Lab 1 Written 1

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1.

1. Consider two random variables X,Y that are not independent. Their probabilities of are given by the following table:

	X=0	X=1
Y=0	1/4	1/4
Y=1	1/6	1/3

(a) What is the probability that $X = 1$?
(b) What is the probability that $X = 1$ conditioned on $Y = 1$?
(c) What is the variance of the random variable X?
(d) What is the variance of the random variable X conditioned that $Y = 1$?
(e) What is $E[X^3 + X^2 + 3Y^7|Y = 1|]$?

a) $P(X=1) =$

$= P(X=1, Y=0) + P(X=1, Y=1)$

$= \frac{1}{4} + \frac{1}{3}$

$= \cancel{4}\frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

$\boxed{P(X=1) = \frac{7}{12}}$

b) $P(X=1 | Y=1) =$

$= \frac{P(Y=1, X=1)}{P(Y=1)} \rightarrow \underbrace{\frac{1}{6} + \frac{1}{3}}_{\frac{1}{6} + \frac{2}{6} = \frac{3}{6}}$

$= \frac{\frac{1}{3}}{\frac{3}{6}}$ $\rightarrow \frac{2}{6}$

$= \frac{2}{3}$

$\boxed{P(X=1 | Y=1) = \frac{2}{3}}$

c) $Var(X) =$

Marginal PMF:

$P_X(x) = \begin{cases} \frac{5}{12} & x = 0 \\ \frac{7}{12} & x = 1 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \sum_{x=0}^{1} x \cdot P_X(x)$

$= (0)\left(\frac{5}{12}\right) + (1)\left(\frac{7}{12}\right)$

$E(x) = \frac{7}{12}$

$Var(x) = \sigma^2 = E(X - E[X])^2$

$= E[X^2] - (E[X])^2$

$= \left[0^2 \cdot \frac{5}{12} + 1^2\left(\frac{7}{12}\right)\right] - \left(\frac{7}{12}\right)^2$

$= \frac{7}{12} - \left(\frac{7}{12}\right)^2$

$= 0.24$

$\boxed{Var(x) = 0.24}$

d) $Var(X | Y=1) =$

$P_X(x) = \begin{cases} \frac{5}{12} & x = 0 \\ \frac{7}{12} & x = 1 \\ 0 & \text{otherwise} \end{cases}$

$P_X(x | Y=1) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{2}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$

$Var(X|Y=1) =$

$E[X^2|Y=1] - (E[X|Y=1])^2$

$0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{2}{3}$ $\qquad$ $\left(0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}\right)^2$

$\frac{2}{3}$ $\qquad$ $\left(\frac{2}{3}\right)^2$

$Var(X|Y=1) = \frac{2}{3} - \left(\frac{2}{3}\right)^2 =$

$\boxed{Var(X | Y=1) = \frac{2}{9}}$

e) $E[X^3 + X^2 + 3Y^7 | Y=1|] =$

- $E[X^3] = 0^3 \cdot \frac{5}{12} + 1^3 \cdot \frac{7}{12} = \frac{7}{12}$
- $E[X^2] = 0^2 \cdot \frac{5}{12} + 1^2 \cdot \frac{7}{12} = \frac{7}{12}$
- $E[Y^7] = 0^7 \cdot \frac{1}{2} + 1^7 \cdot \frac{1}{2} = \frac{1}{2}$

$P_Y(Y) = \begin{cases} \frac{1}{2} & Y = 0 \\ \frac{1}{2} & Y = 1 \\ 0 & \text{otherwise} \end{cases}$

- $|Y=1| = \frac{1}{2}$?

$\frac{1}{2} \cdot \frac{1}{2} \cdot 3 = \frac{3}{4}$

$= \frac{7}{12} \cdot \frac{7}{\cancel{12}_{4}} \cdot \frac{\cancel{3}^{1}}{4} = \frac{49}{192} = 0.26$

$\boxed{E[X^3 + X^2 + 3Y^7 | Y=1|] = 0.26}$

Q2) $v_1 = [1,1,3] \quad v_2 = [1,0,0]$

$p_1 = [3,3,3] \quad p_2 = [1,2,3] \quad p_3 = [0,0,1]$

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \qquad V^T V = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(V^T \cdot V)^{-1} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 1.1 \end{bmatrix}$$

$$(V^T \cdot V)^{-1} V^T = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 1.1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.1 & 0.3 \\ 1 & -0.1 & -0.3 \end{bmatrix}$$

$$\beta_1^* = (V^T \cdot V)^{-1} V^T p_1 = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 1 & -0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1.2 \\ 1.8 \end{bmatrix}}$$

$$\beta_2^* = (V^T \cdot V)^{-1} V^T p_2 = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 1 & -0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1.1 \\ -0.3 \end{bmatrix}}$$

$$\beta_3^* = (V^T \cdot V)^{-1} V^T p_3 = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 1 & -0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}}$$

Lab 1 Written Problems

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3. Consider a coin such that probability of heads is 2/3. Suppose you toss the coin 100 times. Estimate the probability of getting 50 or fewer heads. You can do this in a variety of ways. One way is to use the Central Limit Theorem. Be explicit in your calculations and tell us what tools you are using in these.

$X :=$ the number of heads flipped

$$X = \sum_{i}^{100} x_i \qquad x_i \sim \mathrm{Bern}(2/3)$$

$$E[x_i] = 2/3$$

$$Var[x_i] = 2/9$$

$$E[X] = \sum E[x_i] = \sum 2/3 = 100\,(2/3)$$

$$E[X] = 66.67$$

$$Var[X] = Var\left[\sum x_i\right] = 100\, Var[x_i] = \frac{200}{9}$$

$$\Rightarrow \sigma_X = \frac{\sqrt{200}}{3}$$

Applying Central Limit Theorem

$$\bar{X} \sim N\left(\frac{200}{3}, \frac{\sqrt{200}}{3\sqrt{100}}\right) = N\left(\frac{200}{3}, \frac{\sqrt{2}}{3}\right)$$

$$Z = \frac{50 - \frac{200}{3}}{\frac{\sqrt{2}}{3}} = \frac{150 - 200}{\sqrt{2}} = \frac{-50}{\sqrt{2}}$$

$$Z \approx -35$$

$$P\{X \le 50\} = Q(-35) \approx 0$$

Manual check

$$P\{X \le 50\} = \sum_{k=0}^{50} \left(\frac{2}{3}\right)^{k} \left(\frac{1}{3}\right)^{100-k} \binom{100}{k} \qquad X \sim \mathrm{Binom}(100, 2/3)$$

$$= 4.19\text{E}{-4}$$

$$= 0.000419 \approx 0 \checkmark$$