CS70 HW 12

March 15, 2021

1 Safeway Monopoly Cards

(a) I have no idea about the solution this question. The hint must be something about Collector Coupon Problem and the Expectation and Variance of Geometric(p). But i can't see the connection.

$$\begin{aligned}
Var(X) &= Var(\sum_{i=1}^{n} X_i) \\
&= \sum_{i=1}^{n} Var(X_i) && independence \\
&= \sum_{i=1}^{n} \frac{1 - \frac{n-i+1}{n}}{(\frac{n-i+1}{n})^2} && p = \frac{n-i+1}{n} \\
&= \sum_{j=1}^{n} \frac{1 - j/n}{(j/n)^2} && n - i + 1 = n \to 1 \\
&= \sum_{j=1}^{n} \frac{n9n - j)}{j^2} \\
&= \sum_{j=1}^{n} n^2 j^{-2} - n \sum_{j=1}^{n} i^{-1} \\
&= n^2 \sum_{j=1}^{n} j^{-2} - E(X)
\end{aligned}$$

2 Geometric Distribution

(a) From the description, we can infer that $X = X_1 \cup X_2$. Thus $P(X = 1) = P(X_1 = 1 \cap X_2 = 1) = P(X_1 = 1) + P(X_2 = 1) - P(X_1 = 1 \cap X_2 = 1) = p_1 + p_2 - p_1 p_2$.

Therefore, it's also a geometric distribution with parameter $p' = p_1 + p_2 - p_1p_2$.

It seems that it's not that simple. We should consider this problem more carefully. And $X \neq X_1 \cup X_2$ but $X = min\{X_1, X_2\}$.

Here the best way to solve this is that to show the tail probability is Geometric Contribution.

$$\begin{aligned}
Pr(X \geq k) &= Pr(min\{X_1, X_2\} \geq k) \\
&= Pr(X_1 \geq k \cap X_2 \geq k) \\
&= (1-p_1)^{k-1} * (1-p_2)^{k-1} \\
&= (1-(p_1+p_2-p_1p_2))^{k-1}
\end{aligned}$$

Thus $p' = p_1 + p_2 - p_1p_2$.

(b) From the distribution of X, we can know that the probability for the first person to find is that

$$\sum_{i=0}^{\infty}(1-p')^{2i} \cdot p' = \frac{p'}{1-(1-p')^2}$$

where $p' = p_1 + p_2 - p_1p_2$.

3 Geometric and Poisson

(a)

$$\begin{aligned}
Pr(X > Y) &= \sum_{x=1}^{\infty}\sum_{y=0}^{x-1}\frac{\lambda^y}{y!}e^{-\lambda} \cdot (1-p)^{x-1}p \\
&= \sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty}\frac{\lambda^y}{y!}e^{-\lambda} \cdot (1-p)^{x-1}p \\
&= \sum_{y=0}^{\infty}\frac{\lambda^y}{y!}e^{-\lambda} \cdot \sum_{x=y+1}^{\infty}(1-p)^{x-1}p \\
&= \sum_{y=0}^{\infty}\frac{\lambda^y}{y!}e^{-\lambda} \cdot (1-p)^y \\
&= \frac{\sum_{y=0}^{\infty}\frac{[\lambda(1-p)]^y}{y!}e^{-\lambda(1-p)}}{e^{1-p}} \\
&= \frac{1}{e^{1-p}} = e^{p-1}.
\end{aligned}$$

We are almost right, but a simple mistake here. The last two steps, it should be

$$\begin{aligned} Pr(X > Y) &= \frac{\sum_{y=0}^{\infty} \frac{[\lambda(1-p)]^y}{y!} e^{-\lambda(1-p)}}{e^p} \\ &= \frac{1}{e^{\lambda p}} = e^{-\lambda p} \end{aligned}$$

(b) Since $max\{X,Y\} \geq X$, we infer that $P(max\{X,Y\} \geq X) = 1$.

(c) Since $max\{X,Y\} \geq Y$ and now we want $max\{X,Y\} \leq Y$, we can infer that $max\{X,Y\} = Y$, which is same as $P(X \leq Y) = 1 - P(X > Y) = 1 - e^{p-1}$.

Because of part(a), the right answer shall be $1 - e^{-\lambda p}$.

