1 4.7 Problems: 11, 12, 15

11 We will consider the equation:

$$f(z) = \frac{xy^2(x+iy)}{x^2+x^4}$$
$$f(0) = 0$$

Let us take the limit when $z \to 0$ along any line $z = (a+bi)t$

$$\begin{aligned}
\lim_{t\to 0} \frac{f((a+ib)t)}{(a+ib)t} &= \frac{(at)(bt)^2(a+ib)t}{((at)^2+(at)^4)(a+ib)t} \\
&= \frac{ab^2t^3}{a^2t^2+a^4t^4} \\
&= \frac{ab^2t^1}{a^2+a^4t^2} \\
&= 0
\end{aligned}$$

So the limit goes to zero when $z \to 0$ along a straight path. But what if we take a different path to 0. We will instead let $z(t) = t^2 + it$ as $z \to 0$.

$$\begin{aligned}
\lim_{t\to 0} \frac{f(t^2+it)}{t^2+it} &= \frac{t^2t^2(t^2+it)}{(t^4+t^8)(t^2+it)} \\
&= \frac{t^4}{t^4+t^8} \\
&= \frac{1}{1+t^4} \\
&= 1
\end{aligned}$$

12 (i) First we will find f'

$$f'(z) = (z^2)' = 2z$$

Then we will plug in the path

$$f'(\gamma(t)) = 2(t^3+it^4)$$

We can easily find the derivative of the path

$$\gamma'(t) = 3t^2 + 4it^3$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)'(t) &= \left((t^3+it^4)^2\right)' \\
&= \left(t^6 - t^8 + 2it^7\right)' \\
&= -8t^7 + 14it^6 + 6t^5 \\
f'(\gamma(t))\gamma'(t) &= 2(t^3+it^4)(3t^2+4it^3) \\
&= 2(3t^5 + 7it^6 - 4t^7) \\
&= -8t^7 + 14it^6 + 6t^5
\end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(ii) First we will find f'

$$f'(z) = (1/z)' = -1/z^2$$

Then we will plug in the path

$$f'(\gamma(t)) = -1/(\cos(t) + i\sin(t))^2 = -e^{-2it}$$

We can easily find the derivative of the path

$$\gamma'(t) = -\sin(t) + i\cos(t) = ie^{it}$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)' &= \left(\frac{1}{\cos(t) + i\sin(t)}\right)' \\
&= \left(e^{-it}\right)' \\
&= -ie^{-it} && (1) \\
f'(\gamma(t))\gamma'(t) &= -e^{-2it}ie^{it} \\
&= -ie^{-it} && (2)
\end{aligned}$$

(1)=(2) so $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

(iii) First we will find f'

$$f'(z) = \left(\sum z^n\right)' = \sum nz^{n-1}$$

Then we will plug in the path

$$f'(\gamma(t)) = \sum n(t + it^2)^{n-1}$$

We can easily find the derivative of the path

$$\gamma'(t) = 1 + 2it$$

Now we want to find the derivative of the compostion

$$\begin{aligned}
(f\gamma)' &= \left(\sum (t + it^2)^n\right)' \\
&= \sum n(t + it^2)^{n-1}(1 + 2it) \\
f'(\gamma(t))\gamma'(t) &= (1 + 2it)\sum n(t + it^2)^{n-1} \\
&= \sum n(t + it^2)^{n-1}(1 + 2it)
\end{aligned}$$

So $f'(\gamma(t))\gamma'(t) = (f\gamma)'(t)$.

15 Let's consider the series $s(z) = \sum a_n z^n, c(z) = \sum b_n, z^n$. Let's take the derivative of $s(z)$ and $c(z)$:

$$s'(z) = \sum a_n n z^{n-1}$$

$$c'(z) = \sum b_n n z^{n-1}$$

We can then work out the coefficients because we know $s'(z) = c(z)$ and $c'(z) = -s(z)$:

$$\sum a_n n z^{n-1} = \sum b_{n-1} z^{n-1} \tag{1}$$

$$\sum b_n n z^{n-1} = \sum -a_{n-1} z^{n-1} \tag{2}$$

If we combine (1) and (2) to solve for a_n

$$\begin{aligned}\sum b_{n-1}(n-1)z^{n-2} &= \sum -a_{n-2} z^{n-2} \\ \implies b_{n-1} &= -a_{n-2}/(n-1) \\ \implies \sum a_n n z^{n-1} &= \sum -a_{n-2}/(n-1) z^{n-1} \\ \implies a_n &= -a_{n-2}/(n(n-1))\end{aligned}$$

Then we solve for b_n

$$\begin{aligned}\sum a_{n-1}(n-1)z^{n-2} &= \sum b_{n-2} z^{n-2} \\ \implies a_{n-1} &= b_{n-2}/(n-1) \\ \implies b_n &= -b_{n-2}/(n(n-1))\end{aligned}$$

Now we will assume that $s(0) = 0, c(0) = 1$. I claim that $s(z) = \sin(z)$, meaning even powers have a coefficient of 0. We have $a_0 = 0$

$$\begin{aligned} a_2 &= -a_0/(2 \cdot 1) = 0 \\ \text{Let } a_{2n} = 0 \implies a_{2n+2} &= -a_{2n}/(2n(2n-1)) = 0 \end{aligned}$$

And the odd terms will give us $a_{2n+1} = (-1)^n/(2n+1)!$. We have $b_0 = 1$ so

$$\begin{aligned} a_1 &= b_0 = 1 \\ a_3 &= -a_1/(3 \cdot 2) = (-1)/3! \\ \text{Let } a_{2n+1} &= (-1)^n/(2n+1)! \\ \implies a_{2(n+1)+1} &= \frac{-(-1)^n}{(2n+1)!(2n+3)(2n+2)} \\ &= \frac{(-1)^{n+1}}{(2(n+1)+1)!} \end{aligned}$$

Which gives the sum:

$$\sum \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sin(z)$$

so $s'(z) = \cos(z) = c(z)$ thus $c^2(z) + s^2(z) = 1$

2 5.10 Problems: 1, 5, 15

1 (i)

$$e^i = \cos(1) + i\sin(1)$$

(ii)

$$e^{2+i\pi} = e^2\cos(\pi) + e^2\sin(\pi)$$

(iii)

$$e^{-2-i\pi} = e^{-2}\cos(-\pi) + e^{-2}\sin(-\pi)$$

5

$$\begin{aligned}
\cos(\theta+\phi) + i\sin(\theta+\phi) &= e^{i(\theta+\phi)} \\
&= e^{i\theta}e^{i\phi} \\
&= (\cos(\theta) + i\sin(\theta))(\cos(\phi) + i\sin(\phi)) \\
&= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + i(\cos(\theta)\sin(\phi) + \cos(\phi)\sin(\theta))
\end{aligned}$$

For the next relation:

$$\begin{aligned}
\frac{1}{\cos(\theta) + i\sin(\theta)} &= e^{-i\theta} \\
&= \cos(-\theta) + i\sin(-\theta) \\
&= \cos(\theta) - i\sin(\theta)
\end{aligned}$$

15 (i)

$$\begin{aligned}
\sum_{m=0}^{n} \cos(mx) + i\sin(mx) &= \sum_{m=0}^{n} e^{imx} \\
&= \sum_{m=0}^{n} \left(e^{ix}\right)^m \\
&= \frac{1 - e^{ix(n+1)}}{1 - e^{ix}} \\
&= \frac{1 - \cos(x(n+1)) - i\sin(x(n+1))}{1 - \cos(x) - i\sin(x)}
\end{aligned}$$