108B HW 1

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Problem 2. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map, defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution. We will give $\mathcal{P}_3(\mathbb{R})$ the basis $\{x^3, x^2, x, 1\}$ and $\mathcal{P}_2(\mathbb{R})$ the basis $\{3x^2, 2x, 1\}$. If we have some polynomial $p(x) = ax^3 + bx^2 + cx + d$ we know that $p'(x) = 3ax^2 + 2bx + c$. Rewriting these as matrices we get

$$p(x) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$p'(x) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\implies D \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Problem 3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Solution. First let us suppose that $dim(V) = dim(W)$, and each has basis $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ..., w_n\}$ respectively. For some $v \in V$

$$Iv = T(v)$$
$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_W$$
$$\implies a_i v_i = a_i w_i$$

So they would have the same basis. Assume that $dim(V) > dim(W)$, and each has basis $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ..., w_m\}$ respectively. For some $v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in V$ we would need that

$$M(T) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_W$$

This would require that the basis of W is the first m vectors (with their first m elements) of the basis for V. If we let $dim(V) < dim(W)$, then we need

$$M(T) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_V = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}_W$$

As long as the first n vectors of W are the basis vectors of V (with $m - n$ 0 entries at the end) then this will work.

Problem 4. Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases above) are 0 except for possibly a 1 in the first row, first column.

Solution. First let us consider $m \geq n$. Let us consider some

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{m} \in V$$

and let's say

$$T(v) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} b_i w_i$$

If we have such an $M(T)$ then

$$M(T) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 w_1 + \sum_{i=2}^{n} b_i w_i$$

$$a_1 v_1 = a_1 w_1 \implies v_1 = w_1$$

Thus we need the basis of W to have

Problem 5. Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all the entries in the first row of $\mathcal{M}(T)$ are 0 except for possibly a 1 in the first row, first column.

Solution.

Problem Additional. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$f((1,0)) = (-2,1)$$
$$f((0,1)) = (-1,2).$$

Let T be a change of basis for $\mathbb{R}^2$ given by

$$T((1,0)) = (1,1)$$
$$T((0,1)) = (1,0).$$

(a) Write down the matrix representation A of f under the standard basis $(1,0),(0,1)$.

(b) Find the matrix representation B of f under the new basis $(1,1),(1,0)$.

(c) Compute the eigenvectors and eigenvalues of A and B, respectively.

(d) What is the relation between the eigenvalues of A and B? Explain.

(e) What is the relation between the eigenvectors of A and B? Explain.

Solution.

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Problem 7. Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution.

Problem 11. Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution.

Problem 21. Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Solution.

Problem 24. Suppose A is an n-by-n matrix with entries in $\mathbb{F}$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of $\mathbb{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries of each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries of each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution.

5C

Problem 1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{range} T \oplus \operatorname{null} T$.

Solution.

Problem 7. Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbb{F}$. Prove that λ appears in the diagonal of A precisely $\dim E(\lambda, T)$ times.

Solution.