11 Suppose that Cauchy's estimate is an equality. Let's create a disk around z_0 with radius $r > 0$. Using Cauchy's integral formula we get:

$$\begin{aligned}|f^{(n)}(z_0)| &= \frac{n!}{2\pi i}\left|\int_{C_r}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\ &= \frac{n!}{2\pi i}\int_{C_r}\left|\frac{f(z)}{(z-z_0)^{n+1}}\right|dz \\ &= \frac{Mn!}{r^n}\end{aligned}$$

The second equality only holds if $\frac{f(z)}{(z-z_0)^{n+1}}$ has a constant argument. We can let $z = z_0 + re^{it}$ and $dz = ire^{it}dt$. t therefore must be constant. giving us

$$\begin{aligned}\frac{n!}{2\pi}\int_{C_r}\left|\frac{f(z_0+re^{it})}{(re^{it})^n}\right|dt &= \frac{n!}{2\pi}\int_{C_r}\frac{|f(z_0+re^{it})|}{r^n}dt \\ &= \frac{n!|f(z_0+re^{it})|}{r^n}\end{aligned}$$

12 For the domain D we have a fixed center z_0 and a radius r. ∂D can be written as the path $\partial D(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$. If we integrate $f(\partial D(t))$ we get the sum of all $f(z)$ along the path so to get the average we can divide by the length of the path giving us:

$$\frac{1}{2\pi}\int_0^{2\pi} f(\partial D(t))dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$$

We can then do a substitution with $z = z_0 + re^{it}$ and $dz = ire^{it}dt$.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{f(z)}{ire^{it}}dt &= \frac{1}{2i\pi}\int_0^{2\pi}\frac{f(z)}{re^{it}}dz \\ &= \frac{1}{2i\pi}\int_{\partial D}\frac{f(z)}{z-z_0}dz\end{aligned}$$

by Cauchy's integral formula we can see

$$\frac{1}{2i\pi}\int_{\partial D}\frac{f(z)}{z-z_0}dz = f(z_0)$$

13 Suppose we have a domain D with radius $r > 0$. The maximum would be $|f(z)| \leq Kr^c$. Then using Cauchy's estimate for $n = c$:

$$|f^{(c)}(z)| \leq \frac{Kr^c(c)!}{r^c} = Kc!$$

This means that $f^{(c)}(z) = w$ and w is constant. Let's find the antiderivative of $f^{(c)}(z)$, c times:

$$
\begin{aligned}
f^{(c-1)}(z) &= wz + l_0 (\text{l is a constant of integer}) \\
f^{(c-2)}(z_0) &= \frac{wz^2}{2!} + l_0 z + l_1 \\
f^{(c-(c-1))}(z_0) &= \frac{wz^{c-1}}{(c-1)!} + l_0 \frac{z^{c-2}}{(c-2)!} + l_1 \frac{z^{c-3}}{(c-3)!} + ... \\
f(z_0) &= \frac{wz^c}{(c)!} + l_0 \frac{z^{c-1}}{(c-1)!} + l_1 \frac{z^{c-2}}{(c-2)!} + ...
\end{aligned}
$$

This gives us a polynomial of degree $\leq c$

14 Let's consider the taylor expansion of f, g centered at 0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!}$$

We also know that

$$f^{(n)}(0) = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

if we choose a small enough r that would mean

$$\frac{n!}{2i\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| = \frac{n!}{2i\pi} \left| \int_{C_r} \frac{g(z)}{(z-z_0)^{n+1}} dz \right|$$

Thus:

$$
\begin{aligned}
f^{(n)}(0) &= g^{(n)}(0) \\
\implies f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!} \\
&= g(z)
\end{aligned}
$$

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