

Three special perfect squares 三个特殊的完美方块

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```
In [1]: from math import sqrt
```

We aim to find all triples (a, b, c) of natural numbers such that:

- $a < b < c$;
- a, b and c are perfect squares;
- 0 occurs in none of a, b and c ; 每个非零数字恰好出现在 a, b 和 c 之中一次
- every nonzero digit occurs exactly once in one of a, b and c .

一个糟糕的策略是生成自然数，检查它们是否是完美的正方形，如果不是，则丢弃它们。一个更好的策略是只生成完美的正方形

For instance $(25, 841, 7396)$ is a solution since $25 = 5^2$, $841 = 29^2$, $7396 = 86^2$, and we see that each of 1, 2, 3, 4, 5, 6, 7, 8 and 9 occurs **once and only once** in 25, 841 or 7396.

A bad strategy would be to generate natural numbers, check whether they are perfect squares, and in case they aren't, discard them. A much better strategy is to generate nothing but perfect squares.

- Since we need 3 perfect squares and use up each of the 9 nonzero digits **once and only once**, c can consist of at most 7 digits, leaving one digit to a and 1 digit to b . There are three 1-digit nonzero perfect squares: 1, 4 and 9. So a largest possible value for c is obtained by setting a to 1, b to 4, and c to 9876532.
- A largest possible value for a is 997: otherwise, b would consist of at least 3 digits and c would consist of at least 4 digits, for a total of at least 10 digits.
- A largest possible value for b is 9998: otherwise, c would consist of at least 5 digits and a takes at least one digit, for a total of at least 10 digits.

We could therefore look for solutions with the following code structure:

```
In [2]: for i in range(1, int(sqrt(997)) + 1):
        a = i ** 2
        for j in range(i + 1, int(sqrt(9998)) + 1):
            b = j ** 2
            for k in range(j + 1, int(sqrt(9876532)) + 1):
                c = k ** 2
                # Check whether (a, b, c) is a solution; display it if it is
```

Let us see how many triples would be considered with the previous code structure:

```
In [3]: sum(1 for i in range(1, int(sqrt(997)) + 1)
            for j in range(i + 1, int(sqrt(9998)) + 1)
            for k in range(j + 1, int(sqrt(9876532)) + 1)
            )
```

如果代码生成可被丢弃的值，则不需要考虑b（和c）的候选者，因为它包含至少一次0或至少两次出现的相同数字；如果代码生成可以被丢弃的b的值，则也不需要考虑c的候选者，因为它包含至少一次出现0或者在a和b中一起考虑至少两次出现。所以更好的代码结构是

Out [3]: 7936372

There is no need to consider candidates for b (and c) in case the code generates a value for a that can be discarded because it contains at least one occurrence of 0 or at least two occurrences of the same digit; there is also no need to consider candidates for c in case the code generates a value for b that can be discarded because it contains at least one occurrence of 0 or some digit occurs at least twice in a and b considered together. So a better code structure is:

```
In [4]: for i in range(1, int(sqrt(997)) + 1):
        a = i ** 2 ** 幂 - 返回x的y次幂
        # If a contains 0 or many occurrences of the same digit:
        #     continue
        for j in range(i + 1, int(sqrt(9998)) + 1):
            b = j ** 2
            # If b contains 0 or digits that occur in a
            # or many occurrences of the same digit:
            #     continue
            for k in range(j + 1, int(sqrt(9876532)) + 1):
                c = k ** 2
                # Check whether (a, b, c) is a solution; display it if it is
```

似乎0扮演一个特殊角色，但是人们可以“假装”首先生成0，在这种情况下，我们只想考虑一个只包括不同数字的候选者，其中没有一个属于该集合之前“看到”的数字的 S_0 （即 $\{0\}$ ），然后只考虑b的候选者，其中只包括不同的数字，其中没有一个属于前面看到的数字集合 S_1 （即 S_0 与之结合的数字）a）中出现的一组数字，然后只考虑c的候选者，它们只包括不同的数字，其中没有一个数字属于之前看到的数字集合 S_2 （即， S_1 与一组数字的联合）b）

It seems that 0 plays a special role, but one can “pretend” that 0 is being generated in the first place, in which case we want to consider only candidates for a which consist of nothing but distinct digits none of which belongs to the set S_0 of digits “seen” before (namely, $\{0\}$), and then consider only candidates for b which consist of nothing but distinct digits none of which belongs to the set S_1 of digits seen before (namely, the union of S_0 with the set of digits that occur in a), and then consider only candidates for c which consist of nothing but distinct digits none of which belongs to the set S_2 of digits seen before (namely, the union of S_1 with the set of digits that occur in b).

We will examine three techniques to keep track of the digits seen before, and to check that the candidate for a , b or c under consideration consists of nothing but distinct new digits.

With the first technique, we work with sets of digit characters rather than sets of digits, e.g., with $\{'0'\}$ rather than $\{0\}$, with $\{'0', '2', '4', '5', '7', '8'\}$ rather than $\{0, 2, 4, 5, 7, 8\}$. Having such a set S of digit characters for the digits seen before (or “pretended” to be seen before in the case of 0), and considering a candidate n for a , b or c , we get a string s from n , then a set T of characters from s . Then all digits in n are distinct and different to the digits seen before iff the number of digits in n plus the number of (distinct) digits seen before, the latter being the cardinality of S , is equal to the cardinality of $S \cup T$. The cardinality of a set can be computed with the `len()` function. The number of digits in n is nothing but the length of s , which can also be computed with the `len()` function. The union of two sets can be computed with the `|` operator:

```
In [5]: # Assume that 0, 2 and 5 are the digits seen before
        S = {'0', '2', '5'}

        # All digits in b are distinct and different to
        # the digits seen before, so all good
        b = 784
        str(b), set(str(b)), S | set(str(b))
        len(S) + len(str(b)), len(S | set(str(b)))
```

使用第一种技术，我们使用数字字符串而不是数字组，例如，使用 $\{'0'\}$ 而不是 $\{0\}$ ，使用 $\{'0', '2', '4', '5', '7', '8'\}$ 而不是 $\{0, 2, 4, 5, 7, 8\}$ 。对于之前看到的数字具有这样的数字字符集 S （或者在之前的情况下“假装”以前看到的数字），并考虑 a , b 或 c 的候选 n ，我们从 n 得到一个字符串 s ，然后从一个字符集得到 T 。然后 n 中的所有数字都是不同的，并且与之前看到的数字不同，如果 n 中的数字加上之前看到的（不同）数字的数量，后者是 S 的基数，则等于 $S \cup T$ 的基数。可以使用`len()`函数计算集合的基数。 n 中的位数只是 s 的长度，也可以使用`len()`函数计算。两组的并集可以用`|`来计算运算符：

```

# Not all digits in b are distinct, so not good
b = 676
str(b), set(str(b)), S | set(str(b))
len(S) + len(str(b)), len(S | set(str(b)))

# Some digits in b have been seen before, so not good
b = 729
str(b), set(str(b)), S | set(str(b))
len(S) + len(str(b)), len(S | set(str(b)))

```

```
Out[5]: ('784', {'4', '7', '8'}, {'0', '2', '4', '5', '7', '8'})
```

```
Out[5]: (6, 6)
```

```
Out[5]: ('676', {'6', '7'}, {'0', '2', '5', '6', '7'})
```

```
Out[5]: (6, 5)
```

```
Out[5]: ('729', {'2', '7', '9'}, {'0', '2', '5', '7', '9'})
```

```
Out[5]: (6, 5)
```

Rather than using the `|` operator, one can also call the `union` method, to be distinguished from the `update()` method: the latter does not return a new set but changes the set to which the method is applied. Also note that one can get the union of more than two sets:

```
In [6]: S = {2, 4, 6}
```

```

S.union((3, 5, 7)), S
S.update((3, 5, 7)), S
S.union((10, 11), {20, 21, 22}, [30]), S
S.update((10, 11), {20, 21, 22}, [30]), S
{2, 4, 6} | {10, 11} | {20, 21, 22} | {30}

```

