1 8.8 Problems: 3, 4, 6, 7, 8

3 Let's find the sum of the winding numbers for $z = \pm 1$.

$$w(\gamma, 1) = -1$$

$$w(\gamma_1, 1) = 0$$
$$w(\gamma_2, 1) = 1$$
$$w(\gamma, -1) = -1$$
$$w(\gamma_1, -1) = 1$$
$$w(\gamma_2, -1) = 0$$

So the sum is 0. This means that the sum of the integrals would be 0. Hence we have:

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma} f = 0 \implies \int_{\gamma_1} f + \int_{\gamma_2} f = -\int_{\gamma} f$$

This does not come out to the same as the homework asks due to a typo.

4 Choose the path $\gamma = e^{it}$ with $t \in [0, 2\pi]$. We can take the winding number of this path around the point $z = -1$. This gives us $w(\gamma, z) = 1 \neq 0$ so the domain is not simply connected.
Suppose the domain D_0 is NOT simply connected. That means there is some $z \notin D_0$ where the winding number is not zero. Let's choose the point $z = x + i0$ where $x \leq -1$. In order for the winding number around this point to be non-zero we need a path that goes completely around it. This would require passing through a point $z = a + i0$ where $a < x \leq -1$. But this is not in the domain. The same argument applies for $z = x + i0$ when $x \geq 1$ because there are no points to the right of z on the real axis within the domain. So we cannot create a close path around a point outside the domain, hence a winding number of 0.
The origin is a star domain by the argument from 8.8.1(i). Because the function $\frac{1}{z^2-1}$ is differentiable and in a star domain it also has an antiderivative.

6 There are four cases: The inside of γ might contain $-i, i$, both, or neither. When it it contains neither $w(\gamma, \pm i) = 0$ so $\int_\gamma f = 0$ by Thm 8.8.
Suppose the path γ_1 only contains i. All paths with the same winding number will be equal (See section 8.6). Let us use $\gamma_1(t) = i + e^{it}$ with $t \in [0, 2\pi]$, it has winding number 1. But if we wanted a path with winding number n, then we only need to follow the same path n times.

$$\int_{n\gamma} f = n \int_{\gamma} f \qquad (\dagger)$$

$$\begin{aligned}
\int_{n_1\gamma_1} \frac{1}{z^2+1}dz &= \frac{in_1}{2}\int_{\gamma} \frac{1}{z+i} - \frac{1}{z-i}dz && \text{(by †)}\\
&= \frac{in_1}{2}[\log(z+i) - \log(z-i)]_{\gamma_1}\\
&= \frac{in_1}{2}[\log(e^{it}+2i) - \log(e^{it})]_0^{2\pi}\\
&= \frac{in_1}{2}[\log(e^{i2\pi}+2i) - \log(e^0+2i) - \log(e^{i2\pi}) + \log(e^0)]\\
&= \frac{in_1}{2}[-i2\pi]\\
&= n_1\pi
\end{aligned}$$

Now we can let $\gamma_2(t) = e^{it} - i$ so it contains $-i$. If we want winding number n_2 then integrate along $n_2\gamma_2$. Using previous work:

$$\begin{aligned}
\int_{n_2\gamma_2} \frac{1}{z^2+1}dz &= \frac{in_2}{2}[\log(z+i) - \log(z-i)]_{\gamma_2}\\
&= \frac{in_2}{2}[\log(e^{it}) - \log(e^{it}-2i)]_0^{2\pi}\\
&= \frac{in_2}{2}[\log(e^{i2\pi}) - \log(e^0) - \log(e^{i2\pi}-2i) + \log(e^0+2i)]\\
&= -n_2\pi
\end{aligned}$$

Finally we can choose the path γ_3, which contains both points. Suppose that for this path $w(\gamma_3, i) = n_3$ and $w(\gamma_3, -i) = n_4$. Observe the sum

$$w(n_3\gamma_1, i)+w(n_4\gamma_2, i)-w(\gamma_3, i) = 0 = w(n_3\gamma_1, -i)+w(n_4\gamma_2, -i)-w(\gamma_3, -i)$$

By theorem 8.9 we get the result

$$\begin{aligned}
n_3\int_{\gamma_1} f + n_4\int_{\gamma_2} f - \int_{\gamma_3} f = 0\\
\implies \int_{\gamma_3} \frac{1}{z^2+1}dz = n_3\pi - n_4\pi = (n_3-n_4)\pi
\end{aligned}$$

Let's consider now the contour $\sigma(t) = t$ with $t \in [0,1]$.

