt{14}}{2}
\end{aligned}$$

6. Substitute back to find x and y:

$$\begin{aligned}
x &= \pm\frac{3}{\sqrt{14}} \\
y &= \pm\frac{2}{\sqrt{14}} \\
z &= \pm\frac{1}{\sqrt{14}}
\end{aligned}$$

7. Empirically determine min and max from original $f(x, y)$:

$$\begin{aligned}
\text{Min:} \qquad & f\left(\frac{-3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right) = \frac{-3}{\sqrt{14}} + \frac{-2}{\sqrt{14}} + \frac{-1}{\sqrt{14}} = \frac{-6}{\sqrt{14}} \\
\text{Max:} \qquad & f\left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = \frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{1}{\sqrt{14}} = \frac{6}{\sqrt{14}}
\end{aligned}$$

Problem 38

1. Define Functions:

$$\begin{aligned}
\text{Function:} \qquad & f(x, y) = x^2 + y^2 \\
\text{Constraint:} \qquad & x^2 + xy + y^2 = 3
\end{aligned}$$

2. Rearrange such that the condition is satisfied when $g(x, y) = 0$:

$$g(x, y) = x^2 + xy + y^2 - 3$$

3. Plug into Lagrange Multiplier equation, $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$$

4. Take partials with respect to x, y, and λ, and set to 0:

$$\frac{\partial L}{\partial x} = 2x - 2\lambda x - \lambda y = 0 \Longrightarrow x = \frac{\lambda y}{2-2\lambda}$$
$$\frac{\partial L}{\partial y} = 2y - 2\lambda y - \lambda x = 0 \Longrightarrow y = \frac{\lambda x}{2-2\lambda}$$
$$\frac{\partial L}{\partial \lambda} = x^2 + xy + y^2 - 3 = 0$$

5. Solve as a system of equations:

$$x = \frac{\lambda y}{2-2\lambda} \qquad\qquad y = \frac{\lambda x}{2-2\lambda}$$
$$\frac{x}{y} = \frac{\lambda}{2-2\lambda} \qquad\qquad \frac{y}{x} = \frac{\lambda}{2-2\lambda}$$
$$\frac{x}{y} = \frac{y}{x}$$
$$x^2 = y^2$$
$$y = \pm x$$

$$\begin{aligned} g(x,y) = 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x = y &= \pm 1 \end{aligned} \qquad\qquad \begin{aligned} g(x,y) = x^2 - 3 &= 0 \\ x^2 &= 3 \\ x^2 &= 3 \\ x = -y &= \pm\sqrt{3} \end{aligned}$$

6. Substitute back to find x and y:

$$(x,y) = (1,1), (-1,-1), (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$$

7. Empirically determine min and max from original $f(x,y)$:

$$\begin{aligned} \text{Distance from Origin:} &\qquad D(x,y) = \sqrt{x^2+y^2} \\ \text{Minimums @ } (1,1), (-1,-1): &\qquad D(1,1) = D(-1,-1) = \sqrt{2} \\ \text{Maximums @ } (-\sqrt{3}, \sqrt{3}), (\sqrt{3}, -\sqrt{3}): &\qquad D(-\sqrt{3}, \sqrt{3}) = \sqrt{6} \end{aligned}$$

Problem 52

1. Define Functions:

$$\begin{aligned} \text{Function:} &\qquad f(x,y) = (x-3)^2 + (y-2)^2 \\ \text{Constraint:} &\qquad 4x^2 + 9y^2 = 36 \end{aligned}$$

2. Rearrange such that the condition is satisfied when $g(x, y) = 0$:

$$g(x, y) = 4x^2 + 9y^2 - 36$$

3. Plug into Lagrange Multiplier equation, $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$L(x, y, \lambda) = (x - 3)^2 + (y - 2)^2 - \lambda(4x^2 + 9y^2 - 36)$$

4. Take partials with respect to x, y, and λ, and set to 0:

$$\frac{\partial L}{\partial x} = 2x - 6 - 8\lambda x = 0 \Longrightarrow x = \frac{3}{1 - 4\lambda}$$
$$\frac{\partial L}{\partial y} = 2y - 4 - 18\lambda y = 0 \Longrightarrow y = \frac{2}{1 - 9\lambda}$$
$$\frac{\partial L}{\partial \lambda} = 4x^2 + 9y^2 - 36 = 0$$

5. Solve as a system of equations, using CAS software:

$$\begin{aligned} 4x^2 + 9y^2 - 36 &= 0 \\ 4\left(\frac{3}{1 - 4\lambda}\right)^2 + 9\left(\frac{2}{1 - 9\lambda}\right)^2 - 36 &= 0 \\ \lambda &= [0.5103, -0.0684] \end{aligned}$$

6. Substitute back to find x and y:

$$(x, y) = \left(\frac{3}{1 - 4\lambda}, \frac{2}{1 - 9\lambda}\right)$$

$$\begin{aligned} (x, y) &= \left(\frac{3}{1 - 4(0.5103)}, \frac{2}{1 - 9(0.5103)}\right) \quad \text{OR } (x, y) = \left(\frac{3}{1 - 4(-0.0684)}, \frac{2}{1 - 9(-0.0684)}\right) \\ (x, y) &= (-2.88, -0.557) \quad \text{OR } (x, y) = (2.36, 1.24) \end{aligned}$$

