CS70 HW 13

March 17, 2021

1 Buffon's Needle on a Grid

Before we formally solve the following questions, I think the most important part is to model this problem, maybe with the similar method talked in the lecture.
Setting: center (x, y) where 0 < x < 1/2 ; 0 < y < 1/2 and $f_{X,Y}(x, y) = 4$.

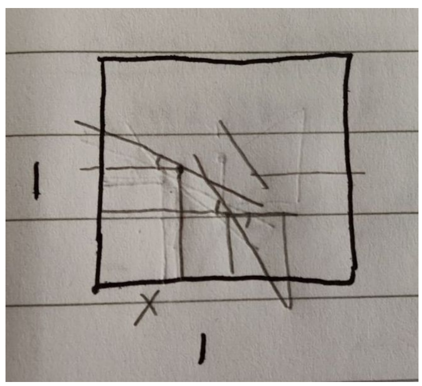

Figure 1: A sample square

And we can also get

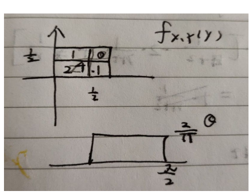

Figure 2: Distribution of Union and θ

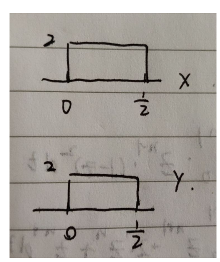

Figure 3: $distribution of X and Y$

Then having those setting, we can compute as following:

(a) $Pr(X > \frac{1}{2}cos\theta \cap Y > \frac{1}{2}sin\theta \mid \Theta = \theta) = 4 * (\frac{1}{2} - \frac{1}{2}cos\theta)(\frac{1}{2} - \frac{1}{2}sin\theta)$.

(b) By the law of total probability, we have

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}} Pr(X > \frac{1}{2}cos\theta \cap Y > \frac{1}{2}sin\theta \mid \Theta = \theta) \cdot Pr(\Theta = \theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} 4 * (\frac{1}{2} - \frac{1}{2}cos\theta)(\frac{1}{2} - \frac{1}{2}sin\theta) \cdot \frac{\pi}{2}\, d\theta \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (1 - cos\theta)(1 - sin\theta)\, d\theta \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} 1 - cos\theta - sin\theta + \frac{1}{2}sin(2\theta)\, d\theta \\
&= \frac{2}{\pi} \cdot [\frac{\pi}{2} - \frac{3}{2}] \\
&= 1 - \frac{3}{\pi}
\end{aligned}
$$

Therefore, Pr(intersects a grid line) $= 1 - (1 - \frac{3}{\pi}) = \frac{3}{\pi}$.

(c) Now we wanna compute Pr(intersects twice)

$$
\begin{aligned}
&= Pr(X < \frac{1}{2}cos\theta \cap Y < \frac{1}{2}sin\theta \mid \Theta = \theta) \cdot Pr(\Theta = \theta)\, d\theta \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} cos\theta sin\theta\, d\theta \\
&= \frac{1}{\pi}
\end{aligned}
$$

Therefore, Pr(twice) $= \frac{1}{\pi}$, Pr(once) $= \frac{3}{\pi} - \frac{1}{\pi} = \frac{2}{\pi}$. Then we compute $E(X) = 1 * \frac{2}{\pi} + 2 * \frac{1}{\pi} = \frac{4}{\pi}$.

An alternative Solutions: Since we know that E(cross a line in a parallel lines) $= \frac{2}{\pi}$. If we define X_1 as vertical, X_2 as horizontal, then E(X) $= E(X_1) + E(X_2) = 2 * \frac{2}{\pi} = \frac{4}{\pi}$.

(d) Pr(X = 1) $= \frac{2}{\pi}$.

(e) If we define R.V Y_i: the number of intersections with length $\frac{1}{3}$ of i^{th} edge. Then we have $Y = \sum_i Y_i$. Therefore, $E(Y) = E(\sum_i Y_i) = \sum_i E(Y_i) = 3 * E(Y_i)$. Now we compute $E(Y_i)$ same as before.