$$\begin{aligned}
\int_{\sigma} \frac{1}{z^2+1}dz &= \frac{i}{2}[\log(z+i) - \log(z-i)]_{\sigma}\\
&= \frac{i}{2}[\log(t+i) - \log(t-i)]_0^1\\
&= \frac{i}{2}[\log(1+i) - \log(1-i) - \log(i) + \log(-i)]\\
&= \frac{i}{2}[1/2\log(2) + i\pi/4 - 1/2\log(2) + i\pi/4 - i\pi/2 - i\pi/2]\\
&= \pi/4
\end{aligned}$$

7 The inside of both γ_1, γ_2 is the same ring or donut.

$$\begin{aligned}
\int_{\gamma_1} \frac{\cos(z)}{z} dz &= \int_{\gamma_1} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!} dz \\
&= \int_{\gamma_1} \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{(2n+2)!} dz \\
&= 0 + \left[\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \right]_1^2 - \left[\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \right]_1^2 = 0
\end{aligned}$$

Now we will use the other path to get:

$$\begin{aligned}
\int_{\gamma_2} \frac{\cos(z)}{z} dz &= \int_{\gamma_2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!} dz \\
&= \int_{\gamma_2} \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n+1}}{(2n+2)!} dz \\
&= 4i\pi + \left[\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \right]_1^2 - \left[\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \right]_1^2 \\
&= 4i\pi
\end{aligned}$$

8 We know that for each S_r we get $w(S_r, z_r) = 1$. If we want winding number n_r we can change the contour to $n_r S_r$ which goes n_r times along S_r. Now if we integrate along this path:

$$\int_{nS_r} f = n \int_{S_r} f \qquad (\dagger)$$

Suppose we choose n_r such that

$$w(n_r S_r, z_r) = w(\gamma, z_r)$$

We know that $w(S_r, z_{j \neq r}) = 0$ because the circles are sufficiently small. Thus we can see that $\forall z_j \notin \mathbb{C}$ the sum

$$w(\gamma, z_j) - \sum_{r=1}^{k} w(n_r S_r, z_j) = 0$$

Which would imply by Theorem 8.9 that

$$\int_{\gamma} f - \sum_{r=1}^{k} \int_{n_r S_r} f = 0$$

Using (†) and some algebra we get the result

$$\int_\gamma f = \sum_{r=1}^{k} n_r \int_{S_r} f$$

Let us suppose that for $r = 1, 2, ..., k$

$$\begin{aligned}
&\lim_{z\to z_r}(z - z_r)f(z) = a_r \in \mathbb{C} \\
\implies &\lim_{z\to z_r}(z - z_r)f(z)\frac{1}{z - z_r} = a_r \lim_{z\to z_r}\frac{1}{z - z_r} \\
\implies &\lim_{z\to z_r} f(z) = a_r \lim_{z\to z_r}\frac{1}{z - z_r}
\end{aligned}$$

Let us note the winding number of S_r

$$\begin{aligned}
w(S_r, z_r) &= \frac{1}{2\pi i}\int_{S_r}\frac{1}{z - z_r}dz \\
\implies 2\pi i &= \int_{S_r}\frac{1}{z - z_r}dz
\end{aligned}$$

We can then use this to compute the integral as S_r shrinks

$$\begin{aligned}
\lim_{z\to z_r}\int_{S_r} f(z)dz &= \lim_{z\to z_r}\int_{S_r}\frac{a_r}{z - z_r}dz \\
&= 2\pi i a_r
\end{aligned}$$

Giving us the result

$$\int_\gamma f = \sum_{r=1}^{k} 2\pi i n_r a_r$$

2 10.9 Problems: 1, 3, 4, 5, 8

1. Let's find the first few terms of the sequence to get an idea of what it would come out to.

$$\begin{aligned}
f(0) &= 0\\
f'(z) &= \frac{1}{z+1}\\
f'(0) &= 1\\
f''(z) &= \frac{-1}{(z+1)^2}\\
f''(z) &= -1\\
f^{(3)}(z) &= \frac{2}{(z+1)^3}\\
f^{(3)}(0) &= 2\\
f^{(4)}(z) &= \frac{-6}{(z+1)^4}\\
f^{(4)}(0) &= -6
\end{aligned}$$

I claim that

$$\begin{aligned}
f(z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ... + \frac{(-1)^{n-1}z^n}{n} + ...\\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{n}
\end{aligned}$$

