5.1

$$f_{xy}(x,y) = \gamma e^{-2x^2 - 8y^2}$$

$$a) f_x(x) = \gamma_1 e^{-2x^2} = \frac{1}{\sqrt{2\pi}(1/2)} e^{-\frac{x^2}{2(1/2)^2}} \Rightarrow X \sim N(0, \frac{1}{2})$$

$$f_y(y) = \gamma_2 e^{-8y^2} = \frac{1}{\sqrt{2\pi}(1/4)} e^{-\frac{y^2}{2(1/4)^2}} \Rightarrow Y \sim N(0, \frac{1}{4})$$

$$b) \gamma = \gamma_1 \gamma_2 = \frac{1}{\sqrt{2\pi}(1/2)} \times \frac{1}{\sqrt{2\pi}(1/4)} = \frac{8}{2\pi} = \frac{4}{\pi}$$

$$c) P\{X \le 0.5, Y \le 0.5\} = P\{X \le 0.5\} P\{Y \le 0.5\} = G(\frac{0.5}{1/2}) G(\frac{0.5}{1/4}) = G(1)G(2) = (0.84134)(0.97726) = 0.822$$

5.2

$a)$

$\mathrm{I}) Y = 2X$

$$F_{xy}(x,y) = P\{X \le x, Y \le y\} = \begin{cases} P\{X \le x\} = F_x(x) & y \ge 2xh \\ P\{Y \le y\} = & \end{cases}$$

II) $Y = -2X$

$F_{xy}(x,y) = P\{X \le x, Y \le y\}$

$X \le x \Rightarrow \ -2X \ge -2x > y \Rightarrow Y > y$ if y < -2x

so $P\{X \le x, Y \le y\} = 0$

$$\underbrace{P\{X \le x\}}_{F_x(x)} = \underbrace{P\{X \le x, Y \le y\}}_{F_{xy}(x,y)} + P\{X \le x, Y > y\} \qquad \text{if } y \ge -2x$$

but $Y > y \Rightarrow -2X > y \Rightarrow X < -\frac{y}{2} \le x$ so :

$P\{X \le x, Y > y\} = P\{Y > y\} = P\{2X < -y\} = F_x(-\frac{y}{2})$

so

$F_{xy}(x,y) = F_x(x) - F_x(-\frac{y}{2}) \qquad y \ge -2x$

as a result : $F_{xy}(x,y) = \begin{cases} 0 & y < -2x \\ F_x(x) - F_x(-\frac{y}{2}) & y \ge -2x \end{cases}$

III) $Y = X^2$

$$F_{xy}(x,y) = P\{X \le x, Y \le y\} = \begin{cases} 0 & \text{if } \ y < 0 \\ P\{X \le x, -\sqrt{y} \le X \le \sqrt{y}\} = 0 & \text{if } \ x^2 > y\ ,\ x \le 0 \\ 0 & \text{if } \ x^2 > y\ ,\ x > 0 \end{cases}$$

$= P\{X \le x, -\sqrt{y} \le X \le \sqrt{y}\} = P\{-\sqrt{y} \le X \le \sqrt{y}\} = F_x(\sqrt{y}) - F_x(-\sqrt{y})$

if $X^2 \le Y$:

$= P\{X \le x, -\sqrt{y} \le X \le \sqrt{y}\} = P\{-\sqrt{y} \le X \le x\} = F_x(x) - F_x(-\sqrt{y})$

$b) P\{X \le x, Y > y\} = P\{X \le x\ ,\ y < Y \le \infty\} = F_{xy}(x,\infty) - F_{xy}(x,\overline{y})$

$= F_{xy}(x,\infty) - F_{xy}(x,y)$ *if* F_{xy} doesn't have any discontinueties

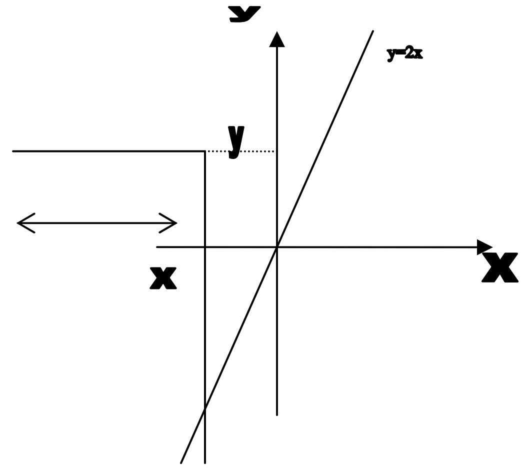
راه دیگر:

