CS 70 Discrete Mathematics and Probability Theory
Fall 2018 Alistair Sinclair and Yun Song

DIS 9A

1 Numbered Balls

$\binom{7}{2} = \frac{7\times 6}{2\times 1} = \frac{1}{21}$

Suppose you have a bag containing seven balls numbered $0,1,1,2,3,5,8$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

$E(X) = \sum x \cdot P_X(x) = (0+1+1+2+3+5+8)\cdot\frac{1}{7} = \frac{20}{7}$

For this sample space 21 $\{(1,2), \ldots\}$ and $\{(1,1)\ldots\}$ both applicable.

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record their total. What is the expected value of the total that you record?

$P_X(x) = \frac{1}{\binom{7}{2}} = \frac{1}{21}$ $E(X) = 6\times(20)\cdot\frac{1}{21} = \frac{40}{7}$

$X = X_1 + X_2$?

$X_1 = \begin{cases} 0 & 1/7 \\ \vdots \\ i \end{cases}$ $X_2 = \begin{cases} 0 & 1/7 \\ \vdots \\ i \end{cases}$

Better think it through.

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$? $\binom{48}{3}/\binom{52}{3}$

(b) What is $\mathbb{P}(X=1)$? $\binom{4}{1}\cdot\binom{48}{2}/\binom{52}{3}$

(c) What is $\mathbb{P}(X=2)$? $\binom{4}{2}\cdot\binom{48}{1}/\binom{52}{3}$

(d) What is $\mathbb{P}(X=3)$? $\binom{4}{3}\cdot\binom{48}{0}/\binom{52}{3}$

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

Yes.

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

$E(X) = \frac{1}{\binom{52}{3}}\cdot\left[4\cdot\binom{48}{2} + 12\cdot\binom{48}{1} + 12\cdot\binom{48}{0}\right]$

(g) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$ using linearity of expectation.

$E(X_i) = \frac{4}{52} = \frac{1}{13}$ $E(X) = 3\times\frac{1}{13} = \frac{3}{13}$

Since $X = \sum_{1\le i\le 3} X_i$

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

$P(X_2 \mid X_1) \neq P(X_2)$ No

No

3 More Aces in a Deck

There are four aces in a deck. Suppose you shuffle the deck; define the random variables:

$X_1 =$ number of non-ace cards before the first ace

$X_2 =$ number of non-ace cards between the first and second ace

$X_3 =$ number of non-ace cards between the second and third ace

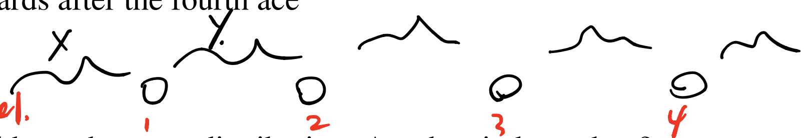
$X_4 =$ number of non-ace cards between the third and fourth ace

$X_5 =$ number of non-ace cards after the fourth ace

or simply use Symmetry.

1,2,3,4 are not restricted.

52



1. What is $X_1 + X_2 + X_3 + X_4 + X_5$?

$x_1, \ldots x_5$ expected number are equal.

2. Argue that the X_i random variables all have the same distribution. Are they independent?

1. 48

2. $P_{X_1}(x) = \dfrac{\binom{52-x-1}{3}}{\binom{52}{4}}$

$\binom{52}{4}$

take out y

$X_4(x) =$

$P_{X_2}(x) = \dfrac{\binom{52-y-1}{3}}{\binom{52}{4}}$

1st 2nd are bonded together. See them as whole.

Symmetry

3. Use the results of the previous parts to compute $\mathbb{E}(X_1)$.

Not independent if $P(X_2 = 1 \mid X_1 = 48) = 0 \neq P(X_2 = 1)$.

$E(X_1) = E(X_2) = E(X_3) = E(X_4) = E(X_5)$

$E(X) = E(X_1 + \cdots X_5) = E(48)$

$\Leftrightarrow 5E(X_1) = 48$

$E(X_1) = \frac{48}{5}$.

3.7