

106 中正

- a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$
- b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)}, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

$$\text{取 } L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

106-中正資工

$$\text{得 } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}.$$

(b)

故 $Ax = b \Leftrightarrow LUx = b$,

$$\text{令 } Ux = y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 先解 } Ly = b: \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \text{ 得 } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix},$$

$$\text{再由 } Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 即 } \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}, \text{ 解得 } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

107 台科

6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A .

令 $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 Gram-Schmidt process 對 $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ 作正交化.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 2,$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \frac{11}{2}.$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = \frac{396}{121}$$

$$\therefore \begin{cases} \mathbf{v}_1 = \mathbf{u}_1 \\ \mathbf{v}_2 = \frac{-1}{2} \mathbf{u}_1 + \mathbf{u}_2 \\ \mathbf{v}_3 = \frac{1}{2} \mathbf{u}_1 + \frac{7}{11} \mathbf{u}_2 + \mathbf{u}_3 \end{cases}, \text{即 } A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3] \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

107-台科資工

$$\therefore A = \left[\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \right] \begin{bmatrix} \|\mathbf{u}_1\| & \frac{-1}{2} \cdot \|\mathbf{u}_1\| & \frac{1}{2} \cdot \|\mathbf{u}_1\| \\ 0 & 1 \cdot \|\mathbf{u}_2\| & \frac{7}{11} \cdot \|\mathbf{u}_2\| \\ 0 & 0 & 1 \cdot \|\mathbf{u}_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_R.$$

108 中央

20. Diagonalize matrix $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$; that is to find matrices P and D such that $A = PDP^T$. Which values are not in P or D matrices?
- A. $1/\sqrt{3}$. B. $-1/\sqrt{8}$. C. $2/\sqrt{8}$. D. $-1/2$. E. $1/3$.

20. 【解】選(b), (d), (e).

解 $\text{char}_A(x) = \det(A - xI) = \dots = (7-x)(4-x)(-4-x)$, 得特徵根: $7, 4, -4$,

考慮各特徵根所對應的特徵空間:

$$V(7) = \ker(A - 7I) = \ker\left(\begin{bmatrix} -6 & 1 & 5 \\ 1 & -2 & 1 \\ 5 & 1 & -6 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & -11 & 11 \\ 1 & -2 & 1 \\ 0 & 11 & -11 \end{bmatrix}\right) = \underbrace{\text{span}(\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\})}_{v_1},$$

$$V(4) = \ker(A - 4I) = \ker\left(\begin{bmatrix} -3 & 1 & 5 \\ 1 & 1 & 1 \\ 5 & 1 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & 4 & 8 \\ 1 & 1 & 1 \\ 0 & -4 & -8 \end{bmatrix}\right) = \underbrace{\text{span}(\{\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}\})}_{v_2},$$

$$V(-4) = \ker(A + 4I) = \ker\left(\begin{bmatrix} 5 & 1 & 5 \\ 1 & 9 & 1 \\ 5 & 1 & 5 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & -44 & 0 \\ 1 & 9 & 1 \\ 0 & -44 & 0 \end{bmatrix}\right) = \underbrace{\text{span}(\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\})}_{v_3},$$

取 $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, 可得 $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

107 中正

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

- a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.
- b) (7%) Compute the Singular Value Decomposition of A .
- c) (3%) Compute the rank 1 approximation of A .

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$$\text{char}_{A^T A}(x) = \det(A^T A - xI) = \det\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix} = x(x-10), \text{得特徵根 } 10, 0.$$

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$$

107-中正資工

$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$$

\therefore 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0$;

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0$.

(b)

由(a), 故得 A 的奇異值 $\sigma_1 = \sqrt{10}$, $\sigma_2 = 0$, $\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$, 且取 $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = [\nu_1 \ \nu_2]$,

$$\text{另外, 取 } u_1 = \frac{1}{\sigma_1} A \nu_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\ker(A^T) = \ker \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \therefore \text{取 } u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, U = [u_1 \ u_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\text{則 } A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^T.$$

(c)

$$A \text{ 的 rank 1-近似為 } \sigma_1 u_1 v_1^T = \sqrt{10} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}.$$

105 中央

20. Find a singular value decomposition $A = U \Sigma V^T$ with U and V being both

orthogonal matrices, where $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. Which values are **not** in U or V matrices?