The disk of convergence is all $|z| \leq 1$. For $g(z)$ we can simplify

$$\begin{aligned}
g(z) &= \left(e^{\operatorname{Log}(z+1)}\right)^{\alpha}\\
&= (z+1)^{\alpha}\\
g(0) &= 1\\
g'(z) &= \alpha(z+1)^{\alpha-1}\\
g'(0) &= \alpha\\
g''(z) &= \alpha(\alpha-1)(z+1)^{\alpha-2}\\
g''(0) &= \alpha(\alpha-1)\\
g^{(3)}(z) &= \alpha(\alpha-1)(\alpha-2)(z+1)^{\alpha-3}\\
g^{(3)}(0) &= \alpha(\alpha-1)(\alpha-2)\\
g^{(n)}(0) &= \prod_{i=0}^{n-1}(\alpha - i)
\end{aligned}$$

Thus the series would be

$$g(z) = 1 + \alpha x^2 + \alpha(\alpha - 1)\frac{x^2}{2!} + .. + \frac{x^n}{n!}\prod_{i=0}^{n-1}(\alpha - i)$$
$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\prod_{i=0}^{n-1}(\alpha - i)$$

Now we will find the radius of convergence

$$\lim_{n\to\infty}\frac{|a_{n-1}|}{|a_n|} = \lim_{n\to\infty}\frac{x^{n-1}n!\prod_{i=0}^{n-2}(\alpha - i)}{x^n(n-1)!\prod_{i=0}^{n-1}(\alpha - i)}$$
$$= \lim_{n\to\infty}\frac{|n|}{|x(\alpha - n + 1)|}$$
$$= \lim_{n\to\infty}\frac{1}{|x\alpha/n - 1 + 1/n|}$$
$$= 1$$

Thus the disk of convergence is z where $|z| < 1$

3 (i) Looking at the first few terms

$$f(z) = \sin^2(z)$$
$$f(0) = 0$$
$$f'(z) = 2\sin(z)\cos(z)$$
$$f'(0) = 0$$
$$f''(z) = 2\cos^2(z) - 2\sin^2(z)$$
$$f''(0) = 2$$
$$f^{(3)}(z) = -8\sin(z)\cos(z) = -4f'(z)$$
$$f^{(4)}(z) = -4f''(z)$$
$$f^{(2n+1)}(0) = 0$$
$$f^{(2n)}(0) = (-4)^{n-1}2$$

We can then see that

$$\sin^2(z) = 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + 32\frac{x^6}{6!} - 128\frac{x^8}{8!} + ...$$
$$= \sum_{n=0}^{\infty}\frac{2(-4)^{n-1}x^{2n}}{(2n)!}$$

(ii) Looking at the first few terms

$$
\begin{aligned}
f(0) &= \frac{0^2}{(0+2)^2} = 0 \\
f'(z) &= 4\frac{z}{(z+2)^3} \\
f'(0) &= 0 \\
f''(z) &= 4(-2!)\frac{z-1}{(z+2)^4} \\
f''(0) &= 1/2 \\
f^{(3)}(z) &= 4(3!)\frac{(z-2)}{(z+2)^5} \\
f^{(3)}(0) &= \frac{-3}{2}
\end{aligned}
$$

I claim that for $n \geq 1$

$$
\begin{aligned}
f^{(n)}(z) &= 4(-1)^{n+1}n!\frac{(z-n+1)}{(z+2)^{n+2}} \\
\implies f^{(n+1)}(z) &= 4(-1)^{n+1}n!\frac{(z-n+1)'(z+2)^{n+2}-(z-n+1)((z+2)^{n+2})'}{(z+2)^{2n+4}} \\
&= 4(-1)^{n+1}n!\frac{z+2-(z-n+1)(n+2)}{(z+2)^{n+3}} \\
&= 4(-1)^{n+2}n!\frac{(1+n)z-n(n+1)}{(z+2)^{n+3}} \\
&= 4(-1)^{n+2}(n+1)!\frac{z-n}{(z+2)^{n+3}}
\end{aligned}
$$

Thus we have:

$$f^{(n)}(0) = (-1)^n n!\frac{n-1}{(2)^n}$$

So we get the sequence

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\frac{(n-1)z^n}{2^n}$$

(iii) First we will compute several terms

$$
\begin{aligned}
f(0) &= \frac{1}{b} \\
f'(z) &= \frac{-a}{(az+b)^2} \\
f''(z) &= \frac{2a^2}{(az+b)^3} \\
f^{(n)}(z) &= \frac{(-1)^n (n!) a^n}{(az+b)^{n+1}} \\
f^{(n+1)}(z) &= \frac{(-1)^n (n!) a^n (-(n+1)a)}{(az+b)^{n+2}} \\
&= \frac{(-1)^{n+1} (n+1)! a^{n+1}}{(az+b)^{n+2}}
\end{aligned}
$$

