素数：(也说质数) 数学上指在大于1的整数中只能被1和它本身整除的数。
合数：正整数，除了1和它本身以外，还能被其他正整数整除，这个数就叫做合数。如4、6、9、10等。

Euler's sieve 欧拉筛

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In [2]:
```python
from math import sqrt
from timeit import timeit
from itertools import zip_longest
import matplotlib.pyplot as plt
```

使用Eratosthenes的筛子，可以不止一次地划掉一个数字。 例如，在n≥12的情况下，12将被作为2的倍数划掉，然后作为3的倍数。欧拉的筛子使用一个数字列表初始化为2和n之间的所有数字的列表，在给定的情况下 阶段，失去了那些被发现不是素数的成员，并最终失去了所有的非主要数字。让我们尝试以下，找出有一个缺陷

With Eratosthenes' sieve, a number can be crossed out more than once. For instance, in case $n \geq 12$, 12 will be crossed out as a multiple of 2 and then as a multiple of 3. Euler's sieve works with a list of numbers initialised as the list of all numbers between 2 and n that, at a given stage, has lost those of its members that have been found out not to be prime, and that eventually has lost all of its nonprime numbers. Let us try the following, to find out that there is a flaw.

列表的第一个成员包含2；将2乘以2以及列表的以下成员直到和排除大于n / 2的第一个数字，应该删除最多等于n的所有适当的2的倍数。
• 结果列表的下一个成员应包含3；将3乘以3以及列表中的以下成员直到并排除大于n / 3的第一个数，应该删除最多等于n的所有适当的3，即剩余的n，即不是2的倍数的那些。
• 结果列表的下一个成员应包含5；将5乘以5和列表中的以下成员直到和排除大于n / 5的第一个数字应该删除最多等于n的所有适当的5的倍数，即那些不是2或3的倍数的那些...

- The first member of the list contains 2; multiplying 2 with 2 and the following members of the list up to and excluding the first number greater than $\lfloor \frac{n}{2} \rfloor$, should remove all proper multiples of 2 at most equal to n.
- The next member of the resulting list should contain 3; multiplying 3 with 3 and the following members of the list up to and excluding the first number greater than $\lfloor \frac{n}{3} \rfloor$, should remove all proper multiples of 3 at most equal to n that remain, namely, those that are not multiple of 2.
- The next member of the resulting list should contain 5; multiplying 5 with 5 and the following members of the list up to and excluding the first number greater than $\lfloor \frac{n}{5} \rfloor$ should remove all proper multiples of 5 at most equal to n that remain, namely, those that are not multiples of 2 or 3...
- ...

For illustration purposes, let us fix n to some value, assign that value to a variable `n`, and define `sieve` accordingly:

为了便于说明，我们将n修复为某个值，将该值赋给变量n，并相应地定义筛子：

In [3]:
```python
n = 25
sieve = list(range(2, n + 1))
```

In [4]:
```python
def print_sieve_contents():
    for p in sieve:
        print(f'{p:3}', end = '')

print_sieve_contents()
```

```
  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
```

To observe how, with `n` set to `25`, proper multiples of 2 are crossed out, we call the following function with `i` set to `0` as argument:

为了观察当n被设置为25时，2的适当倍数是如何被划掉的，我们调用下面的函数，i被设置为0作为参数

```
In [5]: def eliminate_proper_multiples(i):
            print(f'Now eliminating proper multiples of {sieve[i]}')
            j = i
            while j < len(sieve):
                factor = sieve[i] * sieve[j]
                if factor <= n:
                    print(f' Eliminating {sieve[i]} * {sieve[j]} = {factor}')
                    sieve.remove(factor)
                    j += 1
                else:
                    break
```

round() 四舍五入功能

```
In [6]: eliminate_proper_multiples(0)
        print_sieve_contents()
```

```
Now eliminating proper multiples of 2
 Eliminating 2 * 2 = 4
 Eliminating 2 * 3 = 6
 Eliminating 2 * 5 = 10
 Eliminating 2 * 7 = 14
 Eliminating 2 * 8 = 16
 Eliminating 2 * 9 = 18
 Eliminating 2 * 11 = 22
 Eliminating 2 * 12 = 24
  2  3  5  7  8  9 11 12 13 15 17 19 20 21 23 25
```

We see the flaw. Having first eliminated 2×2, 4 is not longer in the list, which prevents 8 from being eliminated, and the same holds for other multiples of 2.

This suggests to, for an arbitrary value of n, amend the procedure as follows.

这就意味着，对于任意的n值，可以按照下面的方法进行修改

- using 2, the first number in the list:
  - remove 2^2, 2^3, 2^4, ..., up to 2^r for the largest r with $2^r \leq n$,
  - remove 2×3, $2^2 \times 3$, $2^3 \times 3$, ..., up to $2^r \times 3$ for the largest r with $2^r \times 3 \leq n$,
  - as 4 is no longer in the list, remove 2×5, $2^2 \times 5$, $2^3 \times 5$, ..., up to $2^r \times 5$ for the largest r with $2^r \times 5 \leq n$,
  - as 6 is no longer in the list, remove 2×7, $2^2 \times 7$, $2^3 \times 7$, ..., up to $2^r \times 7$ for the largest r with $2^r \times 7 \leq n$,
  - as 8 is no longer in the list, remove 2×9, $2^2 \times 9$, $2^3 \times 9$, ..., up to $2^r \times 9$ for the largest r with $2^r \times 9 \leq n$,
  - ...
- using 3, the next number in what remains of the list:
  - remove 3^2, 3^3, 3^4, ..., up to 3^r for the largest r with $3^r \leq n$,
  - as 4 is no longer in the list, remove 3×5, $3^2 \times 5$, $3^3 \times 5$, ..., up to $3^r \times 5$ for the largest r with $3^r \times 5 \leq n$,
  - as 6 is no longer in the list, remove 3×7, $3^2 \times 7$, $3^3 \times 7$, ..., up to $3^r \times 7$ for the largest r with $3^r \times 7 \leq n$,

- as 8, 9 and 10 are no longer in the list, remove 3×11, $3^2 \times 11$, $3^3 \times 11$, ..., up to $3^r \times 11$ for the largest r with $3^r \times 11 \leq n$,
- ...