- (a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) $1/2$. (d) $1/3$. (e) $1/(3\sqrt{2})$.

注: 背面有試題

10. 【解】(b), (c).

【98 中央資工類題】

$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ 令為 B , $\text{char}_B(x) = x^2 - 34x + 225$, 得特徵根 $25, 9$.

故得 A 的奇異值 $\sigma_1 = 5, \sigma_2 = 3$, $\therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$,

$$V(25) = \ker(B - 25I) = \ker\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\};$$

$$V(9) = \ker(B - 9I) = \ker\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\},$$

故取 $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = [\mathbf{u}_1 \ \mathbf{u}_2]$,

$$\text{又 } \mathbf{v}_1 = \frac{1}{\sigma_1} A^T \mathbf{u}_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \text{ 取 } \mathbf{v}_2 = \frac{1}{\sigma_2} A^T \mathbf{u}_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix},$$

再考慮 $N(A)$, $N(A) = \ker\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right\}$,

$$\text{得 } V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}, \text{ 且 } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T = U \Sigma V^T.$$

105 師大

5. (10 分) Let $M = \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix}$. (a) Determine whether $M^T M$ is diagonalizable or not. If $M^T M$

is diagonalizable, find a diagonalization for it. (b) Find the singular value decomposition of

M .

$$(a) M^T M = \begin{bmatrix} 3 & 5 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \text{(令為 } A \text{)為實對稱矩陣, 故可對角化.}$$

$$char_A(x) = \det(A - xI) = \det \begin{bmatrix} 34-x & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16-x \end{bmatrix} = (0-x)[(34-x)(16-x)-144]$$

$$= (0-x)(x^2 - 50x + 400) = (0-x)(10-x)(40-x), \therefore \text{特徵根為 } 0, 10, 40,$$

$$0 \text{ 的特徵空間 } V(0) = \ker(A - 0I) = \ker \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$10 \text{ 的特徵空間 } V(10) = \ker(A - 10I) = \ker \begin{bmatrix} 24 & 0 & 12 \\ 0 & -10 & 0 \\ 12 & 0 & 6 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\},$$

$$40 \text{ 的特徵空間 } V(40) = \ker(A - 40I) = \ker \begin{bmatrix} -6 & 0 & 12 \\ 0 & -40 & 0 \\ 12 & 0 & -24 \end{bmatrix} = span \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\text{故取 } P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ 可得 } P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 40 \end{bmatrix}.$$

(b)

由 (a) 得 $M^T M$ 特徵根 $40, 10, 0$.

$$\text{故得 } M \text{ 的奇異值 } \sigma_1 = \sqrt{40}, \sigma_2 = \sqrt{10}, \therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix},$$

$$\text{且 } M^T M \text{ 的特徵向量單範化得 } V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3],$$

$$\text{取 } \frac{1}{\sqrt{40}}M\mathbf{v}_1 = \frac{1}{\sqrt{40}} \begin{bmatrix} \frac{10}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \end{bmatrix}, \frac{1}{\sqrt{10}}M\mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{-5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix}, \text{ 單範化後得 } \mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\therefore \text{取 } U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ 可得 } M = U\Sigma V^T, \text{ 為 } M \text{ 的奇異值分解.}$$

106 臺北

5. (50%) Give the matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Solve the following questions.

- (a) (10%) Describe the nullspace of M . (The nullspace of M consists of all solutions to $M\mathbf{x} = 0$.)
- (b) (5%) Compute $M^T M$ and MM^T .
- (c) (5%) Explain whether $M^T M$ is similar to MM^T .
- (d) (10%) Compute the eigenvalues and eigenvectors of $M^T M$.
- (e) (10%) Diagonalize $M^T M$.
- (f) (10%) Find the singular value decomposition of M .