7. Empirically determine min and max from original $f(x, y)$:

$$\begin{aligned} \text{Distance from Origin:} &\quad D(x, y) = \sqrt{(x - 3)^2 + (y - 2)^2} \\ \text{Minimum @ } (2.36, 1.24): &\quad D(2.36, 1.24) = 0.993 \\ \text{Maximum @ } (-2.88, -0.557): &\quad D(-2.88, -0.557) = 6.411 \end{aligned}$$

Section 13.10

Problem 4

1. Calculate partials, set to 0 to find critical points:

$$\begin{aligned}\frac{\partial f}{\partial x} &= y + 3 = 0 \Longrightarrow y = -3 \\ \frac{\partial f}{\partial y} &= x - 2 = 0 \Longrightarrow x = 2\end{aligned}$$

2. Evaluate discriminant at critical point, $(2, -3)$:

$$\begin{aligned}\Delta &= f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 0 \cdot 0 - 1 \\ &= -1\end{aligned}$$

The discriminant at $(2, -3)$ is negative, meaning there is a saddle point here, not a minimum or maximum.

Problem 10

1. Calculate partials, set to 0 to find critical points:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3y - 3x^2 = 0 & &\Longrightarrow y = x^2 \\ \frac{\partial f}{\partial y} &= 3x - 3y^2 = 0 & &\Longrightarrow x = y^2 \\ x &= x^4 & &\Longrightarrow (x, y) = (0, 0) \\ 1 &= x^3 & &\Longrightarrow (x, y) = (1, 1)\end{aligned}$$

2. Evaluate discriminant at critical points $(0, 0)$, and $(1, 1)$:

$$\begin{aligned}\Delta(x, y) &= f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= -6x \cdot -6y - [3]^2 \\ \Delta(0, 0) &= 0 \cdot 0 - 9 = -9 \\ \Delta(1, 1) &= -6 \cdot -6 - 9 = 27\end{aligned}$$

The discriminant at $(0, 0)$ is negative, meaning it is a saddle point, not a minima or maxima.

However, the discriminant at $(1, 1)$ is positive, meaning it is either a minima or maxima. Because $f_{xx}(1, 1)$ is negative, $(1, 1)$ is a maxima.

2

Lecture 13

Section 13.7

Problem 2

1. Finding $\frac{dw}{dt}$ via chain rule:

$$
\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} \\
&= \frac{2u}{(u^2+v^2)^2}(-2\sin(2t)) + \frac{2v}{(u^2+v^2)^2}(2\cos(2t)) \\
&= \frac{2\cos(2t)}{(\cos(2t)^2+\sin(2t)^2)^2}(-2\sin(2t)) + \frac{2\sin(2t)}{(\cos(2t)^2+\sin(2t)^2)^2}(2\cos(2t)) \\
&= 2\cos(2t)(-2\sin(2t)) + 2\sin(2t)(2\cos(2t)) \\
&= 0
\end{aligned}
$$

2. Finding $\frac{dw}{dt}$ via substitution:

$$
\begin{aligned}
w(u,v) &= \frac{1}{\cos(2t)^2+\sin(2t)^2} \\
&= \frac{1}{1} = 1 \\
\frac{dw}{dt} &= 0
\end{aligned}
$$

Problem 4

1. Finding $\frac{dw}{dt}$ via chain rule:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \frac{1}{u+v+z}[2\sin(t)\cos(t) + (-2)\cos(t)\sin(t) + 2t] \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

2. Finding $\frac{dw}{dt}$ via substitution:

$$\begin{aligned}
w(u,v,z) &= \ln(\cos^2 t + \sin^2 t + t^2) \\
&= \ln(1+t^2) \\
\frac{dw}{dt} &= \frac{2t}{1+t^2}
\end{aligned}$$

Problem 6

1. Finding $\frac{\partial w}{\partial t}$:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial p}\frac{dp}{dt} + \frac{\partial w}{\partial q}\frac{dq}{dt} + \frac{\partial w}{\partial r}\frac{dr}{dt} \\
&= q\sin(r)\cdot 1 + p\sin(r)\cdot -1 + pq\cos(r)\cdot s \\
&= pqs\cos(r) + q\sin(r) - p\sin(r)
\end{aligned}$$

2. Finding $\frac{\partial w}{\partial s}$:

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial p}\frac{dp}{ds} + \frac{\partial w}{\partial q}\frac{dq}{ds} + \frac{\partial w}{\partial r}\frac{dr}{ds} \\
&= q\sin(r)\cdot 2 + p\sin(r)\cdot 1 + pq\cos(r)\cdot t \\
&= pqt\cos(r) + 2q\sin(r) + p\sin(r)
\end{aligned}$$

Problem 16

1. Finding $\frac{\partial p}{\partial x}$:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial p}{\partial u}\frac{\partial u}{\partial x}$$

2. Finding $\frac{\partial p}{\partial y}$:

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial p}{\partial u}\frac{\partial u}{\partial y}$$

3. Finding $\frac{\partial p}{\partial z}$:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial p}{\partial u}\frac{\partial u}{\partial z}$$

4. Finding $\frac{\partial p}{\partial t}$:

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial p}{\partial u}\frac{\partial u}{\partial t}$$