- ... 当列表中剩下的下一个数字超过时，我们停止

让我们验证一下这个过程是否正确。在阶段k，所有合适的倍数的第k个素数从列表中删除pk最多等于n。这是通过归纳验证的。事实上，如果在阶段k a

数字m在{2,3,…与pkm≤n,n }是不被认为是:

We stop when the next number in what remains in the list exceeds $\lfloor\sqrt{n}\rfloor$.

Let us verify that the procedure is correct. At stage k, all proper multiples of the kth prime number p_k at most equal to n are removed from the list. This is verified by induction. Indeed, if during stage k, a number m in $\{2, 3, \dots, n\}$ with $p_k m \leq n$ is not considered then:

- either m is smaller than p_k, in which case it is a multiple of at least one of p_1, ..., p_{k-1}, which implies that $p_k \times m$ is also a multiple of at least one of p_1, ..., p_{k-1}, so by inductive hypothesis, $p_k \times m$ was removed from the list during one of the previous stages,
- or m is greater than p_k but no longer belongs to the list (it is a number such as 4, 6, 8 at stage 1, or a number such as 4, 6, 8, 9, 10 at stage 2), in which case:
  - either m was removed during one of the previous stages, hence m is a multiple of at least one of p_1, ..., p_{k-1}, which implies as in the previous case that $p_k \times m$ was also removed from the list,
  - or m is a multiple of p_k which was removed earlier in the current stage, so m is a number of the form $p_k^r m'$ for some $r \geq 1$ and some number m' which was then found in what remained of the list, therefore $p_k m = p_k^{r+1} m'$ was also removed from the list earlier in the current stage.

观察n = 25时，2的非零次幂乘以3的非零次幂

消去5 * 5的非零次幂，依次为
以i为0 k为0 1 2 3 4 5为参数，然后i为1 k为0 1和2为参数，i为2 k为0为参数:

To observe how, with `n` set to `25`, nonzero powers of 2 times 2, 3, 5, 7, 9 and 11, nonzero powers of 3 times 3, 5 and 7, and nonzero powers of 5 times 5 are eliminated, the following function is successively called with `i` set to `0` and `k` set to `0`, `1`, `2`, `3`, `4` and `5` as arguments, then with `i` set to `1` and `k` set to `0`, `1` and `2` as arguments, then with `i` set to `2` and `k` set to `0` as arguments:

```
In [7]: def eliminate_proper_multiples(i, k):
            # We assume that this function will be called in the order
            #   eliminate_proper_multiples(0, 0)
            #   eliminate_proper_multiples(0, 1)
            #   ...
            #   eliminate_proper_multiples(1, 0)
            #   eliminate_proper_multiples(1, 1)
            #   ...
            print('Now eliminating multiples of the form '
                  f'a nonzero power of {sieve[i]} times {sieve[i + k]}'
                 )
            factor = sieve[i] * sieve[i + k]
            power = 1
            while factor <= n:
                print(' Eliminating '
                      f'{sieve[i]}^{power} x {sieve[i + k]} = {factor}'
                     )
                sieve.remove(factor)
                factor *= sieve[i]
                power += 1
```

```
In [8]: sieve = list(range(2, n + 1))
        print_sieve_contents()
```

```
  2   3   4   5   6   7   8   9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
```

```
In [9]: eliminate_proper_multiples(0, 0)
        print_sieve_contents()
```

现在消去2乘以2的非零次幂的倍数

```
Now eliminating multiples of the form a nonzero power of 2 times 2
 Eliminating 2^1 x 2 = 4
 Eliminating 2^2 x 2 = 8
 Eliminating 2^3 x 2 = 16
  2   3   5   6   7   9 10 11 12 13 14 15 17 18 19 20 21 22 23 24 25
```

```
In [10]: eliminate_proper_multiples(0, 1)
         print_sieve_contents()
```

现在消去2乘以2的非零次幂的倍数

```
Now eliminating multiples of the form a nonzero power of 2 times 3
 Eliminating 2^1 x 3 = 6
 Eliminating 2^2 x 3 = 12
 Eliminating 2^3 x 3 = 24
  2   3   5   7   9 10 11 13 14 15 17 18 19 20 21 22 23 25
```

消除适当的倍数

```
In [11]: eliminate_proper_multiples(0, 2)
         print_sieve_contents()
```

现在消去2乘以5的非零次幂的倍数

```
Now eliminating multiples of the form a nonzero power of 2 times 5
 Eliminating 2^1 x 5 = 10
 Eliminating 2^2 x 5 = 20
  2   3   5   7   9 11 13 14 15 17 18 19 21 22 23 25
```

```
In [12]: eliminate_proper_multiples(0, 3)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 2 times 7
 Eliminating 2^1 x 7 = 14
  2   3   5   7   9 11 13 15 17 18 19 21 22 23 25
```

```
In [13]: eliminate_proper_multiples(0, 4)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 2 times 9
 Eliminating 2^1 x 9 = 18
  2   3   5   7   9 11 13 15 17 19 21 22 23 25
```

```
In [14]: eliminate_proper_multiples(0, 5)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 2 times 11
 Eliminating 2^1 x 11 = 22
  2   3   5   7   9 11 13 15 17 19 21 23 25
```

```
In [15]: eliminate_proper_multiples(1, 0)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 3 times 3
 Eliminating 3^1 x 3 = 9
  2  3  5  7 11 13 15 17 19 21 23 25
```

```
In [16]: eliminate_proper_multiples(1, 1)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 3 times 5
 Eliminating 3^1 x 5 = 15
  2  3  5  7 11 13 17 19 21 23 25
```

```
In [17]: eliminate_proper_multiples(1, 2)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 3 times 7
 Eliminating 3^1 x 7 = 21
  2  3  5  7 11 13 17 19 23 25
```

```
In [18]: eliminate_proper_multiples(2, 0)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 5 times 5
 Eliminating 5^1 x 5 = 25
  2  3  5  7 11 13 17 19 23
```

