

a) (5%) Find an LU-Decomposition of the following matrix  $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$2x_1 + 6x_2 + 2x_3 = 2$$

$$-3x_1 - 8x_2 = 2$$

$$4x_1 + 9x_2 + 2x_3 = 3$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)}, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故  $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$ , 即  $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$ ,

$$\text{取 } L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

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$$\text{得 } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}.$$

(b)

$$\text{故 } Ax = b \Leftrightarrow LUx = b,$$

$$\text{令 } Ux = y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 先解 } Ly = b: \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \text{ 得 } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix},$$

$$\text{再由 } Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 即 } \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}, \text{ 解得 } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

6. (10%) Let  $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ . Find the  $QR$ -decomposition of  $A$ .

令  $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ ,  $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ , 用 *Gram-Schmidt process* 對  $v_1, v_2, v_3$  作正交化。

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \langle u_1, u_1 \rangle = 2,$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \langle u_2, u_2 \rangle = \frac{11}{2}.$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, \langle u_3, u_3 \rangle = \frac{396}{121}$$

$$\therefore \begin{cases} v_1 = u_1 \\ v_2 = \frac{1}{2} u_1 + u_2 \\ v_3 = \frac{1}{2} u_1 + \frac{7}{11} u_2 + u_3 \end{cases}, \text{ 即 } A = [v_1 \ v_2 \ v_3] = [u_1 \ u_2 \ u_3] \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix} \begin{bmatrix} \|u_1\| & \frac{1}{2} \cdot \|u_1\| & \frac{1}{2} \cdot \|u_1\| \\ 0 & 1 \cdot \|u_2\| & \frac{7}{11} \cdot \|u_2\| \\ 0 & 0 & 1 \cdot \|u_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_R.$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

- a) (5%) Find the eigenvalues and eigenvectors of  $A^T A$ .
- b) (7%) Compute the Singular Value Decomposition of  $A$ .
- c) (3%) Compute the rank 1 approximation of  $A$ .

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$$\text{char}_{A^T A}(x) = \det(A^T A - xI) = \det \begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix} = x(x-10), \text{ 得特徵根 } 10, 0.$$

$$V(10) = \ker(A^T A - 10I) = \ker \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

∴ 特徵根 10, 對應的特徵向量是  $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0;$

特徵根 0, 對應的特徵向量是  $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0.$

(b)

由 (a), 故得  $A$  的奇異值  $\sigma_1 = \sqrt{10}, \sigma_2 = 0, \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$ , 且取  $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2]$ ,

另外, 取  $\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,

$\ker(A^T) = \ker \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \therefore$  取  $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $U = [\mathbf{u}_1 \ \mathbf{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ ,

則  $A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^T$ .

(c)

$A$  的 rank 1-近似為  $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = \sqrt{10} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ .

20. Find a singular value decomposition  $A = U\Sigma V^T$  with  $U$  and  $V$  being both

orthogonal matrices, where  $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ . Which values are not in  $U$  or  $V$  matrices?

(a)  $1/\sqrt{2}$  (b)  $1/\sqrt{3}$  (c)  $1/2$  (d)  $1/3$  (e)  $1/(3\sqrt{2})$ .

注：背面有試題

10. 【解】 (b), (c).

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$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$  令為  $B$ ,  $\text{char}_B(x) = x^2 - 34x + 225$ , 得特徵根 25, 9.

