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a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)},\, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

取 $L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$,

得 $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

(b)

故 $Ax = \mathbf{b} \Leftrightarrow LUx = \mathbf{b}$,

令 $Ux = \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 先解 $L\mathbf{y} = \mathbf{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, 得 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$,

再由 $Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 即 $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$, 解得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A.

令 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, 用 $Gram-Schmidt\ process$ 對 v_1, v_2, v_3 作正交化.

$$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, <u_1, u_1> = 2,$$

$$u_2 = v_2 - \frac{<v_2, u_1>}{<u_1, u_1>} u_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}, \ <u_2, u_2> = \frac{11}{2}.$$

$$u_3 = v_3 - \frac{<v_3, u_1>}{<u_1, u_1>} u_1 - \frac{<v_3, u_2>}{<u_2, u_2>} u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2} \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, <u_3, u_3> = \frac{396}{121}$$

$$\therefore \begin{cases} v_1 = u_1 \\ v_2 = \frac{-1}{2} u_1 + u_2 \\ v_3 = \frac{1}{2} u_1 + \frac{7}{11} u_2 + u_3 \end{cases}, \text{即 } A = [v_1\ v_2\ v_3] = [u_1\ u_2\ u_3] \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = [\frac{u_1}{\|u_1\|}\ \frac{u_2}{\|u_2\|}\ \frac{u_3}{\|u_3\|}] \begin{bmatrix} \|u_1\| & \frac{-1}{2} \cdot \|u_1\| & \frac{1}{2} \cdot \|u_1\| \\ 0 & 1 \cdot \|u_2\| & \frac{7}{11} \cdot \|u_2\| \\ 0 & 0 & 1 \cdot \|u_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_{R}.$$

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20. Diagonalize matrix $\boldsymbol{A}=\begin{bmatrix}1&1&5\\1&5&1\\5&1&1\end{bmatrix}$; that is to find matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{A}=\boldsymbol{PDP}^T$. Which values are **not** in $\boldsymbol{P}$ or $\boldsymbol{D}$ matrices ?

A. $1/\sqrt{3}$. B. $-1/\sqrt{8}$. C. $2/\sqrt{8}$. D. -1/2. E. 1/3.

20.【解】選(b), (d), (e).

解 $char_A(x)=\det(A-xI)=\ldots=(7-x)(4-x)(-4-x)$, 得特徵根 : $7, 4, -4$,

考慮各特徵根所對應的特徵空間:

$$V(7)=\ker(A-7I)=\ker(\begin{bmatrix}-6&1&5\\1&-2&1\\5&1&-6\end{bmatrix})=\ker(\begin{bmatrix}0&-11&11\\1&-2&1\\0&11&-11\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_1}\}),$$

$$V(4)=\ker(A-4I)=\ker(\begin{bmatrix}-3&1&5\\1&1&1\\5&1&-3\end{bmatrix})=\ker(\begin{bmatrix}0&4&8\\1&1&1\\0&-4&-8\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\-1\end{bmatrix}}_{v_2}\}),$$

$$V(-4)=\ker(A+4I)=\ker(\begin{bmatrix}5&1&5\\1&9&1\\5&1&5\end{bmatrix})=\ker(\begin{bmatrix}0&-44&0\\1&9&1\\0&-44&0\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{v_3}\}),$$

取 $P=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}&0\\\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}&\frac{-1}{\sqrt{2}}\end{bmatrix}$, 可得 $D=\begin{bmatrix}7&0&0\\0&4&0\\0&0&-4\end{bmatrix}$.

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.

b) (7%) Compute the Singular Value Decomposition of A.

c) (3%) Compute the rank 1 approximation of A.

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$char_{A^T A}(x) = \det(A^T A - xI) = \det(\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix}) = x(x-10)$, 得特徵根 10, 0.