Problem 20

1. Rearrange to get $z = f(x, y, z) + g(x, y, z) + h(x, y, z)$:

$$\begin{aligned} z &= f(x, y, z) + g(x, y, z) + h(x, y, z) \\ f(x, y, z) &= \frac{x^2}{y} \\ g(x, y, z) &= \frac{y^2}{x} \\ h(x, y, z) &= \frac{z^3}{xy} \end{aligned}$$

2. Use chain rule to find $\frac{\partial z}{\partial x}$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial z}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial z}{\partial g}\frac{\partial g}{\partial x} \\ &= 1 \cdot \frac{2x}{y} + 1 \cdot -\frac{y^2}{x^2} + 1 \cdot -\frac{z^3}{x^2 y} \\ &= \frac{2x}{y} - \frac{y^2}{x^2} - \frac{z^3}{x^2 y} \\ &= \frac{2x^3 - y^3 - z^3}{x^2 y} \end{aligned}$$

3. Use chain rule to find $\frac{\partial z}{\partial y}$:

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial g}\frac{\partial g}{\partial y} + \frac{\partial z}{\partial g}\frac{\partial g}{\partial y} \\
&= 1 \cdot -\frac{x^2}{y^2} + 1 \cdot \frac{2y}{x} + 1 \cdot -\frac{z^3}{xy^2} \\
&= -\frac{x^2}{y^2} + \frac{2y}{x} - \frac{z^3}{xy^2} \\
&= \frac{2y^3 - x^3 - z^3}{xy^2}
\end{aligned}$$

Problem 40

$$\begin{aligned}
\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 &= \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \\
&= \left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}\right)^2 \\
&= \left(\frac{\partial w}{\partial x}\cos(\theta) + \frac{\partial w}{\partial y}\sin(\theta)\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial x}(-r\sin(\theta)) + \frac{\partial w}{\partial y}r\cos(\theta)\right)^2 \\
&= \left(\frac{\partial w}{\partial x}\cos(\theta) + \frac{\partial w}{\partial y}\sin(\theta)\right)^2 + \left(\frac{\partial w}{\partial y}(\cos(\theta)) - \frac{\partial w}{\partial x}\sin(\theta)\right)^2 \\
&= \left(\frac{\partial w}{\partial x}\right)^2\cos^2(\theta) + 2\frac{\partial w}{\partial y}\frac{\partial w}{\partial x}\sin(\theta)\cos(\theta) + \left(\frac{\partial w}{\partial y}\right)^2\sin^2(\theta) + \\
&\qquad \left(\frac{\partial w}{\partial y}\right)^2\cos^2(\theta) - 2\frac{\partial w}{\partial y}\frac{\partial w}{\partial x}\sin(\theta)\cos(\theta) + \left(\frac{\partial w}{\partial x}\right)^2\sin^2(\theta) \\
&= \left(\frac{\partial w}{\partial x}\right)^2\cos^2(\theta) + \left(\frac{\partial w}{\partial y}\right)^2\sin^2(\theta) + \left(\frac{\partial w}{\partial y}\right)^2\cos^2(\theta) + \left(\frac{\partial w}{\partial x}\right)^2\sin^2(\theta) \\
&= \left(\frac{\partial w}{\partial x}\right)^2\left(\cos^2(\theta) + \sin^2(\theta)\right) + \left(\frac{\partial w}{\partial y}\right)^2\left(\sin^2(\theta) + \cos^2(\theta)\right) \\
&= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2
\end{aligned}$$

3

Lecture 14

Section 14.1

Problem 14

$$\begin{aligned}
\int_{-2}^{1}\int_{2}^{4} x^2y^3 \mathrm{d}y\,\mathrm{d}x &= \int_{-2}^{1}\left(\int_{2}^{4} x^2y^3\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{-2}^{1} \left.\frac{x^2y^4}{4}\right|_{y=2}^{4} \mathrm{d}x \\
&= \int_{-2}^{1} 60x^2\mathrm{d}x \\
&= 20x^3\Big|_{-2}^{1} \\
&= 180
\end{aligned}$$

Problem 16

$$\begin{aligned}
\int_{0}^{2}\int_{2}^{4}\left(x^2y^2-17\right)\mathrm{d}x\,\mathrm{d}y &= \int_{0}^{2}\left(\int_{2}^{4}\left(x^2y^2-17\right)\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{0}^{2} \left.\frac{x^3y^2}{3}-17x\right|_{x=2}^{4} \mathrm{d}y \\
&= \int_{0}^{2}\frac{56y^2}{3}-34\mathrm{d}y \\
&= \left.\frac{56y^3}{9}-34y\right|_{0}^{2} \\
&= \frac{-164}{9}
\end{aligned}$$

Problem 22

$$\begin{aligned}
\int_0^1 \int_{-2}^2 x^2 e^y \mathrm{d}x\,\mathrm{d}y &= \int_0^1 \left(\int_{-2}^2 x^2 e^y \mathrm{d}x\right) \mathrm{d}y \\
&= \int_0^1 \left. \frac{x^3 e^y}{3} \right|_{x=-2}^{2} \mathrm{d}y \\
&= \int_0^1 \frac{16e^y}{3} \mathrm{d}y \\
&= \left. \frac{16e^y}{3} \right|_0^1 \\
&= \frac{16e-16}{3}
\end{aligned}$$