故得  $A$  的奇異值  $\sigma_1 = 5, \sigma_2 = 3, \therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$ ,

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$$V(25) = \ker(B - 25I) = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\};$$

$$V(9) = \ker(B - 9I) = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},$$

$$\text{故取 } U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = [u_1 \ u_2],$$

$$\text{又 } v_1 = \frac{1}{\sigma_1} A^T u_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \text{ 取 } v_2 = \frac{1}{\sigma_2} A^T u_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix},$$

$$\text{再考慮 } N(A), N(A) = \ker \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\},$$

$$\text{得 } V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}, \text{ 且 } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T = U\Sigma V^T.$$

5. (10 分) Let  $M = \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix}$ . (a) Determine whether  $M^T M$  is diagonalizable or not. If  $M^T M$

is diagonalizable, find a diagonalization for it. (b) Find the singular value decomposition of  $M$ .

$$(a) M^T M = \begin{bmatrix} 3 & 5 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} \text{ (令為 } A \text{) 為實對稱矩陣, 故可對角化.}$$

$$\text{char}_A(x) = \det(A - xI) = \det \begin{bmatrix} 34-x & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16-x \end{bmatrix} = (0-x)[(34-x)(16-x)-144]$$

$$= (0-x)(x^2 - 50x + 400) = (0-x)(10-x)(40-x), \therefore \text{特徵根為 } 0, 10, 40,$$

$$0 \text{ 的特徵空間 } V(0) = \ker(A - 0I) = \ker \begin{bmatrix} 34 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 16 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$10 \text{ 的特徵空間 } V(10) = \ker(A - 10I) = \ker \begin{bmatrix} 24 & 0 & 12 \\ 0 & -10 & 0 \\ 12 & 0 & 6 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\},$$

$$40 \text{ 的特徵空間 } V(40) = \ker(A - 40I) = \ker \begin{bmatrix} -6 & 0 & 12 \\ 0 & -40 & 0 \\ 12 & 0 & -24 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\text{故取 } P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ 可得 } P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 40 \end{bmatrix}.$$

(b)

由 (a) 得  $M^T M$  特徵根 40, 10, 0.

故得  $M$  的奇異值  $\sigma_1 = \sqrt{40}, \sigma_2 = \sqrt{10}, \therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$ ,

且  $M^T M$  的特徵向量單範化得  $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ ,

取  $\frac{1}{\sqrt{40}} M \mathbf{v}_1 = \frac{1}{\sqrt{40}} \begin{bmatrix} \frac{10}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \end{bmatrix}, \frac{1}{\sqrt{10}} M \mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{-5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix}$ , 單範化後得  $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ ,

$\therefore$  取  $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ , 可得  $M = U \Sigma V^T$ , 為  $M$  的奇異值分解.

3. Let  $C[-1, 1]$  be the vector space over  $R$  of all continuous functions defined on the interval  $[-1, 1]$ . Let  $V = \{f(x) \in C[-1, 1] \mid f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$ .
- (2 points) Prove that  $V$  is a subspace of  $C[-1, 1]$ .
  - (5 points) Prove that  $B = \{e^x, e^{2x}, e^{3x}\}$  is a basis of  $V$ .
  - (5 points) Prove that  $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$  is a basis of  $V$ .

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(a)

零函數  $= 0e^x + 0e^{2x} + 0e^{3x} \in V, \therefore V \neq \emptyset$ ,

任取  $f(x) = ae^x + be^{2x} + ce^{3x} \in V, g(x) = le^x + me^{2x} + ne^{3x} \in V, \alpha \in R$ ,

則  $\alpha f(x) + g(x) = \alpha(ae^x + be^{2x} + ce^{3x}) + (le^x + me^{2x} + ne^{3x})$

$$= (\alpha a + l)e^x + (\alpha b + m)e^{2x} + (\alpha c + n)e^{3x} \in V,$$

即滿足向量加法與純量積封閉性, 故  $V$  為  $C[-1, 1]$  的一個子空間。

(b)

$$\text{Wronskian}[e^x, e^{2x}, e^{3x}] = \det \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} = \dots = 2e^{6x} \neq 0, \forall x \in R,$$

所以  $B$  是線性獨立的

又  $B$  生成  $V$ , 故  $B$  為  $V$  的一個基底

(c)

$$\text{設 } \alpha(e^x - 2e^{3x}) + \beta(e^x + e^{2x} + 2e^{3x}) + \gamma(3e^{2x} + e^{3x}) = 0,$$

$$\text{則 } (\alpha + \beta)e^x + (\beta + 3\gamma)e^{2x} + (-2\alpha + 2\beta + \gamma)e^{3x} = 0,$$

$$\text{但 } \{e^x, e^{2x}, e^{3x}\} \text{ 是線性獨立集, } \therefore \begin{cases} \alpha + \beta = 0 \\ \beta + 3\gamma = 0 \\ -2\alpha + 2\beta + \gamma = 0 \end{cases}$$

可解得  $\therefore \alpha = \beta = \gamma = 0$ .

