106 中正

a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)},\, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)}R_{13}^{(-2)}R_{12}^{(1.5)}A = U$, 即 $A = [R_{23}^{(3)}R_{13}^{(-2)}R_{12}^{(1.5)}]^{-1}U = R_{12}^{(-1.5)}R_{13}^{(2)}R_{23}^{(-3)}U$,

取 $L = R_{12}^{(-1.5)}R_{13}^{(2)}R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$,

得 $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

(b)

故 $Ax = \boldsymbol{b} \Leftrightarrow LUx = \boldsymbol{b}$,

令 $Ux = \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 先解 $L\boldsymbol{y} = \boldsymbol{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, 得 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$,

再由 $Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 即 $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$, 解得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

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6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A.

令 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 $Gram-Schmidt\ process$ 對 v_1, v_2, v_3 作正交化.

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, < u_1, u_1 > = 2,$$

$$u_2 = v_2 - \frac{< v_2, u_1 >}{< u_1, u_1 >} u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix},\ < u_2, u_2 > = \frac{11}{2}.$$

$$u_3 = v_3 - \frac{< v_3, u_1 >}{< u_1, u_1 >} u_1 - \frac{< v_3, u_2 >}{< u_2, u_2 >} u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, < u_3, u_3 > = \frac{396}{121}$$

$$\therefore \begin{cases} v_1 = u_1 \\ v_2 = \frac{-1}{2} u_1 + u_2 \\ v_3 = \frac{1}{2} u_1 + \frac{7}{11} u_2 + u_3 \end{cases}, \text{即 } A = [v_1\ v_2\ v_3] = [u_1\ u_2\ u_3] \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = [\frac{u_1}{\| u_1 \|}\ \frac{u_2}{\| u_2 \|}\ \frac{u_3}{\| u_3 \|}] \begin{bmatrix} \| u_1 \| & \frac{-1}{2} \cdot \| u_1 \| & \frac{1}{2} \cdot \| u_1 \| \\ 0 & 1 \cdot \| u_2 \| & \frac{7}{11} \cdot \| u_2 \| \\ 0 & 0 & 1 \cdot \| u_3 \| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_{R}.$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.

b) (7%) Compute the Singular Value Decomposition of A.

c) (3%) Compute the rank 1 approximation of A.

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$char_{A^T A}(x) = \det(A^T A - xI) = \det(\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix}) = x(x-10)$, 得特徵根 10, 0.

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

$\therefore$ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0;$

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0.$

(*b*)

由 (a), 故得 A 的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}$, 且取 $V=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}=[\boldsymbol{v}_1 \;\; \boldsymbol{v}_2]$,

另外, 取 $\boldsymbol{u}_1=\dfrac{1}{\sigma_1}A\boldsymbol{v}_1=\dfrac{1}{\sqrt{10}}\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}}\end{bmatrix}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$,

$\ker(A^T)=\ker(\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$, $\therefore$ 取 $\boldsymbol{u}_2=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$, $U=[\boldsymbol{u}_1 \;\; \boldsymbol{u}_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$,

則 $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}^T$.

(*c*)

A 的 $rank\,1-$近似為 $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T=\sqrt{10}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.

20. Find a singular value decomposition $A = U\Sigma V^T$ with U and V being both orthogonal matrices, where $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. Which values are **not** in U or V matrices?

(a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) 1/2. (d) 1/3. (e) $1/(3\sqrt{2})$.

注:背面有試題

10.【解】 (b), (c). 【98 中央資工類題】

$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$令為$B$, $char_B(x) = x^2 - 34x + 225$, 得特徵根25, 9.

故得A的奇異值$\sigma_1 = 5, \sigma_2 = 3, \therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$,

$V(25) = \ker(B - 25I) = \ker(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}) = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$;

$V(9) = \ker(B - 9I) = \ker(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = span\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$,

故取$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = [\boldsymbol{u}_1 \ \ \boldsymbol{u}_2]$,

又 $\boldsymbol{v}_1 = \frac{1}{\sigma_1} A^T \boldsymbol{u}_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, 取$\boldsymbol{v}_2 \ \frac{1}{\sigma_2} A^T \boldsymbol{u}_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}$,

再考慮$N(A)$, $N(A) = \ker(\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}) = span\{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\}$,

得$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}$, 且$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T = U\Sigma V^T$.

