Digital Signal Processing

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Ho Chi Minh city - Ngay 29 thang 8 nam 2025

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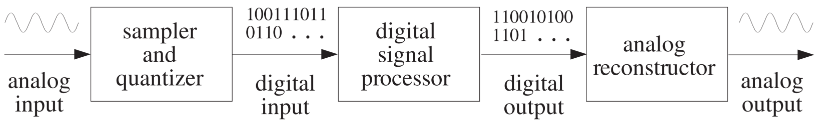
1 Sampling and Reconstruction

1.1 Introduction

A typical signal processing system includes 3 stages:



The analog signal is ***sampled*** and each sample is ***quantized*** to a finite number of bits (A/D converter). The digitalized samples are processed by a digital signal processor.

- The digital processor can be programmed to perform signal processing operations such as filtering, spectrum estimation.
- Digital signal processor can be a general purpose computer, DSP chip or other digital hardware.

The resulting output samples are converted back into analog by an ***analog reconstructor*** (D/A converter).

1.2 Analog to digital conversion

Analog to digital (A/D) conversion is a three-step process.

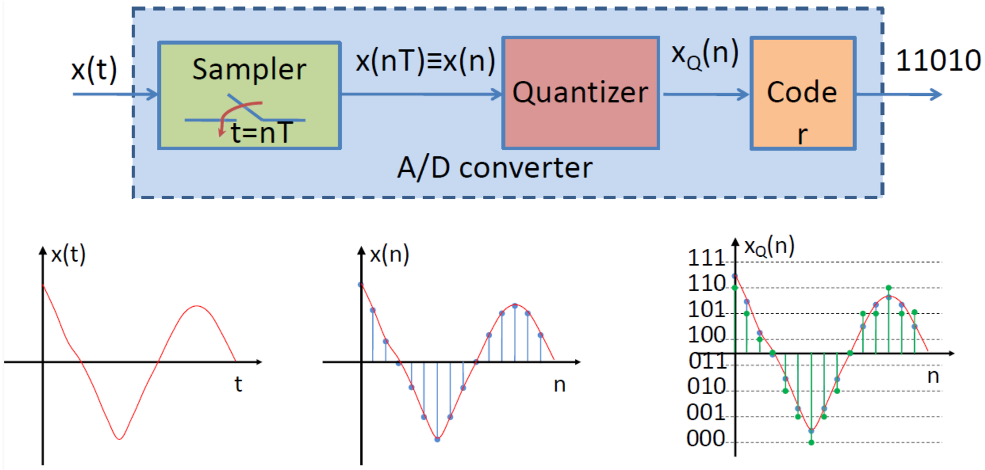
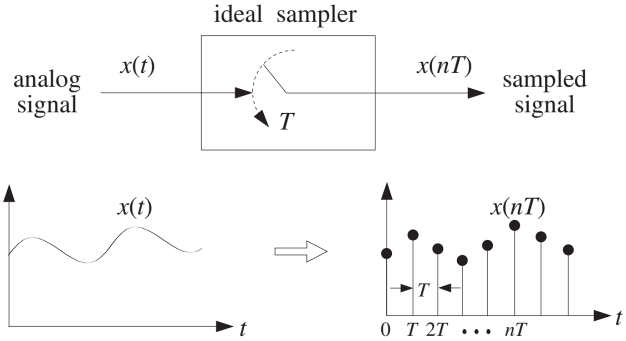


Figure 1: x(n) is discrete time signal but continus in amplitude

* Transform from blue to green is quantizer.

