1 4.7 Problems: 11, 12, 15

11 We will consider the equation:

$$f(z) = \frac{xy^2(x+iy)}{x^2+x^4}$$
$$f(0) = 0$$

Let us take the limit when $z \to 0$ along any line $z = (a+bi)t$

$$\begin{aligned}
\lim_{t\to 0} \frac{f((a+ib)t)}{(a+ib)t} &= \frac{(at)(bt)^2(a+ib)t}{((at)^2+(at)^4)(a+ib)t} \\
&= \frac{ab^2t^3}{a^2t^2+a^4t^4} \\
&= \frac{ab^2t^1}{a^2+a^4t^2} \\
&= 0
\end{aligned}$$

So the limit goes to zero when $z \to 0$ along a straight path. But what if we take a different path to 0. We will instead let $z(t) = t^2 + it$ as $z \to 0$.

$$\begin{aligned}
\lim_{t\to 0} \frac{f(t^2+it)}{t^2+it} &= \frac{t^2t^2(t^2+it)}{(t^4+t^8)(t^2+it)} \\
&= \frac{t^4}{t^4+t^8} \\
&= \frac{1}{1+t^4} \\
&= 1
\end{aligned}$$

12 (i) First we will find f'

$$f'(z) = (z^2)' = 2z$$

Then we will plug in the path

$$f'(\gamma(t)) = 2(t^3+it^4)$$

We can easily find the derivative of the path

$$\gamma'(t) = 3t^2 + 4it^3$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)'(t) &= \left((t^3+it^4)^2\right)' \\
&= \left(t^6 - t^8 + 2it^7\right)' \\
&= -8t^7 + 14it^6 + 6t^5 \\
f'(\gamma(t))\gamma'(t) &= 2(t^3+it^4)(3t^2+4it^3) \\
&= 2(3t^5 + 7it^6 - 4t^7) \\
&= -8t^7 + 14it^6 + 6t^5
\end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(ii) First we will find f'

$$f'(z) = (1/z)' = -1/z^2$$

Then we will plug in the path

$$f'(\gamma(t)) = -1/(\cos(t) + i\sin(t))^2 = -e^{-2it}$$

We can easily find the derivative of the path

$$\gamma'(t) = -\sin(t) + i\cos(t) = ie^{it}$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)' &= \left(\frac{1}{\cos(t) + i\sin(t)}\right)' \\
&= \left(e^{-it}\right)' \\
&= -ie^{-it} \qquad (1) \\
f'(\gamma(t))\gamma'(t) &= -e^{-2it}ie^{it} \\
&= -ie^{-it} \qquad (2)
\end{aligned}$$

(1)=(2) so $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(iii) First we will find f'

$$f'(z) = \left(\sum z^n\right)' = \sum nz^{n-1}$$

Then we will plug in the path

$$f'(\gamma(t)) = \sum n(t + it^2)^{n-1}$$

We can easily find the derivative of the path

$$\gamma'(t) = 1 + 2it$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)' &= \left(\sum (t + it^2)^n\right)' \\
&= \sum n(t + it^2)^{n-1}(1 + 2it) \\
f'(\gamma(t))\gamma'(t) &= (1 + 2it)\sum n(t + it^2)^{n-1} \\
&= \sum n(t + it^2)^{n-1}(1 + 2it)
\end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

15 Let's consider the series $s(z) = \sum a_n z^n, c(z) = \sum b_n, z^n$. Let's take the derivative of $s(z)$ and $c(z)$:

$$s'(z) = \sum a_n n z^{n-1}$$
$$c'(z) = \sum b_n n z^{n-1}$$

We can then work out the coefficients because we know $s'(z) = c(z)$ and $c'(z) = -s(z)$:

$$\sum a_n n z^{n-1} = \sum b_{n-1} z^{n-1} \tag{1}$$
$$\sum b_n n z^{n-1} = \sum -a_{n-1} z^{n-1} \tag{2}$$

If we combine (1) and (2) to solve for a_n

$$\begin{aligned} &\sum b_{n-1}(n-1)z^{n-2} = \sum -a_{n-2} z^{n-2} \\ &\implies b_{n-1} = -a_{n-2}/(n-1) \\ &\implies \sum a_n n z^{n-1} = \sum -a_{n-2}/(n-1) z^{n-1} \\ &\implies a_n = -a_{n-2}/(n(n-1)) \end{aligned}$$

Then we solve for b_n

$$\begin{aligned} &\sum a_{n-1}(n-1)z^{n-2} = \sum b_{n-2} z^{n-2} \\ &\implies a_{n-1} = b_{n-2}/(n-1) \\ &\implies b_n = -b_{n-2}/(n(n-1)) \end{aligned}$$

Now we will assume that $s(0) = 0, c(0) = 1$. I claim that $s(z) = \sin(z)$, meaning even powers have a coefficient of 0. We have $a_0 = 0$

$$\begin{aligned} a_2 &= -a_0/(2 \cdot 1) = 0 \\ \text{Let } a_{2n} = 0 \implies a_{2n+2} &= -a_{2n}/(2n(2n-1)) = 0 \end{aligned}$$