To observe more synthetically how, with `n` set to `25`, proper multiples of 2, proper multiples of 3 that are not multiples of 2, and proper multiples of 5, equal to $\lfloor\sqrt{25}\rfloor$, that are multiples of neither 2 nor 3, are eliminated, we successively call the following function with `i` set to `0`, `1` and `2` as argument:

为了更综合地观察，当n被设为25时，2的固有倍数，3的固有倍数不是2的固有倍数，以及5的固有倍数，是如何等于b的√25既不是2也不是3的倍数

消去后，我们依次调用下面的函数，将i设置为0,1和2作为参数:

```
In [19]: def eliminate_proper_multiples(i):
             # We assume that this function will be called in the order
             #   eliminate_proper_multiples(0)
             #   eliminate_proper_multiples(1)
             #   eliminate_proper_multiples(2)
             #   ...
             k = 0
             while True:
                 factor = sieve[i] * sieve[i + k]
                 if factor > n:
                     break
                 print('Now eliminating multiples of the form '
                       f'a nonzero power of {sieve[i]} times {sieve[i + k]}'
                      )
                 power = 1
                 while factor <= n:
                     print(' Eliminating '
```

```
                        f'{sieve[i]}^{power} x {sieve[i + k]} = {factor}'
                    )
                sieve.remove(factor)
                factor *= sieve[i]
                power += 1
            k += 1
```

```
In [20]: sieve = list(range(2, n + 1))
         print_sieve_contents()
```

```
  2   3   4   5   6   7   8   9  10  11  12  13  14  15  16  17  18  19  20  21  22  23  24  25
```

消除适当的倍数

```
In [21]: eliminate_proper_multiples(0)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 2 times 2
 Eliminating 2^1 x 2 = 4
 Eliminating 2^2 x 2 = 8
 Eliminating 2^3 x 2 = 16
Now eliminating multiples of the form a nonzero power of 2 times 3
 Eliminating 2^1 x 3 = 6
 Eliminating 2^2 x 3 = 12
 Eliminating 2^3 x 3 = 24
Now eliminating multiples of the form a nonzero power of 2 times 5
 Eliminating 2^1 x 5 = 10
 Eliminating 2^2 x 5 = 20
Now eliminating multiples of the form a nonzero power of 2 times 7
 Eliminating 2^1 x 7 = 14
Now eliminating multiples of the form a nonzero power of 2 times 9
 Eliminating 2^1 x 9 = 18
Now eliminating multiples of the form a nonzero power of 2 times 11
 Eliminating 2^1 x 11 = 22
  2   3   5   7   9  11  13  15  17  19  21  23  25
```

```
In [22]: eliminate_proper_multiples(1)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 3 times 3
 Eliminating 3^1 x 3 = 9
Now eliminating multiples of the form a nonzero power of 3 times 5
 Eliminating 3^1 x 5 = 15
Now eliminating multiples of the form a nonzero power of 3 times 7
 Eliminating 3^1 x 7 = 21
  2   3   5   7  11  13  17  19  23  25
```

```
In [23]: eliminate_proper_multiples(2)
         print_sieve_contents()
```

```
Now eliminating multiples of the form a nonzero power of 5 times 5
 Eliminating 5^1 x 5 = 25
  2   3   5   7  11  13  17  19  23
```

Putting it all together:
第一个筛选质数到

```
In [24]: def first_sieve_of_primes_up_to(n):
             sieve = list(range(2, n + 1))
             i = 0
             while sieve[i] <= round(sqrt(n)):
                 k = 0
                 while True:
                     factor = sieve[i] * sieve[i + k]
                     if factor > n:
                         break
                     while factor <= n:
                         sieve.remove(factor)
                         factor *= sieve[i]
                     k += 1
                 i += 1
             return sieve
```

我们可以重用与Eratosthenes的sieve相关的nicely_display()

We can reuse `nicely_display()` as defined in relation to Eratosthenes' sieve:

```
In [25]: def nicely_display(sequence, max_size):
             field_width = max_size + 2
             nb_of_fields = 80 // field_width
             count = 0
             for e in sequence:
                 print(f'{e:{field_width}}', end = '')
                 count += 1
                 if count % nb_of_fields == 0:
                     print()
```

第一个筛选质数到

`first_sieve_of_primes_up_to()` returns precisely the list of numbers that we want to display, so we make it the first argument of `nicely_display()`: 返回要显示的数字列表，因此
我们将它作为nicely_display()的第一个参数:

```
In [26]: primes = first_sieve_of_primes_up_to(1_000)
         nicely_display(primes, len(str(primes[-1])))

    2    3    5    7   11   13   17   19   23   29   31   37   41   43   47   53
   59   61   67   71   73   79   83   89   97  101  103  107  109  113  127  131
  137  139  149  151  157  163  167  173  179  181  191  193  197  199  211  223
  227  229  233  239  241  251  257  263  269  271  277  281  283  293  307  311
  313  317  331  337  347  349  353  359  367  373  379  383  389  397  401  409
  419  421  431  433  439  443  449  457  461  463  467  479  487  491  499  503
  509  521  523  541  547  557  563  569  571  577  587  593  599  601  607  613
  617  619  631  641  643  647  653  659  661  673  677  683  691  701  709  719
  727  733  739  743  751  757  761  769  773  787  797  809  811  821  823  827
  829  839  853  857  859  863  877  881  883  887  907  911  919  929  937  941
  947  953  967  971  977  983  991  997
```

