CS70 HW 7

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1 Bijective or not?

(a) $f(x) = 10^{-5}x$

  (i) Bijection. There exists an inverse function, $g(x) = 10^5 y$.

  (ii) It's injective but not subjective. For any x_1, x_2 that $x_1 \neq x_2, f(x_1) \neq f(x_2)$. But not onto, since elements like irrational numbers, e.g $\sqrt{2}$ can't find x that $f(x) = \sqrt{2}$.

(b) (i) Not one-to-one. For any $x \in \mathbb{N}$ that $x > p, f(x) \equiv p \mod x$.

  Not onto. For $y = p - 1$, there is no $x \in \mathbb{N}$ that $f(x) = p - 1$.

  (ii) For domain $\{(p+1)/2, \ldots, p\}, f(x) = p \pmod{x} = p - \lfloor \frac{p}{x} \rfloor * x$. Since $x \geq (p+1)/2 > p/2$, thus $\lfloor \frac{p}{x} \rfloor = 1$. Thus $f(x) = p - x$, which is one-to-one.

  Since the # of elements in the domain and range are same and the function is one-to-one, it's on-to.

(c) One-to-one. For any x_1, x_2 that $x_1 \neq x_2$, set $\{x_1\} \neq \{x_2\} \implies f(x_1) \neq f(x_2)$.

  Not onto. For any set S in powerset of D that |S| ≥ 2, there is no $x \in D$ that $f(x) = S$.

(d) Bijection.

  After shuffling, the number X' is just a permutation of X. And we can list them as the natural number and their total number are both 10.

2 Counting Tools

(a) Countable. Since A and B are countable, we can view them as natural number and therefore we can define a map $N \times N$ to N^2. Since N^2 is countable, which is shown in the note by spiral ordering. Thus it's countable.

(b) Countable. We can define a map from jth element in ${}_{i\in A}B_i$ to (i, j), which is ultimately as N^2. Therefore it's countable.

(c) Uncountable.

Proof. Proof by contradiction.

Say it's countable, then we can enumerate them where number in the diagonal are non-decreasing

f	f(0) f(1) ...
g	g(0) g(1) ...
$\vdots$	$\vdots$

Then we can define a new map from the Diagonalization by modifying each number d to make $d' = 2 * d$. Then with the similar logic in the note, this map can't be placed in the listing. So it's uncountable. $\square$

Some mistake here, we can't let the listing satisfy some certain property like I did to make the diagonal elements none-decreasing, but we need to construct from our side instead.
For example, here we can def f(i) is $max\{f_0(0), \ldots, f_i(i)\} + 1$. It's still different functions and the property is successfully constructed.

(d) Uncountable.

Proof. Proof by contradiction.

Say it's countable, then we can enumerate them where number in the diagonal are non-increasing

f	f(0) f(1) ...
g	g(0) g(1) ...
$\vdots$	$\vdots$

Then we can define a new map from the Diagonalization by modifying each number d to make $d' = \lfloor d/2 \rfloor$. Then with the similar logic in the note, this map can't be placed in the listing. So it's uncountable. $\square$

Mistake here: f(0), f(1), ...are not infinite. Because it's none-increasing, there must be some finite steps K that make f(K) = 0, which is the destination of the natural number! Thus we can use this decreasing points to represent this function. And $K \leq n$ if we let the subset D of f that D(0) = n, since at each decreasing step, it must at least decrease 1. Then since N^n is countable, then D(n) is countable, which leads to the fact that the function f is $\cup_{i\in N} D(i)$ is countable from part (b).

Therefore, it's countable.

(e) Uncountable.

Proof. Proof by contradiction.

Say it's countable, then we can enumerate them where number in the diagonal are also bijective functions

f	f(0) f(1) ...
g	g(0) g(1) ...
$\vdots$	$\vdots$

Then we can define a new map from the Diagonalization by modifying each number d to make $d' = d + 1$. Then with the similar logic in the note, this map can't be placed in the listing. So it's uncountable. □

Mistake here: the function we construct may not be bijection. **Reason as before: we can't ask certain property in the diagonal direction.**

The rest proof is a little complicated, I just restate here in brief. For more concrete sol, go to see solution.

Total idea: find a injective map from subset of N, aka power set of N to bijective function. For any subset S of Power-set(N) except $N\{X\}$, we can let S set corresponding to identity function f(x) = x, while $\overline{S}$ to the function $g(x) = shuffle(x)$ where $x \neq g(x)$.

Why S must exclude $N\{X\}$? Because $\overline{S} = \{X\}$, there is no shuffle function making $x \neq g(x)$.

3 Impossible Programs

(a) can't exist.

Proof. Proof by contradiction.

Say such program exists, called TESTXY(P, x, y). Then we can use this program as subroutine to def Halt program, namely Halt(P, x).

```
Halt(P, x)
#def P' as: modify P to suppress exit
#and return statements and append return y
If TESTXY(P', x, y): return True.
If not TESTXY(P', x, y): return False.
```

Therefore, if such TESTXY(P, x, y) exists, we can generate Halt problem, which we know doesn't exists. So TESTXY doesn't exists. □

(b) can't exist.

Proof. Proof by contradiction.

Say such program exists, called HALTFG(F, G, x). Then we can use this program as subroutine to def Halt program, namely Halt(P, x).

```
Halt(P, x)
If HALTFG(P, P, x): return True.
If not HALTFG(P, P, x): return False.
```

Therefore, if such HALTFG(F, G, x) exists, we can generate Halt problem, which we know doesn't exists. So such program like HALTFG doesn't exists. □

4 Undecided?[Most abstract one]

But I get it right!!!
This problem can be mapped as a 2-D graph, where (l, j) denotes the state c_j will be executed next by instruction i_l. And if we start at the origin, and go along all the points, it ultimately form a directed graph that each vertex can have at most 1 edge.

(a) $n * k$.

(b) We can view the graph without a circle as a tree. And since the biggest number of edge is nk, after $2n^2k^2$ steps, it will form a circle, and no longer be a tree.

(c) From part (b), we know if this algorithm is still running after $2n^2k^2$ iterations, it will loop forever. Therefore, we can just see the result after after $2n^2k^2$ iterations, if it does loop, then return Loop, Halt otherwise.

Don't contradict the undecidability of halting problem. Here a computer's state is finite, k is also finite, and thus make $2n^2k^2$ also a finite number. But in halting problem, n and k are infinite, which makes it undecided.

5 Clothing Argument

(a) 10^4 outfits. Since by the first rule of counting, each time we have 10 choices, and we have 4 times to choose.

(b) $\binom{4}{2} \cdot 10^2 = 600$. Since each time we we have 10 choices, and we have $\binom{4}{2}$(to determine two catagories) times to choose.

(c) 10*9*8*7 = 5040. Since first we have 10 choices, then since first one is picked, we are left with 9, etc.

(d) $\binom{10}{4}$ possibilities. From part c, we have a map from $4!$ to 1, since $4!$ is permutation of 4 different hats, which leads to the equation

$$\binom{10}{4} = \frac{10!}{6! \cdot 4!}$$

(e) This is the situation where sampling with replacement for the number of hat of each color is greater than 3 and order does not matter. Here $n = 3, k = 3$, answer is $\binom{5}{3} = 10$.

More concretely, there are 3 stars and 2 bars and we assume stars before the 1st bar is red, 2nd is green, behind 2nd is turquoise.