CS70 HW9

March 14, 2021

1 Double-Check Your Intuition

(a) (i) $f_X(x) = Pr(X = x) = \binom{5}{x}(\frac{1}{4})^x(\frac{3}{4})^{5-x}$.
$f_Y(y) = Pr(Y = y) = \binom{5}{y}(\frac{3}{4})^y(\frac{1}{4})^{5-y}$.

(ii) $E(Z^2) = \frac{1}{6} * (1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$.

(b) False. The intuition of this problem is that, if $\omega \in \Omega$, and $A(\omega) = i$ and $\omega' \in \Omega$, and $B(\omega') = i$ but $\omega \cap \omega' = \emptyset$. For example,

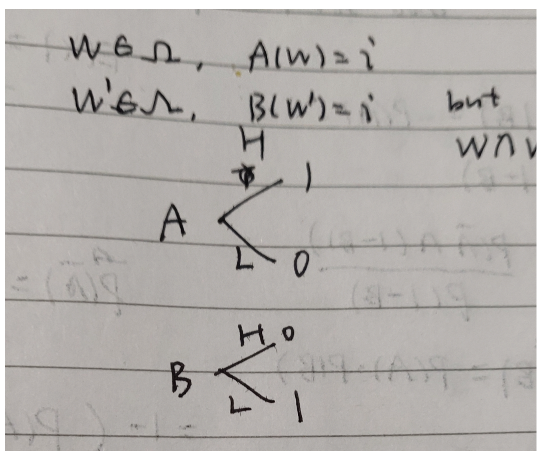

Figure 1: head tail

(c) False. Still from figure 1, $E(C^2) = \frac{1}{2}$, but $E(C)^2 = \frac{1}{4}$.

(d) False. We can generate a case where one point $X(\omega)$ is very very big, and elsewhere is just small. And $Y(\omega)$ are all very small. Give a picture to show:

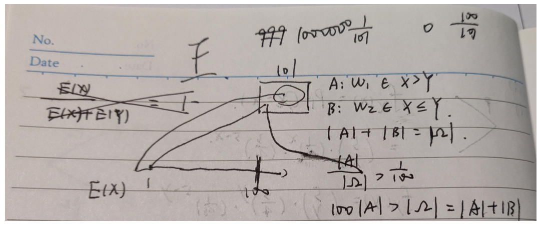

Figure 2: E(X) and E(Y)

(e) False. Still by figure 1, we can assign A as H - 2, T - 1. Event B as H - 3, T - 2. The RHS is $\frac{2}{5} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{11}{30}$. The LHS is $\frac{\frac{3}{2}}{\frac{3}{2}+\frac{5}{2}}$. Thus they are not equal.

(f) False. First according to the note, only if all subsets of A, B, C the equation hold, then they are mutually independent. Then give a counter example,

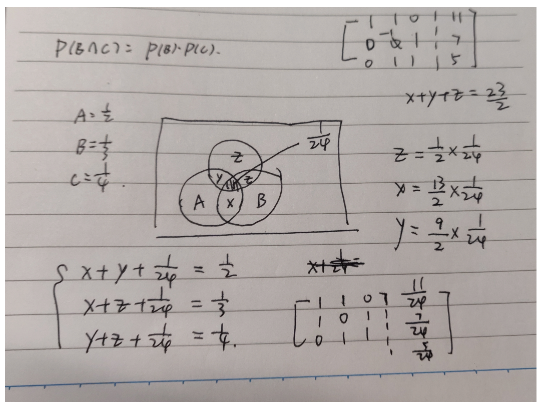

Figure 3:

where $P(A \cap B) \neq P(A) \cdot P(B)$.

Sol: Let A be an event with probability 0 and let B be some event with probability 1/2 and let C = B. Then $P(A \cap B \cap C) \leq P(A) = 0 = P(A)P(B)P(C)$ but B and C are clearly not independent.

(g) For most events, this statement is true. But with an exception is that the possibility space Ω where $P(\Omega) = 1$.

(h) True. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 + P(A) \cdot P(B) - (P(A) + P(B)) = (1 - P(A)) \cdot (1 - P(B)) = P(\overline{A}) \cdot P(\overline{B})$.

2 Airport Revisited

(a) X_i: airport i is empty.
$X = X_1 + X_2 + \cdots + X_n$.
$E(X_i) = P(X_i \text{ is empty}) = \frac{1}{4}$;
Therefore $E(X) = nE(X_i) = \frac{n}{4}$.

(b) X_i: airport i is empty.
$X = X_1 + X_2 + \cdots + X_n$.
$E(X_i) = P(X_i$ is empty). To help us understand this, we draw a picture, and therefore we can conclude that $P(X_i \text{ is empty}) = \sum i \in N(i)(1 - \frac{1}{deg(i)})$.
Therefore $E(X) = nE(X_i) = \sum_{i=1}^{n} \sum_{i \in N(i)} (1 - \frac{1}{deg(i)})$.

3 Fizzbuzz

(a) Set $S = \{1, 2, 4, 7, 8, 11, 13, 14\} \mod 15$. Thus P(integer) $= \frac{8}{15}$. The total number is $\frac{8}{15} * n$.

Or another method from part (b), $\frac{\phi(n)}{n} = (1 - \frac{1}{3})(1 - \frac{1}{5}) = \frac{8}{15}$. Therefore, $\phi(n) = \frac{8}{15} * n$.

(b) *Proof.* Combinational proof.