Euler's sieve's algorithm is more complicated than Eratosthenes' sieve's algorithm, but much less effective: 欧拉筛网算法比埃拉托斯梯尼筛网算法复杂，但效率低得多:

```
In [27]: timeit('first_sieve_of_primes_up_to(20_000)',
                globals = globals(), number = 1
               )
```

```
Out[27]: 1.192367618001299
```

为了理解其中的原因，并尝试解决效率低下的问题，让我们研究一下在列表和集合上执行某些操作的成本，方法是计时这些操作并绘制收集到的数据

plot()函数来自matplotlib。pyplot模块允许轻松绘制数据。,预计

两个参数:x坐标的枚举和y坐标的枚举。通常，我们不
有一个可枚举的对，每个点对应一个，对的第一个和第二个分量是什么
分别是点的x坐标和y坐标。例如,从函数f: x 7→2 x,我们可以
产生点的列表((0,0),(1、2),(3、6)(4、8)、(9日18)];plot()应该将[0,1,3,4,9]作为第一个参数，而[0,2,6,8,18]作为第二个参数(或表示相同数据序列的任何其他枚举)。zip()函数可以很容易地从前一个列表中获得后两个列表:

To understand why, and try and address the inefficiency, let us examine the cost of performing some operations on lists and sets, by timing those operations and plotting collected data.

The `plot()` function from the `matplotlib.pyplot` module allows one to easily plot data. It expects two arguments: an enumerable of x-coordinates and an enumerable of y-coordinates. Usually, we rather have an enumerable of pairs, one for each point to plot, the first and second components of the pair being the point's x- and y-coordinates, respectively. For instance, from the function $f : x \mapsto 2x$, we could generate the list of points `[(0, 0), (1, 2), (3, 6), (4, 8), (9, 18)]`; `plot()` should then be given `[0, 1, 3, 4, 9]` as first argument, and `[0, 2, 6, 8, 18]` as second argument (or any other enumerable representing the same sequence of data). The `zip()` function makes it easy to get the latter two lists from the former list:

```
In [28]: list(zip((0, 0), (1, 2), (3, 6), (4, 8), (9, 18)))
```

```
Out[28]: [(0, 1, 3, 4, 9), (0, 2, 6, 8, 18)]
```

更一般地，zip()接受任意枚举作为参数。这些枚举数的大小可以不同之处:压缩的是如果需要将枚举数截短到最短的大小:

More generally, `zip()` accepts arbitrary enumerables as arguments. The size of those enumerables can differ: what is zipped is the enumerables truncated if needed to the size of the shortest one:

```
In [29]: list(zip(range(1, 5), [11, 12, 13], [21, 22, 23, 24, 25], (31, 32, 33)))
```

```
Out[29]: [(1, 11, 21, 31), (2, 12, 22, 32), (3, 13, 23, 33)]
```

Observe that `zip()` is self-inverse, after some of the arguments have possibly been truncated to the size of the shortest one: 注意zip()是自反转的，在某些参数可能被截断为最短的一个的大小

```
In [30]: list(zip(*zip((0, 0), (1, 2), (3, 6), (4, 8), (9, 18))))
         list(zip(*zip(range(1, 5), [11, 12, 13], [21, 22, 23, 24, 25], (31, 32, 33))))
```

```
Out[30]: [(0, 0), (1, 2), (3, 6), (4, 8), (9, 18)]
```

```
Out[30]: [(1, 2, 3), (11, 12, 13), (21, 22, 23), (31, 32, 33)]
```

In applications where one needs to extend the enumerables to the size of the longest sequence, then the `itertools` module comes to the rescue with the `zip_longest()` function, that accepts an optional `fillvalue` keyword only argument, set to `None` by default:

```
In [31]: list(zip_longest([1], [11, 12], [21, 22, 23, 25], [31, 32]))
         list(zip_longest([1], [11, 12], [21, 22, 23, 25], [31, 32],
                          fillvalue = -1
                         )
             )
```

在需要将枚举数扩展到最长序列大小的应用程序中，那么itertools模块通过zip_longest()函数来解决这个问题，该函数接受一个可选的fillvalue关键字only参数，默认设置为None:

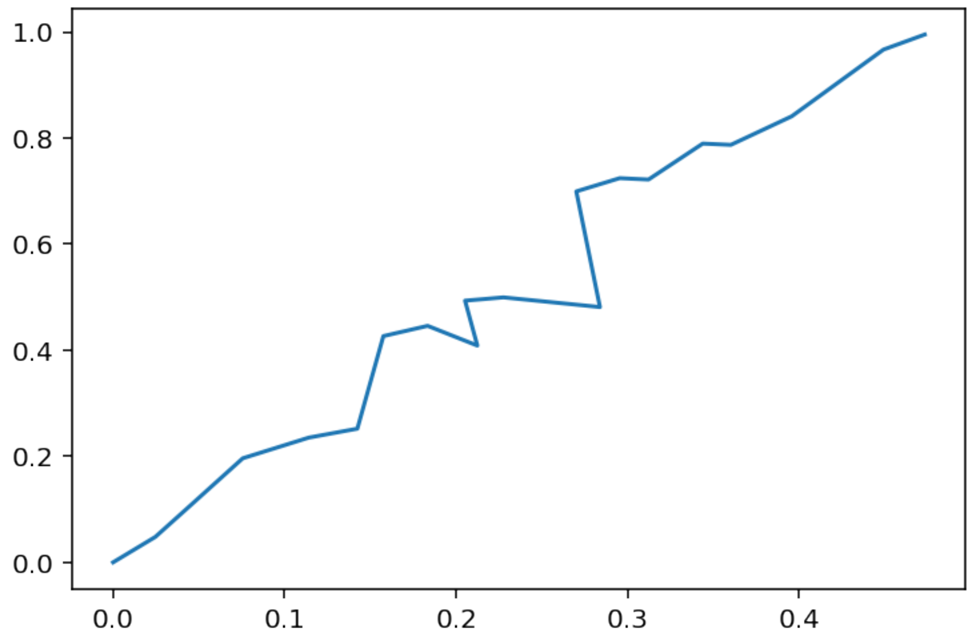
```
Out[31]: [(1, 11, 21, 31),
          (None, 12, 22, 32),
          (None, None, 23, None),
          (None, None, 25, None)]
```

```
Out[31]: [(1, 11, 21, 31), (-1, 12, 22, 32), (-1, -1, 23, -1), (-1, -1, 25, -1)]
```

