CS70 HW 10

March 14, 2021

1 Family Planning

(a) $\{g\} : \frac{1}{2}$
$\{b, g\} : \frac{1}{4}$
$\{b, b, g\} : \frac{1}{8}$
$\{b, b, b\} : \frac{1}{8}$

(b)

	C = 1	C = 2	C = 3
G = 0	0	0	$\frac{1}{8}$
G = 1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

(c)

P(G = 0)	$\frac{1}{8}$
P(G = 1)	$\frac{7}{8}$

P(C = 1)	P(C = 2)	P(C = 3)
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(d) Not indenpendent. $P(C = 1, G = 0) = 0 \neq P(C = 1) * P(G = 0)$.

(e) E(G) $= \frac{7}{8}$.
E(C) $= \frac{1}{2} + 2 * \frac{1}{4} + 3 * \frac{1}{4} = \frac{7}{4}$.

2 Will I Get My Package

(a) Define an indicator $X_i : i^{th}$ person receives his package and unopened. Then event $X = X_1 + X_2 + \cdots + X_n$.
$P(X_i = 1) = \frac{1}{2}\frac{(n-1)!}{n!} = \frac{1}{2n}$. Therefore, $E(X_i) = p = \frac{1}{2n}$. $E(X) = \sum_i E(X_i) = n * p = \frac{1}{2}$.

(b) $Var(X) = E(X^2) - E(x)^2 = E(X^2) - \frac{1}{4}$. Now we compute

$$\begin{aligned} E(X^2) &= \sum_{i=1}^{n}(X_i^2) + \sum_{i\neq j}(X_i)(X_j) \\ &= n * \frac{1}{2n} + n(n-1)\frac{1}{4}\frac{(n-2)!}{n!} = \frac{3}{4}. \end{aligned}$$

Therefore, $Var(X) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

3 Double-Check Your Intuition Again

(a) (i)

$$\begin{aligned} Var(X+Y, X-Y) &= E((X+Y)(X_Y)) - E(X+Y)E(X-Y) \\ &= E(X^2 - Y^2) - E(X-Y)E(X+Y) \\ &= 0 - 0 * E(X+Y) = 0 \end{aligned}$$

(ii) *Proof.* Say $P(X-Y=0) = \frac{6}{36} = \frac{1}{6}$. But $P(X-Y=0 \mid X+Y=12) = 1 \neq P(X-Y=0)$. □

(b) True.

$Var(X) = E((X-E(X)^2)$. Since $(X-E(X)^2 \geq 0 \implies Var(X) \geq 0$. Only when $X - E(X) = 0$ everywhere, which implies that $X = E(X)$ everywhere, $X = E(X) = C$.

(c) False. $Var(cX) = c^2 Var(X)$.

$$\begin{aligned} Var(cX) &= E(c^2X^2) - E(cX)^2 \\ &= c^2E(X^2) - c^2E(X)^2 \\ &= c^2[E(X^2) - E(X)^2] \\ &= c^2Var(X) \end{aligned}$$

(d) False.

Corr(A, B) = 0 and A, B are R.V with nonzero standard deviations $\implies$ Cov(A, B) = 0. But it's not the same that P(AB) = P(A)P(B). Consider (-1, 0) (0, 1) (1,0) (0, -1) with prob $\frac{1}{4}$ where Cov(A, B) = 0 but P(AB) $\neq$ P(A)P(B).

(e) True.

$$\begin{aligned} var(X+Y) &= E((X+Y)^2) - E(X+Y)^2 \\ &= E(X^2 + Y^2 + 2XY) - (E(X) + E(Y))^2 \\ &= E(2XY) - 2E(X)E(Y) \\ &= 0 && \text{based on Corr(X, Y) = 0} \end{aligned}$$

(f) True.

For any w $\in \Omega$, we have either case:

(i) Case 1, X(w) > Y(w).

$$max(X,Y)(w) = X(w) \tag{1}$$
$$min(X,Y)(w) = Y(w) \tag{2}$$

multiply (1)(2) equations, we get exactly what we want.

(ii) Case 2, X(w) $\leq$ Y(w).

$$max(X,Y)(w) = Y(w) \tag{3}$$
$$min(X,Y)(w) = X(w) \qquad \text{based on Corr(X, Y) = 0} \tag{4}$$

multiply (3)(4) equations, we get exactly what we want.

(g) False.

Proof. Since X and Y independent, Corr(X, Y) = 0. Therefore, if Corr(max(X, Y), min(X, Y)) = 0, then we have

$$\begin{aligned} Corr(max(X,Y), min(X,Y)) &= E(max(X,Y)min(X,Y)) - E(max(X,Y))E(min(X,Y)) \\ &= E(XY) - E(max(X,Y))E(min(X,Y)) \\ &= E(X)E(Y) - E(max(X,Y))E(min(X,Y)) \\ &= 0. \end{aligned}$$

But the last equation here is false, we can give a counter-example, E(X)E(Y) $\neq$ 0 but E(min(X, Y)) = 0.

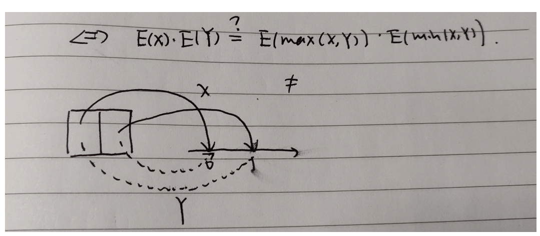

Figure 1: counter-example

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