```
Out[6]: ({2, 3, 4, 5, 6, 7}, {2, 4, 6})
```

```
Out[6]: (None, {2, 3, 4, 5, 6, 7})
```

```
Out[6]: ({2, 3, 4, 5, 6, 7, 10, 11, 20, 21, 22, 30}, {2, 3, 4, 5, 6, 7})
```

```
Out[6]: (None, {2, 3, 4, 5, 6, 7, 10, 11, 20, 21, 22, 30})
```

```
Out[6]: {2, 4, 6, 10, 11, 20, 21, 22, 30}
```

The same distinction applies to intersection, difference, and symmetric difference, which for intersection and difference (but not symmetric difference), also accept many arguments, and which also have their own operators:

```

In [7]: S = {2, 3, 4, 5, 6, 7}

        S.intersection((4, 5, 6, 7, 8, 9)), S
        S.intersection_update((4, 5, 6, 7, 8, 9)), S
        S.intersection((5, 6, 7, 8), {6, 7, 8, 9}, [7, 8, 10]), S
        S.intersection_update((5, 6, 7, 8), {6, 7, 8, 9}, [7, 8, 10]), S
        {2, 3, 4, 5, 6, 7} & {5, 6, 7, 8} & {6, 7, 8, 9} & {7, 8, 10}

Out[7]: ({4, 5, 6, 7}, {2, 3, 4, 5, 6, 7})

Out[7]: (None, {4, 5, 6, 7})

Out[7]: ({7}, {4, 5, 6, 7})

Out[7]: (None, {7})

Out[7]: {7}

In [8]: S = {2, 3, 4, 5, 6, 7}

        S.difference((0, 1, 4, 5)), S
        S.difference_update((0, 1, 4, 5)), S
        S.difference((2, 8, 9), {6, 8, 9, 10}, [10, 11]), S
        S.difference_update((2, 8, 9), {6, 8, 9, 10}, [10, 11]), S
        {2, 3, 4, 5, 6, 7} - {2, 8, 9} - {6, 8, 9, 10} - {10, 11}

Out[8]: ({2, 3, 6, 7}, {2, 3, 4, 5, 6, 7})

Out[8]: (None, {2, 3, 6, 7})

Out[8]: ({3, 7}, {2, 3, 6, 7})

Out[8]: (None, {3, 7})

Out[8]: {3, 4, 5, 7}

In [9]: S = {2, 3, 4, 5, 6}

        S.symmetric_difference((4, 5, 6, 7, 8)), S
        S.symmetric_difference_update((4, 5, 6, 7, 8)), S
        {2, 3, 4, 5, 6} ^ {4, 5, 6, 7, 8} ^ {2, 3, 8, 9} ^ {0, 1}

Out[9]: ({2, 3, 7, 8}, {2, 3, 4, 5, 6})

Out[9]: (None, {2, 3, 7, 8})

Out[9]: {0, 1, 7, 9}

```

Let us write a function `digits_if_ok_1()` with two parameters, `number`, meant to be assigned a natural number, and `digits_seen_before`, meant to be assigned a set of digit characters. The function is expected to return `None` if as characters, it is not the case that the digits in `number` are all distinct and different to those in `digits_seen_before`; otherwise, `digits_if_ok_1()` is expected to return a new set of digit characters that extends `digits_seen_before` with those in `number`:

```
In [10]: def digits_if_ok_1(number, digits_seen_before):
        number_str = str(number)
        digits_seen_now = digits_seen_before | set(number_str)
        if len(digits_seen_now) != len(digits_seen_before) + len(number_str):
            return
        return digits_seen_now

        digits_if_ok_1(784, {'0', '2', '5'})
        print(digits_if_ok_1(676, {'0', '2', '5'}))
        print(digits_if_ok_1(729, {'0', '2', '5'}))
```

```
Out[10]: {'0', '2', '4', '5', '7', '8'}
```

```
None
```

```
None
```

We can now find out how many triples will be considered with the better code structure previously outlined:

```
In [11]: nb_of_triples = 0

        for i in range(1, int(sqrt(997)) + 1):
            a = i ** 2
            a_digits_and_0 = digits_if_ok_1(a, {'0'})
            if not a_digits_and_0:
                continue
            for j in range(i + 1, int(sqrt(9998)) + 1):
                b = j ** 2
                a_b_digits_and_0 = digits_if_ok_1(b, a_digits_and_0)
                if not a_b_digits_and_0:
                    continue
                for k in range(j + 1, int(sqrt(9876532)) + 1):
                    nb_of_triples += 1

        nb_of_triples
```

```
Out[11]: 552226
```

To complete the implementation, it suffices to also check whether the digits in c are distinct and different to those seen before, and if that is the case, check that 10 digits have been seen altogether (since we included 0 to start with). We complement the code with a count for the number of solutions being discovered:

```
In [12]: nb_of_solutions = 0

        for i in range(1, int(sqrt(997)) + 1):
            a = i ** 2
            a_digits_and_0 = digits_if_ok_1(a, {'0'})
```

```

if not a_digits_and_0:
    continue
for j in range(i + 1, int(sqrt(9998)) + 1):
    b = j ** 2
    a_b_digits_and_0 = digits_if_ok_1(b, a_digits_and_0)
    if not a_b_digits_and_0:
        continue
    for k in range(j + 1, int(sqrt(9876532)) + 1):
        c = k ** 2
        a_b_c_digits_and_0 = digits_if_ok_1(c, a_b_digits_and_0)
        if not a_b_c_digits_and_0 or len(a_b_c_digits_and_0) != 10:
            continue
        print(f'{a:7} {b:7} {c:7}')
        nb_of_solutions += 1

print(f'Altogether, {nb_of_solutions} solutions have been discovered.')

```

```

1      4 3297856
1      4 3857296
1      4 5827396
1      4 6385729
1      4 8567329
1      4 9572836
1     49  872356
1     64  537289
1    256  73984
1    625  73984
4     16  537289
4     25  139876
4     25  391876
4    289  15376
9    324  15876
16    25  73984
16   784   5329
25   784  1369
25   784  1936
25   841  7396
36    81  74529
36    81  79524
36   729  5184
81   324  7569
81   576  3249
81   729  4356
361   529   784

```

Altogether, 27 solutions have been discovered.

With the second technique, we work with sets of digits. Having such a set S of digits for the digits seen before (or “pretended” to be seen before in the case of 0), and considering a candidate n for a , b or

c , we extract the rightmost digit d in n , give up on that candidate in case d belongs to S but add d to S otherwise, and proceed this way for all remaining digits in n , if any. Computation of n modulo 10 and integer division by 10 return the rightmost digit and the remaining digits in n , respectively. The function `digits_if_ok_2()` implements this technique:

```
In [13]: def digits_if_ok_2(number, seen_digits):
        while number:
            number, digit = divmod(number, 10)
            if digit in seen_digits:
                return
            seen_digits.add(digit)
        return seen_digits

        digits_if_ok_2(784, {0, 2, 5})
        print(digits_if_ok_2(676, {0, 2, 5}))
        print(digits_if_ok_2(729, {0, 2, 5}))
```

```
Out[13]: {0, 2, 4, 5, 7, 8}
```

```
None
```

```
None
```

Assume that we adjust the code that discovers and outputs all solutions with the second technique by just:

- changing `a_digits_and_0 = digits_if_ok_1(a, {'0'})` to `a_digits_and_0 = digits_if_ok_2(a, {0})`, and
- changing the other two calls to `digits_if_ok_1()` to calls to `digits_if_ok_2()`.

We know that there are solutions for $a = 5^2 = 25$ together with both $b = 28^2 = 784$ and $b = 29^2 = 841$. This means that:

- at some point, `a_digits_and_0` will evaluate to `{0, 2, 5}`,
- `a_b_digits_and_0 = digits_if_ok_2(784, a_digits_and_0)` will be executed, and later
- `a_b_digits_and_0 = digits_if_ok_2(841, a_digits_and_0)` will be executed.

The following code fragment demonstrates that we will not get the expected results, and why:

```
In [14]: a_digits_and_0 = {0, 2, 5}
        a_b_digits_and_0 = digits_if_ok_2(784, a_digits_and_0)
        print(a_b_digits_and_0)
        a_digits_and_0
        a_b_digits_and_0 = digits_if_ok_2(841, a_digits_and_0)
        print(a_b_digits_and_0)
```

```
{0, 2, 4, 5, 7, 8}
```

```
Out[14]: {0, 2, 4, 5, 7, 8}
```

None

In the following code fragment, we see:

- $f(n)$ make a denote what n denotes, namely, the integer 1, represented somewhere in memory, before the statement $a = 10$ makes a denote another integer, namely, 10, represented elsewhere in memory;
- $g(A)$ make A denote what S denotes, namely, the set $\{2\}$, represented somewhere in memory,, before the statement $A = \{20\}$ makes A denote another set, namely, $\{20\}$, represented elsewhere in memory;
- $h(B)$ make B denote what L denotes, namely, the list $[3]$, before the statement $B = [30]$ makes B another list, namely, $[30]$, represented elsewhere in memory.

```
In [15]: n = 1
         S = {2}
         L = [3]

         def f(a):
             print(a)
             a = 10

         def g(A):
             print(A)
             A = {20}

         def h(B):
             print(B)
             B = [30]

         f(n); n
         g(S); S
         h(L); L
```

1

Out[15]: 1

{2}

Out[15]: {2}

[3]

Out[15]: [3]

In the following code fragment, we see:

- `g(A)` make `A` denote what `S` denotes, namely, the set `{2}`, represented somewhere in memory, before the statement `A.add(20)` changes that set to `{2, 20}` by letting the representation of `{2, 20}` replace the representation of `{2}` in memory;
- `h(B)` make `B` denote what `L` denotes, namely, the list `[3]`, represented somewhere in memory, before the statement `B.append(30)` changes that list to `[3, 30]` by letting the representation of `[3, 30]` replace the representation of `[3]` in memory.

```
In [16]: S = {2}
        L = [3]

        def g(A):
            print(A)
            A.add(20)

        def h(B):
            print(B)
            B.append(30)

        g(S); S
        h(L); L
```

```
{2}
```

```
Out[16]: {2, 20}
```

```
[3]
```

```
Out[16]: [3, 30]
```

We infer that `digits_if_ok_2()` should receive as second argument not the set of digits seen before, but a copy of that set; the copy will be extended with the digits in the number passed as first argument in case those digits are all distinct and different to the digits seen before, leaving the first argument unmodified; `set()`, `list()`, `tuple()`, return a set, a list, a tuple, respectively, consisting of the elements that make up their arguments:

```
In [17]: S = {1, 2, 3}
        L = [1, 2, 3]
        T = 1, 2, 3
        s = '123'

        set(S), set(L), set(T), set(s)
        list(S), list(L), list(T), list(s)
        tuple(S), tuple(L), tuple(T), tuple(s)

Out[17]: ({1, 2, 3}, {1, 2, 3}, {1, 2, 3}, {'1', '2', '3'})

Out[17]: ([1, 2, 3], [1, 2, 3], [1, 2, 3], ['1', '2', '3'])
```

```
Out[17]: ((1, 2, 3), (1, 2, 3), (1, 2, 3), ('1', '2', '3'))
```

The problematic code fragment can then be amended as follows:

```
In [18]: a_digits_and_0 = {0, 2, 5}
         a_b_digits_and_0 = digits_if_ok_2(784, set(a_digits_and_0))
         print(a_b_digits_and_0)
         a_digits_and_0
         a_b_digits_and_0 = digits_if_ok_2(841, set(a_digits_and_0))
         print(a_b_digits_and_0)
```

```
{0, 2, 4, 5, 7, 8}
```

```
Out[18]: {0, 2, 5}
```

```
{0, 1, 2, 4, 5, 8}
```

The solution based on the second technique is then a simple modification of the implementation based on the first technique:

```
In [19]: nb_of_solutions = 0

         for i in range(1, int(sqrt(997)) + 1):
             a = i ** 2
             a_digits_and_0 = digits_if_ok_2(a, {0})
             if not a_digits_and_0:
                 continue
             for j in range(i + 1, int(sqrt(9998)) + 1):
                 b = j ** 2
                 a_b_digits_and_0 = digits_if_ok_2(b, set(a_digits_and_0))
                 if not a_b_digits_and_0:
                     continue
                 for k in range(j + 1, int(sqrt(9876532)) + 1):
                     c = k ** 2
                     a_b_c_digits_and_0 = digits_if_ok_2(c, set(a_b_digits_and_0))
                     if not a_b_c_digits_and_0 or len(a_b_c_digits_and_0) != 10:
                         continue
                     print(f'{a:7} {b:7} {c:7}')
                     nb_of_solutions += 1

         print(f'Altogether, {nb_of_solutions} solutions have been discovered.')

1      4 3297856
1      4 3857296
1      4 5827396
1      4 6385729
1      4 8567329
```

1	4	9572836
1	49	872356
1	64	537289
1	256	73984
1	625	73984
4	16	537289
4	25	139876
4	25	391876
4	289	15376
9	324	15876
16	25	73984
16	784	5329
25	784	1369
25	784	1936
25	841	7396
36	81	74529
36	81	79524
36	729	5184
81	324	7569
81	576	3249
81	729	4356
361	529	784

Altogether, 27 solutions have been discovered.

With the third technique, we work with natural numbers smaller than 2^{10} to encode sets of digits: given $k < 10$ and $0 \leq n_0 < \dots < n_k \leq 9$, we let $2^{n_0} + \dots + 2^{n_k}$ encode the set $\{n_0, \dots, n_k\}$. In other words, we let the positions of the bits set to 1 in the representation of a number e smaller than 2^{10} as a 10-bit number encode the members of the set encoded by e , the rightmost position being position 0, the second rightmost position being position 1, etc.:

```
In [20]: e = 0
         e, f'{e:010b}', {i for i in range(10) if f'{e:010b}'[9 - i] == '1'}

         e = 2 ** 0 + 2 ** 3 + 2 ** 5 + 2 ** 8
         e, f'{e:010b}', {i for i in range(10) if f'{e:010b}'[9 - i] == '1'}

         e = 2 ** 3 + 2 ** 5 + 2 ** 9
         e, f'{e:010b}', {i for i in range(10) if f'{e:010b}'[9 - i] == '1'}

         e = 2 ** 10 - 1
         e, f'{e:010b}', {i for i in range(10) if f'{e:010b}'[9 - i] == '1'}
```

```
Out[20]: (0, '0000000000', set())
```

```
Out[20]: (297, '0100101001', {0, 3, 5, 8})
```

```
Out[20]: (552, '1000101000', {3, 5, 9})
```

```
Out[20]: (1023, '1111111111', {0, 1, 2, 3, 4, 5, 6, 7, 8, 9})
```

Having a number e encoding the digits seen before (or “pretended” to be seen before in the case of 0), and considering a candidate n for a , b or c , we extract the rightmost digit d in n , give up on that candidate in case d is one of the digits encoded in e , but let a new number encode d together with the digits encoded in e otherwise, and proceed this way for all remaining digits in n , if any. Before we see a possible implementation of the third technique, let us see binary bitwise operations in action.

Dividing a number n by $2^0, 2^1, 2^3, 2^4 \dots$ can be seen as “pushing” n written in binary by 0, 1, 2, 3, 4... positions to the right, which can also be achieved with the $>>$ operator:

```
In [21]: n = 22
```

```
for k in range(5):
    print(k, f'{n // 2 ** k:2}', f'{n >> k:2}', f'{n >> k:010b}')
```

```
0 22 22 0000010110
1 11 11 0000001011
2  5  5 0000000101
3  2  2 0000000010
4  1  1 0000000001
```

Multiplying a number n by $2^0, 2^1, 2^3, 2^4 \dots$ can be seen as “pushing” n written in binary by 0, 1, 2, 3, 4... positions to the left, “filling the gaps” with 0’s, which can also be achieved with the $<<$ operator:

```
In [22]: n = 22
```

```
for k in range(5):
    print(k, f'{n * 2 ** k:3}', f'{n << k:3}', f'{n << k:010b}')
```