LHS: the probability to get a number relatively prime to the number n.
RHS: given a prime factor p_j, there are $\frac{n}{p_j}$ numbers that are multiple of p_j, therefore the probability to get a number relatively prime to the number p_j is $(1 - \frac{n}{p_j}/n) = 1 - \frac{1}{p_j}$. Thus for all P_j, to get a number relatively prime to n, it's equivalent to get a number relatively prime to each P_j. So we get the RHS $\prod_{j=1}^{k}(1 - \frac{1}{p_j})$. □

4 Cliques in Random Graphs

(a) $2^{\binom{n}{2}}$.

For each edge, exist or not, therefore 2 choices.
Total edge number, $\binom{n}{2}$.
Therefore, $2^{\binom{n}{2}}$. By first counting rule.

(b) Like a complete graph, total edge number $\frac{n(n-1)}{2}$. Thus P(k-clique of a set k) $= (\frac{1}{2})^{\frac{k(k-1)}{2}}$.

(c)

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!} < \frac{n!}{(n-k)!}$$
$$= n * (n-1) * \ldots (n-k+1)$$
$$< n * n * * \ldots n = n^k.$$

(d) Def event X: graph contains a k-clique.
X_i: this particular k vertices have a k-clique.
$X = X_1 \cup X_2 \ldots X_a$. where $a = \binom{n}{k}$.
Therefore, applying the union bound,

$$P(X) < \sum X_i = \binom{n}{k} \cdot (\frac{1}{2})^{\frac{k(k-1)}{2}}$$
$$\leq n^k \cdot 2^{\frac{k(1-k)}{2}}$$
$$\leq n^k \cdot (\sqrt{2}n^2)^{1-k}$$
$$= (\sqrt{2})^{1-k} \cdot n^{2-k}$$
$$\leq n^{2-k}$$

To prove n^{2-k} is less than n^{-1}, which is equivalent to prove $k \geq 3$. Since $k \geq 4logn + 1$, when $k = 3$, this equation doesn't hold, k must be larger than 3. Therefore, $n^{2-k} \leq n^{-1}$. The proof is done.

5 Balls and Bins, All Day Every Day

(a) If no constrain, # of bars: n - 1. # of stars: 2n - 1.
Now exactly k balls in first bin, # of bars: n - 2. # of stars: 2n - k - 1.
Therefore, $\binom{2n-k-1}{n-2}$.

(b) at least half of balls in the first bin $\implies \frac{n}{2}$ or $\frac{n}{2} + 1$ ...or n balls in the first bin.
Therefore,

$$\sum_{k=\frac{n}{2}}^{n} \binom{2n-k-1}{n-2}.$$

(c) Def event X: some bins contain at least half of the balls.
Event event X_i: i^{th} bin contain at least half of the balls.
We know that $X = X_1 \cup X_2 \ldots X_n$. Thus applying union bound, $P(X) \leq \sum_i X_i = n * p$.

(d) Def event X: at least half of the balls land in the first bin.
Event Y: at least half of the balls land in the second bin.
$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = p + p - \frac{1}{\binom{2n-1}{n-1}}$.

(e) First, we simplify the problem by assuming the first ball is in the first bin. Then by part (a), we know the probability of exactly k balls in the first bin. Therefore, conditioning on the number of balls in the first bin, we have

$$\sum_{k=1}^{n} \frac{1}{k} \binom{2n-k-1}{n-2}$$

Now we cancel the simplification, the first balls may uniformly at any balls, therefore, the answer is just multiply of above equation.

$$n \sum_{k=1}^{n} \frac{1}{k} \binom{2n-k-1}{n-2}$$

I don't think my solution in wrong, the question itself doesn't tell whether the balls are different or not. In usual sense, if only caring about the number of balls in a particular bin, it shouldn't matter. But the solution is such one.

So I just stick some pictures here.

(a) The probability that a particular ball lands in the first bin is $1/n$. We need exactly k balls to land in the first bin, which occurs with probability $(1/n)^k$, and we need exactly $n-k$ balls to land in a different bin, which occurs with probability $(1-1/n)^{n-k}$, and there are $\binom{n}{k}$ ways to choose which of the n balls land in first bin. Thus, the probability is $\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$.

Figure 4: part (a)

(d) The probability that the first bin has at least half of the balls is p; similarly, the probability that the second bin has at least half of the balls is also p. There is overlap between these two events, however: the first bin has half of the balls and the second bin has the second half of the balls. The probability of this event is $\binom{n}{n/2}n^{-n}$: there are n^n total possible configurations for the n balls to land in the bins, but if we require exactly $n/2$ of the balls to land in the first bin and the remaining balls to land in the second bin, there are $\binom{n}{n/2}$ ways to choose which balls land in the first bin. By the principle of inclusion-exclusion, our desired probability is $p+p-\binom{n}{n/2}n^{-n} = 2p-\binom{n}{n/2}n^{-n}$.

Figure 5: part (d)

an additional $k-1$ of the other $n-1$ balls landed in this bin, which by the reasoning in Part (a) has probability

$$\mathbb{P}(A_k) = \binom{n-1}{k-1} (1/n)^{k-1} (1-1/n)^{n-k} \ .$$

If we let B be the event that we pick up the first ball we threw, then

$$\mathbb{P}(B \mid A_k) = 1/k$$

since we are equally likely to pick any of the k balls in the bin. Thus the overall probability we are looking for is, by an application of the law of total probability,

$$\mathbb{P}(B) = \sum_{k=1}^{n} \mathbb{P}(A_k \cap B) = \sum_{k=1}^{n} \mathbb{P}(A_k)\mathbb{P}(B \mid A_k) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \left(\frac{1}{n}\right)^{k-1} \left(1 - \frac{1}{n}\right)^{n-k}.$$

Figure 6: part (e)