Let us first examine the relative cost of creating, from a given collection, a list versus a set:

让我们首先研究一下从给定集合创建列表与创建集合的相对成本：

```
In [32]: data = []
         for i in range(0, 10_000_000, 500_000):
             data.append((timeit(f'list(range({i}))', number = 1),
                          timeit(f'set(range({i}))', number = 1)
                         )
                        )
         plt.plot(*zip(*data));
```



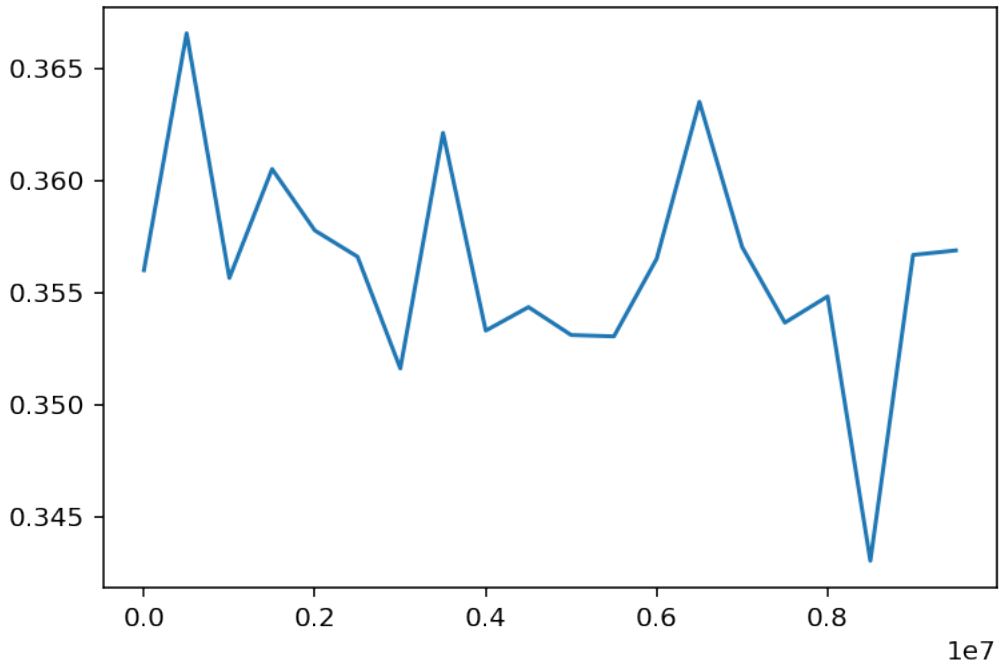
Creating these lists and sets takes time, all the more so that their size is larger. For a given large enough size, creating a list and creating a set do not take the same time. When estimating operations on lists or sets, we want to measure the time it takes to apply that operation to the list or set, without including the extra time needed to create them. When comparing the performance of a given operation on a list versus that same operation on a set, it is even more important not to take into account the time needed to create them, since as observed, that time differs. The `setup` argument of the `timeit()` function allows us to execute some statements prior to executing the statements whose running time we want to estimate.

Retrieving, in a list of size 10 million, the element whose index is a multiple of 500,000, does not significantly depend on the value of the index. As that operation is so efficient, we perform it 10 million times to make the evaluation more precise:

```
In [33]: data = []
         for i in range(0, 10_000_000, 500_000):
             data.append((i, timeit(f'L[{i}]',
                                    setup = 'L = [None] * 10_000_000',
                                    number = 10_000_000
                                    )
```

创建这些列表和集合需要时间，因此它们的大小更大。对于给定的足够大　大小、创建列表和创建集合不需要相同的时间。在估算列表或集合，我们想测量将该操作应用到列表或集合所花费的时间，不包括创建它们所需的额外时间。在比较列表上给定操作的性能时　与对集合进行相同的操作相比，更重要的是不考虑所需的时间　根据观察，创建它们的时间是不同的。timeit()函数的setup参数允许我们这样做在执行我们希望估算其运行时间的语句之前执行一些语句。在大小为1,000万的列表中检索索引为500,000的倍数的元素则不是主要取决于索引的值。由于这个操作非常有效，我们执行了1000万次　使评估更精确的时间：

