1 Worksheet Problems:

1. Let us consider the function $g(z) = f(z)f(-z)$. If z were to tend to zero from the positive direction then we would get

$$\begin{aligned}\lim_{z\to 0^+} g(z) &= \lim_{z\to 0^+} f(z)f(-z)\\ &\leq \lim_{z\to 0^+} 2\cdot 3\\ &= 6\end{aligned}$$

Obviously $g(0) = f(0)^2$ thus $|f(0)| \leq \sqrt{6}$

2. Consider some $z = x + iy$, let us find the maximum and minimum of the function.

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can then see that the modulus of the function is

$$|e^z| = \left|e^x e^{iy}\right| = e^x$$

This means that the maximum of the modulus is the rightmost point in the set. This must be on the boundary. If we had a right most point inside the boundary then we could simply move right until we reach the boundary. The minimum is when we have the greatest negative value of x. This must also be on the boundary by the same logic.

3. First we shall factor to get

$$f(z) = z(z-1)$$

The modulus:

$$|z(z-1)| = |z||z-1|$$

This would mean that the maximum is when both moduli are at a maximum. At all point on the boundary of the disk $|z|$ is a maximum. But only at the point farthest from 1 for $|z-1|$ to be at a maximum. Thus the point $z_0 = -1$ and $|f(-1)| = |-1||-1-1| = 2$. The minimum is 0 and this is at points $z_0 = 0, 1$.

4. Assume that the polynomial $p(z)$ is non constant and differentiable. Consider a closed disk D with radius r centered at the origin. By the maximum modulus theorem we know that if $|p(z_0)| \geq |p(z)|$ for all $z \in D$ then z_0 must be on the boundary. If we increase r then $|z_0| = r$ will also increase. This means that as $|z_0| \to \infty$ then $|p(z_0)| \to \infty$. Hence we can choose some r such that

$$\forall z \in \mathbb{C} - D, |p(z)| > \max_D |p|$$

The minimum then must be insides of D, which by the minimum modulus theorem means they are zeros.

17 Consider the function $f(z) = 1 + z$, in a disk D around the origin with radius a. We know that the maximum must be on the boundary because it is not constant. The modulus is

$$|f(z)| = \sqrt{(1+x)^2 + y^2}$$

It must be that

$$y^2 = a^2 - x^2$$

so

$$\sqrt{(1+x)^2 + a^2 - x^2} = \sqrt{1 + 2x + a^2}$$

which is clearly a maximum when $x = a$.

2 Chapter 11 Problems:

1. (i) $f(z) = (z-3)^{-1}$
   Let us first rewrite
   $$\frac{1}{-3} \cdot \frac{1}{1 - z/3}$$
   By the binomial expansion we get the result
   $$\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$
   in the annulus $0 < |z| < 3$

   (ii) $f(z) = (z-a)^{-k}$
   Let us rewrite this as
   $$\frac{1}{(-a)^k (1 - z/a)^k}$$
   we can then use the binomial series
   $$\frac{1}{(-a)^k}(1 - z/a)^{-k} = (-1)^k \sum_{n=0}^{\infty} \binom{k+n-1}{n} \left(\frac{z^n}{a^{k+n}}\right)$$
   Giving us the series in the annulus $0 < |z| < a$

   (iii) We will partial fraction
   $$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$
   Then we can use the binomial expansion
   $$\frac{1}{z} + \sum_{n=0}^{\infty} z^n$$
   The annulus is $0 < |z| < 1$

(iv) Again we can use partial fractions.

$$\frac{1}{(z-a)(z-b)} = \frac{1}{z(a-b)(1-a/z)} + \frac{1}{(ab-b^2)(1-z/b)}$$

Let us note that either $|a| \leq |b|$ or $|b| \leq |a|$, both cases would be identical so we will assume $|a| \leq |b|$. Then we will use the binomial expansion.

$$\begin{aligned}\frac{1}{z(a-b)(1-a/z)} &= \frac{1}{(a-b)}\sum_{n=0}^{\infty}\left(\frac{a^n}{z^{n+1}}\right)\\ \frac{1}{(ab-b^2)(1-z/b)} &= \frac{1}{ab-b^2}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n f(z) \quad = \frac{1}{(a-b)}\sum_{n=0}^{\infty}\left(\frac{a^n}{z^{n+1}}\right) + \frac{1}{ab-b^2}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n\end{aligned}$$

and this is valid in the annulus $|a| < |z| < |b|$

(v) It is known that $e^{1/z}$ has the power series

$$\sum_{n=0}^{\infty}\frac{1}{n!}z^{-n}$$

so we can easily obtain

$$\sum_{n=0}^{\infty}\frac{1}{n!}z^{3-n} = z^3 + z^2 + \frac{1}{2}z + \sum_{n=0}^{\infty}\frac{1}{(n+3)!}z^{-n}$$