$$Y = g(X) \Rightarrow fxy(x, y) = fx(x)\,\delta(y - g(x))$$

I) $Y = 2X$

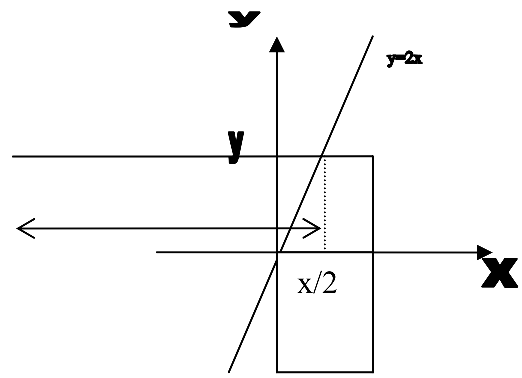
$if \quad y \geq 2x$

$F_{xy}(x, y) = F_x(x)$



$if \;\; y < 2x$

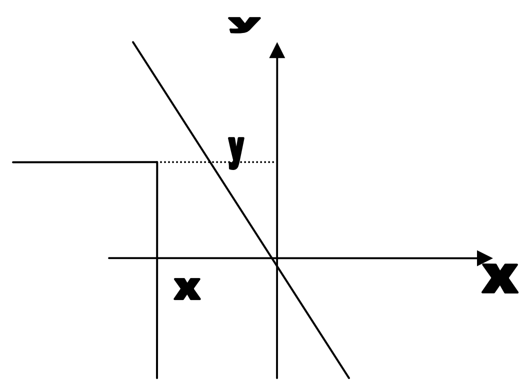
$$F_{xy}(x, y) = F_y(y) = F_x(\frac{x}{2})$$



II)

If y<-2x

$F_{xy}(x, y) = 0$



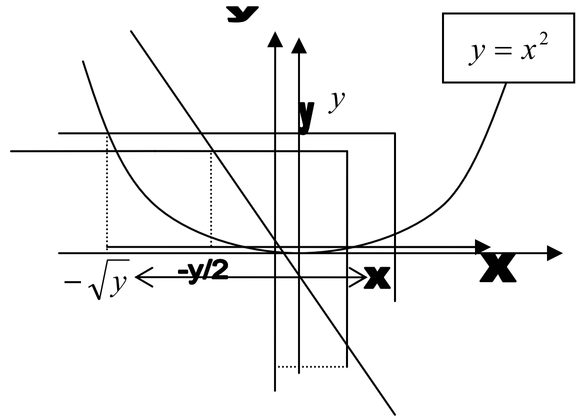
$$if\ \ y \geq -2x$$

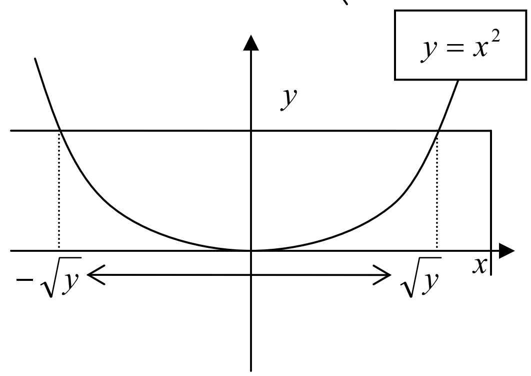
$$F_{xy}(x,y) = F_x(x) - F_x(-\frac{y}{2})$$

