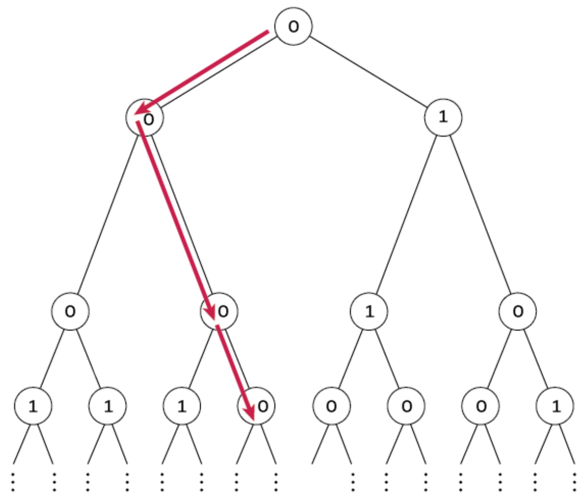

For a fixed p, independently label the nodes of an infinite complete binary tree 0 with probability p, and 1 otherwise. For what p is there exactly a 1/2 probability that there exists an infinite path down the tree that sums to at most 1 (that is, all nodes visited, with the possible exception of one, will be labeled 0). Find this value of p accurate to 10 decimal places.

Solution: Let A be the probability that there exists a path of sum 0, and let B be the probability that there exists a path of sum at most 1. By definition, $B = 1/2$.

We can let $A = p(1-(1-A)^2)$, knowing that we must be in the case where the root is 0 (with probability p), and at least one of the subtrees must contain a path of sum 0. Since the probability that neither contains a path of sum 0 is $(1-A)^2$, this occurs with probability $1-(1-A)^2$.

Likewise, for B, if the root of the tree is 1, occuring with probability $1-p$, we require one of the subtrees to contain a path sum of 0, putting us in the case above. If the root is 0, then we still need a sum of at most 1; essentially, we still have an "allowance" of a 1 to place further down the tree. Thus, we can set up the equation: $B = (1-p)(1-(1-A)^2) + p(1-(1-B)^2)$ and have a system:

$$\begin{aligned} B &= 1/2 \\ A &= p(1-(1-A)^2) \\ B &= (1-p)(1-(1-A)^2) + p(1-(1-B)^2) \end{aligned}$$

We can solve this via the Mathematica code:

```
N[Solve[
    {
      B == 1/2,
      A == p*(1 - (1 - A)^2),
      B == (1 - p)*(1 - (1 - A)^2) + p*(1 - (1 - B)^2)
    },
    {A, B, p}, Reals, Assumptions -> {A > 0, B > 0, p > 0}
  ], 10]
```

This results in $p = \boxed{\mathbf{0.5306035754}}$.