Thus the zeros are:

$$
\begin{aligned}
f^{(n)}(0) &= \frac{(-1)^n (n!) a^n}{b^{n+1}} \\
f(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n a^n z^n}{b^{n+1}}
\end{aligned}
$$

(iv) First few terms:

$$
\begin{aligned}
f(0) &= 0 \\
f'(z) &= e^{z^2} \\
f'(0) &= 1 \\
f''(z) &= 2ze^{z^2} \\
f''(z) &= 0 \\
f^{(3)}(z) &= 2(1+2z^2)e^{z^2} \\
f^{(3)}(0) &= 2 \\
f^{(4)}(z) &= 4(3z+2z^3)e^{z^2} \\
f^{(4)}(0) &= 0 \\
f^{(5)}(z) &= 4(3+12z^2+4z^4)e^{z^2} \\
f^{(5)}(0) &= 12 \\
f^{(2n+1)}(0) &= \frac{(2n)!}{n!}
\end{aligned}
$$

So

$$
f(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(n!)(2n+1)}
$$

(v) First few terms:

$$\begin{aligned}
f(0) &= 1 \\
f'(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) z^{2n-1}}{(2n+1)!} \\
f''(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) z^{2n-2}}{(2n+1)!} \\
f^{(3)}(z) &= \sum_{n=2}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2) z^{2n-3}}{(2n+1)!} \\
f^{(4)}(z) &= \sum_{n=2}^{\infty} \frac{(-1)^n (2n)(2n-1)(2n-2)(2n-3) z^{2n-4}}{(2n+1)!} \\
f^{(2k)}(z) &= \sum_{n=k}^{\infty} \frac{(-1)^n (2n)! z^{2n-2k}}{(2n-2k)!(2n+1)!} \\
f^{(2k)}(0) &= \frac{(-1)^k (2k)!}{(2k+1)!}
\end{aligned}$$

Thus we get the series:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

(vi) I cannot figure this out

4 We know that $\sec(z) = 1/\cos(z)$ thus $\sec(z)\cos(z) = 1$. So if we take the product of the power series:

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} \frac{(-1)^n c_{2n} z^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right) &= \sum_{n=0}^{\infty} (-1)^n z^{2n} \left(\sum_{i=0}^{n} \frac{c_{2i}}{(2n-2i)!(2i)!}\right) \\
&= c_0 + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{n!} \left(\sum_{i=0}^{n} \binom{2n}{2i} c_{2i}\right) = 1 \\
&\implies c_0 = 1 \\
&\implies \sum_{i=0}^{n} \binom{2n}{2i} c_{2i} = 0, \forall n \geq 1
\end{aligned}$$

Now we must show that c_{2n} is an integer. By the series we know

$$\begin{aligned}
\sum_{i=0}^{1} \binom{2n}{2i} c_{2i} &= 0 \\
c_0 + \binom{2}{2} c_2 &= 0 \\
c_2 &= -1
\end{aligned}$$

Now let's assume that c_2n is an integer for $n \leq k$

$$\sum_{i=0}^{k+1} \binom{2k+2}{2i} c_{2i} = c_{2k+2} + \sum_{i=0}^{k} \binom{2k+2}{2i} c_{2i} = 0$$
$$\implies c_{2k+2} = \sum_{i=0}^{k} (-1) \binom{2k+2}{2i} c_{2i}$$

We know that the binomial coefficient is an integer and since we are just adding and multiplying integers to get c_{2k+2} it must be an integer. Here are 5 terms of the sequence.

$$\begin{aligned} c_0 &= 1 \\ c_2 &= -1 \\ c_4 &= 5 \\ c_6 &= -61 \\ c_8 &= 1385 \\ c_{10} &= -50521 \end{aligned}$$

5 First we will do some algebra to make the derivatives simpler

$$\begin{aligned} \frac{1}{1-z-z^2} &= \frac{1}{(\sqrt{5}/2 - z - 1/2)(z + 1/2 + \sqrt{5}/2)} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{5}/2 - z - 1/2} + \frac{1}{z + 1/2 + \sqrt{5}/2} \right) \end{aligned}$$