III) $F_{xy} = 0 \qquad \text{if } y < 0$

$$F_{xy} = 0 \qquad \text{if } y < x^2, x \leq 0$$

$$F_{xy}(x,y) = F_x(x) - F_x(-\sqrt{y}) \qquad \text{if } y \geq x^2$$





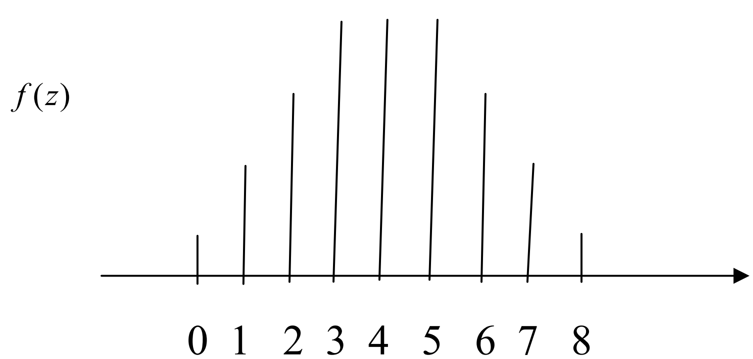
If $\quad y<x^2$, $x>0$

$$F_{xy}(x,y) = F_x(\sqrt{y}) - F_x(-\sqrt{y})$$

5.3

Z=X+Y

Z_i	تعداد زوجهای مطلوب	
0	1	(0و0)
1	2	(1و0) و (0و1)
2	3	(0و2) و (2و0) و (1و1)
3	4	(0و3) و (3و0) و (1و2) و (2و1)
4	4	(0و4) و (1و3) و (3و1) و (2و2)
5	4	(0و5) و (1و4) و (3و2) و (2و3)
6	3	(1و5) و (3و3) و (2و4)
7	2	(2و5) و (3و4)
8	1	(3و5)



5.4)

$$prob\ 2.8 \to P^2(A \cap B) \le P(A)P(B) \quad , P(A \cap B) \le \frac{P(A)+P(B)}{2}$$

با انتخاب $A = \{X \le x\}$, $B = \{Y \le y\}$ داریم:

$$F_{xy}^2(x,y) \le F_x(x)F_y(y) \quad , F_{xy}(x,y) \le \frac{F_x(x)+F_y(y)}{2}$$

(3

5.5)

a)

$$P\{X \le y\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{y} f_{xy}(x,y)dxdy$$

$$P\{X \ge y\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} f_{xy}(x,y)dydx \underbrace{=}_{changing\ the\ role\ of\ x\ and\ y} \int_{-\infty}^{+\infty}\int_{-\infty}^{y} f_{xy}(y,x)dxdy$$

$$(f_{xy}(x,y) = f_x(x)f_y(y) = f_x(x)f_x(y) = f_x(y)f_x(x) = f_{xy}(y,x))$$

$$\Rightarrow P\{X \ge y\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{y} f_{xy}(x,y)dxdy = p\{X \le y\}$$

b)

نشان می دهیم که رابطه برقرار نیست! :

we suppose X and Z are independent and :

$$f_x(x) = \delta(x-1)$$

$$f_z(z) = \begin{cases} 1 & 2 < z < 3 \\ 0 & else \end{cases}$$

$$EX^3 = 1$$

$$EZ^3 = \int_2^3 z^3 dz = (\frac{z^4}{4})\begin{matrix} 3 \\ 2 \end{matrix} = \frac{65}{4}$$

$$Y = \frac{X}{Z} = \frac{1}{Z} \to f_y(y) = \frac{1}{y^2} f_z(\frac{1}{y}) = \begin{cases} \frac{1}{y^2} & \frac{1}{3} < y < \frac{1}{2} \\ 0 & else \end{cases}$$

$$EY^3 = \int y^3 f_y(y) dy = \int_{\frac{1}{3}}^{\frac{1}{2}} y dy = (\frac{y^2}{2})\Bigg|_{\frac{1}{3}}^{\frac{1}{2}} = \frac{5}{72}$$

$$\frac{65}{4} \neq \frac{1}{\frac{5}{72}}$$

صورت مساله باید به این شکل تصحیح شود :

اگر Y, $Z = \frac{X}{Y}$ مستقل باشند آنگاه $E(\frac{X^3}{Y^3}) = \frac{E(X^3)}{E(Y^3)}$

زیرا

$$X = YZ$$

$$E(X^3) = E(Y^3 Z^3) = E(Y^3)E(Z^3)$$

Z and Y are independent so Y^3 and Z^3 are independent so Y^3 and Z^3 are uncorrelated

$$\Rightarrow E(\frac{X^3}{Y^3}) = E(Z^3) = \frac{E(X^3)}{E(y^3)}$$

(4

5.7)

$$\mu_n = E(X-\eta)^2 = E\left[\sum_{k=0}^{n}\binom{n}{k}X^k(-\eta)^{n-k}\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}(-\eta)^{n-k}E(X^k) = \sum_{k=0}^{n}\binom{n}{k}(-\eta)^{n-k}m_k$$
$$m_n = E(X)^n = E((X-\eta)+\eta)^n = E\left[\sum_{k=0}^{n}\binom{n}{k}(X-\eta)^k\eta^{n-k}\right]$$
$$=\sum_{k=0}^{n}\binom{n}{k}\eta^{n-k}E(X-\eta)^k = \sum_{k=0}^{n}\binom{n}{k}\eta^{n-k}\mu_k$$