(a) Nullspace of M

The nullspace $N(M)$ is the set of all vectors \mathbf{x} such that $M\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \implies x_1 = -x_2$$

$$x_2 + x_3 = 0 \implies x_3 = -x_2$$

Let $x_2 = c$ (free variable). Then $\mathbf{x} = \begin{bmatrix} -c \\ c \\ -c \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The nullspace is the span of the vector $[1 \ -1 \ 1]^T$.

(b) Compute $M^T M$ and MM^T

$$M^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$M^T M$ (3×3):

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

MM^T (2×2):

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(c) Is $M^T M$ similar to MM^T ?

No. Similar matrices must have the same dimensions. $M^T M$ is a 3×3 matrix, while MM^T is a 2×2 matrix. Therefore, they cannot be similar. However, they do share the same non-zero eigenvalues.

(d) Eigenvalues and Eigenvectors of $M^T M$

To find the eigenvalues λ , we solve $\det(M^T M - \lambda I) = 0$:

$$\begin{vmatrix} 1 - \lambda & 1 & 0 \\ 1 & 2 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0$$

Expanding along the first row: $(1 - \lambda)[(2 - \lambda)(1 - \lambda) - 1] - 1(1 - \lambda) = 0(1 - \lambda)[\lambda^2 - 3\lambda + 1] - (1 - \lambda) = 0(1 - \lambda)(\lambda^2 - 3\lambda) = 0 \implies -\lambda(\lambda - 1)(\lambda - 3) = 0$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$.

Eigenvectors:

For $\lambda_1 = 3$: $\mathbf{v}_1 = [1 \ 2 \ 1]^T$ (normalized: $\frac{1}{\sqrt{6}} [1 \ 2 \ 1]^T$)

For $\lambda_2 = 1$: $\mathbf{v}_2 = [1 \ 0 \ -1]^T$ (normalized: $\frac{1}{\sqrt{2}} [1 \ 0 \ -1]^T$)

For $\lambda_3 = 0$: $\mathbf{v}_3 = [1 \ -1 \ 1]^T$ (normalized: $\frac{1}{\sqrt{3}} [1 \ -1 \ 1]^T$)

(e) Diagonalize $M^T M$

Since $M^T M$ is symmetric, it is diagonalized by $M^T M = V \Lambda V^T$, where V is the orthogonal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues.

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) Singular Value Decomposition (SVD) of M

The SVD is $M = U\Sigma V^T$.

Σ : Square roots of non-zero eigenvalues of $M^T M$. $\sigma_1 = \sqrt{3}, \sigma_2 = \sqrt{1}$. Thus, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

V : The eigenvectors of $M^T M$ found in part (d).

U : Found via $u_i = \frac{1}{\sigma_i} M v_i$.

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\text{Result: } M = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^T$$

108 交大

3. Let $C[-1, 1]$ be the vector space over \mathbb{R} of all continuous functions defined on the interval $[-1, 1]$. Let $V : \{f(x) \in C[-1, 1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in \mathbb{R}\}$.

a. (2 points) Prove that V is a subspace of $C[-1, 1]$.

b. (5 points) Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V .

c. (5 points) Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V .

108-交大資工

(a)

零函數 $= 0e^x + 0e^{2x} + 0e^{3x} \in V, \therefore V \neq \emptyset$,

任取 $f(x) = ae^x + be^{2x} + ce^{3x} \in V, g(x) = le^x + me^{2x} + ne^{3x} \in V, \alpha \in \mathbb{R}$,

則 $\alpha f(x) + g(x) = \alpha(ae^x + be^{2x} + ce^{3x}) + (le^x + me^{2x} + ne^{3x})$

$$= (\alpha a + l)e^x + (\alpha b + m)e^{2x} + (\alpha c + n)e^{3x} \in V,$$

即滿足向量加法與純量積封閉性，故 V 為 $C[-1, 1]$ 的一個子空間。

(b)

$$\text{Wronskian}[e^x, e^{2x}, e^{3x}] = \det \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = \dots = 2e^{6x} \neq 0, \forall x \in \mathbb{R},$$

所以 B 是線性獨立的。

又 B 生成 V ，故 B 為 V 的一個基底。

(c)