Now we take the derivatives

$$\begin{aligned} f(0) &= 1 \\ f'(z) &= \frac{1}{\sqrt{5}} \left(\frac{1}{(\sqrt{5}/2 - 1/2 - z)^2} + \frac{(-1)^n}{(1/2 + \sqrt{5}/2 + z)^2} \right) \\ f'(0) &= 1 \\ f''(z) &= \frac{2!}{\sqrt{5}} \left(\frac{1}{(\sqrt{5}/2 - 1/2 - z)^3} + \frac{(-1)^n}{(1/2 + \sqrt{5}/2 + z)^3} \right) \\ f''(0) &= 2 \cdot 2 \\ f^{(n)}(0) &= \frac{n!}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{1+n} + (-1)^n \left(\frac{\sqrt{5}-1}{2} \right)^{1+n} \right) \end{aligned}$$

Thus we have the sequence.

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{1+n} + (-1)^n \left(\frac{\sqrt{5}-1}{2} \right)^{1+n} \right) z^n \end{aligned}$$

Thus

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{1+n} + (-1)^n \left(\frac{\sqrt{5}-1}{2} \right)^{1+n} \right)$$

and you can clearly see from plugging in this formula for F_n that $F_n = F_{n-1} + F_{n-2}$

8 Let us first rewrite $g(z)$ with power series:

$$\begin{aligned} g(z) &= \frac{1}{3} \left(\sum a_n z^n + \sum a_n \omega^n z^n + \sum a_n \omega^{2n} z^n \right) \\ &= \frac{1}{3} \left(\sum_{n=0}^{\infty} (1 + \omega^n + \omega^{2n}) a_n z^n \right) \end{aligned}$$

Let us show that $w^4 = w$ and $w^3 = 1$

$$\begin{aligned} w^4 &= e^{4 \cdot 2i\pi/3} = e^{8i\pi/3} = e^{2i\pi/3} = w \\ w^3 &= e^{2i\pi 3/3} = 1 \end{aligned}$$

Now we can take derivatives

$$
\begin{aligned}
g(0) &= a_0 \\
g'(z) &= \frac{1}{3}\left(\sum_{n=1}^{\infty}(1+\omega^n+\omega^{2n})na_n z^{n-1}\right) \\
g'(0) &= \frac{1}{3}\left((1+\omega+\omega^2)a_1\right) = 0 \\
g''(z) &= \frac{1}{3}\left(\sum_{n=2}^{\infty}(1+\omega^n+\omega^{2n})n(n-1)a_n z^{n-2}\right) \\
g''(0) &= \frac{1}{3}\left((1+\omega^2+\omega^4)2a_2\right) = 0 \\
g^{(3)}(z) &= \frac{1}{3}\left(\sum_{n=3}^{\infty}(1+\omega^n+\omega^{2n})n(n-1)(n-2)a_n z^{n-3}\right) \\
g^{(3)}(0) &= \frac{1}{3}\left((1+\omega^3+\omega^6)3!a_3\right) = 3!a_3 \\
g^{(k)}(z) &= \frac{1}{3}\left(\sum_{n=k}^{\infty}(1+\omega^n+\omega^{2n})\frac{n!a_n z^{n-k}}{(n-k)!}\right) \\
g^{(k)}(0) &= \frac{1}{3}\left((1+\omega^k+\omega^{2k})k!a_k\right)
\end{aligned}
$$

We can see that when k is divisible by 3 we get $1/3(1+\omega^k+\omega^{2k}) = 1$. Otherwise we get 0 because:

$$1+\omega^{3k+1}+\omega^{6k+2} = 1+\omega\omega^{3k}+\omega^2\omega^{6k} = 1+\omega+\omega^2 = 0$$

$$1+\omega^{3k+2}+\omega^{6k+4} = 1+\omega^2\omega^{3k}+\omega^4\omega^{6k} = 1+\omega^2+\omega = 0$$

Thus we have $g^{(3n)}(0) = (3n)!a_{3n}$ so

$$g(z) = \sum_{n=0}^{\infty} a_{3n} z^{3n}$$

We can create the functions:

$$
\begin{aligned}
h_1(z) &= \frac{1}{3}\left(\sum_{n=0}^{\infty}(1+\omega^{n+2}+\omega^{2n+2})a_n z^n\right) = \sum_{n=0}^{\infty} a_{3n+1} z^{3n+1} \\
h_2(z) &= \frac{1}{3}\left(\sum_{n=0}^{\infty}(1+\omega^{n+1}+\omega^{2n+1})a_n z^n\right) = \sum_{n=0}^{\infty} a_{3n+2} z^{3n+2}
\end{aligned}
$$