

a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{R_{12}^{(1.5)}, R_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{R_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

$$\text{取 } L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

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$$\text{得 } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

(b)

故 $Ax = b \Leftrightarrow LUx = b$,

$$\text{令 } Ux = y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{先解 } Ly = b: \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \text{得 } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix},$$

$$\text{再由 } Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{即 } \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}, \text{解得 } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A .

令 $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 Gram-Schmidt process 對 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ 作正交化.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 2,$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \frac{11}{2}.$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = \frac{396}{121}$$

$$\therefore \begin{cases} \mathbf{v}_1 = \mathbf{u}_1 \\ \mathbf{v}_2 = \frac{-1}{2} \mathbf{u}_1 + \mathbf{u}_2 \\ \mathbf{v}_3 = \frac{1}{2} \mathbf{u}_1 + \frac{7}{11} \mathbf{u}_2 + \mathbf{u}_3 \end{cases}, \text{即 } A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = \left[\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \right] \begin{bmatrix} \|\mathbf{u}_1\| & \frac{-1}{2} \cdot \|\mathbf{u}_1\| & \frac{1}{2} \cdot \|\mathbf{u}_1\| \\ 0 & 1 \cdot \|\mathbf{u}_2\| & \frac{7}{11} \cdot \|\mathbf{u}_2\| \\ 0 & 0 & 1 \cdot \|\mathbf{u}_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_R.$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

- a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.
- b) (7%) Compute the Singular Value Decomposition of A .
- c) (3%) Compute the rank 1 approximation of A .

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$$\text{char}_{A^T A}(x) = \det(A^T A - xI) = \det\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix} = x(x-10), \text{ 得特徵根 } 10, 0.$$

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$$

\therefore 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0$;

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0$.

(b)

由(a),故得A的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$,且取 $V=\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}=[v_1 \ v_2]$,

$$\text{另外,取 } u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\ker(A^T) = \ker \left(\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \therefore \text{取 } u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, U = [u_1 \ u_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\text{則 } A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^T.$$

(c)

$$A \text{的 rank 1-近似為 } \sigma_1 u_1 v_1^T = \sqrt{10} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}.$$

20. Find a singular value decomposition $A = U\Sigma V^T$ with U and V being both

orthogonal matrices, where $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. Which values are not in U or V matrices?

- (a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) $1/2$. (d) $1/3$. (e) $1/(3\sqrt{2})$.

注：背面有試題

10. 【解】(b), (c).

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$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ 令為 B , $\text{char}_B(x) = x^2 - 34x + 225$, 得特徵根 $25, 9$.

故得 A 的奇異值 $\sigma_1 = 5, \sigma_2 = 3, \therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$,

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$$V(25) = \ker(B - 25I) = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\};$$

$$V(9) = \ker(B - 9I) = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

$$\text{故取 } U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = [\mathbf{u}_1 \ \mathbf{u}_2],$$

$$\text{又 } \mathbf{v}_1 = \frac{1}{\sigma_1} A^T \mathbf{u}_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \text{ 取 } \mathbf{v}_2 = \frac{1}{\sigma_2} A^T \mathbf{u}_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix},$$

$$\text{再考慮 } N(A), N(A) = \ker \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \text{span} \left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\},$$

$$\text{得 } V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}, \text{ 且 } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T = U\Sigma V^T.$$

5. (10 分) Let $M = \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix}$. (a) Determine whether $M^T M$ is diagonalizable or not. If $M^T M$

is diagonalizable, find a diagonalization for it. (b) Find the singular value decomposition of M .

$$(a) M^T M = \begin{bmatrix} 3 & 5 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \quad (\text{令為 } A) \text{ 為實對稱矩陣, 故可對角化.}$$

$$\text{char}_A(x) = \det(A - xI) = \det \begin{bmatrix} 34 - x & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 - x \end{bmatrix} = (0 - x)[(34 - x)(16 - x) - 144]$$

$$= (0 - x)(x^2 - 50x + 400) = (0 - x)(10 - x)(40 - x), \therefore \text{特徵根為 } 0, 10, 40,$$

$$0 \text{ 的特徵空間 } V(0) = \ker(A - 0I) = \ker \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$10 \text{ 的特徵空間 } V(10) = \ker(A - 10I) = \ker \begin{bmatrix} 24 & 0 & 12 \\ 0 & -10 & 0 \\ 12 & 0 & 6 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\},$$

$$40 \text{ 的特徵空間 } V(40) = \ker(A - 40I) = \ker \begin{bmatrix} -6 & 0 & 12 \\ 0 & -40 & 0 \\ 12 & 0 & -24 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\text{故取 } P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ 可得 } P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 40 \end{bmatrix}.$$

(b)

由(a)得 $M^T M$ 特徵根 40, 10, 0.

故得 M 的奇異值 $\sigma_1 = \sqrt{40}$, $\sigma_2 = \sqrt{10}$, $\therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$,

且 $M^T M$ 的特徵向量單範化得 $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$,

取 $\frac{1}{\sqrt{40}} M\mathbf{v}_1 = \frac{1}{\sqrt{40}} \begin{bmatrix} \frac{10}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \end{bmatrix}$, $\frac{1}{\sqrt{10}} M\mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{-5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix}$, 單範化後得 $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$,

\therefore 取 $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, 可得 $M = U \Sigma V^T$, 為 M 的奇異值分解.

