CS70 HW2

February 22, 2021

1 Hit or Miss?

(a) Incorrect. What the proof proves is that for all positive integer $n \in \mathbb{R}, n^2 >= n$, but the claim is to prove for any positive number $\in \mathbb{R}$.

For a counter-example, when $n = 0.5, n^2 = 0.25, n = 0.5$, thus $n^2 < n$.

(b) Correct.

(c) Incorrect. From the reference to the Well Ordering Principle, the first counter-example is when $n = 0$, where $0 < a, b <= n \implies a = 1, b = 0$, and we don't know whether $2 * 1 = 0$ or not.

2 A Coin Game

Proof by Strong Induction on the total coin number N.
Base Case: when $N = 2$, which can only split into 1 and 1. Thus $1 * 1 = \frac{2*(2-1)}{2} = 1$.
Induction Hypothese: Assume total score will be $\frac{n*(n-1)}{2}$ when $2 <= n <= k$ no matter how to split;
Induction Step: We must show when $n = k + 1$, the equation still holds. Say we split the stack into a and b parts where $a+b = k+1, a \in \mathbb{R}, b \in \mathbb{R}$. Use induction hypothesis on these two parts, we gain the equation

$$\frac{a * (a-1)}{2} + \frac{b * (b-1)}{2} + a * b = \frac{(a+b)^2 - (a+b)}{2} = \frac{(k+1) * k}{2}$$

and since $a \in \mathbb{R}, b \in \mathbb{R}$, so proof is done.

3 Grid Induction

Claim: Pacman needs i + j steps to reach (0, 0).
Proof by induction on the sum of its coordinates $N = i + j$.

Base Case: when $N = 0 \implies i = 0, j = 0$, which takes no step to reach (0, 0);
Induction Hypothesis: Assume pacman at (i, j) needs exactly $N = i + j = k$ steps to reach origin.
Induction Step: when N = k + 1, there are two cases:

(a) Pacman walks one step down if allowed. Thus the new sum $N' = k$, by induction hypothesis, the total step is $1 + k$.

(b) Pacman walks one step left if allowed. Thus the new sum $N' = k$, by induction hypothesis, the total step is $1 + k$.

In both cases, the total steps needed are $k + 1$, so the proof is done.

4 Stable Merriage

(a)

day 1

women	men
1	A, B, C
2	
3	D
4	

day 2

women	
1	A
2	
3	D, B, C
4	

day 3

women	
1	A, D
2	C
3	B
4	

day 4

women	
1	D
2	A, C
3	B
4	

day 5

women	
1	D
2	A
3	B
4	C

So the final result is {(D, 1), (A, 2), (B, 3), (C, 4)};

(b) The proof consists of two parts : First is to prove it's stable and section is to prove it puts out a male-optimal pairing.

(a) Stability.
Say there is a rough couple in the result produced by this Algorithm, namely in (W, M*) and (W-, M), W and M are rough couple.

By the definition of rough couple, for M: W > W-; for W: M > M*. Since M proposes, he proposes to W earlier than W-. But W ends up with M*, so for W: M* > M, which contradicts the earlier fact.

So it's stable.

(b) Male-optimal pairing.
Proof similar in the Notebook. Proof by Well Ordering Principle: first introduce a first man M who is rejected by his optimal woman W, ...

And it's Male-optimal pairing.

In short, proving stability and optimality needn't the precise day to propose. So the output remains the same.

5 Optimal Partners

Proof by Contradiction.

Assume both M and M* 's optimal woman are W. And we have two sets where M* and M end up with his optimal women W:
Set T : (M , W), (M*, W*);
Set S : (M, W^{-}), (M*, W);
And for women W, there are two cases:

(a) If woman W likes M* more than M. Then in the Set S, (M*, W) is a rough couple.

(b) If woman W likes M more than M*. Then in the Set T, (M, W) is a rough couple.

In both cases, there exists a contradiction.
So no two men can have the same optimal partner.

6 Examples or It's Impossible

(a) Possible.

men	preferences
A	1, 2, 3
B	2, 1, 3
C	3, 2, 1

women	preferences
1	A, B, C
2	B, A, C
3	C, B, A

(b) Possible.

men	preferences
A	2, 1, 3
B	1, 2, 3
C	2, 3, 1

women	preferences
1	A, B, C
2	B, C, A
3	C, A, B

(c) Possible.

men	preferences
A	1, 2, 3
B	2, 1, 3
C	3, 2, 1

men	preferences
1	C, B, A
2	C, A, B
3	A, B, C

(d) Possible.
But I can't think of an instance.

Impossible.

The proof is quite beautiful written by a student. I will write it our tomorrow to see if I have really understood it.

Proof. Proof by contradiction.

Say every man pairs with his last choice, namely $\{(M_1, W_1), (M_2, W_2), \ldots, (M_n, W_n)\}$. Then assume the algorithm ends at the K^{th} day, on that day M_1 proposes to W_1.

And on $K-1^{th}$, M_1 proposes to another woman $W*$ and is rejected, since $W*$ has already someone $M*$ on her string.

Having these observation,

1. M_1 proposes to W_1 at the last day $\implies$ W_1 has nobody on her string and never was proposed by any man before.
2. $W*$ has $M*$ on her string and $W*$ is $M*$ last choice $\implies$ $M*$ must propose to W_1 before.

Contradiction. Thus the proof is done.

□

(e) Impossible.

Say a man M ends up with his last choice woman W. Next we wanna prove that on W's list, M is her first choice.

Let M pairs up with other $n-1$ women, say $W_1, W_2, \ldots, W_{n-1}$ whose responding men are $M_1, M_2, \ldots, M_{n-1}$.

M like $W_1, W_2, \ldots, W_{n-1}$ more than W, and since they are not rough couple, which implies $W_1, W_2, \ldots, W_{n-1}$ like their $M_1, M_2, \ldots, M_{n-1}$ more than M.

Besides M is on $W_1, W_2, \ldots, W_{n-1}$ second place, so $M_1, M_2, \ldots, M_{n-1}$ are all $W_1, W_2, \ldots, W_{n-1}$ first choice.

Thus the remaining couple, for W, her first choice is not determined.

All the above is not related.
Answer should be Possible.

men	preferences
A	2, 3, 1
B	2, 1, 3
C	3, 2, 1

men	preferences
1	C, A, B
2	B, A, C
3	C, A, B