(5

5.8)

$$E(|X|^n) = \int_{-\infty}^{+\infty}|X|^n\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-x^2}{2\sigma^2}}dx$$
$$n = 2k:$$
$$m_{2k} = \int_{-\infty}^{+\infty}X^{2k}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-x^2}{2\sigma^2}}dx$$

برای محاسبه این انتگرال از رابطه زیر نسبت به $\alpha = \dfrac{1}{2\sigma^2}$. k بار مشتق می گیریم

$$\int_{-\infty}^{+\infty}e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$$
$$\rightarrow \int_{-\infty}^{+\infty}x^{2k}e^{-\alpha x^2}dx = \frac{1\times 3\times\cdots\times(2k-1)}{2^k}\sqrt{\frac{\pi}{\alpha^{2k+1}}}$$
$$(by\ substitution\ of\ \alpha = \frac{1}{2\sigma^2}):$$

$$m_n = m_{2k} = \int_{-\infty}^{+\infty} x^{2k} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} dx = 1 \times 2 \times \cdots \times (n-1)\sigma^{2k}$$

$n = 2k+1$

$$m_n = m_{2k+1} = \int_{-\infty}^{+\infty} |x|^{2k+1} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} dx = 2\int_{0}^{+\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}} dx$$

$(y = \frac{x^2}{2\sigma^2})$

$$m_n = \sqrt{\frac{2}{\pi}} \frac{(2\sigma^2)^{k+1}}{2\sigma} \underbrace{\int_0^{\infty} y^k e^{-y} dy}_{k!} \Rightarrow m_n = m_{2k+1} = \sqrt{\frac{2}{\pi}} 2^k k! \sigma^{2k+1}$$

so we have:

$$m_n = \begin{cases} 1 \times 3 \times \cdots \times (n-1)\sigma^n & n = 2k \\ \sqrt{\frac{2}{\pi}} 2^k k! \sigma^n & n = 2k+1 \end{cases}$$

$$E(Y) = E(X^2) = E|X|^2 = \sigma^2 \qquad E(Y^2) = E(|X|^4) = 3\sigma^4 \quad \to \mathrm{var}(Y) = 3\sigma^4 - \sigma^4 = 2\sigma^4$$

(6 5-10)

$$Z = aX + b \to \eta_z = a\eta_x + b$$

$$\sigma_z^2 = a^2 \sigma_x^2 \Rightarrow \sigma_z = |a|\sigma_x$$

$$W = cY + d \to \eta_w = c\eta_y + d$$

$$\sigma_w = |c|\sigma_y$$

$$r_{zw} = \frac{\mu_{zw}}{\sigma_z \sigma_w} = \frac{E(Z-\eta_z)(W-\eta_w)}{\sigma_z \sigma_w} = \frac{E\{[a(X-\eta_x)][c(Y-\eta_y)]\}}{|a|\sigma_x |c|\sigma_y} = \frac{ac\mu_{xy}}{|ac|\sigma_x \sigma_y}$$

$$= \frac{ac}{|ac|} r_{xy} \Rightarrow r_{zw}^2 = r_{xy}^2$$

(7 5.18)

$$\Phi(s) = \int_{-\infty}^{+\infty} e^{sx} f_x(x) dx = \int_{-\infty}^{+\infty} \frac{c}{2} e^{sx} e^{-c|x|} dx$$

$$= \frac{c}{2} \int_{-\infty}^{+\infty} e^{sx} e^{-cx} dx + \frac{c}{2} \int_{-\infty}^{0} e^{sx} e^{cx} dx = \frac{1}{2} \frac{c}{\underbrace{c-s}_{\mathrm{Re}\, s<c}} + \frac{1}{2} \frac{c}{\underbrace{c+s}_{\mathrm{Re}\, s>0}}$$