1.3 Sampling

Sampling is to convert a continuous time signal into a discrete time signal. The analog signal is periodically measured at every T seconds.



$$x(n) \equiv x(nT) = x(t = nT),\ n = \ldots -2, -1, 0, 1, 2, 3\ldots$$

- T: sampling interval or sampling period (second);
- Fs=1/T: sampling rate or frequency (samples/second or Hz)

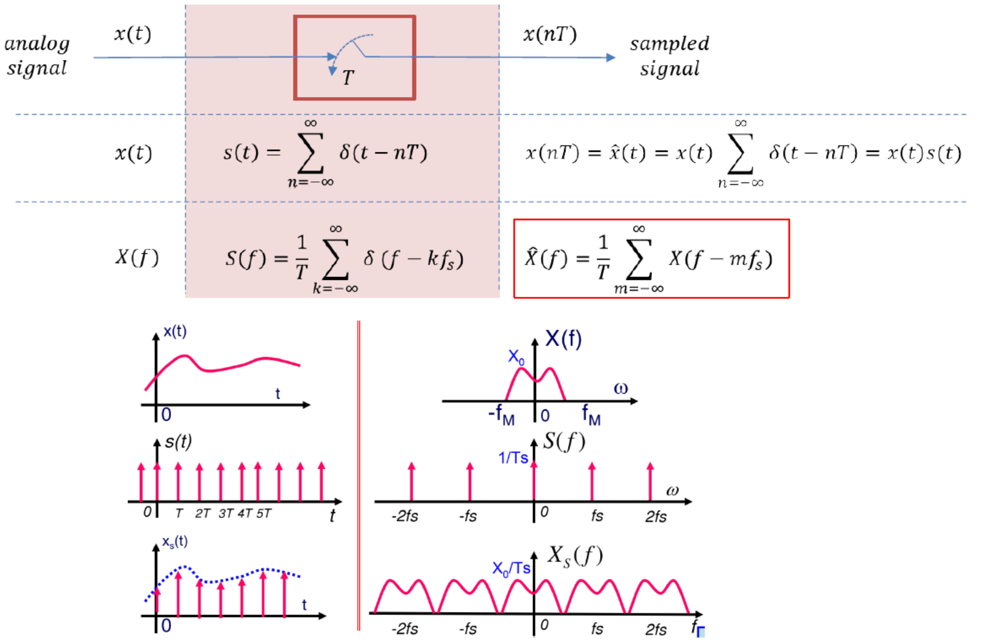
1.4 Aliasing of Sinusoids

In general, the sampling of a continuous-time sinusoidal signal $x(t) = A\cos(2\pi F_0 t + \theta)$ at a sampling rate $F_s = 1/T$ results in a discrete-time signal $x(n)$.

The sinusoids $x_k(t) = A\cos(2\pi F_k t + \theta)$ is sampled at F_s, resulting in a discrete time signal $x_k(n)$.

If $F_k = F_0 + kF_s, k = 0, \pm1, \pm2, \ldots,$ then $x(n) = x_k(n)$.

1.5 Spectrum Replication



Observation: The spectrum of discrete-time signal is a sum of the original spectrum of analog signal and its periodic replication at the interval F_s.

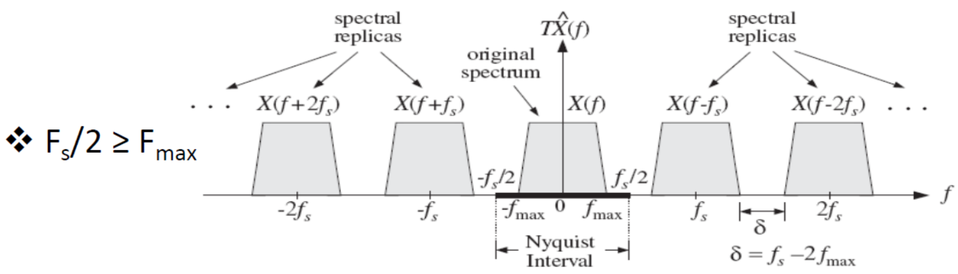


Fig: Spectrum replication caused by sampling

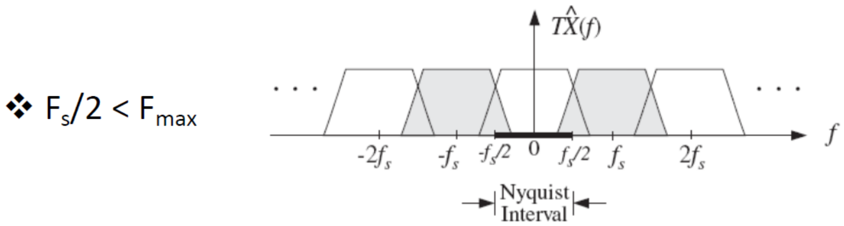


Fig: Aliasing caused by overlapping spectral replicas

1.6 Sampling Theorem

For accurate representation of a signal $x(t)$ by its time samples $x(nT)$, two conditions must be met:

1) The signal $x(t)$ must be band-limited, i.e., its frequency spectrum must be limited to $F_{\max}$.
2) The sampling rate F_s must be chosen at least twice the maximum frequency $F_{\max}$. $F_s \geq 2F_{\max}$
   - $F_s = 2F_{\max}$ is called Nyquist rate.
   - $F_s/2$ is called Nyquist frequency.
   - $[-F_s/2, F_s/2]$ is Nyquist interval.

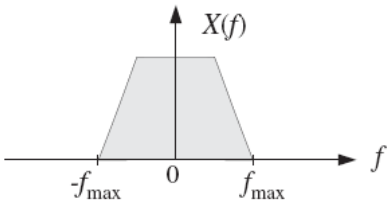


Figure 2: Typical band-limited spectrum

1.7 Ideal analog reconstruction

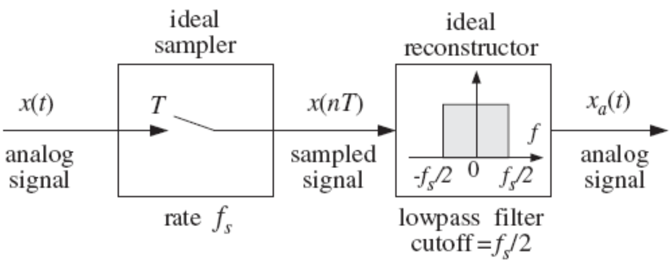


Figure 3: Ideal reconstructor as a lowpass filter

An ideal reconstructor acts as a lowpass filter with cutoff frequency equal to the Nyquist frequency $F_s/2$. An ideal reconstructor (lowpass filter) $H(F) = \begin{cases} T \in [-F_s/2, F_s/2] \\ 0 \quad otherwise \end{cases}$. Then $\widehat{X}_a(F) = \widehat{X}(F)H(F) = X(F)$.

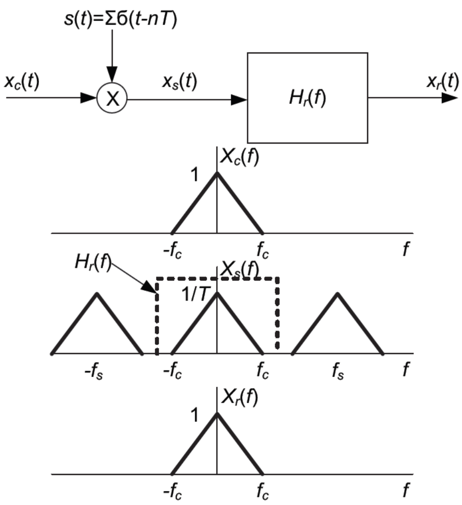


Figure 4: Example Demonstration

1.8 Ideal antialiasing prefilter

The signals in practice may not band-limited, thus they must be filtered by a lowpass filter