Pr(X = 0)

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} Pr(X > \frac{1}{6}cos\theta \cap Y > \frac{1}{6}sin\theta \mid \Theta = \theta) \cdot Pr(\Theta = \theta)\, d\theta \\
&= \int_0^{\frac{\pi}{2}} 4 * (\frac{1}{2} - \frac{1}{6}cos\theta)(\frac{1}{2} - \frac{1}{6}sin\theta) \cdot \frac{\pi}{2}\, d\theta \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (1 - \frac{1}{3}cos\theta)(1 - \frac{1}{3}sin\theta)\, d\theta \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} 1 - \frac{1}{3}cos\theta - \frac{1}{3}sin\theta + \frac{1}{18}sin(2\theta)\, d\theta \\
&= \frac{2}{\pi} \cdot [\frac{\pi}{2} - \frac{5}{9}] = \frac{2}{\pi} \cdot [\frac{\pi}{2} - \frac{2}{3} + \frac{1}{18}] \\
&= 1 - \frac{10}{9\pi} = 1 - \frac{11}{9\pi}
\end{aligned}
$$

Therefore, Pr(intersects a grid line) $= 1 - (1 - \frac{10}{9\pi}) = \frac{10}{9\pi}$.

Correct: Pr(intersects a grid line) $= 1 - (1 - \frac{11}{9\pi}) = \frac{11}{9\pi}$

Now we wanna compute Pr(intersects twice)

$$
\begin{aligned}
&= Pr(X < \frac{1}{6}cos\theta \cap Y < \frac{1}{6}sin\theta \mid \Theta = \theta) \cdot Pr(\Theta = \theta)\, d\theta \\
&= \frac{1}{\pi}\frac{1}{9} \int_0^{\frac{\pi}{2}} 2cos\theta sin\theta\, d\theta \\
&= \frac{1}{9\pi}
\end{aligned}
$$

Therefore, Pr(twice) $= \frac{1}{9\pi}$, Pr(once) $= \frac{10}{9\pi} - \frac{1}{9\pi} = \frac{1}{\pi}$. Then we compute $E(X) = 1 * \frac{1}{\pi} + 2 * \frac{1}{9\pi} = \frac{11}{9\pi}$.

Correct: Pr(twice) $= \frac{1}{9\pi}$, Pr(once) $= \frac{11}{9\pi} - \frac{1}{9\pi} = \frac{10}{9\pi}$. Then we compute

$$E(X_i) = 1 * \frac{10}{9\pi} + 2 * \frac{1}{9\pi} = \frac{4}{3\pi}$$

Then

$$E(X) = 3(X_i) = \frac{4}{3\pi} * 3 = \frac{4}{\pi}.$$

Sol: Actually it's not that difficult, we needn't compute all again.

Let Y_i be the number of times the i-th side of the triangle intersects a grid line, for i = 1, 2, 3. Thus, the total number of times this triangle intersects a grid line is $Y_1 + Y_2 + Y_3$.

Let us revisit part (c), however. Let Z_1 be the number of times the first $\frac{1}{3}$ of the needle intersects a grid line, and let Z_2 be the number of times the second $\frac{1}{3}$ of the needle intersects the grid line, and similarly for Z_3. We know that

$$E(Z_1 + Z_2 + Z_3) = \frac{4}{\pi}$$

But we also know that $E(Y_i) = E(Z_i)$. Therefore,

$$E(Y_1 + Y_2 + Y_3) = E(Z_1 + Z_2 + Z_3) = \frac{4}{\pi}$$

2 Variance of the Minimum of Uniform Random Variables

We obtain pdf first by its CDF

$$\begin{aligned} Pr(Y \leq y) &= 1 - Pr(Y > y) \\ &= 1 - (\frac{1-y}{1})^n \\ &= 1 - (1-y)^n \end{aligned}$$

Then if we differentiate it, we obtain

$$f_Y(y) = n \cdot (1-y)^{n-1}$$

And we can obtain

$$\begin{aligned} E(Y) &= \int_0^1 yn \cdot (1-y)^{n-1}\, dz \\ &= n \int_0^1 (1-z) z^{n-1}\, dz && z = 1 - y \\ &= n \int_0^1 z^{n-1} - z^n\, dz \\ &= n(1/n - 1/(n+1)) \\ &= \frac{1}{1+n} \end{aligned}$$

$$
\begin{aligned}
E(Y^2) &= \int_0^1 y^2 n \cdot (1-y)^{n-1}\, dz \\
&= n\int_0^1 (1-z)^2 z^{n-1}\, dz && z = 1-y \\
&= n\int_0^1 z^{n+1} - 2z^n + z^{n-1}\, dz \\
&= n(1/(n+2) - 2/(n+1) + 1/n) \\
&= \frac{2}{(1+n)(2+n)}
\end{aligned}
$$

Therefore, $Var(Y) = E(X^2) - E(X)^2 = \frac{2}{(1+n)(2+n)} - \frac{1}{(1+n)(1+n)}$.