```
0  22  22 0000010110
1  44  44 0000101100
2  88  88 0001011000
3 176 176 0010110000
4 352 352 0101100000
```

In particular, $2^0, 2^1, 2^3, 2^4 \dots$ are also 1 pushed 0, 1, 2, 3, 4... positions to the left: these are the numbers that in binary, have a single 1 at position 0, 1, 2, 3, 4...:

```
In [23]: for k in range(5):
          print(k, f'{2 ** k:2}', f'{1 << k:2}', f'{1 << k:010b}')
```

```
0  1  1 0000000001
1  2  2 0000000010
2  4  4 0000000100
3  8  8 0000001000
4 16 16 0000010000
```

With bitwise or, |, we get a number with a bit of 1 at a given position iff at least one operand has a bit of 1 at that position:

```
In [24]: x = 0
y = 2 ** 0 + 2 ** 3 + 2 ** 4
print(f'{x:6}', f'{y:6}', f'{x | y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x | y:06b}')

x = 2 ** 0 + 2 ** 3 + 2 ** 4
y = 2 ** 6 - 1
print(f'{x:6}', f'{y:6}', f'{x | y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x | y:06b}')

x = 2 ** 0 + 2 ** 2
y = 2 ** 1 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x | y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x | y:06b}')

x = 2 ** 0 + 2 ** 2 + 2 ** 4
y = 2 ** 3 + 2 ** 4 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x | y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x | y:06b}')
```

0	25	25
000000	011001	011001
25	63	63
011001	111111	111111
5	34	39
000101	100010	100111
21	56	61
010101	111000	111101

With bitwise and, &, we get a number with a bit of 1 at a given position iff both operands have a bit of 1 at that position:

```
In [25]: x = 0
y = 2 ** 0 + 2 ** 3 + 2 ** 4
print(f'{x:6}', f'{y:6}', f'{x & y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x & y:06b}')

x = 2 ** 0 + 2 ** 3 + 2 ** 4
y = 2 ** 6 - 1
print(f'{x:6}', f'{y:6}', f'{x & y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x & y:06b}')

x = 2 ** 0 + 2 ** 2
y = 2 ** 1 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x & y:6}')
```

```

print(f'{x:06b}', f'{y:06b}', f'{x & y:06b}')

x = 2 ** 0 + 2 ** 2 + 2 ** 4
y = 2 ** 3 + 2 ** 4 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x & y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x & y:06b}')

    0    25    0
000000 011001 000000
    25    63    25
011001 111111 011001
     5    34     0
000101 100010 000000
    21    56    16
010101 111000 010000

```

With bitwise xor, ^, we get a number with a bit of 1 at a given position iff exactly one operand has a bit of 1 at that position:

```

In [26]: x = 0
         y = 2 ** 0 + 2 ** 3 + 2 ** 4
         print(f'{x:6}', f'{y:6}', f'{x ^ y:6}')
         print(f'{x:06b}', f'{y:06b}', f'{x ^ y:06b}')

x = 2 ** 0 + 2 ** 3 + 2 ** 4
y = 2 ** 6 - 1
print(f'{x:6}', f'{y:6}', f'{x ^ y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x ^ y:06b}')

x = 2 ** 0 + 2 ** 2
y = 2 ** 1 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x ^ y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x ^ y:06b}')

x = 2 ** 0 + 2 ** 2 + 2 ** 4
y = 2 ** 3 + 2 ** 4 + 2 ** 5
print(f'{x:6}', f'{y:6}', f'{x ^ y:6}')
print(f'{x:06b}', f'{y:06b}', f'{x ^ y:06b}')

    0    25    25
000000 011001 011001
    25    63    38
011001 111111 100110
     5    34    39
000101 100010 100111
    21    56    45
010101 111000 101101

```

So to check whether a digit d is one of the digits encoded in e , and get a new number f that encodes d together with the digits encoded in e in case d is a new digit, it suffices to take the bitwise or of e with 1 pushed d positions to the left; either d is not a new digit in which case this is e , or d is a new digit in which case this is the desired f . The following code fragment illustrates with e encoding the set $\{0, 2, 4\}$, and d ranging all possible digits:

```
In [27]: e = 2 ** 0 + 2 ** 2 + 2 ** 5
         print(f' {e:3}', ' ' * 10, f'{e:010b}')
         for d in range(10):
             x = 1 << d
             y = e | 1 << d
             print(d, f'{x:3}', f'{x:010b}', f'{y:010b}', y == e)
```

```
37          0000100101
0   1 0000000001 0000100101 True
1   2 0000000010 0000100111 False
2   4 0000000100 0000100101 True
3   8 0000001000 0000101101 False
4  16 0000010000 0000110101 False
5  32 0000100000 0000100101 True
6  64 0001000000 0001100101 False
7 128 0010000000 0010100101 False
8 256 0100000000 0100100101 False
9 512 1000000000 1000100101 False
```

Based on these considerations, we can implement the third technique in the form of the function `digits_if_ok_3()`:

```
In [28]: def digits_if_ok_3(number, digits_seen_before):
         while number:
             number, digit = divmod(number, 10)
             digits_seen_now = digits_seen_before | 1 << digit
             if digits_seen_now == digits_seen_before:
                 return
             digits_seen_before = digits_seen_now
         return digits_seen_before

k = 2 ** 0 + 2 ** 2 + 2 ** 5
k, f'{k:010b}', {i for i in range(10) if f'{k:010b}'[9 - i] == '1'}

e = digits_if_ok_3(784, 2 ** 0 + 2 ** 2 + 2 ** 5)
e, f'{e:010b}', {i for i in range(10) if f'{e:010b}'[9 - i] == '1'}
print(digits_if_ok_3(676, 2 ** 2 + 2 ** 4))
print(digits_if_ok_3(729, 2 ** 2 + 2 ** 4))
```

```
Out[28]: (37, '0000100101', {0, 2, 5})
```

```
Out[28]: (437, '0110110101', {0, 2, 4, 5, 7, 8})
```

None
None

We can then adjust the code that discovers and outputs all solutions with the first technique by just

- changing `a_digits_and_0 = digits_if_ok_1(a, {'0'})` to `a_digits_and_0 = digits_if_ok_3(a, 1)` since 1 encodes 0 as a singleton set,
- changing the other two calls to `digits_if_ok_1()` to calls to `digits_if_ok_3()`, and
- eventually checking whether `a_b_c_digits_and_0` is not None and evaluates to $2^{10} - 1$, which in base 2, reads as 10 consecutive 1's, therefore encoding all 10 digits:

```
In [29]: nb_of_solutions = 0
```

```
for i in range(1, int(sqrt(997)) + 1):
    a = i ** 2
    a_digits_and_0 = digits_if_ok_3(a, 1)
    if not a_digits_and_0:
        continue
    for j in range(i + 1, int(sqrt(9998)) + 1):
        b = j ** 2
        a_b_digits_and_0 = digits_if_ok_3(b, a_digits_and_0)
        if not a_b_digits_and_0:
            continue
        for k in range(j + 1, int(sqrt(9876532)) + 1):
            c = k ** 2
            a_b_c_digits_and_0 = digits_if_ok_3(c, a_b_digits_and_0)
            if not a_b_c_digits_and_0 or a_b_c_digits_and_0 != 2 ** 10 - 1:
                continue
            print(f'{a:7} {b:7} {c:7}')
            nb_of_solutions += 1

print(f'Altogether, {nb_of_solutions} solutions have been discovered.')
```

```
1      4 3297856
1      4 3857296
1      4 5827396
1      4 6385729
1      4 8567329
1      4 9572836
1     49 872356
1     64 537289
1    256 73984
1    625 73984
4     16 537289
4     25 139876
4     25 391876
4    289 15376
9    324 15876
```


16	25	73984
16	784	5329
25	784	1369
25	784	1936
25	841	7396
36	81	74529
36	81	79524
36	729	5184
81	324	7569
81	576	3249
81	729	4356
361	529	784

Altogether, 27 solutions have been discovered.