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a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)},\, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

取 $L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$,

得 $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

(b)

故 $Ax = \boldsymbol{b} \Leftrightarrow LUx = \boldsymbol{b}$,

令 $Ux = \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 先解 $Ly = \boldsymbol{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, 得 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$,

再由 $Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 即 $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$, 解得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

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6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A.

令 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 $Gram-Schmidt\ process$ 對 v_1, v_2, v_3 作正交化.

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, < u_1, u_1 > = 2,$$

$$u_2 = v_2 - \frac{< v_2, u_1 >}{< u_1, u_1 >} u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \ < u_2, u_2 > = \frac{11}{2}.$$

$$u_3 = v_3 - \frac{< v_3, u_1 >}{< u_1, u_1 >} u_1 - \frac{< v_3, u_2 >}{< u_2, u_2 >} u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, < u_3, u_3 > = \frac{396}{121}$$

$$\therefore \begin{cases} v_1 = u_1 \\ v_2 = \frac{-1}{2} u_1 + u_2 \\ v_3 = \frac{1}{2} u_1 + \frac{7}{11} u_2 + u_3 \end{cases}, \text{即 } A = [v_1\ v_2\ v_3] = [u_1\ u_2\ u_3] \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = [\frac{u_1}{\| u_1 \|}\ \frac{u_2}{\| u_2 \|}\ \frac{u_3}{\| u_3 \|}] \begin{bmatrix} \| u_1 \| & \frac{-1}{2} \cdot \| u_1 \| & \frac{1}{2} \cdot \| u_1 \| \\ 0 & 1 \cdot \| u_2 \| & \frac{7}{11} \cdot \| u_2 \| \\ 0 & 0 & 1 \cdot \| u_3 \| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_{R}.$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.

b) (7%) Compute the Singular Value Decomposition of A.

c) (3%) Compute the rank 1 approximation of A.

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$char_{A^T A}(x) = \det(A^T A - xI) = \det(\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix}) = x(x-10)$, 得特徵根 10, 0.

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

$\therefore$ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0$;

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0$.

(*b*)

由 (a), 故得 A 的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}$, 且取 $V=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}=[\boldsymbol{v}_1\ \ \boldsymbol{v}_2]$,

另外, 取 $\boldsymbol{u}_1=\dfrac{1}{\sigma_1}A\boldsymbol{v}_1=\dfrac{1}{\sqrt{10}}\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}}\end{bmatrix}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$,

$\ker(A^T)=\ker(\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$, ∴ 取 $\boldsymbol{u}_2=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$, $U=[\boldsymbol{u}_1\ \ \boldsymbol{u}_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$,

則 $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}^T$.

(*c*)

A 的 $rank\ 1-$ 近似為 $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T=\sqrt{10}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.