And the odd terms will give us $a_{2n+1} = (-1)^n/(2n+1)!$. We have $b_0 = 1$ so

$$\begin{aligned} a_1 &= b_0 = 1 \\ a_3 &= -a_1/(3 \cdot 2) = (-1)/3! \\ \text{Let } a_{2n+1} &= (-1)^n/(2n+1)! \\ \implies a_{2(n+1)+1} &= \frac{-(-1)^n}{(2n+1)!(2n+3)(2n+2)} \\ &= \frac{(-1)^{n+1}}{(2(n+1)+1)!} \end{aligned}$$

Which gives the sum:

$$\sum \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sin(z)$$

so $s'(z) = \cos(z) = c(z)$ thus $c^2(z) + s^2(z) = 1$

2 5.10 Problems: 1, 5, 15

1 (i)

$$e^i = \cos(1) + i\sin(1)$$

(ii)

$$e^{2+i\pi} = e^2\cos(\pi) + e^2\sin(\pi)$$

(iii)

$$e^{-2-i\pi} = e^{-2}\cos(-\pi) + e^{-2}\sin(-\pi)$$

5

$$\begin{aligned}
\cos(\theta+\phi) + i\sin(\theta+\phi) &= e^{i(\theta+\phi)} \\
&= e^{i\theta}e^{i\phi} \\
&= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\
&= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + i(\cos(\theta)\sin(\phi) + \cos(\phi)\sin(\theta))
\end{aligned}$$

For the next relation:

$$\begin{aligned}
\frac{1}{\cos(\theta) + i\sin(\theta)} &= e^{-i\theta} \\
&= \cos(-\theta) + i\sin(-\theta) \\
&= \cos(\theta) - i\sin(\theta)
\end{aligned}$$

15 (i)

$$\begin{aligned}
\sum_{m=0}^{n} \cos(mx) + i\sin(mx) &= \sum_{m=0}^{n} e^{imx} \\
&= \sum_{m=0}^{n} \left(e^{ix}\right)^m \\
&= \frac{1 - e^{ix(n+1)}}{1 - e^{ix}} \\
&= \frac{(1 - e^{ix(n+1)})(1 - e^{-ix})}{(1 - e^{ix})(1 - e^{-ix})} \\
&= \frac{1 - e^{-ix} - e^{ixn+ix} + e^{ixn}}{2 - 2\cos(x)} \\
&= \frac{1 - \cos(x) + i\sin(x) - \cos(xn+x) - i\sin(xn+x) + \cos(xn) + i\sin(xn)}{2 - 2\cos(x)}
\end{aligned}$$

This gives us

$$\sum_{m=0}^{n}\cos(mx) = \frac{1-\cos(x)-\cos(nx+x)+\cos(nx)}{2-2\cos(x)}$$

(ii) From the previous problem we have:

$$\sum_{m=0}^{n}\sin(mx) = \frac{\sin(x)+\sin(nx)-\sin(nx+x)}{2-2\cos(x)}$$

(iii)

$$\begin{aligned}\sum_{m=1}^{n}\cos((2m-1)x)+i\sin((2m-1)x) &= \sum_{m=0}^{n} e^{2ixm}e^{-ix}\\ &= e^{-ix}\sum_{m=0}^{n}\left(e^{i(2x)}\right)^m\\ &= e^{-ix}\frac{1-e^{-i2x}-e^{i2xn+i2x}+e^{i2xn}}{2-2\cos(2x)}+e^{-ix}\\ &= \frac{e^{-ix}-e^{-i3x}-e^{i2xn+ix}+e^{i2xn-ix}}{2-2\cos(2x)}+e^{-ix}\\ \sum_{m=0}^{n}\cos((2m-1)x) &= \frac{\cos(x)-\cos(3x)-\cos(2xn+x)+\cos(2xn-x)}{2-2\cos(2x)}+\cos(x)\end{aligned}$$

(iv) By the previous problem:

$$\sum_{m=1}^{n}\sin((2m-1)x) = \frac{-\sin(x)+\sin(3x)-\sin(2xn+x)+\sin(2xn-x)}{2-2\cos(2x)}-\sin(x)$$

(v) The problem uses $(-1)^n$ as a coefficient. This is constant unlike $(-1)^m$ so we can just multiply it by our previous answer.

$$\sum_{m=0}^{n}(-1)^n(\cos(mx)+i\sin(mx)) = (-1)^n\sum_{m=0}^{n}e^{imx}$$

$$\implies \sum_{m=0}^{n}(-1)^n\sin(mx) = (-1)^n\frac{\sin(x)+\sin(nx)-\sin(nx+x)}{2-2\cos(x)}$$

