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a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A=\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}\xrightarrow{r_{12}^{(1.5)},\, r_{13}^{(-2)}}\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix}\xrightarrow{r_{23}^{(3)}}\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}=U$$

故 $R_{23}^{(3)}R_{13}^{(-2)}R_{12}^{(1.5)}A=U$, 即 $A=[R_{23}^{(3)}R_{13}^{(-2)}R_{12}^{(1.5)}]^{-1}U=R_{12}^{(-1.5)}R_{13}^{(2)}R_{23}^{(-3)}U$,

取 $L=R_{12}^{(-1.5)}R_{13}^{(2)}R_{23}^{(-3)}=\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$,

得 $A=LU=\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

(b)

故 $Ax=\mathbf{b}\Leftrightarrow LUx=\mathbf{b}$,

令 $Ux=\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 先解 $L\mathbf{y}=\mathbf{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, 得 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$,

再由 $Ux=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 即 $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$, 解得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

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6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A.

令 $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$，用 $Gram-Schmidt\ process$ 對 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ 作正交化.

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, <\boldsymbol{u}_1, \boldsymbol{u}_1> = 2,$$

$$\boldsymbol{u}_2 = \boldsymbol{v}_2 - \frac{<\boldsymbol{v}_2, \boldsymbol{u}_1>}{<\boldsymbol{u}_1, \boldsymbol{u}_1>}\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix},\ <\boldsymbol{u}_2, \boldsymbol{u}_2> = \frac{11}{2}.$$

$$\boldsymbol{u}_3 = \boldsymbol{v}_3 - \frac{<\boldsymbol{v}_3, \boldsymbol{u}_1>}{<\boldsymbol{u}_1, \boldsymbol{u}_1>}\boldsymbol{u}_1 - \frac{<\boldsymbol{v}_3, \boldsymbol{u}_2>}{<\boldsymbol{u}_2, \boldsymbol{u}_2>}\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{7/2}{11/2}\begin{bmatrix} \frac{-1}{2} \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 9 \\ -3 \\ 15 \\ -9 \end{bmatrix}, <\boldsymbol{u}_3, \boldsymbol{u}_3> = \frac{396}{121}$$

$$\therefore \begin{cases} \boldsymbol{v}_1 = \boldsymbol{u}_1 \\ \boldsymbol{v}_2 = \frac{-1}{2}\boldsymbol{u}_1 + \boldsymbol{u}_2 \\ \boldsymbol{v}_3 = \frac{1}{2}\boldsymbol{u}_1 + \frac{7}{11}\boldsymbol{u}_2 + \boldsymbol{u}_3 \end{cases}，\text{即 } A = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3] = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3]\begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{7}{11} \\ 0 & 0 & 1 \end{bmatrix},$$

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$$\therefore A = \left[\frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}\ \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|}\ \frac{\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|}\right]\begin{bmatrix} \|\boldsymbol{u}_1\| & \frac{-1}{2}\cdot\|\boldsymbol{u}_1\| & \frac{1}{2}\cdot\|\boldsymbol{u}_1\| \\ 0 & 1\cdot\|\boldsymbol{u}_2\| & \frac{7}{11}\cdot\|\boldsymbol{u}_2\| \\ 0 & 0 & 1\cdot\|\boldsymbol{u}_3\| \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{22}} & \frac{3}{\sqrt{44}} \\ 0 & \frac{4}{\sqrt{22}} & \frac{-1}{\sqrt{44}} \\ 0 & \frac{2}{\sqrt{22}} & \frac{5}{\sqrt{44}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{22}} & \frac{-3}{\sqrt{44}} \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} \sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{11}}{2} & \frac{7\sqrt{11}}{22} \\ 0 & 0 & \frac{\sqrt{396}}{11} \end{bmatrix}}_{R}.$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.

b) (7%) Compute the Singular Value Decomposition of A.

c) (3%) Compute the rank 1 approximation of A.

(a)

$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}$,

$char_{A^T A}(x) = \det(A^T A - xI) = \det(\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix}) = x(x-10)$, 得特徵根 $10, 0$.

$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$.

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$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

$\therefore$ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0$;

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0$.

(b)

由 (a), 故得 A 的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}$, 且取 $V=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}=[\boldsymbol{v}_1 \ \ \boldsymbol{v}_2]$,

另外,取 $\boldsymbol{u}_1=\dfrac{1}{\sigma_1}A\boldsymbol{v}_1=\dfrac{1}{\sqrt{10}}\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}}\end{bmatrix}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$,

$\ker(A^T)=\ker(\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$, ∴ 取 $\boldsymbol{u}_2=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$, $U=[\boldsymbol{u}_1 \ \ \boldsymbol{u}_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$,

則 $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}^T$.

(c)

A 的 $rank\,1-$近似為 $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T=\sqrt{10}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.

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20. Find a singular value decomposition $A = U\Sigma V^{T}$ with U and V being both orthogonal matrices, where $A=\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix}$. Which values are **not** in U or V matrices?

(a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) 1/2. (d) 1/3. (e) $1/(3\sqrt{2})$.

注:背面有試題

10.【解】 (b), (c). 【98 中央資工類題】

$AA^{T}=\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix}$令為$B$, $char_B(x)=x^2-34x+225$, 得特徵根25, 9.

故得A的奇異值$\sigma_1=5,\sigma_2=3, \therefore \Sigma=\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}$,

$V(25)=\ker(B-25I)=\ker(\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$;

$V(9)=\ker(B-9I)=\ker(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ -1\end{bmatrix}\}$,

故取$U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}=[\boldsymbol{u}_1\ \ \boldsymbol{u}_2]$,

又$\boldsymbol{v}_1=\dfrac{1}{\sigma_1}A^{T}\boldsymbol{u}_1=\dfrac{1}{5}\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix}$, 取$\boldsymbol{v}_2\ \dfrac{1}{\sigma_2}A^{T}\boldsymbol{u}_2=\dfrac{1}{3}\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{3\sqrt{2}}\\ \frac{-1}{3\sqrt{2}}\\ \frac{4}{3\sqrt{2}}\end{bmatrix}$,

再考慮$N(A)$, $N(A)=\ker(\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix})=span\{\begin{bmatrix}-2\\ 2\\ 1\end{bmatrix}\}$,

得$V=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3}\\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3}\\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3}\end{bmatrix}$, 且$A=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3}\\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3}\\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3}\end{bmatrix}^{T}=U\Sigma V^{T}$.

3. (10 分) Determine an explicit description of $T(\boldsymbol{x})$ using the given basis $\mathcal{B}$ and the matrix representation of T respective to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: *It needs to find the standard matrix of T.*)

3.【解】

令 $S=\{e_1,e_2,e_3\}$ 為 R^3 上的標準基底, 則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+6y-5z \\ -4x+4y+5z \\ -x+3y+z \end{bmatrix}.$$