設 $\alpha(e^x - 2e^{3x}) + \beta(e^x + e^{2x} + 2e^{3x}) + \gamma(3e^{2x} + e^{3x}) = 0$,

則 $(\alpha + \beta)e^x + (\beta + 3\gamma)e^{2x} + (-2\alpha + 2\beta + \gamma)e^{3x} = 0$,

但 $\{e^x, e^{2x}, e^{3x}\}$ 是線性獨立集， $\therefore \begin{cases} \alpha + \beta = 0 \\ \beta + 3\gamma = 0 \\ -2\alpha + 2\beta + \gamma = 0 \end{cases}$

可解得 $\therefore \alpha = \beta = \gamma = 0$.

$\therefore B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ 是線性獨立集，

又由 (b) 可得 $\dim(V) = 3$ ，故 B' 可為 V 的一基底。

106 師大

3. (10 分) Determine an explicit description of $T(x)$ using the given basis \mathcal{B} and the matrix representation of T respective to \mathcal{B} .

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: It needs to find the standard matrix of T .)

3. 【解】

令 $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 為 R^3 上的標準基底，則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ，

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix}.$$

$$\therefore T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 6y - 5z \\ -4x + 4y + 5z \\ -x + 3y + z \end{pmatrix}.$$

110 台科

4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix A into a product of three simpler matrices $A = U\Sigma V^T$, where T denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
- (a) $UU^T = I$
 - (b) $U^TU \neq I$
 - (c) $VV^T \neq I$
 - (d) $V^TV = I$
 - (e) $V^TA^T = \Sigma U^T$
 - (f) $\Sigma V^T = AU^T$
 - (g) The result of performing SVD for a given matrix is unique
 - (h) $\text{rank}(A) = \text{rank}(\Sigma)$
 - (i) The column vectors of V are eigenvectors of A^TA
 - (j) The row vectors of U are eigenvectors of AA^T

<sol> A,D,H,I

108 中正

4. (7%) Find a 3×3 symmetric matrix whose eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$ and for which the corresponding eigenvectors are $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (0, 0, 1)^T$, $\mathbf{v}_3 = (-1, 1, 0)^T$.

4. 【解】

$$\begin{aligned} \text{因 } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ 為正交集, 故可取 } A &= 4 \cdot \frac{1}{2} \cdot \mathbf{v}_1 \mathbf{v}_1^T + 2 \cdot \mathbf{v}_2 \mathbf{v}_2^T + 0 \cdot \frac{1}{2} \cdot \mathbf{v}_3 \mathbf{v}_3^T \\ &= 2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ 滿足所求.} \end{aligned}$$

Spectral Decomposition/Eigen-decomposition

當矩陣 A 滿足：

- 是 實對稱矩陣 ($A = A^T$) · 或
- 是 Hermitian 矩陣 (複數情況)

就一定可以寫成：

$$A = \sum_i \lambda_i v_i v_i^T$$

其中：

- λ_i ：特徵值
- v_i ：正交且單位長度的特徵向量
- $v_i v_i^T$ ：投影矩陣

若 A 是 對稱矩陣 (或可正交對角化)，則：

$$A = \lambda_1 v v^T + \lambda_2 u u^T + \lambda_3 s s^T$$

條件是：

- v, u, s 是 單位特徵向量
 $\|v\| = \|u\| = \|s\| = 1$
- 彼此正交
- $\lambda_1, \lambda_2, \lambda_3$ 為對應的 eigenvalues

如果 v, u, s 不是單位向量

那正確寫法應該是：

$$A = \lambda_1 \frac{v v^T}{v^T v} + \lambda_2 \frac{u u^T}{u^T u} + \lambda_3 \frac{s s^T}{s^T s}$$

110 師大

3. (12 points) Find an explicit formula for the reflection operator T_W of \mathbb{R}^3 about the plane W with equation $x - 2y + 3z = 0$.