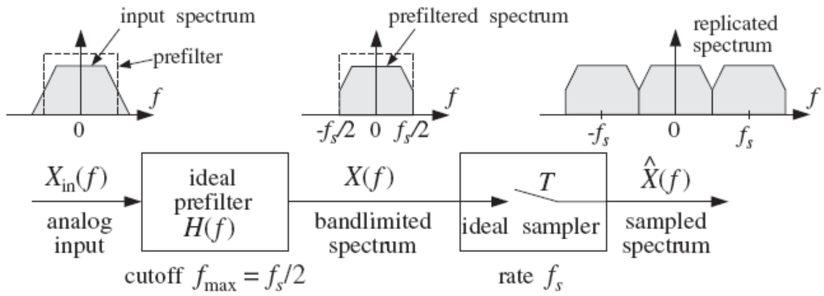


Figure 5: Ideal antialiasing prefilter

1.9 Practical antialiasing prefilter

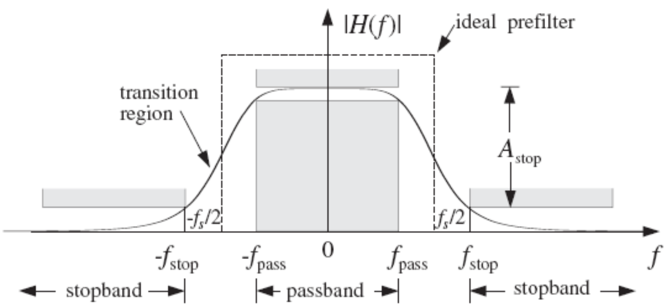
- A lowpass filter: Passband $[-F_{pass}, F_{pass}]$ is the frequency range of interest for the application ($F_{max} = F_{pass}$).
- The stopband frequency F_{stop} and the minimum stopband attenuation Astop dB must be chosen appropriately to minimize the aliasing effects.
- The Nyquist frequency $F_s/2$ is in the middle of transition region.

$$F_s = F_{pass} + F_{stop}$$

- The attenuation of the filter in decibels is defined as (where f_0 is a convenient reference frequency, typically taken to be at DC for a lowpass filter):

$$A(F) = -2\log_{10}\left|\frac{H(F)}{H(F_0)}\right| \quad (dB)$$

- $\alpha_{10} = A(10F) - A(F)$ **(dB/decade)**: the increase in attenuation when F is changed by a factor of ten.
- $\alpha_2 = A(2F) - A(F)$ **(dB/octave)**: the increase in attenuation when F is changed by a factor of two.



1.10 Practical analog reconstructors

The ideal reconstructor has the impulse response: $h(t) = \dfrac{\sin(\pi F_s t)}{\pi F_s t}$, which is not realizable since its impulse response is not casual. It is practical to use a staircase reconstructor.