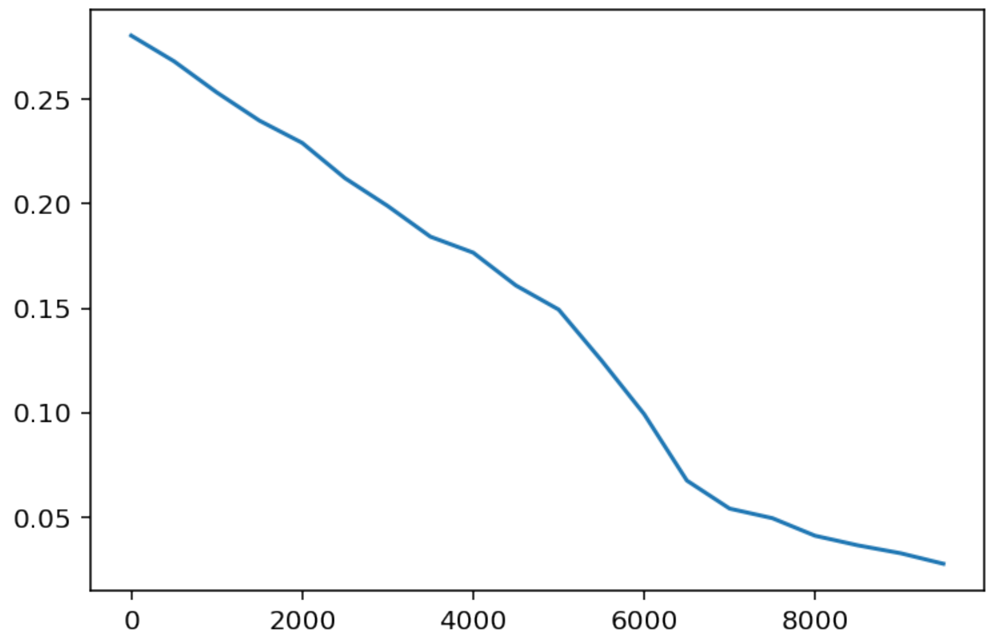
```
                  )
              )
plt.plot(*zip(*data));
```



The next experiment consists in popping, in a list of size 10 thousand, the element whose index is a multiple of 500. The operation is performed 100,000 times, again to make the evaluation more precise, but the setup code is executed only once, so after each call to `pop()`, we append an element to the end of the list so as to maintain its size. The smaller the index is, that is, the closer the element being popped is to the beginning of the list, the larger is the number of elements that have to be "shifted" from right to left to fill the vacancy left by that element:

```
In [34]: data = []
         for i in range(0, 10_000, 500):
             data.append((i, timeit(f'L.pop({i}); L.append(None)',
                                    setup = 'L = [None] * 10_000',
                                    number = 100_000
                                   )
                         )
                        )
         plt.plot(*zip(*data));
```

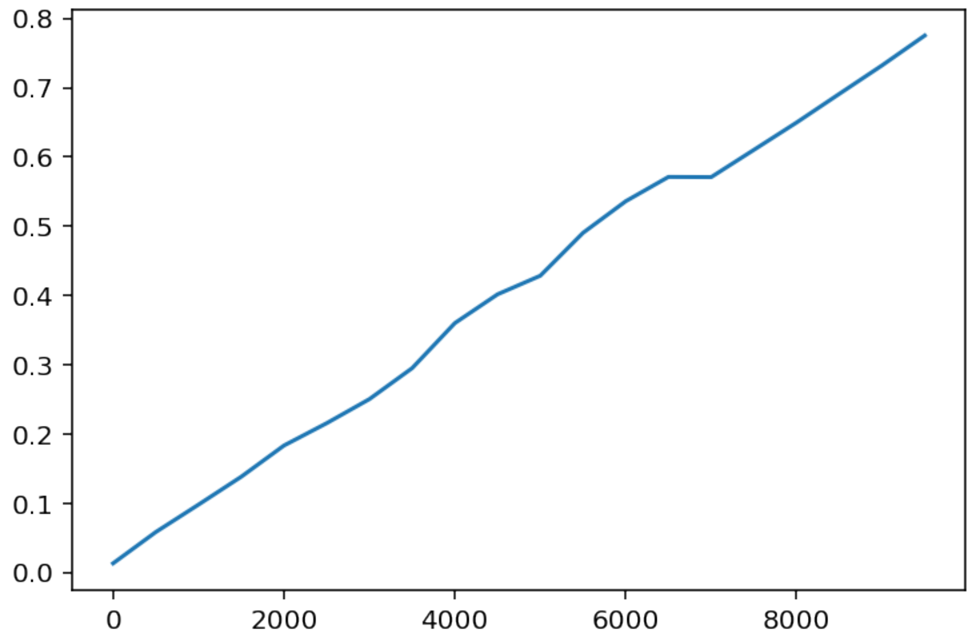
下一个实验是在一个大小为10,000的列表中弹出索引为a的元素
500多个。该操作执行了100,000次，再次使计算更加精确，但是
设置代码只执行一次，因此在每次调用pop()之后，我们将一个元素追加到
列表以保持其大小。索引越小，即元素弹出的距离越近
在列表的开头，需要从右向左 " 移动 " 到的元素的数量越大
填补该部分留下的空缺:



The next experiment consists in removing, in a list of size 10,000, an element (0), whose first occurrence has an index that is a multiple of 500, up to and including 9,500. The operation is performed 5,000 times, again to make the evaluation more precise; we again append an element to the end of the list so as to maintain its size. The cost of finding an element e that is further and further away from the beginning of the list increases more than the cost of “shifting” fewer and fewer elements from right to left to fill the vacancy left by removing e decreases:

```
In [35]: data = []
         for i in range(0, 10_000, 500):
             data.append((i, timeit(f'L.remove(0); L.append(None)',
                                    setup = 'L = [None] * 10_000; '
                                            f'L[{i}: {i} + 5_000] = [0] * 5_000',
                                    number = 5_000
                                   )
                         )
                        )
         plt.plot(*zip(*data));
```

