1. Let us consider the function $g(z) = f(z)f(-z)$. If z were to tend to zero from the positive direction then we would get

$$\begin{aligned}\lim_{z\to 0^+} g(z) &= \lim_{z\to 0^+} f(z)f(-z)\\ &\leq \lim_{z\to 0^+} 2\cdot 3\\ &= 6\end{aligned}$$

Obviously $g(0) = f(0)^2$ thus $|f(0)| \leq \sqrt{6}$

2. Consider some $z = x + iy$, let us find the maximum and minimum of the function.

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can then see that the modulus of the function is

$$|e^z| = \left|e^x e^{iy}\right| = e^x$$

This means that the maximum of the modulus is the rightmost point in the set. This must be on the boundary. If we had a right most point inside the boundary then we could simply move right until we reach the boundary. The minimum is when we have the greatest negative value of x. This must also be on the boundary by the same logic.

3. First we shall factor to get

$$f(z) = z(z-1)$$

The modulus:

$$|z(z-1)| = |z||z-1|$$

This would mean that the maximum is when both moduli are at a maximum. At all point on the boundary of the disk $|z|$ is a maximum. But only at the point farthest from 1 for $|z-1|$ to be at a maximum. Thus the point $z_0 = -1$ and $|f(-1)| = |-1||-1-1| = 2$. The minimum is 0 and this is at points $z_0 = 0, 1$.

4. Assume that the polynomial $p(z)$ is non constant and differentiable. Consider a closed disk D with radius r centered at the origin. By the maximum modulus theorem we know that if $|p(z_0)| \geq |p(z)|$ for all $z \in D$ then z_0 must be on the boundary. If we increase r then $|z_0| = r$ will also increase. This means that as $|z_0| \to \infty$ then $|p(z_0)| \to \infty$. Hence we can choose some r such that

$$\forall z \in \mathbb{C} - D, |p(z)| > \max_D |p|$$

The minimum then must be insides of D, which by the minimum modulus theorem means they are zeros.

5. Consider the function $f(z) = 1 + z$, in a disk D around the origin with radius a. We know that the maximum must be on the boundary because it is not constant. The modulus is

$$|f(z)| = \sqrt{(1+x)^2 + y^2}$$

It must be that

$$y^2 = a^2 - x^2$$

so

$$\sqrt{(1+x)^2 + a^2 - x^2} = \sqrt{1 + 2x + a^2}$$

which is clearly a maximum when $x = a$.