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}.$$

$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

$\therefore$ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0;$

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0.$

(b)

由 (a), 故得 A 的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}$, 且取 $V=\begin{bmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ \frac{-1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}=[\boldsymbol{v}_1\ \ \boldsymbol{v}_2]$,

另外, 取 $\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{10}}\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{-1}{\sqrt5}\end{bmatrix}=\frac{1}{\sqrt2}\begin{bmatrix}1\\ -1\end{bmatrix}$,

$\ker(A^T)=\ker(\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$, $\therefore$ 取 $\boldsymbol{u}_2=\frac{1}{\sqrt2}\begin{bmatrix}1\\ 1\end{bmatrix}$, $U=[\boldsymbol{u}_1\ \ \boldsymbol{u}_2]=\begin{bmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}$,

則 $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ \frac{-1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}^T$.

(c)

A 的 $rank\ 1-$ 近似為 $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T=\sqrt{10}\begin{bmatrix}\frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5} & \frac{-1}{\sqrt5}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.

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20. Find a singular value decomposition $A = U\Sigma V^T$ with U and V being both orthogonal matrices, where $A=\begin{bmatrix}3&2&2\\2&3&-2\end{bmatrix}$. Which values are **not** in U or V matrices?

(a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) 1/2. (d) 1/3. (e) $1/(3\sqrt{2})$.

注意:背面有試題

10.【解】 (b), (c). 【98 中央資工類題】

$AA^T=\begin{bmatrix}17&8\\8&17\end{bmatrix}$令為$B$, $char_B(x)=x^2-34x+225$, 得特徵根25, 9.

故得A的奇異值$\sigma_1=5,\sigma_2=3$, $\therefore \Sigma=\begin{bmatrix}5&0&0\\0&3&0\end{bmatrix}$,

$V(25)=\ker(B-25I)=\ker(\begin{bmatrix}-1&1\\1&-1\end{bmatrix})=span\{\begin{bmatrix}1\\1\end{bmatrix}\}$;

$V(9)=\ker(B-9I)=\ker(\begin{bmatrix}1&1\\1&1\end{bmatrix})=span\{\begin{bmatrix}1\\-1\end{bmatrix}\}$,

故取$U=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix}=[\boldsymbol{u}_1\ \boldsymbol{u}_2]$,

又$\boldsymbol{v}_1=\frac{1}{\sigma_1}A^T\boldsymbol{u}_1=\frac{1}{5}\begin{bmatrix}3&2\\2&3\\2&-2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{bmatrix}$, 取$\boldsymbol{v}_2\ \frac{1}{\sigma_2}A^T\boldsymbol{u}_2=\frac{1}{3}\begin{bmatrix}3&2\\2&3\\2&-2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}\end{bmatrix}=\begin{bmatrix}\frac{1}{3\sqrt2}\\\frac{-1}{3\sqrt2}\\\frac{4}{3\sqrt2}\end{bmatrix}$,

再考慮$N(A)$, $N(A)=\ker(\begin{bmatrix}3&2&2\\2&3&-2\end{bmatrix})=span\{\begin{bmatrix}-2\\2\\1\end{bmatrix}\}$,

得$V=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{3\sqrt2}&\frac{-2}{3}\\\frac{1}{\sqrt2}&\frac{-1}{3\sqrt2}&\frac{2}{3}\\0&\frac{4}{3\sqrt2}&\frac{1}{3}\end{bmatrix}$, 且$A=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{bmatrix}\begin{bmatrix}5&0&0\\0&3&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{3\sqrt2}&\frac{-2}{3}\\\frac{1}{\sqrt2}&\frac{-1}{3\sqrt2}&\frac{2}{3}\\0&\frac{4}{3\sqrt2}&\frac{1}{3}\end{bmatrix}^T=U\Sigma V^T$.

5. (10 分) Let $M=\begin{bmatrix}3&0&4\\5&0&0\end{bmatrix}$. (a) Determine whether M^TM is diagonalizable or not. If M^TM is diagonalizable, find a diagonalization for it. (b) Find the singular value decomposition of M.