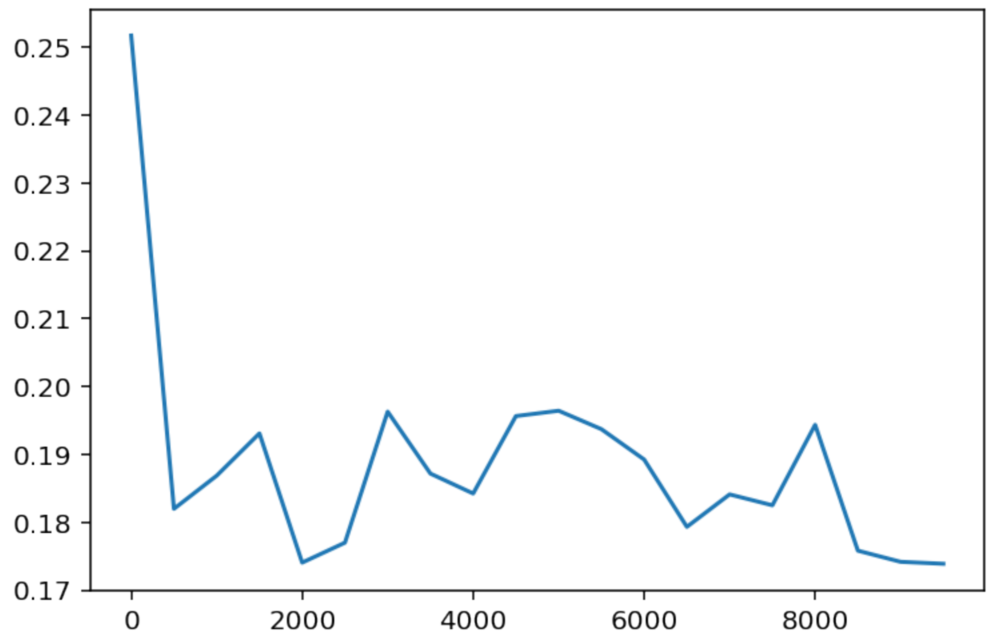
下一个实验是在大小为10,000的列表中删除第一个出现的元素(0)
有一个指数是500的倍数，直到包括9500。操作执行5000次，让评估更加精确;我们
再次将元素追加到列表的末尾，以便于
保持它的大小。找到一个距离开始越来越远的元素e的代价
的值，而将越来越少的元素从右向左“ 移动 ”以填充
去除e后留下的空位减少:



The next experiment consists in removing, in the set of all numbers between 0 included and 10 thousand excluded, the multiples of 500. The operation is performed 1,000,000 times, again to make the evaluation more precise; right after we have removed an element from the set, we bring it back. We see the benefit of working with sets rather than lists, when we can afford it, when we often need to remove elements:

```
In [36]: data = []
         for i in range(0, 10_000, 500):
             data.append((i, timeit(f'S.remove({i}); S.add({i})',
                                    setup = 'S = set(range(10_000))',
                                    number = 1_000_000
                                   )
                         )
                        )
         plt.plot(*zip(*data));
```

下一个实验是删除0到10000之间的所有数字
不包括500的倍数。执行100万次操作，再次进行评估
更精确;就在我们从集合中移除一个元素之后，我们把它带回来。我们看到了好处
当我们能够负担得起的时候，当我们经常需要删除元素的时候，使用集合而不是列表:



So though Euler's sieve sounded as a good idea and a possible improvement over Eratosthenes' sieve, it turned out to perform much worse because often applying the `remove()` method to a list is not efficient: it is on average linear in the length of the list. But applying the `remove()` method to a set is efficient: it has constant complexity. This suggests to, at every stage when all proper multiples of a (prime) number p have to be eliminated, convert the list of numbers that are still left to a set, remove those multiples from the set, and then convert the resulting set back to a list. Sorting is costly, but being performed only as many times as there are prime numbers smaller than $\lfloor\sqrt{n}\rfloor$, it could still bring an improvement.

Also, that allows one to get back to the original idea of eliminating all proper multiples at most equal to n of a given prime number p that are not multiples of smaller prime numbers by multiplying p with all numbers at least equal to p in `sieve`, until the product exceeds n. The approach is flawed if numbers are eliminated from `sieve`, but it is valid if numbers are eliminated from a copy of `sieve` as a set. To demonstrate it, we define a new version of `eliminate_proper_multiples_from_set()` that makes use of a `global` **declaration**, a notion that we now introduce.

The value of a global variable can be accessed from within a function:

```
In [37]: a = 1

         def f():
             print(a)

         f()
         a
```

所以虽然欧拉的筛子听起来是个好主意可能比埃拉托斯提尼的筛子更好，事实证明，它的性能要差得多，
因为经常对列表应用remove()方法是低效的:它的长度平均是线性的。但是将remove()方法应用于一个集合
是有效的:it有恒定的复杂性。这意味着，在a(素数)p的所有适当倍数的每个阶段
必须消去，把剩下的数字转换成一个集合，把这些倍数从然后将结果集转换回列表。排序是很昂贵的，但
只能作为因为质数比b小很多倍√n，它仍然可以带来改进。同时，这也让我们回到了最初的想法，去除了
所有的倍数对于n个给定的质数p它们不是较小质数的倍数乘以p在sieve中，所有数字至少等于p，直到乘积
超过n从筛选器中删除，但如果以集合的形式从筛选器的副本中删除编号，则为有效演示一下，我们定义
了一个使用的新版本的cancate_proper_multiples_from_set ()一个全球性的宣言，一个我们现在引入的

```
1
```

```
Out[37]: 1
```

A function can define a local variable with the same name as a global variable, the former then hiding the latter within the function:

```
In [38]: a = 1

         def f():
             a = 2
             print(a)

         f()
         a
```

```
2
```

```
Out[38]: 1
```