(vi)

$$\begin{aligned}\sum_{m=0}^{n}\cos(\theta+m\phi)+i\sin(\theta+m\phi) &= e^{i\theta}\sum_{m=0}^{n}e^{im\phi}\\ &= \frac{e^{i\theta}-e^{-i\phi+i\theta}-e^{in\phi+i\phi+i\theta}+e^{in\phi+i\theta}}{2-2\cos(\phi)}\\ \sum_{m=0}^{n}\cos(\theta+m\phi) &= \frac{\cos(\theta)-\cos(\theta-\phi)-\cos(n\phi+\phi+\theta)+\cos(n\phi+\theta)}{2-2\cos(\phi)}\end{aligned}$$

(vii) From the previous problem:

$$\sum_{m=0}^{n} \sin(\theta + m\phi) = \frac{\sin(\theta) - \sin(\theta - \phi) - \sin(n\phi + \phi + \theta) + \sin(n\phi + \theta)}{2 - 2\cos(\phi)}$$

3 6.14 Problems: 1, 2, 4, 5, 12, 13

1 We have $f(a, b) = a$ and $\gamma(t) = it$ for $t \in [0, 1]$

$$f(\gamma(t)) = 0$$

$$\int_{\gamma} re(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt = 0$$

$$\begin{aligned}
\sigma_1(t) = t &\implies \sigma_1'(t) = 1 \\
\sigma_2(t) = 1 - t + it &\implies \sigma_2'(t) = -1 + i \\
\int_{\sigma_1} f(\sigma_1(t))\sigma_1'(t)dt + \int_{\sigma_2} f(\sigma_2(t))\sigma_2'(t)dt &= \int_0^1 tdt + \int_0^1 (1-t)(i-1)dt \\
&= 1/2 + \int_0^1 i - it - 1 + tdt \\
&= 1/2 + i(1) - i(1)^2/2 - (1) + (1)^2/2 \\
&= i/2
\end{aligned}$$

2 We have the contour $\gamma(t) = it$ for $t \in [-1, 1]$. We will find the derivative $\gamma'(t) = i$. If we find the magnitude of the contour then $|\gamma(t)| = t$ so we can evaluate the integral

$$\int_{\gamma} |z|dz = \int_{-1}^{1} itdt = \frac{i(1)^2}{2} - \frac{i(-1)^2}{2} = 0$$

We can take the derivative of the second contour to get $\sigma'(t) = ie^{it}$. We also have $|\sigma(t)| = 1$ so we can evaluate the integral

$$\int_{\sigma} |\sigma(t)|\sigma'(t)dt = \int_{-\pi/2}^{\pi/2} ie^{it}dt = e^{i(\pi/2)} - e^{i(-\pi/2)} = i - (-i) = 2i$$

4 We must use the basic formula. Plugging in the contour we get $\frac{1}{z_0 + re^{it} - z_0} = e^{-it}/r$. And the derivative is $\gamma'(t) = ire^{it}$ Giving us the integral:

$$\int_{\gamma} \frac{ire^{it}}{re^{it}}dt = \int_0^{2n\pi} idt = 2in\pi$$

5 For all problems we have $\gamma(t) = e^{it}$ and $\gamma'(t) = ie^{it}$.

(i) We can solve the integral to get

$$\int_1^{-1} 1/z^2 dz = -1/(-1) - (-1/(1)) = 2$$

(ii) Plug in the contour

$$f(\gamma(t)) = e^{-it}$$

Then solve the integral

$$\int_0^{\pi} e^{-it} \cdot ie^{it} dt = i\pi$$

(iii) We will simply take the integral to get

$$\int_1^{-1} \cos(z) dz = \sin(-1) - \sin(1) = -2\sin(1)$$

(iv) Simply take the integral

$$\int_1^{-1} \sinh(z) dz = \cosh(-1) - \cosh(1) = -2\cosh(1)$$

(v) We will need a u-substitution

$$\int_1^{-1} \sin(z)/\cos(z) dz$$
$$u = \cos(z) \implies du = -\sin(z) dz$$
$$\int_1^{-1} \frac{1}{u} du = \ln(\cos(-1)) - \ln(\cos(1)) = -2\ln(\cos(1))$$

(vi) We will plug in the contour

$$f(e^{it}) = e^{(e^{it})^3} = e^{e^{3it}}$$

Now we integrate

$$\int_1^{-1} \sum_{n=0}^{\infty} \frac{z^{3n}}{n!} dz = \sum_{n=0}^{\infty} \int_1^{-1} \frac{z^{3n}}{n!}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}}{n!(3n-1)} - \sum_{n=1}^{\infty} \frac{(1)^{3n-1}}{n!(3n-1)}$$

12 (i) First we must find the length of the contour. Because it is a circle we can just find the perimeter and get $2\pi r$. Suppose we have some M such that $|f(z)| \leq M$. Given some $\epsilon > 0$ we will then let $\delta = \frac{\epsilon}{2M\pi}$

$$|r| < \delta \implies$$
$$\left| \int_{c_r} f(z) dz \right| \leq |2Mr\pi| < \epsilon$$

(ii) We will let the