1 7.9 Problems: 20, 21, 22

20 (i) With the function $\gamma(t) = 2e^{-it} - 1$ we must find θ at the end points. We can see that $\arg(2e^{-i0} - 1) = \arg(1)$ so $\theta(0) = 0$. Next we get $\arg(2e^{-i2\pi} - 1) = \arg(1)$, to make this continuous we must choose $\theta(2\pi) = 2\pi$. Thus we get

$$\omega(\gamma, 1) = \frac{2\pi - 0}{2\pi} = 1$$

Now with $z = 3i$, we have $\arg(2e^{-i0} - 3i) = \arg(1 - 3i)$. We also have $\arg(2e^{-i2\pi} - 3i) = \arg(1 - 3i)$. These are the same point and unlike last time they must have the same argument because θ oscillates while less than 2π. So the winding number is 0.

(ii) Let us consider the path $\gamma(t) = t + i(1 - t)$. if we wind around the point $1 + i$ we get argument π when $t = 0$ and $3\pi/2$ when $t = 1$. This gives us a winding number of $1/4$.
Now let $z = -i$. When $t = 0$ we get the angle $\pi/2$ and when $t = 1$ we get the angle $\pi/4$. So we get winding number $-\pi/4$.
Let's use the point $z = 10i$. The angle when $t = 0$ is just $3\pi/2$. When we let $t = 1$ then we are at the point $(1, -10)$.

$$\tan^{-1}\left(\frac{-10}{1}\right) + 2\pi = 4.812$$

so we get the winding number

$$\frac{4.812 - 3\pi/2}{2\pi} = 0.01585$$

21 (i)

$$\begin{aligned}
\int_\gamma \frac{1}{z} dz &= [\log|te^{-it}| + i\arg(te^{-it})]_\pi^{5\pi} \\
&= [\log|t| + i(-t)]_\pi^{5\pi} \\
&= (\log(\pi) - \log(5\pi) - i\pi + 5i\pi) \\
&= 4i\pi - \log(5)
\end{aligned}$$

(ii)

$$\begin{aligned}
\int_\gamma \frac{1}{z-1} dz &= [\log(-it - 1)]_0^1 \\
&= [\log|-it - 1| + i\arg(-it - 1)]_0^1 \\
&= [1/2\log(t^2 + 1) + i\tan^{-1}(t) + i\pi]_0^1 \\
&= 1/2\log(1^2 + 1) + i\tan^{-1}(1) - 1/2\log(0^2 + 1) - i\tan^{-1}(0) \\
&= 1/2\log(2) + i\pi/4
\end{aligned}$$

(iii)

$$\begin{aligned}\int_\gamma \frac{1}{z-1}dz &= [\log(it-1)]_{-1}^{1} \\ &= [\log|it-1| + i\arg(it-1)]_{-1}^{1} \\ &= [1/2\log(t^2+1) + i\tan^{-1}(-t) + i\pi]_{-1}^{1} \\ &= 1/2\log(2) + i\tan^{-1}(-1) - 1/2\log(2) - i\tan^{-1}(1) \\ &= -0.5i\end{aligned}$$

22 (i) Half circle. $A = 0, B = 1$
(ii) Donut. $A = 0, B = 1, C = 2$
(iii) Bean Pod. $A = 0, B = 1, C = 2, D = 2, E = 2$
(iv) Squiggles. $A = 0, B = 1, C = 2, D = 1, E = 0, F = 1$

2 8.8 Problems: 1,2

1. (i) The origin (z_*) would be a star center of this domain. First let's choose some $z_0 \in D$ where $0 \neq \operatorname{im}(z_0)$. Alternatively: $z_0 = a+bi, b \neq 0$. We can then find a line this point from the origin.

$$\gamma_{z_0}(t) = (t(\operatorname{re}(z_0) - \operatorname{re}(z_*)), t(\operatorname{im}(z_0) - \operatorname{im}(z_*))) = (at, bt), t \in [0,1]$$

We can see that $bt = 0$ only when $t = 0$, which is $\gamma_{z_0}(0) = (0,0)$. So it never goes through $z = x + 0i$ with $|x| \geq 1$.
Now let's choose some $z_1 = a + bi \in D$ where $b = 0$. In order for this to be in the domain we need $|a| < 1$. If we find a line from this point to the origin we get:

$$\gamma_{z_1}(t) = (at, bt), t \in [0,1]$$

We must check if this is always in bounds. $|at| < |a| < 1$ so $(\forall t)\gamma_{z_1}(t) \in D$ so z_* is star center and D is a star domain. Observation: all points such that $z = a + 0i, |a| < 1$ might be star centers.

(ii) I will prove the origin is a star center. Take any $z = a + bi \in D$ and connect it to the origin. We get the path:

$$(at, bt), t \in [0,1]$$

We can now see if (at, bt) is always in the domain.

$$|(at, bt)| = \sqrt{(at)^2 + (bt)^2} < \sqrt{a^2 + b^2} < 1$$

So the origin is a star center and the domain is a star domain. Observation: Any point in this disc is a star center because a line between two points in the disk will still be in the disk.

(iii) This is not a star domain. Let's suppose we choose a point z_0 "inside" this half circle. We cannot draw a straight line from z_0 to a point outside the half circle and above the real axis. For the same reason a star center cannot be outside the half circle and above the real axis. Let's choose a point z_1 below the real axis but outside the unit circle. Then $[-z_1, z_1]$ passes through the half circle. If z_1 is inside the unit circle then $-2z_1/|z_1|$ is outside the unit circle and cannot connect to z_1 by a straight line in the domain.

(iv) This is not a star domain. This domain is in 2 sections that do not shar any common points. So any point from the upper left quadrant cannot be connected to one in the lower right. So it is not a star domain.

2. (i)

$$\log(z)$$

(ii)

$$\frac{-1}{z}$$

(iii)

$$\frac{z}{z^2} + \frac{1}{z^2} \implies \log(z) + \frac{-1}{z}$$

(iv)

$$\begin{aligned}\int \sum \frac{(-1)^n z^{2n-1}}{(2n)!} dz &= \sum \frac{(-1)^n \int z^{2n-1} dz}{(2n)!} \\ &= \sum \frac{(-1)^n z^{2n}}{(2n+1)! - (2n)!}\end{aligned}$$

(v)

$$\begin{aligned}\int \sum \frac{(-1)^n z^{2n}}{(2n+1)!} dz &= \sum \frac{(-1)^n \int z^{2n} dz}{(2n+1)!} \\ &= \sum \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}\end{aligned}$$