$(a)M^TM=\begin{bmatrix}3&5\\0&0\\4&0\end{bmatrix}\begin{bmatrix}3&0&4\\5&0&0\end{bmatrix}=\begin{bmatrix}34&0&12\\0&0&0\\12&0&16\end{bmatrix}$(令為 A) 為實對稱矩陣, 故可對角化.

$char_A(x)=\det(A-xI)=\det(\begin{bmatrix}34-x&0&12\\0&0&0\\12&0&16-x\end{bmatrix})=(0-x)[(34-x)(16-x)-144]$

$=(0-x)(x^2-50x+400)=(0-x)(10-x)(40-x)$, $\therefore$ 特徵根為 0, 10, 40,

0 的特徵空間 $V(0)=\ker(A-0I)=\ker(\begin{bmatrix}34&0&12\\0&0&0\\12&0&16\end{bmatrix})=span\{\begin{bmatrix}0\\1\\0\end{bmatrix}\}$,

10 的特徵空間 $V(10)=\ker(A-10I)=\ker(\begin{bmatrix}24&0&12\\0&-10&0\\12&0&6\end{bmatrix})=span\{\begin{bmatrix}1\\0\\-2\end{bmatrix}\}$,

40 的特徵空間 $V(40)=\ker(A-40I)=\ker(\begin{bmatrix}-6&0&12\\0&-40&0\\12&0&-24\end{bmatrix})=span\{\begin{bmatrix}2\\0\\1\end{bmatrix}\}$,

故取 $P=\begin{bmatrix}0&1&2\\1&0&0\\0&-2&1\end{bmatrix}$ 可得 $P^{-1}AP=D=\begin{bmatrix}0&0&0\\0&10&0\\0&0&40\end{bmatrix}$.

(b)

由 (a) 得 $M^T M$ 特徵根 40, 10, 0.

故得 M 的奇異值 $\sigma_1 = \sqrt{40}, \sigma_2 = \sqrt{10}, \therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$,

且$M^T M$ 的特徵向量單範化得 $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{bmatrix} = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \boldsymbol{v}_3]$,

取 $\frac{1}{\sqrt{40}} M\boldsymbol{v}_1 = \frac{1}{\sqrt{40}} \begin{bmatrix} \frac{10}{\sqrt{5}} \\ \frac{10}{\sqrt{5}} \end{bmatrix}, \frac{1}{\sqrt{10}} M\boldsymbol{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} \frac{-5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix}$, 單範化後得 $\boldsymbol{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$,

$\therefore$ 取 $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, 可得 $M = U \Sigma V^T$, 為M的奇異值分解.

108 交大

3. Let $C[-1,1]$ be the vector space over R of all continuous functions defined on the interval $[-1,1]$. Let $V: \{f(x) \in C[-1,1] | f(x) = ae^x + be^{2x} + ce^{3x}, a, b, c \in R\}$.

a. **(2 points)** Prove that V is a subspace of $C[-1,1]$.

b. **(5 points)** Prove that $B = \{e^x, e^{2x}, e^{3x}\}$ is a basis of V.

c. **(5 points)** Prove that $B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ is a basis of V.

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(a)

零函數 $= 0e^x + 0e^{2x} + 0e^{3x} \in V, \therefore V \neq \varnothing,$

任取 $f(x) = ae^x + be^{2x} + ce^{3x} \in V, g(x) = le^x + me^{2x} + ne^{3x} \in V, \alpha \in R,$

則 $\alpha f(x) + g(x) = \alpha(ae^x + be^{2x} + ce^{3x}) + (le^x + me^{2x} + ne^{3x})$

$$= (\alpha a + l)e^x + (\alpha b + m)e^{2x} + (\alpha c + n)e^{3x} \in V,$$

即滿足向量加法與純量積封閉性, 故 V 為 $C[-1,1]$ 的一個子空間.