Problem 26

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{\pi/2} (y-1)\cos(x) \mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/2} \left(\int_0^{\pi/2} (y-1)\cos(x) \mathrm{d}x\right) \mathrm{d}y \\
&= \int_0^{\pi/2} \left. (y-1)\sin(x) \right|_{x=0}^{\pi/2} \mathrm{d}y \\
&= \int_0^{\pi/2} (y-1) \mathrm{d}y \\
&= \left. \frac{y^2}{2} - y \right|_0^{\pi/2} \\
&= \frac{\pi^2}{8} - \frac{\pi}{2}
\end{aligned}$$

Section 14.2

Problem 2

$$\begin{aligned}
\int_0^2 \int_0^{2x} (y+1) \mathrm{d}y\,\mathrm{d}x &= \int_0^2 \left(\int_0^{2x} (y+1) \mathrm{d}y\right) \mathrm{d}x \\
&= \int_0^2 \left. y + \frac{y^2}{2} \right|_{y=0}^{2x} \mathrm{d}x \\
&= \int_0^2 2x + 2x^2 \mathrm{d}x \\
&= \left. x^2 + \frac{2x^3}{3} \right|_0^2 \\
&= \frac{28}{3}
\end{aligned}$$

Problem 6

$$
\begin{aligned}
\int_0^1 \int_y^{\sqrt{y}} (y+x)\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \left(\int_y^{\sqrt{y}} (y+x)\mathrm{d}x \right) \mathrm{d}y \\
&= \int_0^1 \frac{x^2}{2} + xy \Big|_{x=y}^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^1 \frac{y}{2} + \sqrt{y^3} - \frac{y^2}{2} - y^2 \mathrm{d}y \\
&= \frac{y^2}{4} + \frac{2y^{5/2}}{5} - \frac{y^3}{6} - \frac{y^3}{3} \Big|_0^1 \\
&= \frac{3}{20}
\end{aligned}
$$

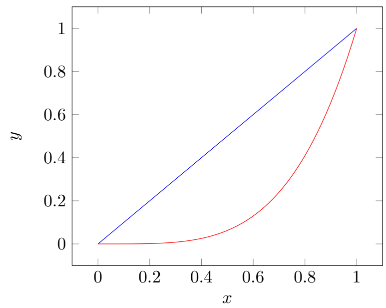
Problem 12

$$
\begin{aligned}
\int_0^\pi \int_0^{\sin(x)} y\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\pi \frac{y^2}{2} \Big|_0^{\sin(x)} \mathrm{d}x \\
&= \int_0^\pi \frac{\sin^2(x)}{2} \mathrm{d}x \\
&= \frac{1}{4} \int_0^\pi 1 - \cos(2x)\mathrm{d}x \\
&= \frac{1}{4} \int_0^\pi 1\,\mathrm{d}x - \frac{1}{8} \int_0^{2\pi} \cos(u)\mathrm{d}u \\
&= \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4}
\end{aligned}
$$

Problem 18

$$
\begin{aligned}
\int_{-1}^1 \int_{y^2-1}^{1-y^2} y\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^1 xy \Big|_{x=y^2-1}^{1-y^2} \mathrm{d}y \\
&= \int_{-1}^1 \left[\left(1-y^2\right) - \left(y^2-1\right) \right] y\,\mathrm{d}y \\
&= \int_{-1}^1 2y - y^3\,\mathrm{d}y \\
&= y^2 - \frac{y^4}{2} \Big|_{-1}^1 \\
&= (1 - 1/2) - (1 - 1/2) \\
&= 0
\end{aligned}
$$

Problem 26



$$\begin{aligned}
\int_0^1 \int_{x^4}^{x} (x-1)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_y^{y^{1/4}} (x-1)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \frac{x^2}{2} - x \bigg|_y^{y^{1/4}} \mathrm{d}y \\
&= \int_0^1 \left[\frac{y^{1/2}}{2} - y^{1/4}\right] - \left[\frac{y^2}{2} - y\right] \mathrm{d}y \\
&= \frac{y^{3/2}}{3} - \frac{4y^{5/4}}{5} - \frac{y^3}{6} + \frac{y^2}{2}\bigg|_0^1 \\
&= \frac{1}{3} - \frac{4}{5} - \frac{1}{6} + \frac{1}{2} \\
&= -\frac{2}{15}
\end{aligned}$$

4

Lecture 15

Section 14.4

Problem 2

$$\begin{aligned}
\int_0^{2\pi}\int_0^{3\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^{3\sin\theta} \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{9\sin^2\theta}{2}\mathrm{d}\theta \\
&= \frac{9}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\mathrm{d}\theta \\
&= \frac{9}{4}\left[2\pi - \left.\frac{\sin\phi}{2}\right|_0^{4\pi}\right] \\
&= \frac{9\pi}{2}
\end{aligned}$$

Problem 6

Assuming the problem meant to ask for area inside $r = 2$ and outside $r = 2\cos\theta$, as $r = 2\cos\theta$ lies entierly within $r = 2$.

$$\begin{aligned}
A = A_{outer} - A_{inner} &= \int_0^{2\pi}\int_0^{2} r\,\mathrm{d}r\,\mathrm{d}\theta - \int_0^{\pi}\int_0^{3\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 2\,\mathrm{d}\theta - \int_0^{\pi} \frac{4\cos^2\theta}{2}\mathrm{d}\theta \\
&= 4\pi - 2\int_0^{\pi} \cos^2\theta\,\mathrm{d}\theta \\
&= 4\pi - \pi \\
&= 3\pi
\end{aligned}$$

Care must be taken to not simply integrate $\int_0^{2\pi}\int_{2\cos\theta}^{2} r\,\mathrm{d}r\,\mathrm{d}\theta$, as this double counts the area of the interiaor cricle, which is of period π, rather than 2π.

