

1 4.7 Problems: 11, 12, 15

11 We will consider the equation:

$$f(z) = \frac{xy^2(x + iy)}{x^2 + x^4}$$

$$f(0) = 0$$

Let us take the limit when $z \rightarrow 0$ along any line $z = (a + bi)t$

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f((a + ib)t)}{(a + ib)t} &= \frac{(at)(bt)^2(a + ib)t}{((at)^2 + (at)^4)(a + ib)t} \\ &= \frac{ab^2t^3}{a^2t^2 + a^4t^4} \\ &= \frac{ab^2t^1}{a^2 + a^4t^2} \\ &= 0 \end{aligned}$$

So the limit goes to zero when $z \rightarrow 0$ along a straight path. But what if we take a different path to 0. We will instead let $z(t) = t^2 + it$ as $z \rightarrow 0$.

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f(t^2 + it)}{t^2 + it} &= \frac{t^2t^2(t^2 + it)}{(t^4 + t^8)(t^2 + it)} \\ &= \frac{t^4}{t^4 + t^8} \\ &= \frac{1}{1 + t^4} \\ &= 1 \end{aligned}$$

12 (i) First we will find f'

$$f'(z) = (z^2)' = 2z$$

Then we will plug in the path

$$f'(\gamma(t)) = 2(t^3 + it^4)$$

We can easily find the derivative of the path

$$\gamma'(t) = 3t^2 + 4it^3$$

Now we want to find the derivative of the composition

$$\begin{aligned} (f\gamma)'(t) &= ((t^3 + it^4)^2)' \\ &= (t^6 - t^8 + 2it^7)' \\ &= -8t^7 + 14it^6 + 6t^5 \\ f'(\gamma(t))\gamma'(t) &= 2(t^3 + it^4)(3t^2 + 4it^3) \\ &= 2(3t^5 + 7it^6 - 4t^7) \\ &= -8t^7 + 14it^6 + 6t^5 \end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(ii) First we will find f'

$$f'(z) = (1/z)' = -1/z^2$$

Then we will plug in the path

$$f'(\gamma(t)) = -1/(\cos(t) + i \sin(t))^2 = -e^{-2it}$$

We can easily find the derivative of the path

$$\gamma'(t) = -\sin(t) + i \cos(t) = ie^{it}$$

Now we want to find the derivative of the composition

$$\begin{aligned} (f\gamma)' &= \left(\frac{1}{\cos(t) + i \sin(t)} \right)' \\ &= (e^{-it})' \\ &= -ie^{-it} \end{aligned} \tag{1}$$

$$\begin{aligned} f'(\gamma(t))\gamma'(t) &= -e^{-2it}ie^{it} \\ &= -ie^{-it} \end{aligned} \tag{2}$$

(1)=(2) so $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(iii) First we will find f'

$$f'(z) = \left(\sum z^n \right)' = \sum n z^{n-1}$$

Then we will plug in the path

$$f'(\gamma(t)) = \sum n(t + it^2)^{n-1}$$

We can easily find the derivative of the path

$$\gamma'(t) = 1 + 2it$$

Now we want to find the derivative of the composition

$$\begin{aligned} (f\gamma)' &= \left(\sum (t + it^2)^n \right)' \\ &= \sum n(t + it^2)^{n-1}(1 + 2it) \\ f'(\gamma(t))\gamma'(t) &= (1 + 2it) \sum n(t + it^2)^{n-1} \\ &= \sum n(t + it^2)^{n-1}(1 + 2it) \end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

- 15 Let's consider the series $s(z) = \sum a_n z^n, c(z) = \sum b_n z^n$. Let's take the derivative of $s(z)$ and $c(z)$:

$$s'(z) = \sum a_n n z^{n-1}$$

$$c'(z) = \sum b_n n z^{n-1}$$

We can then work out the coefficients because we know $s'(z) = c(z)$ and $c'(z) = -s(z)$:

$$\sum a_n n z^{n-1} = \sum b_{n-1} z^{n-1} \quad (1)$$

$$\sum b_n n z^{n-1} = \sum -a_{n-1} z^{n-1} \quad (2)$$

If we combine (1) and (2) to solve for a_n

$$\begin{aligned} \sum b_{n-1} (n-1) z^{n-2} &= \sum -a_{n-2} z^{n-2} \\ \implies b_{n-1} &= -a_{n-2} / (n-1) \\ \implies \sum a_n n z^{n-1} &= \sum -a_{n-2} / (n-1) z^{n-1} \\ \implies a_n &= -a_{n-2} / (n(n-1)) \end{aligned}$$

Then we solve for b_n

$$\begin{aligned} \sum a_{n-1} (n-1) z^{n-2} &= \sum b_{n-2} z^{n-2} \\ \implies a_{n-1} &= b_{n-2} / (n-1) \\ \implies b_n &= -a_{n-2} / (n(n-1)) \end{aligned}$$

Now we will assume that $s(0) = 0, c(0) = 1$. I claim that $s(z) = \sin(z)$, meaning even powers have a coefficient of 0. We have $a_0 = 0$

$$a_2 = -a_0 / (2 \cdot 1) = 0$$

$$\text{Let } a_{2n} = 0 \implies a_{2n+2} = -a_{2n} / (2n(2n-1)) = 0$$

And the odd terms will give us $a_{2n+1} = (-1)^n / (2n+1)!$. We have $b_0 = 1$ so

$$\begin{aligned} a_1 &= b_0 = 1 \\ a_3 &= -a_1 / (3 \cdot 2) = (-1)/3! \\ \text{Let } a_{2n+1} &= (-1)^n / (2n+1)! \\ \implies a_{2(n+1)+1} &= \frac{-(-1)^n}{(2n+1)!(2n+3)(2n+2)} \\ &= \frac{(-1)^{n+1}}{(2(n+1)+1)!} \end{aligned}$$

Which gives the sum:

$$\sum \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sin(z)$$

so $s'(z) = \cos(z) = c(z)$ thus $c^2(z) + s^2(z) = 1$

2 5.10 Problems: 1, 5, 15

1 (i)

$$e^i = \cos(1) + i \sin(1)$$

(ii)

$$e^{2+i\pi} = e^2 \cos(\pi) + e^2 \sin(\pi)$$

(iii)

$$e^{-2-i\pi} = e^{-2} \cos(-\pi) + e^{-2} \sin(-\pi)$$

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$$\begin{aligned} \cos(\theta + \phi) + i \sin(\theta + \phi) &= e^{i(\theta + \phi)} \\ &= e^{i\theta} e^{i\phi} \\ &= (\cos(\theta) + i \sin(\theta))(\cos(\phi) + i \sin(\phi)) \\ &= \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi) + i(\cos(\theta) \sin(\phi) + \cos(\phi) \sin(\theta)) \end{aligned}$$

For the next relation:

$$\begin{aligned} \frac{1}{\cos(\theta) + i \sin(\theta)} &= e^{-i\theta} \\ &= \cos(-\theta) + i \sin(-\theta) \\ &= \cos(\theta) - i \sin(\theta) \end{aligned}$$

15 (i)

$$\begin{aligned} \sum_{m=0}^n \cos(mx) + i \sin(mx) &= \sum_{m=0}^n e^{imx} \\ &= \sum_{m=0}^n (e^{ix})^m \\ &= \frac{1 - e^{ix(n+1)}}{1 - e^{ix}} \\ &= \frac{1 - \cos(x(n+1)) - i \sin(x(n+1))}{1 - \cos(x) - i \sin(x)} \\ &= \end{aligned}$$