108 交大

3. Let $C[-1,1]$ be the vector space over R of all continuous functions defined on the interval $[-1,1]$. Let $V:\{f(x)\in C[-1,1] | f(x)=ae^x+be^{2x}+ce^{3x}, a,b,c\in R\}$.

a. **(2 points)** Prove that V is a subspace of $C[-1,1]$.

b. **(5 points)** Prove that $B=\{e^x,e^{2x},e^{3x}\}$ is a basis of V.

c. **(5 points)** Prove that $B'=\{e^x-2e^{3x}, e^x+e^{2x}+2e^{3x}, 3e^{2x}+e^{3x}\}$ is a basis of V.

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(a)

零函數 $=0e^x+0e^{2x}+0e^{3x}\in V, \therefore V\neq\varnothing,$

任取 $f(x)=ae^x+be^{2x}+ce^{3x}\in V, g(x)=le^x+me^{2x}+ne^{3x}\in V, \alpha\in R,$

則 $\alpha f(x)+g(x)=\alpha(ae^x+be^{2x}+ce^{3x})+(le^x+me^{2x}+ne^{3x})$

$=(\alpha a+l)e^x+(\alpha b+m)e^{2x}+(\alpha c+n)e^{3x}\in V,$

即滿足向量加法與純量積封閉性, 故 V 為 $C[-1,1]$ 的一個子空間.

(b)

$$Wronskian[e^x,e^{2x},e^{3x}]=\det\left(\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x}\end{bmatrix}\right)=\ldots=2e^{6x}\neq 0, \forall x\in R,$$

所以 B 是線性獨立的

又 B 生成 V, 故 B 為 V 的一個基底

(c)

設 $\alpha(e^x-2e^{3x})+\beta(e^x+e^{2x}+2e^{3x})+\gamma(3e^{2x}+e^{3x})=0,$

則 $(\alpha+\beta)e^x+(\beta+3\gamma)e^{2x}+(-2\alpha+2\beta+\gamma)e^{3x}=0,$

但 $\{e^x,e^{2x},e^{3x}\}$ 是線性獨立集, $\therefore\begin{cases}\alpha+\beta=0\\ \beta+3\gamma=0\\ -2\alpha+2\beta+\gamma=0\end{cases}$

可解得 $\therefore \alpha=\beta=\gamma=0.$

$\therefore B'=\{e^x-2e^{3x}, e^x+e^{2x}+2e^{3x}, 3e^{2x}+e^{3x}\}$ 是線性獨立集,

又由 (b) 可得 $\dim(V)=3$, 故 B' 可為 V 的一基底.

106 師大

3. (10 分) Determine an explicit description of $T(\boldsymbol{x})$ using the given basis $\mathcal{B}$ and the matrix representation of T respective to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: *It needs to find the standard matrix of T.*)

3.【解】

令 $S=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ 為 R^3 上的標準基底, 則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+6y-5z \\ -4x+4y+5z \\ -x+3y+z \end{bmatrix}.$$

4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix A into a product of three simpler matrices $A = U\Sigma V^{T}$, where T denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
   (a) $UU^{T} = I$
   (b) $U^{T}U \neq I$
   (c) $VV^{T} \neq I$
   (d) $V^{T}V = I$
   (e) $V^{T}A^{T} = \Sigma U^{T}$
   (f) $\Sigma V^{T} = AU^{T}$
   (g) The result of performing SVD for a given matrix is unique
   (h) $\text{rank}(A) = \text{rank}(\Sigma)$
   (i) The column vectors of V are eigenvectors of $A^{T}A$
   (j) The row vectors of U are eigenvectors of AA^{T}

<sol> A,D,H,I