4 Darts

(a) $f_X(x) = exp(-x)$. First we check that the integral is 1.

$$\int_0^{\infty} e^{-x}\, dx = \left.\frac{e^{-x}}{-1}\right|_{x=0}^{x=\infty} = 1$$

Then Pr(dart will stay within the board) =

$$\int_0^{4} e^{-x}\, dx = \left.\frac{e^{-x}}{-1}\right|_{x=0}^{x=4} = 1 - e^{-4}.$$

(b) Pr(within 1 unit | within 4 unit) $= \frac{\text{Pr(within 1 unit)}}{\text{P(within 4 unit)}} =$

$$\frac{\int_0^1 e^{-x}\, dx}{\int_0^4 e^{-x}\, dx} = \frac{1-e^{-1}}{1-e^{-4}}$$

(c) Pr(Score) =

$$\begin{cases} 1 - e^{-1} & 0 \leq x \leq 1, S = 4 \\ e^{-1} - e^{-2} & 1 \leq x \leq 2, S = 3 \\ e^{-2} - e^{-3} & 2 \leq x \leq 3, S = 2 \\ e^{-3} - e^{-4} & 3 \leq x \leq 4, S = 1 \end{cases}$$

Thus we have the E(S) =

$$4(1-e^{-1}) + 3(e^{-1} - e^{-2}) + 2(e^{-2} - e^{-3}) + 1(e^{-3} - e^{-4}) = 4 - \sum_{i=1}^{4} e^{-i}$$

5 Exponential Practice

(a) We can compute as following:

$$\begin{aligned} F_Y(y) &= \int_0^y \int_0^{c-x_1} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2}\, dx_1\, dx_2 \\ &= \int_0^y e^{-\lambda x_1} \cdot \int_0^{c-x_1} \lambda e^{-\lambda x_2}\, dx_2\, dx_1 \\ &= \int_0^y e^{-\lambda x_1} \cdot \frac{e^{-\lambda x_2}}{-\lambda}\Bigg|_0^{c-x_1} dx_1 \\ &= \int_0^y e^{-\lambda x_1} \cdot (1 - e^{-\lambda(c-x_1)})\, dx_1 \\ &= \int_0^y e^{-\lambda x_1}\, dx_1 - \int_0^y \lambda e^{-\lambda y}\, dx_1 \\ &= 1 - e^{-\lambda y} - \lambda y e^{-\lambda y} \end{aligned}$$

Then differentiate it, we obtain

$$\begin{aligned} f_Y(y) &= \lambda e^{-\lambda y} - [\lambda e^{-\lambda y} - \lambda^2 y e^{-\lambda y}] \\ &= \lambda^2 y e^{-\lambda y} \end{aligned}$$

(b)

$$\begin{aligned} \frac{Pr(X_1 \le x \cap x_1 + x_2 = t)}{Pr(x_1 + x_2 = t)} &= \frac{\text{Same as part(a)}}{f_T(t)\, dt} \\ &= \frac{1 - e^{-\lambda x} - \lambda x e^{-\lambda x}}{\lambda^2 t e^{-\lambda t}\, dt} \\ &= \text{I don't know the next} \\ &= \text{now maybe I know} \\ &= \text{this is wrong} \end{aligned}$$

$$\begin{aligned} \frac{Pr(X_1 \le x \cap x_1 + x_2 = t)}{Pr(x_1 + x_2 = t)} &= \frac{\int_0^x \lambda^2 t e^{-\lambda t}\, dt\, dx}{\int_0^t \lambda^2 t e^{-\lambda t}\, dt\, dx} \\ &= \frac{x \lambda^2 t e^{-\lambda t}\, dt}{t \lambda^2 t e^{-\lambda t}\, dt} \\ &= \frac{x}{t} \end{aligned}$$

Therefore, pdf of X_1 conditioned on $X_1 + X_2 = t$ is $\frac{1}{t}$.

6 Uniform Means

(a) By tail sum formula, $E(X) = \int_0^\infty P(X > x)\,dx$, we have

$$\begin{aligned}
E(Y) &= \int_0^\infty P(Y > y)\,dy \\
&= \int_0^1 P(Y > y)\,dy \\
&= \int_0^1 (\frac{1-y}{1})^n\,dy \\
&= -\int_0^1 (\frac{1-y}{y})^n\,d(1-y) \\
&= -\frac{(1-y)^{n+1}}{n+1}\bigg|_0^1 \\
&= \frac{1}{n+1}
\end{aligned}$$

(b) Find the CDF, $F_Z(z) = Pr(Z \leq z) = (\frac{z}{1})^n = z^n$. Therefore, $f_Z(z) = n \cdot z^{n-1}$.

$$\begin{aligned}
E(Z) &= \int_0^1 zn \cdot z^{n-1}\,dz \\
&= n \cdot \int_0^1 z^n\,dz \\
&= n \cdot \frac{z^{n+1}}{n+1}\bigg|_0^1 \\
&= \frac{n}{n+1}
\end{aligned}$$