

a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$2x_1 + 6x_2 + 2x_3 = 2$$

$$-3x_1 - 8x_2 = 2$$

$$4x_1 + 9x_2 + 2x_3 = 3$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)}, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

$$\text{取 } L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

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$$\text{得 } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}.$$

(b)

$$\text{故 } Ax = b \Leftrightarrow LUx = b,$$

$$\text{令 } Ux = y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 先解 } Ly = b: \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \text{ 得 } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix},$$

$$\text{再由 } Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \text{ 即 } \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}, \text{ 解得 } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR -decomposition of A .

令 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 *Gram-Schmidt process* 對 v_1, v_2, v_3 作正交化。

$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\langle u_1, u_1 \rangle = 2$,

$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}$, $\langle u_2, u_2 \rangle = \frac{11}{2}$.

$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}$, $\langle u_3, u_3 \rangle = \frac{396}{121}$

$\therefore \begin{cases} v_1 = u_1 \\ v_2 = \frac{1}{2} u_1 + u_2 \\ v_3 = \frac{1}{2} u_1 + \frac{7}{11} u_2 + u_3 \end{cases}$, 即 $A = [v_1 \ v_2 \ v_3] = [u_1 \ u_2 \ u_3] \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix}$,

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$\therefore A = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix} \begin{bmatrix} \|u_1\| & \frac{1}{2} \cdot \|u_1\| & \frac{1}{2} \cdot \|u_1\| \\ 0 & 1 \cdot \|u_2\| & \frac{7}{11} \cdot \|u_2\| \\ 0 & 0 & 1 \cdot \|u_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_R$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

- a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.
- b) (7%) Compute the Singular Value Decomposition of A .
- c) (3%) Compute the rank 1 approximation of A .

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$$\text{char}_{A^T A}(x) = \det(A^T A - xI) = \det \begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix} = x(x-10), \text{ 得特徵根 } 10, 0.$$

$$V(10) = \ker(A^T A - 10I) = \ker \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

∴ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0;$

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0.$

(b)

由 (a), 故得 A 的奇異值 $\sigma_1 = \sqrt{10}, \sigma_2 = 0, \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$, 且取 $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2]$,

另外, 取 $\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

$\ker(A^T) = \ker \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \therefore$ 取 $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $U = [\mathbf{u}_1 \ \mathbf{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

則 $A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^T$.

(c)

A 的 rank 1-近似為 $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = \sqrt{10} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$.

20. Find a singular value decomposition $A = U\Sigma V^T$ with U and V being both

orthogonal matrices, where $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. Which values are not in U or V matrices?

(a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$ (c) $1/2$ (d) $1/3$ (e) $1/(3\sqrt{2})$.

注：背面有試題

10. 【解】 (b), (c).

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$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ 令為 B , $\text{char}_B(x) = x^2 - 34x + 225$, 得特徵根 25, 9.

故得 A 的奇異值 $\sigma_1 = 5, \sigma_2 = 3, \therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$,

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$$V(25) = \ker(B - 25I) = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\};$$

$$V(9) = \ker(B - 9I) = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},$$

$$\text{故取 } U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = [u_1 \ u_2],$$

$$\text{又 } v_1 = \frac{1}{\sigma_1} A^T u_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \text{ 取 } v_2 = \frac{1}{\sigma_2} A^T u_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix},$$

$$\text{再考慮 } N(A), N(A) = \ker \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \text{span} \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\},$$

$$\text{得 } V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}, \text{ 且 } A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^T = U\Sigma V^T.$$

3. Let $C[-1, 1]$ be the vector space over R of all continuous functions defined on the interval $[-1, 1]$. Let $V = \{f(x) \in C[-1, 1] \mid f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$.

- (2 points) Prove that V is a subspace of $C[-1, 1]$.
- (5 points) Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V .
- (5 points) Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V .

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(a)

零函數 $= 0e^x + 0e^{2x} + 0e^{3x} \in V, \therefore V \neq \emptyset$,

任取 $f(x) = ae^x + be^{2x} + ce^{3x} \in V, g(x) = le^x + me^{2x} + ne^{3x} \in V, \alpha \in R$,

則 $\alpha f(x) + g(x) = \alpha(ae^x + be^{2x} + ce^{3x}) + (le^x + me^{2x} + ne^{3x})$

$$= (\alpha a + l)e^x + (\alpha b + m)e^{2x} + (\alpha c + n)e^{3x} \in V,$$

即滿足向量加法與純量積封閉性, 故 V 為 $C[-1, 1]$ 的一個子空間。

(b)

$$\text{Wronskian}[e^x, e^{2x}, e^{3x}] = \det \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} = \dots = 2e^{6x} \neq 0, \forall x \in R,$$

所以 B 是線性獨立的

又 B 生成 V , 故 B 為 V 的一個基底

(c)

$$\text{設 } \alpha(e^x - 2e^{3x}) + \beta(e^x + e^{2x} + 2e^{3x}) + \gamma(3e^{2x} + e^{3x}) = 0,$$

$$\text{則 } (\alpha + \beta)e^x + (\beta + 3\gamma)e^{2x} + (-2\alpha + 2\beta + \gamma)e^{3x} = 0,$$

$$\text{但 } \{e^x, e^{2x}, e^{3x}\} \text{ 是線性獨立集, } \therefore \begin{cases} \alpha + \beta = 0 \\ \beta + 3\gamma = 0 \\ -2\alpha + 2\beta + \gamma = 0 \end{cases}$$

可解得 $\therefore \alpha = \beta = \gamma = 0$.

$\therefore B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ 是線性獨立集,

又由 (b) 可得 $\dim(V) = 3$, 故 B' 可為 V 的一基底。

3. (10 分) Determine an explicit description of $T(\mathbf{x})$ using the given basis \mathcal{B} and the matrix representation of T respective to \mathcal{B} .

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: It needs to find the standard matrix of T .)

3. 【解】

令 $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 為 R^3 上的標準基底, 則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_{\mathcal{B}} [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 6y - 5z \\ -4x + 4y + 5z \\ -x + 3y + z \end{bmatrix}.$$