3 Erasures, Bounds, and Probabilities

(a) $X = X_1 + X_2 + \cdots + X_{1000}$ where X_i is an indicator with p prob of 1 value, 1 - p prob of 0 value. $E(X_i) = p, E(X) = E(\sum_i X_i) = \sum_i E(X_i) = 1000p$.

Thus by Markov's inequality,

$$Pr(X \geq 200) < \frac{E(X)}{200} = \frac{1000p}{200} = 5p \leq 10^{-6}$$

Therefore, $p \leq 2 * 10^{-7}$.

(b) $Var(X_i) = p(1-p)$. $Var(X) = Var(\sum_i X_i) = \sum_i Var(X_i) = 1000p(1-p)$.

Thus by Chebyshev's inequality,

$$Pr(X > 200) \leq Pr(|X - 1000p| \geq 200 - 1000p) \leq \frac{1000p(1-p)}{(200-1000p)^2} \leq 10^{-6}$$

Then we have

$$
\begin{aligned}
1001p^2 - 1000.4p + 0.04 \geq 0 \\
p \leq 3.99856 \times 10^{-5}
\end{aligned}
$$

(c) $E(X) = 1000p. Var(X) = 1000p(1-p)$. Therefore, we define a new R.V

$$Y = \frac{X - 1000p}{\sqrt{1000p(1-p)}}$$

When X = 200, $y = \frac{200-1000p}{\sqrt{1000p(1-p)}}$ and we want $\Phi(y) \geq 1 - 10^{-6}$. From Normal Accumulative Distribution table, we know that $\Phi(4)$ satisfy this equation. Therefore, we have

$$\frac{200-1000p}{\sqrt{1000p(1-p)}} \geq 4$$
$$(10^6+16000)p^2+(-4*10^5-16*10^3)p+4*10^4 \geq 0$$
$$p \leq 0.1543$$

4 Sampling a Gaussian With Uniform

(a) $Expo(1) = \lambda \cdot e^{-\lambda t}$. Now we wanna prove that the tail summation of $-lnU_1$ and Expo(1) are same. For Expo(1), we know that $Pr(X > t) = e^{-\lambda t} = e^{-t}$. For $-lnU_1$,

$$\begin{aligned} Pr(-lnU_1 \geq t) &= Pr(lnU_1 \leq -t) \\ &= Pr(U_1 \leq e^{-t}) \\ &= \frac{e^{-t}}{1} \\ &= e^{-t} \end{aligned}$$

Therefore, $-lnU_1 \to Expo(1)$.

(b) Still from CDF, we have

$$\begin{aligned} Pr(N_1^2+N_2^2 \leq r^2) &= \int\int_R \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-y^2}{2}}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^r \frac{1}{2\pi} \cdot e^{-r^2(cos(\theta)^2+sin(\theta)^2/2)}\, dr\, d\theta \quad x = rcos(\theta), y = rsin(\theta) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r re^{\frac{-r^2}{2}}\, dr\, d\theta \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^r re^{\frac{-r^2}{2}}\, d(\frac{-r^2}{2})\, d\theta \\ &= -\frac{1}{2\pi} * 2\pi * e^{\frac{-r^2}{2}}\Big|_0^r \\ &= 1-e^{\frac{-r^2}{2}} \end{aligned}$$

And for Expo(1/2), we know that $Pr(X < r) = 1-e^{\frac{-r^2}{2}}$. Therefore, $N_1^2+N_2^2 \to Expo(1/2)$.

(c) Since we know the point (N_1, N_2) will have a distance from the origin that is distributed as the square root of an exponential distribution. So we can make $Expo(1/2)$ by $1/2\sqrt{-lnU_1}$, denoted as R. And if we let $2\pi U_2$ denotes θ, since

$N_1 = Rcos(\theta), N_2 = Rsin(\theta)$, we can get a normal distribution by $Rcos(\theta) = 1/2\sqrt{-lnU_1} \cdot 2\pi U_2$.

