1 3.6 Problems: 6, 8, 12, 15

6 (i) $\sum z^n/n$
We can use this equation to find the radius of convergence.

$$\begin{aligned}
1/R &= \limsup |1/n|^{1/n} \\
&= \limsup e^{1/n \ln(1/n)} \\
&= \limsup e^0 \\
&= 1 \\
\implies R &= 1
\end{aligned}$$

(ii) $\sum z^n/n!$

$$\begin{aligned}
1/R &= \limsup |1/n!|^{1/n} \\
&= \limsup e^{ln(1/n!)/n} \\
&= \limsup e^{-ln(n!)/n} \\
&= 0 \\
\implies R &= \infty
\end{aligned}$$

(iii) $\sum n!z^n$

$$\begin{aligned}
1/R &= \limsup |n!|^{1/n} \\
&= \limsup e^{\ln(n!)/n} \\
&= e^{\infty} \\
\implies R &= 0
\end{aligned}$$

(iv) $\sum n^k z^n$

$$\begin{aligned}
1/R &= \limsup |n^k|^{1/n} \\
&= \limsup |n|^{k/n} \\
&= \limsup e^{k\ln(n)/n} \\
&= e^{k \limsup \ln(n)/n} \\
&= e^{k\cdot 0} \\
&= 1 \\
\implies R &= 1
\end{aligned}$$

(v) $\sum z^{n!}$
Consider the series $\sum z^n$. It has radius of convergence $R = 1$. If we let $|z| < 1$ then clearly $z^{n!} \leq z^n$, so the series will converge. If we let $|z| > 1$ then we get $z^{n!} \geq z^n$, which means that the series diverges.

8 (i) first we shall create a formula for this series.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}$$

now we can try to find a radius of convergence.

$$\begin{aligned} 1/R &= \limsup |1/(2n-1)!|^{1/n} \\ &= \limsup e^{-\ln((2n-1)!)/n} \\ &= e^{-\infty} \\ &= 0 \\ \implies R &= \infty \end{aligned}$$

(ii)

$$\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

The radius of convergence would be given by

$$\begin{aligned} 1/R &= \limsup |1/(2n)!|^{1/n} \\ &= 0 \\ \implies R &= \infty \end{aligned}$$

(iii) We will use the ratio test to see if the series converges. First let's assume that a is a non negative integer. Then at some point the terms will become 0 because there exists some n such that $a-n+1=0$. Now we will use the ratio test to check all other cases

$$\begin{aligned} &\lim \left|\frac{(n-1)!a(a-1)\dots(a-n+2)(a-n+1)}{n!a(a-1)\dots(a-n+2)}z\right| \\ &= \lim \left|\frac{(a-n+1)}{n}z\right| \\ &= \lim |z|\left|\frac{a+1}{n}-1\right| \\ &= |z| \end{aligned}$$

so the series will converge when $|z|<1$ for all other values of a.

12 Assume that for $\sum a_n z^n, \sum b_n z^n$, and $\sum a_n b_n z^n$ we have $R=1$. let's find the radius of convergence for $\sum a_n^2 b_n z^n$:

$$\begin{aligned} 1/R &= \limsup |a_n^2 b_n|^{1/n} && (1) \\ &= \limsup |a_n b_n|^{1/n} \cdot \limsup |a_n|^{1/n} && (2) \\ &= 1 \cdot 1 && (3) \end{aligned}$$

The implication (2) $\implies$ (3) is true because each term is $1/R$ for the series mentioned above. If we switch a_n and b_n in this proof then it proves the other case. Thus $\sum a_n^2 b_n z^n, \sum a_n b_n^2 z^n$ have radius of convergence 1.