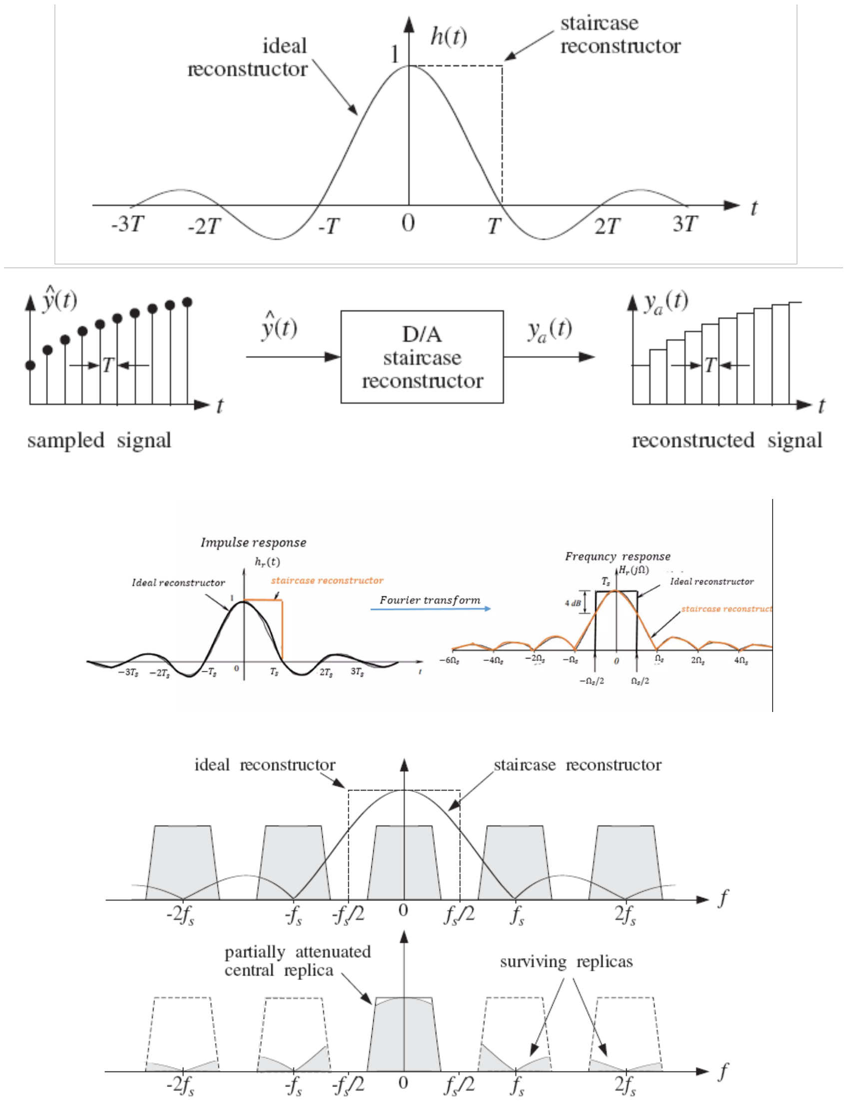


Figure 6: Frequency response of staircase reconstructor

1.11 Anti-image postfilter

An analog lowpass postfilter whose cutoff is Nyquist frequency $Fs/2$ is used to remove the surviving spectral replicas.

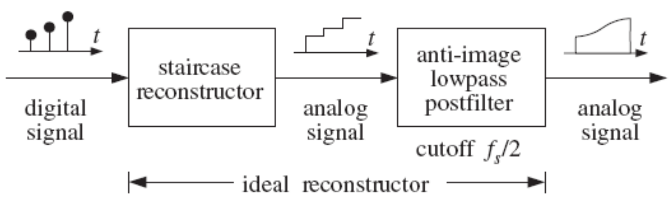


Figure 7: Analog anti-image postfilter

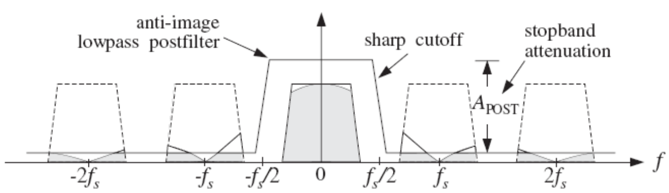


Figure 8: Spectrum after postfilter

2 Quantization

2.1 Quantization process

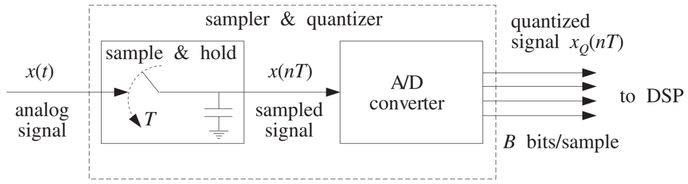


Figure 9: Analog to digital conversion

The quantized sample $x_Q(nT)$ is represented by **B bit**, which can take 2^B possible values.

An A/D is characterized by a **full-scale range R** which is divided into 2^B quantization levels. Typical values of R in practice are between 1-10 volts.

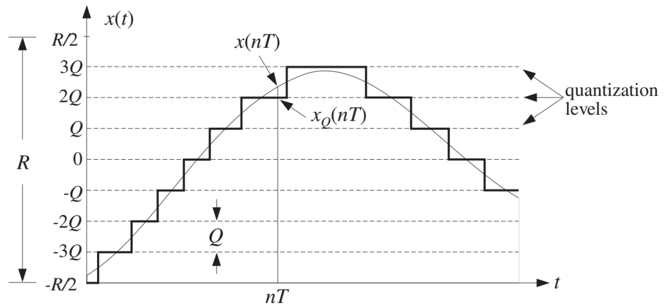


Figure 10: Signal quantization

Quantizer resolution or quantization width (step): $Q = \dfrac{R}{2^B}$

- A **bipolar** ADC: $-\dfrac{R}{2} < x_Q(nT) < \dfrac{R}{2}$
- A **unipolar** ADC: $0 < x_Q(nT) < R$

Quantization by ***rounding***: replace each value x(nT) by the ***nearest*** quantization level.

Quantization by ***truncation***: replace each value x(nT) by its ***below nearest*** quantization level.