What is the wrong? We can't get $Expo(1/2)$ by $1/2\sqrt{-lnU_1}$. Here I made a mistake to have pdf multiplied by each other. The right answer shall be $-2lnU_1 \to Expo(1/2)$. Then $R = \sqrt{-2lnU_1}, Rcos\theta = \sqrt{-2lnU_1} \cdot cos(2\pi U_2)$.

5 Markov Chain Terminology

(a) irreducible: $0 < a \leq 1 \wedge 0 < b \leq 1$.
reducible: $a = 0 \vee b = 0$.

(b) *Proof.* Direct proof. If we start at a, there must be 2*k steps to get back, the same as to start at b. So $gcd(n \mid p^n(i,i)) = 2$. □

(c) *Proof.* Direct proof. Since there exists a circle at node 0 and 1. So $n_{min} \mid P^n(i,i) = 1 \implies gcd(n \mid p^n(i,i)) = 1$ for any $i \in K$. □

(d)

$$P = \begin{pmatrix} 1-b & b \\ a & 1-a \end{pmatrix}$$

(e) Since $\pi = \pi \cdot P$. We have

$$\begin{aligned} (1-b)\pi_0 + a\pi_1 &= \pi_0 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

Solving these equations, we obtain

$$\pi_0 = \frac{b}{a+b}, \pi_1 = \frac{a}{a+b}$$

6 Analyze a Markov Chain

(a) For each node:
Node 0: a cycle.
Node 1: $n \mid p^n(1,1) = 2or3\ldots$.
Node 2: $n \mid p^n(2,2) = 2or4or5\ldots$.
Therefore, we have $gcd(n \mid p^n(i,i)) = 1 \, \forall i \in K$.

(b)

$$\begin{aligned} &P[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0] \\ &= P[X(1) = 1 \mid X(0) = 0] \cdot P[X(2) = 0 \mid X(1) = 1] \ldots \\ &= a \cdot (1-b) \cdot (1-a) \cdot a. \end{aligned}$$

(c) $\pi = \pi \cdot P$.

$$\begin{cases} \pi_0 = \pi_0(1-a) + \pi_1(1-b) + \pi_2 * 0 \\ \pi_1 = \pi_0 a + \pi_2 * 1 \\ \pi_2 = b\pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Solving these equations, we have

$$\pi_1 = \frac{a}{a-b+a+ab}$$
$$\pi_2 = \frac{ab}{a-b+a+ab}$$
$$\pi_3 = \frac{1-b}{a-b+a+ab}$$

(d) Given we start at Node 1. Now we define $\mu_i : E[T_2 \mid X(0) = i]$ and then we have following equations

$$\begin{cases} \mu_1 = 1 + b\mu_2 + (1-b)\mu_0 \\ \mu_0 = 1 + a\mu_1 + (1-a)\mu_0 \\ \mu_2 = 0 \end{cases}$$

Solving these equations, we have

$$\mu_0 = \frac{1+a}{ab}$$
$$\mu_1 = \frac{1+a-b}{ab}$$

7 Boba in a Straw

(a) Draw the procession

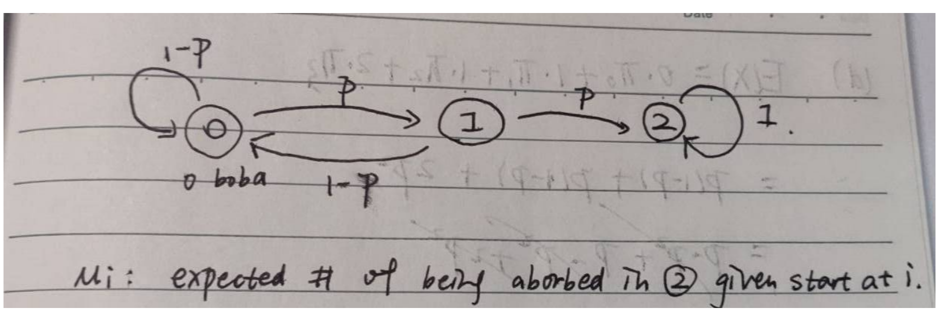

Figure 4: Markov's Chain

Then if let μ_i denote the expected # of being absorbed in node 2 starting at i. We have following equations:

$$\begin{cases} \mu_0 &= 1 + p\mu_1 + (1-p)\mu_0 \\ \mu_1 &= 1 + p\mu_2 + (1-p)\mu_0 \\ \mu_2 &= 0 \end{cases}$$