Problem 16

$$
\begin{aligned}
\int_0^1 \int_x^1 x^2 \,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 x^2 y\Big|_x^1 \,\mathrm{d}x \\
&= \int_0^1 x^2 - x^3 \,\mathrm{d}x \\
&= \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 \\
&= 1/12
\end{aligned}
$$

This doesn't fulfill the requirement of first substituting to polar coordiantes. But whatever.

Problem 22

$$
\begin{aligned}
\int_0^{2\pi} \int_0^{1+\cos\theta} 1 + x \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi} \int_0^{1+\cos\theta} 1 + r\cos\theta \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 1 + \cos\theta + \frac{r^2\cos\theta}{2}\Big|_0^{1+\cos\theta} \,\mathrm{d}\theta \\
&= \int_0^{2\pi} 1 + \cos\theta + \frac{[1 + 2\cos\theta + \cos^2\theta]\cos\theta}{2} \,\mathrm{d}\theta \\
&= \int_0^{2\pi} 1 + \frac{2\cos^2\theta}{2} \,\mathrm{d}\theta \\
&= 3\pi
\end{aligned}
$$

It is convienent to make use of the fact that sin and cos will always have an integral of 0 when their operand iterates over an integral number of periods.

Problem 24

We must first determine an appropriate bounding surface:

$$
z_{hi} = 12 - 2x^2 - y^2 \qquad\qquad z_{lo} = x^2 + 2y^2
$$

$$
\begin{aligned}
z_{hi} &= z_{lo} \\
12 - 2x^2 - y^2 &= x^2 + 2y^2 \\
4 &= x^2 + y^2
\end{aligned}
$$

The enclosed volume is thus bounded by $x^2 + y^2 \leq 4$ on the $x - y$ plane, thus: $r \leq 2$

$$\begin{aligned}
\int_0^{2\pi}\int_0^2 \left[\left(12 - 2x^2 - y^2\right) - \left(x^2 + 2y^2\right)\right] r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^2 \left[12 - 3x^2 - 3y^2\right] r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 \left[12 - 3\left(x^2 + y^2\right)\right] r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 \left[12 - 3\left(r^2\cos^2\theta + r^2\sin^2\theta\right)\right] r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 \left[12 - 3r^2\right] r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 12r - 3r^3 \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 6r^2 - \frac{3r^4}{4}\bigg|_0^2 \,\mathrm{d}\theta \\
&= \int_0^{2\pi} 12 \,\mathrm{d}\theta \\
&= 24\pi
\end{aligned}$$

Problem 30

$$\begin{aligned}
\int_0^{\pi}\int_0^{2a\sin\theta} r^2 \cdot r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{\pi}\int_0^{2a\sin\theta} r^3 \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi} \frac{r^4}{4}\bigg|_0^{2a\sin^4\theta} \,\mathrm{d}\theta \\
&= \int_0^{\pi} \frac{16a^4\sin^4\theta}{4} \,\mathrm{d}\theta \\
&= 4a^4 \int_0^{\pi} \sin^4\theta \,\mathrm{d}\theta \\
&= \frac{4a^4}{2}\int_0^{\pi} 3 - 4\cos 2\theta + \cos 4\theta \,\mathrm{d}\theta \\
&= \frac{4a^4}{2}\int_0^{\pi} 3 \,\mathrm{d}\theta \\
&= \frac{3\pi a^4}{2}
\end{aligned}$$

It is once more convienent to make use of the fact that sin and cos will always have an integral of 0 when their operand iterates over an integral number of periods.

Section 14.5

Problem 2

$$m = \int_2^4 \int_1^3 1 \, \mathrm{d}x \, \mathrm{d}y = 4$$

$$\bar{x} = \frac{1}{m} \int_2^4 \int_1^3 x \, \mathrm{d}x \, \mathrm{d}y$$
$$\bar{x} = \frac{1}{4} \int_2^4 \left. \frac{x^2}{2} \right|_1^3 \mathrm{d}y$$
$$\bar{x} = \frac{1}{4} \int_2^4 4 \, \mathrm{d}y$$
$$\bar{x} = 2$$

$$\bar{y} = \frac{1}{m} \int_2^4 \int_1^3 y \, \mathrm{d}x \, \mathrm{d}y$$
$$\bar{y} = \frac{1}{4} \int_2^4 2y \, \mathrm{d}y$$
$$\bar{y} = \left. \frac{1}{4} y^2 \right|_2^4 \mathrm{d}y$$
$$\bar{y} = 3$$

$$(\bar{x}, \bar{y}) = (2, 3)$$

Problem 6

The problem is made simpler by invoking the argument of symmetry, and only evaluating the y centroid of one half of the reigion, knowing that the x centriod will be at the point of symmetry: $\bar{x} = 1$.