Quantization error: $e(nT) = x_Q(nT) - x(nT)$. Consider rounding quantization: $-\dfrac{Q}{2} < e < \dfrac{Q}{2}$

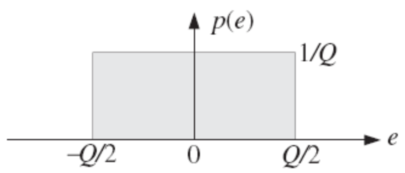


Figure 11: Uniform probability density of quantization error

The **mean value** of quantization error: $\overline{e} = \displaystyle\int_{-Q/2}^{Q/2} ep(e)de = \int_{-Q/2}^{Q/2} e\frac{1}{Q}de = 0$

The **mean-square error**: $\sigma_q^2 = \overline{e^2} = \displaystyle\int_{-Q/2}^{Q/2} (e-\overline{e})^2 p(e)de = \int_{-Q/2}^{Q/2} e^2\frac{1}{Q}de = \frac{Q^2}{12}$ (power)

Root-mean-square (rms) error: $e_{rms} = \sigma_q = \sqrt{\overline{e^2}} = \dfrac{Q}{\sqrt{12}}$

R and Q are the ranges of the signal and quantization noise, then the **signal to noise ratio (SNR)** or **dynamic range** of the quantizer is defined as

$$SNR_{dB} = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_q^2}\right) = 20\log_{10}\left(\frac{R}{Q}\right) = 20\log_{10}(2^B) = 6B\ dB$$

which is referred to as **6 dB bit rule.**

2.2 Digital to Analog Converters (DACs)

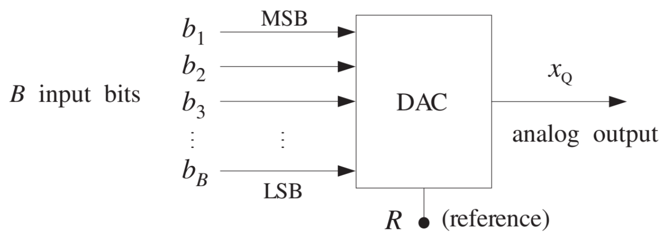


Figure 12: B-bit D/A converter

Vector B input bits: $b = [b_1, b_2, \dots, b_B]$. Note that b_B is the least significant bit (LSB) while b_1 is the most significant bit (MSB).

For unipolar signal, $x_Q \in [0, R)$; for bipolar $x_Q \in [-R/2, R/2)$.

- ***Unipolar natural binary*** $x_Q = R(b_1 2^{-1} + b_2 2^{-2} + ... + b_B 2^{-B}) = Qm$ where m is the integer whose binary representation is $b = [b_1, b_2, \dots, b_B]$.

$$m = b_1 2^{B-1} + b_2 2^{B-2} + ... + b_B 2^0$$

- ***Bipolar offset binary***: obtained by shifting the x_Q of unipolar natural binary converter by half-scale R/2:

$$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + ... + b_B 2^{-B}) - \frac{R}{2} = Qm - \frac{R}{2}$$

- ***Two's complement code***: obtained from the offset binary code by complementing the most significant bit, i.e., replacing b_1 by $\overline{b_1} = 1 - b_1$.

$$x_Q = R(\overline{b}_1 2^{-1} + b_2 2^{-2} + ... + b_B 2^{-B}) - \frac{R}{2}$$

2.3 A/D converters

A/D converters quantize an analog value x so that is is represented by B bits $b = [b_1, b_2, \dots, b_B]$