(b)

$$Wronskian[e^x, e^{2x}, e^{3x}] = \det(\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix}) = \ldots = 2e^{6x} \neq 0, \forall x \in R,$$

所以 B 是線性獨立的

又 B 生成 V, 故 B 為 V 的一個基底

(c)

設 $\alpha(e^x - 2e^{3x}) + \beta(e^x + e^{2x} + 2e^{3x}) + \gamma(3e^{2x} + e^{3x}) = 0,$

則 $(\alpha + \beta)e^x + (\beta + 3\gamma)e^{2x} + (-2\alpha + 2\beta + \gamma)e^{3x} = 0,$

但 $\{e^x, e^{2x}, e^{3x}\}$ 是線性獨立集, $\therefore \begin{cases} \alpha + \beta = 0 \\ \beta + 3\gamma = 0 \\ -2\alpha + 2\beta + \gamma = 0 \end{cases}$

可解得 $\therefore \alpha = \beta = \gamma = 0.$

$\therefore B' = \{e^x - 2e^{3x}, e^x + e^{2x} + 2e^{3x}, 3e^{2x} + e^{3x}\}$ 是線性獨立集,

又由 (b) 可得 $\dim(V) = 3$, 故 B' 可為 V 的一基底.

106 師大

3. (10 分) Determine an explicit description of $T(\boldsymbol{x})$ using the given basis $\mathcal{B}$ and the matrix representation of T respective to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: *It needs to find the standard matrix of T.*)

3.【解】

令 $S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ 為 R^3 上的標準基底, 則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 6y - 5z \\ -4x + 4y + 5z \\ -x + 3y + z \end{bmatrix}.$$

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4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix A into a product of three simpler matrices $A = U\Sigma V^T$, where T denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
   (a) $UU^T = I$
   (b) $U^TU \neq I$
   (c) $VV^T \neq I$
   (d) $V^TV = I$
   (e) $V^TA^T = \Sigma U^T$
   (f) $\Sigma V^T = AU^T$
   (g) The result of performing SVD for a given matrix is unique
   (h) $\text{rank}(A) = \text{rank}(\Sigma)$
   (i) The column vectors of V are eigenvectors of A^TA
   (j) The row vectors of U are eigenvectors of AA^T

<sol> A,D,H,I

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4. (7%) Find a 3×3 symmetric matrix whose eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$ and for which the corresponding eigenvectors are $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (0, 0, 1)^T$, $\mathbf{v}_3 = (-1, 1, 0)^T$.

4.【解】

因$\{v_1, v_2, v_3\}$為正交集,故可取 $A = 4\cdot\frac{1}{2}\cdot v_1v_1^T + 2\cdot v_2v_2^T + 0\cdot\frac{1}{2}\cdot v_3v_3^T$

$$= 2\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} + 0\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}2 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$ 滿足所求.

Spectral Decomposition/Eigen-decomposition

當矩陣 A 滿足：

- 是 **實對稱矩陣**（$A = A^T$），或
- 是 **Hermitian 矩陣**（複數情況）

就一定可以寫成：

$$A = \sum_i \lambda_i v_i v_i^T$$

其中：

- λ_i：特徵值
- v_i：**正交且單位長度**的特徵向量
- $v_i v_i^T$：投影矩陣

若 A 是 **對稱矩陣（或可正交對角化）**，則：

$$\boxed{A = \lambda_1\, vv^T + \lambda_2\, uu^T + \lambda_3\, ss^T}$$

條件是：

- v, u, s 是 **單位特徵向量**
  $\|v\| = \|u\| = \|s\| = 1$
- 彼此正交
- $\lambda_1, \lambda_2, \lambda_3$ 為對應的 eigenvalues

如果 v, u, s **不是單位向量**

那正確寫法應該是：

$$\boxed{A = \lambda_1 \frac{vv^T}{v^T v} + \lambda_2 \frac{uu^T}{u^T u} + \lambda_3 \frac{ss^T}{s^T s}}$$