$$\begin{aligned} m &= \int_0^2 \int_y^1 1 \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 1 - y \, \mathrm{d}y \\ &= 1/2 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m} \int_0^2 \int_y^1 y \, \mathrm{d}x \, \mathrm{d}y \\ &= 2 \int_0^2 y(1 - y) \, \mathrm{d}y \\ &= 2 \int_0^2 y - y^2 \, \mathrm{d}y \\ &= 2 \left. \frac{y^2}{2} - \frac{y^3}{3} \right|_0^1 \\ &= 1/3 \\ (\bar{x}, \bar{y}) &= (1, 1/3) \end{aligned}$$

Problem 14

The problem is made simpler by invoking the argument of symmetry, and only evaluating the x centroid of one half of the reigion, knowing that the y centriod will be at the point of symmetry: $\bar{y} = 0$.

$$\begin{aligned} m &= \int_0^9 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} x^2 \,\mathrm{d}y\,\mathrm{d}x \\ &= -2\int_9^0 (9-u)^2\sqrt{u}\,\mathrm{d}u \\ &= -2\int_9^0 81u^{1/2} - 18u^{3/2} + u^{5/2}\,\mathrm{d}u \\ &= \frac{23328}{35} \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{1}{m}\int_0^9 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} x^3 \,\mathrm{d}y\,\mathrm{d}x \\ &= -2\frac{35}{23328}\int_9^0 (9-u)^3\sqrt{u}\,\mathrm{d}u \\ &= -2\frac{35}{23328}\int_9^0 729u^{1/2} - 243u^{3/2} + 27u^{5/2} - u^{7/2}\,\mathrm{d}u \\ &= 6 \\ (\bar{x},\bar{y}) &= (6,0) \end{aligned}$$

Problem 24

$$\begin{aligned} m &= \int_{-1}^3 \int_{x^2}^{2x+3} x^2 \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^3 \left[2x+3-x^2\right] x^2\,\mathrm{d}x \\ &= \frac{96}{5} \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{1}{m}\int_{-1}^3 \int_{x^2}^{2x+3} x^3 \,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{5}{96}\int_{-1}^3 \left[2x+3-x^2\right] x^3\,\mathrm{d}x \\ &= \frac{17}{9} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m}\int_{-1}^3 \int_{x^2}^{2x+3} x^2 y \,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{5}{96}\int_{-1}^3 \left[2x+3-x^2\right] x^3\,\mathrm{d}x \\ &= \frac{5}{2\cdot 96}\int_{-1}^3 \left[(2x+3)^2-(x^2)^2\right] x^2\,\mathrm{d}x \\ &= \frac{379}{168} \end{aligned}$$

Thus $(\bar{x},\bar{y}) = (\frac{17}{9}, \frac{379}{168})$, or approximately $(1.89, 2.26)$.

Part II

Online

5

Problem 1

We start by simpifying the problem to the maximum volume of a right cuboid with a diagonal between the origin and a point on the unit sphere in the first octant. We then attempt to maximize the volume, $f(x, y, z) = xyz$, under the condition $x^2 + y^2 + z^2 = 1$, or $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$, using Lagrange Multipliers:

$$L(x, y, z, \lambda) = xyz - \lambda \left(x^2 + y^2 + z^2 - 1\right)$$

Setting x, y, z partials to 0 yields:

$$\frac{\partial L}{\partial x} = 1 - 2x\lambda = 0 \qquad \frac{\partial L}{\partial y} = 1 - 2y\lambda = 0 \qquad \frac{\partial L}{\partial z} = 1 - 2z\lambda = 0$$

$$x = y = z = \frac{1}{2\lambda}$$

Setting λ partial to 0 yields:

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

$$3\left(\frac{1}{2\lambda}\right)^2 = 1$$

$$\lambda = \frac{\sqrt{3}}{2}$$

$$x = y = z = \frac{1}{2\lambda} = \frac{1}{\sqrt{3}}$$

Volume is maximized when $x = y = z = \frac{1}{\sqrt{3}}$, when the box is a cube, with volume $8\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{8}{3\sqrt{3}}$

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Problem 2

Function:

$$f(x, y) = x^2 - 2xy + 7y^2$$

Condition:

$$x^2 + 4y^2 = 1 \implies g(x, y) = x^2 + 4y^2 - 1 = 0$$

In Lagrange equation:

$$\begin{aligned} L(x, y, \lambda) &= f(x, y) - \lambda g(x, y) \\ &= x^2 - 2xy + 7y^2 - \lambda \left(x^2 + 4y^2 - 1 \right) \end{aligned}$$

Setting partials to 0:

$$\frac{\partial L}{\partial x} = 2x - 2y - 2\lambda x = 0 \qquad \frac{\partial L}{\partial y} = 14y - 2x - 8\lambda y = 0$$

$$x = \frac{y}{1 - \lambda} \qquad y = \frac{x}{7 - 4\lambda}$$

$$\begin{aligned} x &= \frac{\frac{x}{2-4\lambda}}{1 - \lambda} \\ (7 - 4\lambda)(1 - \lambda) &= 1 \\ 6 - 11\lambda + 4\lambda^2 &= 0 \\ \lambda &= [3/4, 2] \end{aligned}$$

$$\begin{aligned} \lambda &= 2 \\ x &= \frac{y}{1 - 2} \\ x &= -y \\ 5y^2 - 1 &= 0 \\ y = -x &= \pm\sqrt{1/5} \end{aligned} \qquad \begin{aligned} \lambda &= 3/4 \\ y &= \frac{x}{7 - 3} \\ x &= 4y \\ 20y^2 &= 1 \\ y &= \pm\sqrt{1/20} \\ x = 4y &= \pm\sqrt{16/20} \end{aligned}$$

Plugging back into original equation, maximums are at $f(\sqrt{1/5}, -\sqrt{1/5})$ and $f(-\sqrt{1/5}, \sqrt{1/5})$, where $f(x, y) = 2$, and minumums are at $f(\sqrt{16/20}, \sqrt{1/20})$ and $f(-\sqrt{16/20}, -\sqrt{1/20})$, where $f(x, y) = 3/4$.