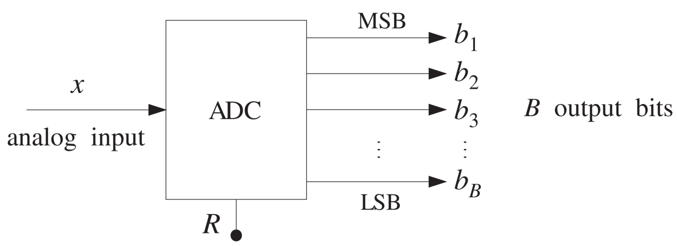


Figure 13: B-bit A/D converter

One of the most popular converters is the successive approximation A/D converter

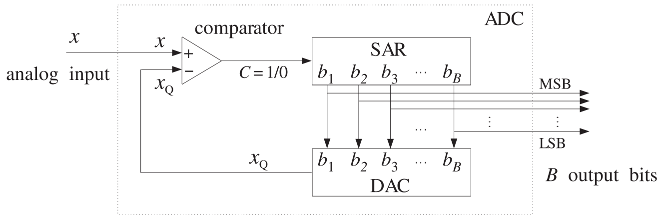
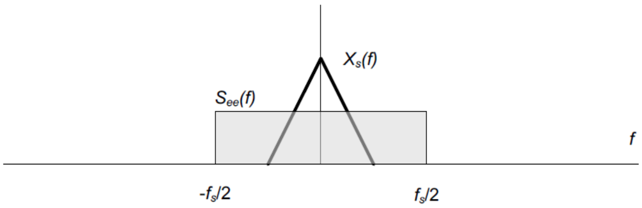


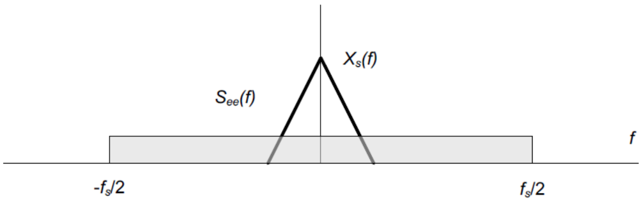
Figure 14: Successive approximation A/D converter

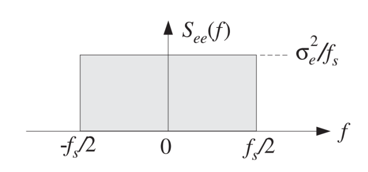
After B tests, the **successive approximation register (SAR)** will hold the correct bit vector b.

2.4 Oversampling and Noise shaping

Because the white noise is equally distributed over the Nyquist interval with an unchanged total average power, the noise power per unit frequency interval will be reduced if the Nyquist interval is enlarged.







$$S_{ee}(f) = \frac{\sigma_e^2}{f_s}, \quad \text{for} \quad -\frac{f_s}{2} \le f \le \frac{f_s}{2}$$

The noise power within any Nyquist subinterval $[f_a, f_b]$ of width $\Delta f = f_b - f_a$ is given by:

$$S_{ee}(f)\,\Delta f = \sigma_e^2 \frac{\Delta f}{f_s} = \sigma_e^2 \frac{f_b - f_a}{f_s}$$

As expected, the total power over the entire interval $\Delta f = f_s$ will be

$$\frac{\sigma_e^2}{f_s} f_s = \sigma_e^2$$

Consider two cases, one with sampling rate f_s and B bits per sample, and the other with higher sampling rate f'_s and B' bits per sample.

$$\boxed{L = \frac{f'_s}{f_s}} \qquad Q = R2^{-B}, \quad Q' = R2^{-B'} \qquad \sigma_e^2 = \frac{Q^2}{12}, \quad \sigma_e'^2 = \frac{Q'^2}{12}$$

L: Over sampling ratio

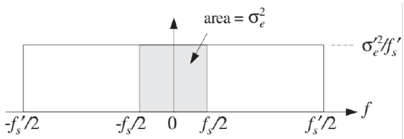
To maintain the same quality in the two cases, we require that the power spectral densities remain the same:

Noise shaping quantizers reshape the spectrum of the quantization noise into a more convenient shape

$$\frac{\sigma_e^2}{f_s} = \frac{\sigma_e'^2}{f_s'} \qquad \sigma_e^2 = f_s\,\frac{\sigma_e'^2}{f_s'} = \frac{\sigma_e'^2}{L}$$

$$L = \frac{\sigma_e'^2}{\sigma_e^2} = 2^{2(B-B')} = 2^{2\Delta B}$$

$$\boxed{\Delta B = 0.5\log_2 L}$$



L = 4 → ΔB = 1
L = 16 → ΔB = 2