$\therefore B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$  是線性獨立集,

又由 (b) 可得  $\dim(V) = 3$ , 故  $B'$  可為  $V$  的一基底。



3. (10 分) Determine an explicit description of  $T(\mathbf{x})$  using the given basis  $\mathcal{B}$  and the matrix representation of  $T$  respective to  $\mathcal{B}$ .

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: It needs to find the standard matrix of  $T$ .)

### 3. 【解】

令  $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  為  $R^3$  上的標準基底, 則  $B$  到  $S$  的轉換矩陣  $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_{\mathcal{B}} [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 6y - 5z \\ -4x + 4y + 5z \\ -x + 3y + z \end{bmatrix}.$$

4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix  $A$  into a product of three simpler matrices  $A = U\Sigma V^T$ , where  $T$  denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
- (a)  $UU^T = I$
  - (b)  $U^TU \neq I$
  - (c)  $VV^T \neq I$
  - (d)  $V^TV = I$
  - (e)  $V^TA^T = \Sigma U^T$
  - (f)  $\Sigma V^T = AU^T$
  - (g) The result of performing SVD for a given matrix is unique
  - (h)  $\text{rank}(A) = \text{rank}(\Sigma)$
  - (i) The column vectors of  $V$  are eigenvectors of  $A^TA$
  - (j) The row vectors of  $U$  are eigenvectors of  $AA^T$

<sol> A,D,H,I

4. (7%) Find a  $3 \times 3$  symmetric matrix whose eigenvalues are  $\lambda_1 = 4$ ,  $\lambda_2 = 2$ ,  $\lambda_3 = 0$  and for which the corresponding eigenvectors are  $\mathbf{v}_1 = (1, 1, 0)^T$ ,  $\mathbf{v}_2 = (0, 0, 1)^T$ ,  $\mathbf{v}_3 = (-1, 1, 0)^T$ .

4. 【解】

因  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  為正交集, 故可取  $A = 4 \cdot \frac{1}{2} \cdot \mathbf{v}_1 \mathbf{v}_1^T + 2 \cdot \mathbf{v}_2 \mathbf{v}_2^T + 0 \cdot \frac{1}{2} \cdot \mathbf{v}_3 \mathbf{v}_3^T$

$$= 2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ 滿足所求.}$$

# Spectral Decomposition/Eigen-decomposition

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當矩陣  $A$  滿足：

- 是 實對稱矩陣 ( $A = A^T$ )，或
- 是 Hermitian 矩陣 (複數情況)

就一定可以寫成：

$$A = \sum_i \lambda_i v_i v_i^T$$

其中：

- $\lambda_i$ ：特徵值
- $v_i$ ：正交且單位長度的特徵向量
- $v_i v_i^T$ ：投影矩陣

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若  $A$  是 對稱矩陣 (或可正交對角化)，則：

$$A = \lambda_1 v v^T + \lambda_2 u u^T + \lambda_3 s s^T$$

條件是：

- $v, u, s$  是 單位特徵向量  
 $\|v\| = \|u\| = \|s\| = 1$
- 彼此正交
- $\lambda_1, \lambda_2, \lambda_3$  為對應的 eigenvalues

如果  $v, u, s$  不是單位向量

那正確寫法應該是：

$$A = \lambda_1 \frac{v v^T}{v^T v} + \lambda_2 \frac{u u^T}{u^T u} + \lambda_3 \frac{s s^T}{s^T s}$$