7.9 Problems: 1,2,3,4,8,9,19

1. $\arg(1+i) = \pi/4$
   $\arg(1/2+\sqrt(3)/2) = 1.047$
   $\arg((1+i)^3) = 3\arg(1+i) = 3\pi/4$
   $\arg((1/2+\sqrt(3)/2)^2 43) = 243\arg(1/2+\sqrt(3)/2) = 254.42$
   $\arg((1+i)^2(1/2+\sqrt(3)/2)^3) = 2\arg(1+i)3\arg(1/2+\sqrt(3)/2) = 2*\pi/4*3*1.047 = 4.934$

2. (i) If we can make the argument continuous in this range we can simplify the expression. We can write the limit instead as $\lim_{y\to0^+}\arg(x+iy)$. We have $y>0$ so we can work in the complex plane $\mathbb{C}_{y>0}$ with this restriction. The function $\cos^{-1}:(-1,1)\to(0,\pi)$ is continuous on this domain.

$$\begin{aligned}\lim_{y\to0^+}\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) &= \lim_{y\to0^+}\cos^{-1}\left(\frac{x}{\sqrt{x^2}}\right)\\ &= \lim_{y\to0^+}\cos^{-1}\left(\frac{x}{|x|}\right)\\ &= \lim_{y\to0^+}\cos^{-1}(-1)\end{aligned}$$

   As $x/|x|\to-1$ the function $\cos^{-1}(x/|x|)\to\pi$ because it is continous.

   (ii) We can write the limit as $\lim_{y\to0^+}\arg(x-iy)$. With $y<0$ we can work in the domain $\mathbb{C}_{y<0}$. In this plane this function is continous:

$$\cos^{-1}:(-1,1)\to(-\pi,0)$$

   So we can find the limit:

$$\begin{aligned}\lim_{y\to0^+}\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) &= \lim_{y\to0^+}\cos^{-1}\left(\frac{x}{\sqrt{x^2}}\right)\\ &= \lim_{y\to0^+}\cos^{-1}\left(\frac{x}{|x|}\right)\\ &= \lim_{y\to0^+}\cos^{-1}(-1)\end{aligned}$$

   As $x/|x|\to-1$ the function $\cos^{-1}(x/|x|)\to-\pi$ because it is continous.

3.

$$
\begin{aligned}
&\log(3i) = \log(3) + i\pi/2 \\
&\log(-2i) = \log(2) + i3\pi/2 \\
&\log(1+i) = \log(\sqrt(2)) + i\pi/4 \\
&\log(-1) = i\pi \\
&\log((e^{i\pi/3})^{10}) = 10(\log|e^{i\pi/3}| + i\pi/3) = 10i\pi/3 \\
&\log(x) = \begin{cases} x, & \text{if } 0 < x \\ \log(-x) + i\pi, & \text{if } 0 > x \end{cases}
\end{aligned}
$$

4.

$$
\begin{aligned}
\mathrm{Log}(z_1 z_2) &= \log|z_1 z_2| + i\arg(z_1 z_2) \\
&= \log|z_1| + i\arg(z_1) + 2iq\pi + \log|z_2| + i\arg(z_2) + 2ip\pi \\
&= \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2(q+p)i\pi
\end{aligned}
$$

We can let q+p=n to satisfy the equation. n can be any integer.

$$
\begin{aligned}
\mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2i(q+r)\pi &= \mathrm{Log}(z_1) + 2qi\pi + \mathrm{Log}(z_2) + 2ip\pi \\
&= \log(z_1) + \log(z_2)
\end{aligned}
$$

8 (i) Let's find all values of n such that $(re^{i\theta})^n = 1$. suppose that $r = e^a$

$$
\begin{aligned}
\log(1) &= \log(re^{in\theta}) \\
2im\pi &= \log(r) + in\theta + 2ip\pi \\
2im\pi - 2ip\pi &= 0 + in\theta \\
2\pi(m-p)/n &= \theta
\end{aligned}
$$

So it will the nth root of unity if $\theta = \frac{2im\pi}{n}$

(ii) First we will consider $n = 2$.

$$
\begin{aligned}
\theta &= 2im\pi/2 \\
&\implies \cos(m\pi) + i\sin(m\pi) \\
&\implies (-1)^m
\end{aligned}
$$

Next $n = 3$.

$$
\begin{aligned}
\theta &= 2im\pi/3 \\
&\implies \cos(2m\pi/3) + i\sin(2m\pi/3)
\end{aligned}
$$

So we have $(-.5 - 0.866i), (-0.5 + 0.866i), (1)$ Finally $n = 4$

$$\begin{aligned}\theta &= 2im\pi/4 = im\pi/2 \\ &\implies \cos(m\pi/2) + i\sin(m\pi/2)\end{aligned}$$

This gives us $1, i, -1, -i$

(iii) Suppose that ω_1, ω_2 are the nth roots of unity.

$$\begin{aligned}((\omega_1)^m)^n &= ((\omega_1)^n)^m = 1^m = 1 \\ (\omega_1 \cdot \omega_2)^n &= \omega_1^n \cdot \omega_2^n = 1 \\ \left(\frac{\omega_1}{\omega_2}\right)^n &= \frac{\omega_1^n}{\omega_2^n} = 1\end{aligned}$$

