1 Monday 4/5/2021

$\mathbb{R}$ is a totally ordered field satisfying the least upper bound axiom. $\mathbb{R}$ is complete. A Cauchy sequence is a sequence $\{x_j\}_{j=1}^{\infty}$ such that

$$\lim_{n,m\to\infty} d(x_n, x_m) \to 0$$

It is a sequence of points that crowd together and a complete metric space is one where every Cauchy sequence is convergent. The completeness axiom is that the real numbers have this property (this is the basis for $\epsilon - \delta$ proofs). If the space is incomplete then cauchy sequences will not have limits and there will be 'holes' in the space. In $\mathbb{Q}$there is not always a limit of a cauchy sequence.

A field has {+,-,/,x}, you also have identity elements such as 0 and 1. There are many examples of fields. A vector space requires a field, rational numbers are a field. The set of all functions can be a field. Obviously the real numbers. The complex numbers are a cool field as well.

$\mathbb{R}$has the operation $\leq$ which provides total order on the set. $\forall x, y$

$$\begin{aligned} x \leq y &\text{ or } y \leq x \\ x \leq y \leq x &\implies x = y \\ x \leq y \leq z &\implies x \leq z \end{aligned}$$

Addition and multiplication both work over the inequality.

The Least Upper Bound Axiom: if S is a subset of $\mathbb{R}$ and $S \neq \emptyset$ and $\exists x \in \mathbb{R}, \forall s \in S, x \geq s$ then there is a supremum of S. $\sup S \in \bar{S}$.

$\mathbb{R}^n$ always has the usual metric.

Theorem 1.1 *Every convergent series is cauchy.*

Let $\epsilon > 0$ and N be such that $\forall n > N, d(x_n, x) < \frac{\epsilon}{2}$. Everything starts to get close to x as we get farther.

$$\forall n, m > N, d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq \epsilon$$

$\mathbb{Q}$.$\mathbb{E}$.$\mathbb{D}$.

Theorem 1.2 *If a cauchy sequence has a convergent subsequence then it is convergent.*

$\{x_j\}_{j=1}^{\infty}$ is a cauchy sequence. $\{x_{j_k}\}_{k=1}^{\infty}$ converges to x. Where the indexes are increasing. Suppose that $\epsilon > 0$ and let N be such that $\forall k > N$ we have $d(x_{j_k}, x) < \frac{\epsilon}{2}$. Let M be such that $\forall k, l > M$ then $d(x_k, x_l) < \frac{\epsilon}{2}$. Let K be $max(M, j_N)$. Suppose $k > K$, we want $d(x_k, x) < \epsilon$. Pick $x_l > k$ then $d(x_{j_l}, x) < \frac{\epsilon}{2}$. $d(x_k, x_{j_l}) < \frac{\epsilon}{2}$ by choice of M from cauchyness. SO $d(x_k, x) \leq d(x_k, x_{j_l}) + d(x_{j_l}, x) < \epsilon$
$\mathbb{Q}$.$\mathbb{E}$.$\mathbb{D}$.

Theorem 1.3 *A closed subset of a complete metric space is complete.*

Say X is complete, $A \subseteq X$, A is closed. 'Sequential' definition of closed. If $\{a_1, a_2, ..., a_n\}$ is a sequence of points in A, convergent in X $\implies$ convergent in A. If $\{a_1, a_2, ..., a_n\}$ is Cauchy then it converges in X and hence in A. (Cauchy does not depend on what space that you are in, just the distance between the points matters).
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Theorem 1.4 *If A is a complete subspace of any metric space X then A is closed.*

Use the Sequential definition of closed. Suppose that $\{a_1, a_2, ..., a_n\}$ is a sequence in A and converges to $x \in X$. Goal is $x \in A$. $\{a_1, a_2, ..., a_n\}$ is convergent in X $\implies$ Cauchy $\implies$ convergent in A. Convergent to $a \in A$. Limits are unique so $x = a \in A$.
$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$S is a set and X is a complete metric space {function S→X} can be turned into a complete metric space. Each 'point' is a function.

2 Wednesday 4/7/21

Theorem 2.1 *Baire Category Theorem. In a complete metric space X, any intersection of countably many dense open sets is dense*

(U is dense iff $\overline{u}$=x, open sets are 'big') $\mathbb{Q} \subseteq \mathbb{R}$ is dense not open. First we are going to look at some equivalent statements. In a complete metric space X, any intersection of countably many nowhere dense sets has empty interior. **nowhere dense** iff $int(\overline{E}) = \emptyset$. Compare Cantors famous fact that $[0, 1] \neq \{e_1, e_2, ..., e_n\}$ (not countable) and $E_i = e_i$. We want $x \in [0, 1] \ \cup E_i$. X is a decimal using only two digits, say 1 and 3 (arbitrary). eg $x = 0.113131313131 31$ such that nth digit of x $\neq$ nth digit of e_n.
Proof 2 more similar to general Baire Categroy Theorem. Contrust closed balls 'converging' to 0.131331. Nested

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq ...$$
$$B_n \cap E_n = \emptyset$$

The diameters go to 0. We know the intersection is a single of x. and $x \in B_n \implies x \notin E_n$. A nowhere dense set in 2D space is a line and you cannot fill in the plane with a countable set of lines.

2.1 Products

There are three popular ways to put a metric on $X \times Y = \{(x, y) : x \in X, y \in Y\}$. We can construct a metric $d_{X\times Y}((x, y), (x', y'))$.

$$d((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$$
$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$
$$d((x, y), (x', y')) = \max(d_X(x, x'), d_Y(y, y'))$$

Whatever metric is chosen needs to satisfy the axioms of a metric space. The triangle inequality is often the most difficult to prove.
Now let's look at $\mathbb{R} \times \mathbb{R}$ with these 3 metrics. what do the open balls look like?

- Pythagoras:
  We will just get a classic open ball centered around the point.
- Max Metric:
  This will give us a square of side length r.
- Manhatten Metric:
  This will give us a diamond shape where the distance from the center to a vertex is r.

Property 4.4 A sequence $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$ that converges to (x, y) iff $x_n \to x$ in X and $y_n \to y$ in Y.