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Problem 3

$$\begin{aligned} r &= \sqrt{x^2+y^2} & \theta &= \tan^-1\frac{y}{x} \\ \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} & \frac{\partial \theta}{\partial x} &= \frac{-y}{y^2+x^2} \\ \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} & \frac{\partial \theta}{\partial y} &= \frac{x}{y^2+x^2} \end{aligned}$$

$$\begin{aligned} |\nabla f|^2 &= \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 \\ &= \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right)^2 \\ &= \left(\frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial f}{\partial \theta}\frac{y}{y^2+x^2}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\frac{x}{y^2+x^2}\right)^2 \end{aligned}$$

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$\begin{aligned} x^2+y^2 &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 \end{aligned}$$

$$\begin{aligned} |\nabla f|^2 &= \left(\frac{\partial f}{\partial r}\frac{x}{\sqrt{r^2}} - \frac{\partial f}{\partial \theta}\frac{y}{r^2}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{y}{\sqrt{r^2}} + \frac{\partial f}{\partial \theta}\frac{x}{r^2}\right)^2 \\ &= \left(\frac{\partial f}{\partial r}\frac{r\cos\theta}{r} - \frac{\partial f}{\partial \theta}\frac{r\sin\theta}{r^2}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{r\sin\theta}{r} + \frac{\partial f}{\partial \theta}\frac{r\cos\theta}{r^2}\right)^2 \\ &= \left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right)^2 + \left(\frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\right)^2 \end{aligned}$$

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Problem 4

$$
\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= -2xF_0e^{-\left(x^2+y^2+z^2\right)} * (-1) + -2yF_0e^{-\left(x^2+y^2+z^2\right)} * (0) + -2zF_0e^{-\left(x^2+y^2+z^2\right)} * \left(-2\left(1-t\right)^2\right) \\
&= -2xF_0e^{-\left(x^2+y^2+z^2\right)} * (-1) + -2zF_0e^{-\left(x^2+y^2+z^2\right)} * \left(-2\left(1-t\right)^2\right) \\
&= 2\left(1-t\right)F_0e^{-\left((1-t)^2+\left((1-t)^2\right)^2\right)} + 4\left(1-t\right)^2F_0e^{-\left((1-t)^2+\left((1-t)^2\right)^2\right)} * \left(1-t\right)^2 \\
&= 2\left(1-t\right)F_0e^{-\left((1-t)^2+\left((1-t)^2\right)^2\right)}\left(1+2(1-t)\right) \\
&= 2\left(1-t\right)F_0e^{-\left((1-t)^2+\left((1-t)^2\right)^2\right)}\left(3-t\right) \\
&= 2\left(3-6t+t^2\right)F_0e^{-\left((1-t)^2+\left((1-t)^2\right)^2\right)}
\end{aligned}
$$

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Problem 5

Assume base starts at origin and has diagonal going from $(0,0)$ to (j,k). Through adjusting a, b, c, d, j, and k, this model can be shown to represent any possible rectangular prism with varying side lengths.

$$\begin{aligned}
V &= \int_0^j \int_0^k ax + by + c \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^j (k-0)(ax+c) + \frac{bk^2}{2} \,\mathrm{d}x \\
&= kcj + \frac{jbk^2}{2} \int_0^j kax \,\mathrm{d}x \\
&= kcj + \frac{jbk^2}{2} + \frac{kaj^2}{2} \\
&= kj\left(\frac{aj}{2} + \frac{bk}{2} + c\right)
\end{aligned}$$

$$\begin{aligned}
l_0 &= z(0,0) = a(0) + b(0) + c = c \\
l_1 &= z(j,0) = a(j) + b(0) + c = aj + c \\
l_2 &= z(0,k) = a(0) + b(k) + c = bk + c \\
l_3 &= z(j,j) = a(j) + b(k) + c = aj + bk + c \\
l_{avg} &= \frac{1}{4} \sum l_i \\
&= \frac{4c + 2aj + 2bk}{4} \\
&= c + \frac{aj}{2} + \frac{bk}{2}
\end{aligned}$$

$$\begin{aligned}
A_{base} &= kj \\
A_{base} \cdot l_{avg} &= kj \cdot \left(c + \frac{aj}{2} + \frac{bk}{2}\right) = \int_0^j \int_0^k ax + by + c \,\mathrm{d}y \,\mathrm{d}x
\end{aligned}$$