(iv) Let's suppose that $z = de^{i\phi}$

$$\begin{aligned}z^n &= re^{i\theta} \\ d^n e^{in\phi} &= re^{i\theta} \\ \log(d^n) + in\phi + 2im\pi &= \log(r) + i\theta + 2ip\pi \\ n\log(d) + in\phi &= \log(r) + i\theta + 2i(p-m)\pi \\ \log(d) + i\phi &= (\log(r) + i\theta + 2i(p-m)\pi)/n \\ z &= re^{i(\theta + 2\pi(p-m))/n}\end{aligned}$$

(v) Assume that $z_1^n = z_2^n$.

$$\begin{aligned}\log(z_1^n) &= \log(z_2^n) \\ \log(z_1^n) &= \log(z_2^n) \\ n\log|z_1| + in\arg(z_1) + 2im\pi &= n\log|z_1| + in\arg(z_2) + 2ip\pi \\ \log|z_1| + i\arg(z_1) &= \log|z_2| + i\arg(z_2) + 2i(p-m)\pi/n \\ z_1 = z_2 e^{2i(p-m)\pi/n}\end{aligned}$$

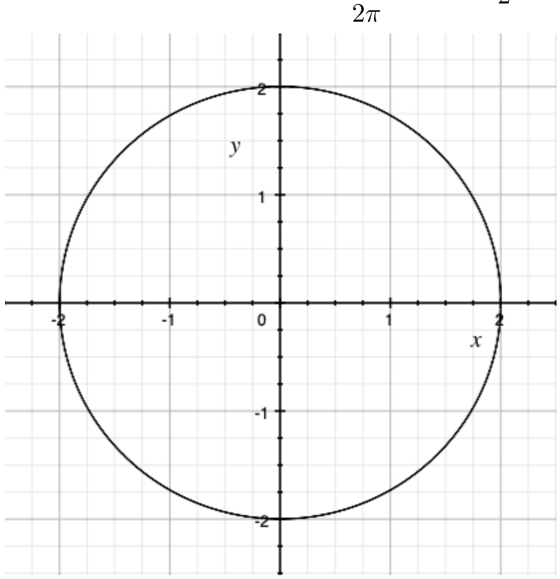
$e^{2i(p-m)\pi/n}$ is the nth root of unity.

9

$$\begin{aligned}
1^{\sqrt{2}} &= e^{\log(1^{\sqrt{2}})} \\
&= e^{\sqrt{2}\log(1)} = 1 \\
(-2)^{\sqrt{2}} &= e^{\sqrt{2}\log(-2)} \\
&= e^{\sqrt{2}\log(2)} e^{i\sqrt{2}\pi)} \\
&= e^{\sqrt{2}\log(2)}(\cos(\sqrt{2}\pi) + i\sin(\sqrt{2}\pi)) \\
i^i &= e^{i\log(i)} \\
&= e^{i(\log|i| + i\arg(i))} \\
&= e^{-\pi/2} \\
2^i &= e^{\log(2^i)} \\
&= e^{i\log(2)} \\
&= \cos(\log(2)) + i\sin(\log(2)) \\
(3-4i)^{1+i} &= e^{(1+i)(\log(5) + i\arg(3-4i))} \\
&= e^{\log(5) - \arg(3-4i) + i\arg(3-4i) + i\log(5)} \\
&= e^{\log(5) - \arg(3-4i)}(\cos(\arg(3-4i) + \log(5)) + i\sin(\arg(3-4i) + \log(5))) \\
(3+4i)^5 &= e^{5(\log(5) + i\arg(3+4i))} \\
&= e^{5\log(5)}(\cos(\arg(3+4i)) + i\sin(\arg(3+4i)))
\end{aligned}$$

19 (i) The choice of argument will be $\theta(t) = -t$. Thus the winding number is

$$\frac{\theta(4\pi) - \theta(0)}{2\pi} = -2$$

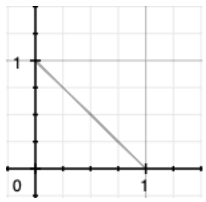


(ii) We will use

$$\cos^{-1}\left(\frac{t}{\sqrt{t^2+(1-t)^2}}\right)$$

as the the continous choice of argument. Then we get the winding number

$$\frac{\cos^{-1}(0)-\cos^{-1}(1)}{2\pi}=1/4$$

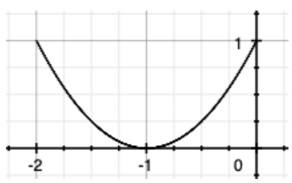


(iii) The choice of argument will be

$$\cos^{-1}\left(\frac{t-1}{\sqrt{(t-1)^2+t^4}}\right)$$

So we get the winding number:

$$\frac{\cos^{-1}\left(-2/\sqrt{5}\right)-\cos^{-1}(0)}{2\pi}=\frac{2.67794504459-1.5707963267}{2\pi}=.176$$



(iv) The continous choice of argument will be

$$\begin{cases}\cos^{-1}\left(\frac{t}{\sqrt{t^2+(1-t)^2}}\right), & \text{if } t\in[0,1]\\ \cos^{-1}\left(\frac{1}{\sqrt{1+(t-1)^2}}\right), & \text{if } t\in[1,2]\end{cases}$$

So we end up with the winding number

$$\frac{\pi/2-\pi/4}{2\pi}=1/8$$