11 Suppose that Cauchy's estimate is an equality. Let's create a disk around the origin of radius $r > 0$. Using Cauchy's estimate we get

$$|f^{(0)}(0)| = |f(0)| = M$$

This means that the origin is a maximum. Suppose we let $f(z) = Kz^n$ then in the disk $|f(z)| = |K||z^n| \leq M = |f(0)|$. But this gives us

$$|K| \leq \frac{|f(0)|}{|z^n|}$$

We can let the disk be arbitrarily large as well as z thus $|K| \leq \frac{|f(0)|}{|z^n|} \to 0$. So $f(z) = 0$
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

12 For the domain D we have a fixed center z_0 and a radius r. ∂D can be written as the path $\partial D(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$. If we integrate $f(\partial D(t))$ we get the sum of all $f(z)$ along the path so to get the average we can divide by the length of the path giving us:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\partial D(t))dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

We can then do a substitution with $z = z_0 + re^{it}$ and $dz = ire^{it}dt$.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{f(z)}{ire^{it}}dt &= \frac{1}{2i\pi}\int_0^{2\pi} \frac{f(z)}{re^{it}}dz \\ &= \frac{1}{2i\pi}\int_{\partial D} \frac{f(z)}{z - z_0}dz\end{aligned}$$

by Cauchy's integral formula we can see

$$\frac{1}{2i\pi}\int_{\partial D} \frac{f(z)}{z - z_0}dz = f(z_0)$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

13 Suppose we have a domain D with radius $r > 0$. The maximum would be $|f(z)| \leq Kr^c$. Then using Cauchy's estimate for $n = c$:

$$|f^{(c)}(z)| \leq \frac{Kr^c(c)!}{r^c} = Kc!$$

This means that $f^{(c)}(z) = w$ and w is constant. Let's find the antiderivative of $f^{(c)}(z)$, c times:

$$\begin{aligned}
f^{(c-1)}(z) &= wz + l_0 (\text{l is a constant of integer}) \\
f^{(c-2)}(z_0) &= \frac{wz^2}{2!} + l_0 z + l_1 \\
f^{(c-(c-1))}(z_0) &= \frac{wz^{c-1}}{(c-1)!} + l_0 \frac{z^{c-2}}{(c-2)!} + l_1 \frac{z^{c-3}}{(c-3)!} + ... \\
f(z_0) &= \frac{wz^c}{(c)!} + l_0 \frac{z^{c-1}}{(c-1)!} + l_1 \frac{z^{c-2}}{(c-2)!} + ...
\end{aligned}$$

This gives us a polynomial of degree $\leq c$
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

14 Let's consider the taylor expansion of f, g centered at 0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!}$$

We also know that

$$f^{(n)}(0) = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

if we choose a small enough r that would mean

$$\frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{g(z)}{(z-z_0)^{n+1}} dz \right|$$

Thus:

$$\begin{aligned}
f^{(n)}(0) &= g^{(n)}(0) \\
\implies f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!} \\
&= g(z)
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

15 Suppose that $f(z) = \sum a_n(z-z_0)^n$ in a disk D center z_0 radius R. Assume $0 \leq r < R$.

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0+re^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\overline{f(z_0+re^{i\theta})}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta}\right)\left(\sum_{n=0}^{\infty} \overline{a_n} r^n e^{-in\theta}\right) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{n} (a_k r^k e^{ik\theta})(\overline{a_{n-k}} r^{n-k} e^{-i(n-k)\theta})d\theta \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_0^{2\pi} (a_k\overline{a_{n-k}} r^n e^{i(2k-n)\theta})d\theta
\end{aligned}$$

Let us consider different terms in this sequence. Suppose that $2k-n \neq 0$, so the integral would be:

$$\left[a_k\overline{a_{n-k}}r^n \frac{e^{i(2k-n)\theta}}{i(2k-n)}\right]_0^{2\pi} = a_k\overline{a_{n-k}}r^n \frac{e^{i(2k-n)2\pi} - e^{i(2k-n)}}{i(2k-n)} = 0$$

But if instead $2k-n=0$

$$\begin{aligned}
\int_0^{2\pi} a_k\overline{a_{n-k}}r^n e^{i(2k-n)\theta}d\theta &= \int_0^{2\pi} |a_k|^2 r^n d\theta \\
&= 2\pi|a_k|^2 r^n
\end{aligned}$$

Now let us see when each case occurs. When n is even then

$$\sum_{k=0}^{n}\int_0^{2\pi}(a_k\overline{a_{n-k}}r^n e^{i(2k-n)\theta})d\theta = 2\pi|a_{n/2}|^2 r^n$$

if n is odd then all terms become 0. Thus the series is

$$\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$$

We know that $|f(z_0+re^{i\theta})|^2 \leq \sup_\theta |f(z_0+re^{i\theta})|$, so clearly

$$\begin{aligned}
\sum_{n=0}^{\infty}|a_n|^2 r^{2n} &= \frac{1}{2\pi}\int_0^{2\pi}\left|f(z_0+re^{i\theta})\right|^2 d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\sup_\theta\left|f(z_0+re^{i\theta})\right|^2 d\theta \\
&= \sup_\theta\left|f(z_0+re^{i\theta})\right|^2
\end{aligned}$$

Now let us assume that we have a local maximum $z_0 = 0$. We can create a disk around z_0 within the radius of convergence. We can see that $|f(0)| = |a_0|$ so

$$\begin{aligned} &\left|\sum a_n z^n\right| \leq |a_0| \\ \implies &\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq |a_0|^2 \\ \implies &\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 0 \end{aligned}$$

Thus the function must be constant.
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

19 Suppose that there exists some z_0 such that $f(z_0) = 0$. Because $f(z)$ is differentiable in it's domain there is a Taylor expansion around z_0

$$\begin{aligned} \text{Let } a_n &= \frac{f^{(n)}(z_0)}{n!} \\ f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \end{aligned}$$

Suppose $f^{(m)}(z_0) \neq 0$ thus we have

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n$$

then we know that the function is not identically zero. But if it is NOT of finite order then $\forall n \in \mathbb{N}, f^{(n)}(z_0) = 0$. This would mean that $a_n = 0$ which gives the sequence

$$f(z) = \sum_{n=0}^{\infty} 0 \cdot z^n = 0$$

So f is identically 0.
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$