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3. Let $C[-1, 1]$ be the vector space over \mathbb{R} of all continuous functions defined on the interval $[-1, 1]$. Let $V : \{f(x) \in C[-1, 1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in \mathbb{R}\}$.
- (2 points) Prove that V is a subspace of $C[-1, 1]$.
 - (5 points) Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V .
 - (5 points) Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V .

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(a)

$$\text{零函數} = 0e^x + 0e^{2x} + 0e^{3x} \in V, \therefore V \neq \emptyset,$$

$$\text{任取 } f(x) = ae^x + be^{2x} + ce^{3x} \in V, g(x) = le^x + me^{2x} + ne^{3x} \in V, \alpha \in \mathbb{R},$$

$$\text{則 } \alpha f(x) + g(x) = \alpha(ae^x + be^{2x} + ce^{3x}) + (le^x + me^{2x} + ne^{3x})$$

$$= (\alpha a + l)e^x + (\alpha b + m)e^{2x} + (\alpha c + n)e^{3x} \in V,$$

即滿足向量加法與純量積封閉性，故 V 為 $C[-1, 1]$ 的一個子空間。

(b)

$$\text{Wronskian}[e^x, e^{2x}, e^{3x}] = \det \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = \dots = 2e^{6x} \neq 0, \forall x \in \mathbb{R},$$

所以 B 是線性獨立的

又 B 生成 V ，故 B 為 V 的一個基底

(c)

$$\text{設 } \alpha(e^x - 2e^{3x}) + \beta(e^x + e^{2x} + 2e^{3x}) + \gamma(3e^{2x} + e^{3x}) = 0,$$

$$\text{則 } (\alpha + \beta)e^x + (\beta + 3\gamma)e^{2x} + (-2\alpha + 2\beta + \gamma)e^{3x} = 0,$$

$$\text{但 } \{e^x, e^{2x}, e^{3x}\} \text{ 是線性獨立集, } \therefore \begin{cases} \alpha + \beta = 0 \\ \beta + 3\gamma = 0 \\ -2\alpha + 2\beta + \gamma = 0 \end{cases}$$

可解得 $\therefore \alpha = \beta = \gamma = 0$.

$\therefore B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ 是線性獨立集，

又由 (b) 可得 $\dim(V) = 3$ ，故 B' 可為 V 的一基底。

3. (10 分) Determine an explicit description of $T(\mathbf{x})$ using the given basis \mathcal{B} and the matrix representation of T respective to \mathcal{B} .

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: It needs to find the standard matrix of T .)

3. 【解】

令 $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 為 R^3 上的標準基底，則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ，

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix}.$$

$$\therefore T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+6y-5z \\ -4x+4y+5z \\ -x+3y+z \end{pmatrix}.$$

4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix A into a product of three simpler matrices $A = U\Sigma V^T$, where T denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
- (a) $UU^T = I$
 - (b) $U^TU \neq I$
 - (c) $VV^T \neq I$
 - (d) $V^TV = I$
 - (e) $V^TA^T = \Sigma U^T$
 - (f) $\Sigma V^T = AU^T$
 - (g) The result of performing SVD for a given matrix is unique
 - (h) $\text{rank}(A) = \text{rank}(\Sigma)$
 - (i) The column vectors of V are eigenvectors of A^TA
 - (j) The row vectors of U are eigenvectors of AA^T

<sol> A,D,H,I

4. (7%) Find a 3×3 symmetric matrix whose eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$ and for which the corresponding eigenvectors are $v_1 = (1, 1, 0)^T$, $v_2 = (0, 0, 1)^T$, $v_3 = (-1, 1, 0)^T$.

4. 【解】

因 $\{v_1, v_2, v_3\}$ 為正交集, 故可取 $A = 4 \cdot \frac{1}{2} \cdot v_1 v_1^T + 2 \cdot v_2 v_2^T + 0 \cdot \frac{1}{2} \cdot v_3 v_3^T$

$$= 2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

滿足所求.

Spectral Decomposition/Eigen-decomposition

當矩陣 A 滿足：

- 是 實對稱矩陣 ($A = A^T$) · 或
- 是 Hermitian 矩陣 (複數情況)

就一定可以寫成：

$$A = \sum_i \lambda_i v_i v_i^T$$

其中：

- λ_i : 特徵值
 - v_i : 正交且單位長度的特徵向量
 - $v_i v_i^T$: 投影矩陣
-

若 A 是 對稱矩陣 (或可正交對角化) · 則：

$$A = \lambda_1 v v^T + \lambda_2 u u^T + \lambda_3 s s^T$$

條件是：

- v, u, s 是 單位特徵向量
- $\|v\| = \|u\| = \|s\| = 1$
- 彼此正交
- $\lambda_1, \lambda_2, \lambda_3$ 為對應的 eigenvalues

如果 v, u, s 不是單位向量

那正確寫法應該是：

$$A = \lambda_1 \frac{v v^T}{v^T v} + \lambda_2 \frac{u u^T}{u^T u} + \lambda_3 \frac{s s^T}{s^T s}$$