1 Problems 2, 4, 6, 7, 8

2 Let us consider the discrete metric on a set X, for each x, y in X:

$$d(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

The first three axioms follow immediately from the definition. Now suppose we have 3 points x,y,z that are in X. If $x = z$ then

$$d(x,z) = 0 \leq d(x,y) + d(y,z)$$

The righthand side is always ≥ 0 by definition. If $x \neq z$ then $d(x,z) = 1$. The point y is equal to either one or neither of x, z.

$$d(x,y) + d(y,z) = \begin{cases} 0, & x \neq y \neq z \\ 1, & x = y \\ 1, & y = z \end{cases} \geq 1 = d(x,z)$$

An open ball in this space is

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

But what if $r = 1$? Then $B(x,1) = \{x\}$. If we create a set S and take some element x then $B(x,1) \subset S$ so it is open. Consider the complement of S, it is a subset of X so it is open thus S is closed.

4 Let us consider the interval (0,1]. The point 1 is in this interval but it is not an interior point, so it is not open. The point 0 is not in this set but it also is an adherent point, so it is not closed.

6 We need it to be the case that $int(Y) = \bigcup S_i$ and each open $S_i \subset Y$. If we take any $x \in S_i$, we know there exist $\epsilon > 0$ where $B(x,\epsilon) \subset S_i$, because it is open. Hence $B(x,\epsilon) \subset int(Y)$ and $x \in int(Y)$. So $\bigcup S_i \subset int(Y)$.
Let us now take some $x \in int(Y)$, which has an open ball $B(x,\epsilon) \subset int(Y)$. An open set and hence in $\bigcup S_i$. Thus they are equal.

7 By definition we know that $Y \subset \cap S_i$, each subset is closed so all points adherent to Y are adherent to a set in the union. So the closure is a subset of the union. For a contradiction let's assume that there is some $s \in \cap S_i$ where s is not adherent to Y. So for some $\epsilon > 0$ we get $B(s,\epsilon) \cap Y = \emptyset$. Then we can create a closed set in X (most notable the closure of Y) that contains Y and does not contain s, hence it cannot be in the union. A contradiction so $\cap S_i = \overline{Y}$.

8 Take some open ball $B(x,r)$ and some adherent point y. If $d(x,y) > r$ then we can create $B(y, d(x,y) - r) \cap B(x,r) = \emptyset$. So we know that if a point is adherent to y then $d(x,y) \leq r$, the definition of a closed ball.