Here I don't think my answer is wrong, the Markov chain of solution for part(a)(b) are all same as part(c).

$$\begin{aligned} &\mathbb{E}_{(0,0)}[T] = 1 + (1-p)\,\mathbb{E}_{(0,0)}[T] + p\,\mathbb{E}_{(0,1)}[T], \\ &\mathbb{E}_{(0,1)}[T] = 1 + (1-p)\,\mathbb{E}_{(1,0)}[T] + p\,\mathbb{E}_{(1,1)}[T], \\ &\mathbb{E}_{(1,0)}[T] = 1 + (1-p)\,\mathbb{E}_{(0,0)}[T] + p\,\mathbb{E}_{(0,1)}[T], \\ &\mathbb{E}_{(1,1)}[T] = 0. \end{aligned}$$

Figure 5: Sol of part(a)

(b) Similar to part(a), we have

$$\begin{cases} \mu_0 = p(2 + \mu_1) + (1-p)(1 + \mu_0) \\ \mu_1 = p(3 + \mu_2) + (1-p)(2 + \mu_0) \\ \mu_2 = 0 \end{cases}$$

Solution:

$$\begin{aligned} &\mathbb{E}_{(0,0)}[T] = (1-p)(1 + \mathbb{E}_{(0,0)}[T]) + p(2 + \mathbb{E}_{(0,1)}[T]), \\ &\mathbb{E}_{(0,1)}[T] = (1-p)(2 + \mathbb{E}_{(1,0)}[T]) + p(3 + \mathbb{E}_{(1,1)}[T]), \\ &\mathbb{E}_{(1,0)}[T] = (1-p)(1 + \mathbb{E}_{(0,0)}[T]) + p(2 + \mathbb{E}_{(0,1)}[T]), \\ &\mathbb{E}_{(1,1)}[T] = 0. \end{aligned}$$

Figure 6: Sol of part(b)

(c) Still we draw the procession,

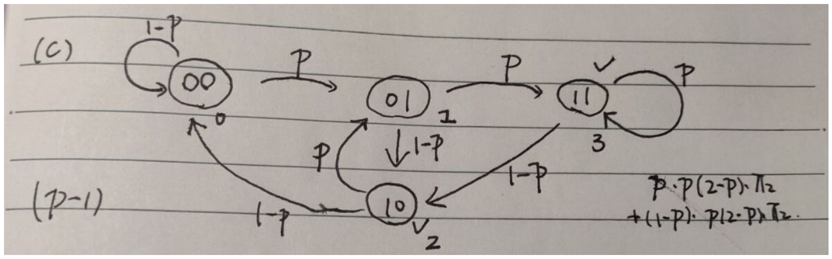

Figure 7: Markov Chain

According to Balanced Equation, $\pi = \pi \cdot P$. We have

$$\begin{cases} \pi_0 = (1-p)\pi_0 + (1-p)\pi_2 \\ \pi_1 = p\pi_0 + p\pi_2 \\ \pi_2 = (1-p)\pi_1 + (1-p)\pi_3 \\ \pi_3 = p\pi_1 + p\pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Then solving these equations, we have

$$\begin{aligned} \pi_0 &= (1-p)^2 \\ \pi_1 &= p(1-p) \\ \pi_2 &= p(1-p) \\ \pi_3 &= p^2 \end{aligned}$$

Then since state 2 and 3 will get one boba, $V_{average\,calorie} = 10 * Pr(state\,at\,2\,or\,3) = 10 * (\pi_2 + \pi_3) = 10 * p = 10p$.

(d) Now since we already have our distribution, E(X)

$$\begin{aligned} &= 0 * \pi_0 + 1\pi_1 + 1\pi_2 + 2\pi_3 \\ &= p(1-p) + p(1-p) + 2p^2 \\ &= 2p \end{aligned}$$