11 Suppose that Cauchy's estimate is an equality. Let's create a disk around the origin of radius $r > 0$. We know that the maximum is either on a boundary or at a zero. Using Cauchy's estimate we get

$$|f^{(0)}(0)| = |f(0)| = M(0!)/r^0 = M$$

This means that the origin is a maximum and thus a zero. So the maximum of the entire function must be 0. Thus $f(z) = 0$

12 For the domain D we have a fixed center z_0 and a radius r. ∂D can be written as the path $\partial D(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$. If we integrate $f(\partial D(t))$ we get the sum of all $f(z)$ along the path so to get the average we can divide by the length of the path giving us:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\partial D(t))dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

We can then do a substitution with $z = z_0 + re^{it}$ and $dz = ire^{it}dt$.

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z)}{ire^{it}}dt &= \frac{1}{2i\pi}\int_0^{2\pi} \frac{f(z)}{re^{it}}dz \\ &= \frac{1}{2i\pi}\int_{\partial D} \frac{f(z)}{z - z_0}dz \end{aligned}$$

by Cauchy's integral formula we can see

$$\frac{1}{2i\pi}\int_{\partial D} \frac{f(z)}{z - z_0}dz = f(z_0)$$

13 Suppose we have a domain D with radius $r > 0$. The maximum would be $|f(z)| \leq Kr^c$. Then using Cauchy's estimate for $n = c$:

$$|f^{(c)}(z)| \leq \frac{Kr^c(c)!}{r^c} = Kc!$$

This means that $f^{(c)}(z) = w$ and w is constant. Let's find the antiderivative of $f^{(c)}(z)$, c times:

$$\begin{aligned} f^{(c-1)}(z) &= wz + l_0 (\text{l is a constant of integer}) \\ f^{(c-2)}(z_0) &= \frac{wz^2}{2!} + l_0 z + l_1 \\ f^{(c-(c-1))}(z_0) &= \frac{wz^{c-1}}{(c-1)!} + l_0\frac{z^{c-2}}{(c-2)!} + l_1\frac{z^{c-3}}{(c-3)!} + ... \\ f(z_0) &= \frac{wz^c}{(c)!} + l_0\frac{z^{c-1}}{(c-1)!} + l_1\frac{z^{c-2}}{(c-2)!} + ... \end{aligned}$$

This gives us a polynomial of degree $\leq c$

14 Let's consider the taylor expansion of f, g centered at 0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)z^n}{n!}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)z^n}{n!}$$

We also know that

$$f^{(n)}(0) = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

if we choose a small enough r that would mean

$$\frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{g(z)}{(z-z_0)^{n+1}} dz \right|$$

Thus:

$$\begin{aligned} f^{(n)}(0) &= g^{(n)}(0) \\ \implies f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)z^n}{n!} \\ &= g(z) \end{aligned}$$

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