Which converges on the annulus $z \neq 0$

(vi) We can put $z + 1/z$ into the series for e^z and use the binomial expansion.

$$\sum_{k=0}^{\infty}\frac{(z+1/z)^k}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=0}^{k}\binom{k}{n}z^{2n-k}\right)$$

We can now try to find the coefficients of negative powers with this series. Consider when $2n-k=a$ and try to find the coefficient for z^a.

$$\begin{aligned}2n-k=a &\implies k=2n-a\\ \frac{1}{k!}\binom{k}{n}z^{2n-k} &\implies \frac{1}{(2n-a)!}\binom{2n-a}{n}z^a\\ &\implies e^{z+1/z}\sum_{a=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{z^a}{(2n-a)!}\binom{2n-a}{n}\end{aligned}$$

and $0 < |z|$

(vii) We will use the regular series for $\cos(z)$ but use $1/z$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{(2n)!}$$

The annulus would be $0 < |z|$

(viii) Similarly to $e^{1/z}$ we will put z^{-5} into the expansion of e^z,

$$\sum_{n=0}^{\infty} \frac{z^{-5n}}{n!}$$

This also has annulus $0 < |z|$

2. (i) We will first use partial fractions

$$\frac{1}{(z-1)^2(z-2)} = \frac{-3z}{(z-1)^2} + \frac{4}{(z-1)^2} + \frac{2}{z-2}$$

From 11.7.1(ii)

$$\frac{-3z}{(z-1)^2} = -3\sum_{n=0}^{\infty}(n+1)z^{n+1}$$

$$\frac{4}{(z-1)^2} = 4\sum_{n=0}^{\infty}(n+1)z^n$$

$$\frac{2}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty}(n+4-2^{-n-1})z^n$$

(ii) Similarly to (i)

$$\frac{1}{(z-1)^2(z-2)} = \frac{-3z}{(z-1)^2} + \frac{4}{(z-1)^2} + \frac{2}{z-2}$$

The first two series must change slightly

$$\frac{-3z}{z^2(1-1/z)^2} = -3\sum_{n=0}^{\infty}(n+1)\frac{2^n}{z^{n+1}}$$

$$\frac{4}{z^2(1-1/z)^2} = 4\sum_{n=0}^{\infty}(n+1)\frac{2^n}{z^{n+2}}$$

$$\frac{2}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty}(n2^{n+1} - 3(n+1)2^n)z^{-n-1} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

(iii) Similarly to (i) and (ii)

$$\frac{1}{(z-1)^2(z-2)} = \frac{-3z}{(z-1)^2} + \frac{4}{(z-1)^2} + \frac{2}{z-2}$$

The first two series must change slightly

$$\frac{-3z}{z^2(1-1/z)^2} = -3\sum_{n=0}^{\infty}(n+1)\frac{2^n}{z^{n+1}}$$
$$\frac{4}{z^2(1-1/z)^2} = 4\sum_{n=0}^{\infty}(n+1)\frac{2^n}{z^{n+2}}$$
$$\frac{2}{2(1-2/z)} = \sum_{n=0}^{\infty}\frac{2^{n+1}}{z^{n+1}}$$
$$f(z) = \sum_{n=0}^{\infty}((n+1)2^{n+1} - 3(n+1)2^n)z^{-n-1}$$

(iv) We can use the series for e^z.

$$e^{-z^{-2}} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{-2n}}{n!}$$

(v) Consider the disk $0 < |z-1| < 1 + (1-\sqrt{5})/2$. We can factor to get

$$\frac{1}{(1-z-z^2)} = \frac{1}{(\frac{\sqrt{5}-1}{2} - z)(\frac{\sqrt{5}+1}{2} + z)}$$

We can then split this with partial fractioning

$$\frac{1/\sqrt{5}}{(\frac{\sqrt{5}-1}{2} - z)} + \frac{1/\sqrt{5}}{(\frac{\sqrt{5}+1}{2} + z)}$$

We can try to find a series of $z-1$ (whatever that means) for each of these fractions.

(vi) Using partial fractions we get

$$\frac{1}{(1+z^2)} = \frac{1}{(z-i)(z+i)} = \frac{i/2}{(i+z)} + \frac{i/2}{(i-z)}$$

We can find the series for each of these

$$\frac{1/2}{(1-zi)} = \frac{1}{2}\sum_{n=0}^{\infty} i^n z^n$$
$$\frac{1/2}{1+iz} = \frac{1}{2}\sum_{n=0}^{\infty} i^{-n} z^n$$
$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

Now we must find

$$
\begin{aligned}
e^z/(1+z^2) &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(i^n+i^{-n})z^n\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\frac{i^k+i^{-k}}{(n-k)!}\right)
\end{aligned}
$$