A function can change the value of a global variable. It cannot then define a local variable with the same name, and the global variable has to be declared as `global` within the function:

函数可以改变全局变量的值。然后，它不能使用。定义局部变量
相同的名称，并且全局变量必须在函数中声明为全局变量

```
In [39]: a = 1

         def f():
             global a
             print(a)
             a = 2

         f()
         a
```

```
1
```

```
Out[39]: 2
```

In the following code fragment, the assignment of `2` to `a` makes `a` a **local** variable of the function `f()`. Within a function, a variable is either global (the `global` declaration being necessary if and only if the function has a statement that assigns some value to that variable) or local; it is not global in parts of the function, and local in other parts. And since one cannot get the value of a variable before that variable has been assigned a value, the definition of `f()` below is incorrect:

在下面的代码片段中，将2赋值给a使得a成为函数f()的一个局部变量。在函数中，变量要么
是全局的(当且仅当函数有一个语句，该语句为该变量赋值)或局部;它不是全球性的函数，
在其他部分是局部的。因为我们不能在变量之前得到它的值如果给f()赋值，下面f()的定义
是错误的:

```
In [40]: a = 1

         def f():
             print(a)
             a = 2

         f()
```

```
UnboundLocalError                         Traceback (most recent call last)

<ipython-input-40-1ce06d5ec6b3> in <module>()
  5     a = 2
  6
----> 7 f()


<ipython-input-40-1ce06d5ec6b3> in f()
  2
  3 def f():
----> 4     print(a)
  5     a = 2
  6


UnboundLocalError: local variable 'a' referenced before assignment
```

Now, to observe how, with `n` set to `25`, proper multiples of 2, proper multiples of 3 that are not multiples of 2, and proper multiples of 5, equal to $\lfloor\sqrt{25}\rfloor$, that are multiples of neither 2 nor 3, are eliminated, we successively call the following function with `i` set to `0`, `1` and `2` as argument:

```
In [41]: def eliminate_proper_multiples_from_set(i):
             # We assume that this function will be called in the order
             #   eliminate_proper_multiples(0)
             #   eliminate_proper_multiples(1)
             #   eliminate_proper_multiples(2)
             #   ...
             global sieve
             sieve_as_set = set(sieve)
             print(f'Now eliminating proper multiples of {sieve[i]}')
             j = i
             while j < len(sieve):
                 factor = sieve[i] * sieve[j]
                 if factor <= n:
                     print(f'  Eliminating {sieve[i]} * {sieve[j]} = {factor}')
                     sieve_as_set.remove(factor)
                     j += 1
                 else:
                     break
             sieve = sorted(sieve_as_set)

In [42]: sieve = list(range(2, n + 1))
         print_sieve_contents()

  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

In [43]: eliminate_proper_multiples_from_set(0)
         print_sieve_contents()
```

```
Now eliminating proper multiples of 2
 Eliminating 2 * 2 = 4
 Eliminating 2 * 3 = 6
 Eliminating 2 * 4 = 8
 Eliminating 2 * 5 = 10
 Eliminating 2 * 6 = 12
 Eliminating 2 * 7 = 14
 Eliminating 2 * 8 = 16
 Eliminating 2 * 9 = 18
 Eliminating 2 * 10 = 20
 Eliminating 2 * 11 = 22
 Eliminating 2 * 12 = 24
  2  3  5  7  9 11 13 15 17 19 21 23 25
```

```
In [44]: eliminate_proper_multiples_from_set(1)
         print_sieve_contents()
```

```
Now eliminating proper multiples of 3
 Eliminating 3 * 3 = 9
 Eliminating 3 * 5 = 15
 Eliminating 3 * 7 = 21
  2  3  5  7 11 13 17 19 23 25
```

```
In [45]: eliminate_proper_multiples_from_set(2)
         print_sieve_contents()
```

```
Now eliminating proper multiples of 5
 Eliminating 5 * 5 = 25
  2  3  5  7 11 13 17 19 23
```

Putting it all together:

```
In [46]: def second_sieve_of_primes_up_to(n):
             sieve = list(range(2, n + 1))
             i = 0
             while sieve[i] <= round(sqrt(n)):
                 sieve_as_set = set(sieve)
                 k = 0
                 while True:
                     factor = sieve[i] * sieve[i + k]
                     if factor > n:
                         break
                     sieve_as_set.remove(factor)
                     k += 1
                 sieve = sorted(sieve_as_set)
                 i += 1
             return sieve
```

```
In [47]: primes = second_sieve_of_primes_up_to(1_000)
         nicely_display(primes, len(str(primes[-1])))
```

```
   2    3    5    7   11   13   17   19   23   29   31   37   41   43   47   53
  59   61   67   71   73   79   83   89   97  101  103  107  109  113  127  131
 137  139  149  151  157  163  167  173  179  181  191  193  197  199  211  223
 227  229  233  239  241  251  257  263  269  271  277  281  283  293  307  311
 313  317  331  337  347  349  353  359  367  373  379  383  389  397  401  409
 419  421  431  433  439  443  449  457  461  463  467  479  487  491  499  503
 509  521  523  541  547  557  563  569  571  577  587  593  599  601  607  613
 617  619  631  641  643  647  653  659  661  673  677  683  691  701  709  719
 727  733  739  743  751  757  761  769  773  787  797  809  811  821  823  827
 829  839  853  857  859  863  877  881  883  887  907  911  919  929  937  941
 947  953  967  971  977  983  991  997
```

This second version of Euler's sieve is indeed more efficient than the first version, but it is still less efficient than Eratosthenes' sieve:

```
In [48]: timeit('second_sieve_of_primes_up_to(500_000)',
                globals = globals(), number = 1
               )
```

```
Out[48]: 1.7024993670056574
```