15 We will assume that $|z| < 1$ and $\sum_{n=0}^{\infty} a_n z^n = p(z)/q(z)$. Let's create k different sequences with constant coefficients. Each of these series is a convergent geometric series. Geometric series converge:

$$a_0 \sum_{n=0}^{\infty} z^{nk} = \frac{a_0}{1-z}$$
$$a_1 \sum_{n=0}^{\infty} z^{1+nk} = \frac{a_1}{1-z}$$
$$\vdots$$
$$a_{k-1} \sum_{n=0} z^{nk+k-1} = \frac{a_{nk+k-1}}{1-z}$$

When we add these all back together we get a rational function.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{k} \frac{a_n}{1-z} \\
&= \sum_{n=0}^{k} \frac{a_n + a_n z}{1-z^2} = p(z)/q(z) \\
p(z) &= \sum_{n=0}^{k} a_n + a_n z && \text{(some constant)} \\
q(z) &= 1-z
\end{aligned}$$

Now we must figure out where the zeros are for p, q.

$$\begin{aligned}
p(z) &= 1-z = 0 \\
&\implies z = 1
\end{aligned}$$

$$\begin{aligned}
q(z) = 0 &= \sum_{n=0}^{k} a_n(1+z) \\
&= (1+z) \sum_{n=0}^{k} a_n \\
\implies z &= -1
\end{aligned}$$

So the zeros are on the unit circle. Now let's try a slight variation to this problem. Instead of $a_n = a_{n+k}$ let's let $a_{n+k} = a_n/k$. We can still split it into different series:

$$a_0 \sum_{n=0}^{\infty} \frac{z^{nk}}{k^n}$$
$$a_1 \sum_{n=0}^{\infty} \frac{z^{1+nk}}{k^n}$$
$$\vdots$$
$$a_{k-1} \sum_{n=0} \frac{z^{nk+k-1}}{k^n}$$

Each of these can actually be manipulated into geometric series.

$$\begin{aligned} a_{k-1} \sum_{n=0}^{\infty} \frac{z^{nk+k-1}}{k^n} &= a_{k-1} \sum_{n=0}^{\infty} \frac{z^{nk} z^{k-1}}{k^n} \\ &= a_{k-1} z^{k-1} \sum_{n=0}^{\infty} \left(\frac{z^k}{k}\right)^n \\ &= \frac{a_{k-1} z^{k-1}}{1-(z^k/k)} \end{aligned}$$

They are convergent for all z such that $|z^k/k| < 1$

2 4.7 Problems: 2, 4, 6, 7, 9

2 Find the derivative

(i) $f(z) = z^2 + 2z$

$$f'(z) = 2z + 2$$

(ii) $f(z) = 1/z$
We will use the quotient rule. let $h(z) = 1, g(z) = z$ which would mean $h'(z) = 0, g'(z) = 1$

$$\begin{aligned} \left(\frac{h(z)}{g(z)}\right)' &= \frac{h'(z)g(z) - h(z)g'(z)}{g^2(z)} \\ &= \frac{-1}{z^2} \end{aligned}$$

(iii) $f(z) = z^3 + z^2$

$$f'(z) = 3z^2 + 2z$$

4 Let
$$f_n(z) = \left(1 + \frac{z}{n}\right)^n$$
We will solve using induction. Base Case:
$$f_2(z) = (1 + z/2)^2 \implies f_2'(z) = 1 + z/2$$
which matches $f_2'(z) = f_1(z/2) = 1 + z/2$. Now we can move onto induction by letting
$$f_n'(z) = f_{n-1}\left(\frac{(n-1)z}{n}\right)$$
We need to get
$$f_{n+1}'(z) = f_n\left(\frac{nz}{n+1}\right) = \left(1 + \frac{z}{n+1}\right)^n$$
But we have
$$f_{n+1}'(z) = (n+1)\left(1 + \frac{z}{n+1}\right)^n * \frac{1}{n+1}$$
$$= \left(1 + \frac{z}{n+1}\right)^n$$
So the property holds.