1. 核心概念

給定平面的單位法向量 \mathbf{u} , 反射算子 T_W 可以表示為:

$$T_W(\mathbf{x}) = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)\mathbf{x}$$

其中:

\mathbf{I} 是單位矩陣。

\mathbf{u} 是平面的單位法向量 (必須將長度正規化為 1)。

$\mathbf{u}\mathbf{u}^T$ 是投影矩陣, 將向量投影到法向量的方向上。

2. 具體推導步驟

第一步: 找出法向量並正規化

平面的法向量為 $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ 。其長度為 $\|\mathbf{n}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$ 。

因此, 單位法向量 \mathbf{u} 為:

$$\mathbf{u} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

第二步: 計算 $\mathbf{u}\mathbf{u}^T$

$$\mathbf{u}\mathbf{u}^T = \frac{1}{14} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

第三步: 計算反射矩陣 $H = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{14} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

$$H = \frac{1}{7} \left(\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix} \right) = \frac{1}{7} \begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{bmatrix}$$

3. 最後答案 (Explicit Formula)

將矩陣與向量 $[x \quad y \quad z]^T$ 相乘，得到：

$$T_W \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6x+2y-3z}{7} \\ \frac{2x+3y+6z}{7} \\ \frac{-3x+6y-2z}{7} \end{bmatrix}$$

110 師大-2

5. (12 points) Find an orthogonal operator T on \mathbb{R}^3 such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$, where $\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{w}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{1}{7} \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$.

已知：

$$\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{w}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \mathbf{w}_2 = \frac{1}{7} \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

第一步：驗證長度與正交性

一個算子要是正交的，前提是映射前後的向量性質必須一致：

長度 (Norms):

$$\|v_1\| = \frac{1}{3}\sqrt{1^2 + 2^2 + 2^2} = 1, \|w_1\| = \frac{1}{7}\sqrt{2^2 + 3^2 + 6^2} = 1 \text{ (符合)}$$

$$\|v_2\| = \frac{1}{3}\sqrt{2^2 + 1^2 + (-2)^2} = 1, \|w_2\| = \frac{1}{7}\sqrt{6^2 + 2^2 + (-3)^2} = 1 \text{ (符合)}$$

夾角 (Dot Product):

$$v_1 \cdot v_2 = \frac{1}{9}(1 \cdot 2 + 2 \cdot 1 + 2 \cdot (-2)) = 0$$

$$w_1 \cdot w_2 = \frac{1}{49}(2 \cdot 6 + 3 \cdot 2 + 6 \cdot (-3)) = 0$$

兩組向量皆為單位正交向量組 (Orthonormal sets)，這表示該正交算子 T 確實存在。

第二步：找出第三個基底向量

為了建立完整的正交矩陣 Q ，我們需要利用外積 (Cross Product) 找出與前兩者垂直的第三個單位向量。

計算 $v_3 = v_1 \times v_2$:

$$v_3 = \frac{1}{9} \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{9} \begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

計算 $w_3 = w_1 \times w_2$:

$$w_3 = \frac{1}{49} \begin{vmatrix} i & j & k \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{vmatrix} = \frac{1}{49} \begin{bmatrix} -21 \\ 42 \\ -14 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}$$

第三步：求正交矩陣 T

我們知道正交矩陣 T 滿足 $TV = W$ ，其中 $V = [v_1, v_2, v_3]$ 且 $W = [w_1, w_2, w_3]$ 。

由於 V 與 W 都是正交矩陣，其反矩陣等於轉置矩陣（即 $V^{-1} = V^T$ ）。

$$T = WV^T = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

矩陣相乘計算：

$$T = \frac{1}{7} \begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ 6 & -3 & -2 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$T = \frac{1}{21} \begin{bmatrix} (2+12+6) & (4+6-6) & (4-12+3) \\ (3+4-12) & (6+2+12) & (6-4-6) \\ (6-6+4) & (12-3-4) & (12+6+2) \end{bmatrix}$$

最終答案：

$$T = \frac{1}{21} \begin{bmatrix} 20 & 4 & -5 \\ -5 & 20 & -4 \\ 4 & 5 & 20 \end{bmatrix}$$

這個矩陣 T 即為所求的正交算子，它將 v_1, v_2 映射至 w_1, w_2 。