Which Was What We Wanted

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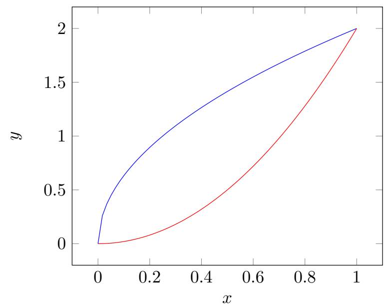
Problem 6



Plot for $y^2 = 4ax$ and $x^2 = {}^{ay}/_2$ when $a = 1$.

$$y^2 = 4ax \implies y = \sqrt{4ax} \text{ or, equivalently } x = \frac{y^2}{4a}$$
$$x^2 = \frac{ay}{2} \implies x = \sqrt{\frac{ay}{2}} \text{ or, equivalently } y = \frac{2x^2}{a}$$

$$\begin{aligned} A &= \int_0^{2a} \sqrt{\frac{ay}{2}} - \frac{y^2}{4a}\,\mathrm{d}y \\ &= \sqrt{\frac{a}{2}} \int_0^{2a} \sqrt{y}\,\mathrm{d}y - \frac{1}{4a}\int_0^{2a} y^2\,\mathrm{d}y \\ &= \sqrt{\frac{a}{2}} \left[\frac{2y^{3/2}}{3}\right]_0^{2a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{2a} \\ &= \sqrt{\frac{a}{2}}\frac{4a\sqrt{2a}}{3} - \frac{2a^2}{3} \\ &= \frac{2a^2}{3} \end{aligned}$$

$$\begin{aligned} A &= \int_0^{a} \sqrt{4ax} - \frac{2x^2}{a}\,\mathrm{d}x \\ &= 2\sqrt{a} \int_0^{a} \sqrt{x}\,\mathrm{d}x - \frac{2}{a}\int_0^{a} x^2\,\mathrm{d}x \\ &= 2\left[\frac{2}{3}x\sqrt{ax}\right]_0^{a} - \frac{2}{a}\left[\frac{y^3}{3}\right]_0^{a} \\ &= \frac{4a^2}{3} - \frac{2a^2}{3} \\ &= \frac{2a^2}{3} \end{aligned}$$

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Problem 7

$$
\begin{aligned}
I^2 = \int_0^\infty e^{-x^2}\,\mathrm{d}x \int_0^\infty e^{-y^2}\,\mathrm{d}y &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= -\frac{1}{2}\int_0^{\pi/2} \int_0^{-\infty} e^{u}\,\mathrm{d}u\,\mathrm{d}\theta \\
&= -\frac{1}{2}\int_0^{\pi/2} e^{-\infty} - e^{0}\,\mathrm{d}\theta \\
&= -\frac{1}{2}\int_0^{\pi/2} -1\,\mathrm{d}\theta \\
I^2 &= \pi/4 \\
I = \int_0^\infty e^{-x^2}\,\mathrm{d}x &= \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

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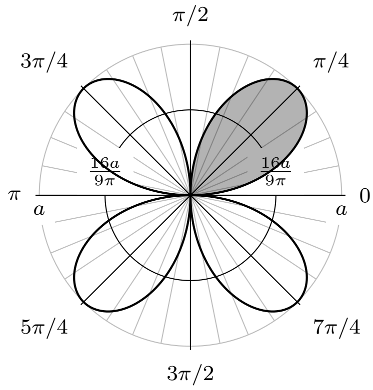
Problem 8



The area of entire curve will be that of shaded reigion, times 4. Average distance from origin of reigion will trivially be 0, as can be proven by invoking the argument of symmetry about $x = 0$ and $y = 0$. Perhaps the question is asking the average distance from origin for a single lobe? This can be shown to be $\frac{16a}{9\pi}$, and is marked on the above plot.

$$
\begin{aligned}
A &= 4\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\frac{1}{2}\int_0^{\pi/2} r^2\Big|_0^{a\sin 2\theta}\,\mathrm{d}\theta \\
&= 4\frac{a^2}{2}\int_0^{\pi/2}\sin^2(2\theta)\,\mathrm{d}\theta \\
&= 4\frac{a^2}{2}\int_0^{\pi/2}\frac{1-2\cos 4\theta}{2}\,\mathrm{d}\theta \\
&= 4\frac{a^2}{4}\int_0^{\pi/2} 1\,\mathrm{d}\theta - \frac{a^2}{2}\int_0^{\pi/2}(-2)\cos 4\theta\,\mathrm{d}\theta \\
&= 4\frac{a^2}{4}\int_0^{\pi/2} 1\,\mathrm{d}\theta - 0 \\
A &= 4\frac{a^2\pi}{8} = \frac{a^2\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\bar{D} &= \frac{1}{\frac{A}{4}}\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\sqrt{x^2+y^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{8}{a^2\pi}\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{8}{a^2\pi}\int_0^{\pi/2}\int_0^{a\sin 2\theta} r^2\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{8}{a^2\pi}\frac{a^3}{3}\int_0^{\pi/2}\sin^3 2\theta\,\mathrm{d}\theta \\
&= \frac{4a}{3\pi}\int_0^{\pi/2} 3\sin\theta - \sin 3\theta\,\mathrm{d}\theta \\
&= \frac{2a}{3\pi}\left(3-\frac{1}{3}\right) \\
\bar{D} &= \frac{2a}{3\pi}\cdot\frac{8}{3} = \frac{16a}{9\pi}
\end{aligned}
$$