6 Suppose we have a polynomial $f(z)$. If we take the the conjugate of it we get
$$\overline{f(\bar{z})} = \overline{\sum a_n \bar{z}^n}$$
$$= \sum \overline{a_n \bar{z}^n}$$
$$= \sum a_n z^n$$
$$= f(z)$$
So the derivative of g would be $g'(z) = \sum a_n n z^{n-1}$. Now we consider $h(z) = \overline{f(z)}$. Suppose $f'(0) \neq 0$. Let's let $f(z) = z$. If we choose a real z_0 then
$$\lim \frac{\bar{z} - \bar{z_0}}{z - z_0} = 1$$
But if we instead choose an imaginary z_0 we get
$$\lim \frac{-z + z_0}{z - z_0} = -1$$
Now let's consider when $f'(0) = 0$
$$\lim_{z \to 0} \frac{\overline{\sum a_n z^n}}{z} = \frac{\sum a_n \bar{z}^n}{z} = 0$$

So it is differentiable at the origin when $f'(0) = 0$

7 (a)

$$\begin{aligned} f(z) &= \frac{1}{x+iy} \\ &= \frac{x-iy}{x^2-y^2} \\ &= \frac{x}{x^2-y^2} - i\frac{y}{x^2-y^2} \end{aligned}$$

Now we need to find the partial derivatives

$$\begin{aligned} u(x,y) &= \frac{x}{x^2-y^2} \\ \frac{\partial u}{\partial x} &= \frac{y^2+x^2}{2x^2y^2-x^4-y^4} \\ \frac{\partial u}{\partial y} &= \frac{2yx}{x^4-2x^2y^2+y^4} \\ v(x,y) &= \frac{y}{x^2-y^2} \\ \frac{\partial v}{\partial x} &= \frac{-2yx}{x^4-2x^2y^2+y^4} \\ \frac{\partial v}{\partial y} &= \frac{y^2+x^2}{2x^2y^2-x^4-y^4} \end{aligned}$$

This satisfies the Cauchy-Riemann equation.

(b)

$$\begin{aligned} f(z) = |z| &= \sqrt{x^2+y^2} \\ u(x,y) &= \sqrt{x^2+y^2} \\ \frac{\partial u}{\partial x} &= \frac{2x}{\sqrt{x^2+y^2}} \qquad (1) \\ \frac{\partial u}{\partial y} &= \frac{2y}{\sqrt{x^2+y^2}} \qquad (2) \\ v(x,y) &= 0 \end{aligned}$$

The partial derivatives of $v(x,y)$ are both zero. If we want (1)=0 then we need x=0. To have (2)=0 we need y=0. But when we let x,y=0 neither function is defined. So it is never differentiable.

(c)

$$
\begin{aligned}
f(z) &= \bar{z} = x - iy \\
u(x,y) &= x \\
\frac{\partial u}{\partial x} &= 1 \\
\frac{\partial u}{\partial y} &= 0 \\
v(x,y) &= y \\
\frac{\partial v}{\partial x} &= 0 \\
\frac{\partial v}{\partial y} &= 1
\end{aligned}
$$

This does not satisfy the Cauchy-Riemann equation anywhere.

9

$$
\begin{aligned}
f(z) &= \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} \\
\frac{\partial u}{\partial x} &= \frac{3x^4 + 3x^2y^2 - 2x^4 + 2xy^3}{(x^2 + y^2)^2} \\
\frac{\partial u}{\partial y} &= \frac{-3y^2(x^2 + y^2) - 2y(x^3 - y^3)}{(x^2 + y^2)^2} \\
\frac{\partial v}{\partial x} &= \frac{3x^2(x^2 + y^2) - 2x(x^3 + y^3)}{(x^2 + y^2)^2} \\
\frac{\partial v}{\partial y} &= \frac{3y^2x^2 + 3y^4 - 2yx^3 - 2y^4}{(x^2 + y^2)^2}
\end{aligned}
$$

These satisfy the Cauchy-Reimann equations because they approach 0 at the origin but they are not actually defined there.