$$\frac{c^2}{c^2 - s^2} \qquad -c < \mathrm{Re}\, s < c$$

$$\eta_x = \frac{d\Phi(s)}{ds}\Big|_{s=0} = \frac{2sc^2}{(c^2-s^2)^2}\Big|_{s=0} = 0$$

$$\sigma^2 = E(x^2) = \frac{d^2\Phi(s)}{ds^2}\Big|_{s=0} = 2c^2\,\frac{(c^2-s^2)^2+4(c^2-s^2)s(2sc^2)}{(c^2-s^2)^4}\Big|_{s=0} = 2c^2\,\frac{c^2-s^2+8s^2c^2}{(c^2-s^2)^3}\Big|_{s=0} = \frac{2}{c^2}$$

5.21)

a) $k_0 = \Psi(0) = \ln\Phi(0) = \ln 1 = 0$

$$k_1 = \frac{d}{ds}\Psi(s)\Big|_{s=0} = \frac{d}{ds}\ln\Phi(s)\Big|_{s=0} = \frac{\Phi'(0)}{\Phi(0)} = \frac{\eta}{1} = \eta$$

$$k_2 = \frac{d^2}{ds^2}\Psi(s)\Big|_{s=0} = \frac{d^2}{ds^2}\ln\Phi(s)\Big|_{s=0} = \frac{\Phi''(0)\Phi(0)-\Phi'^2(0)}{\Phi^2(0)} = E(x^2)-E^2(x) = \sigma^2$$

$$k_3 = \frac{d^3}{ds^3}\ln\Phi(s)\Big|_{s=0} = \frac{(\Phi'''\Phi+\Phi''\Phi'-2\Phi'\Phi'')\Phi^2-(\Phi''\Phi-\Phi'^2)(2\Phi\Phi')}{\Phi^4}$$

$$= \frac{\Phi^2\Phi'''-3\Phi\Phi'\Phi''+2\Phi'^3}{\Phi^3} = m_3-3m_2\eta+2\eta^3 = \mu_3$$

از 5.7

$$k_4 = \frac{d^4}{ds^4}\ln\Phi(s)\Big|_{s=0} = \frac{[2\Phi\Phi'\Phi'''+\Phi^2\Phi^{(4)}-2(\Phi'^2\Phi''+\Phi\Phi''^2+\Phi\Phi'\Phi''')+6\Phi'^2\Phi'']\Phi^3-3\Phi^2\Phi'(\Phi^2\Phi'''-3\Phi\Phi'\Phi''+2\Phi'^3)}{\Phi^6}$$

$$= \frac{-4\Phi^3\Phi'\Phi'''+\Phi^4\Phi^{(4)}+12\Phi\Phi'^2\Phi''-3\Phi^2\Phi''^2-6\Phi'^4}{\Phi^4}$$

$$= -4\eta m_3+m_4+12\eta^2m_2-3m_2^2-6\eta^4$$

طبق 5.7 داریم

$$\mu_4 = \eta^4-4\eta^4+6m_2\eta^2-4m_3\eta+m_4$$

$\sigma^2 = m_2-\eta^2$ و داریم

$$\Rightarrow \mu_4-3\sigma^4 = -3\eta^4+6m_2\eta^2-4m_3\eta+m_4-3m_2^2-3\eta^4+6m_2\eta^2$$

$$= -6\eta^4+12m_2\eta^2-4m_3\eta+m_4-3m_2^2$$

$k_4 = \mu_4 - 3\sigma^4$ پس

b) $f(k) = e^{-a}\dfrac{a^k}{k!}$, $k = 0, 1, 2, ---$

$$\Phi(s) = \sum_{k=0}^{\infty} e^{sk} f(k) = \sum_{k=0}^{\infty} e^{sk} e^{-a}\frac{a^k}{k!} = e^{-a}\sum_{k=0}^{\infty}\frac{(ae^s)^k}{k!}$$

$$\Rightarrow \Phi(s) = e^{-a} e^{ae^s}$$

$$\Psi(s) = \ln \Phi(s) = -a + ae^s$$

$$\eta = \Psi'(s)\Big|_{s=0} = ae^s\Big|_{s=0} = a$$

$$\sigma^2 = \Psi''(s)\Big|_{s=0} = ae^s\Big|_{s=0} = a$$

— گشتاورهای توزیع ریلی

$$m_n = E(x^n) = \int_0^{\infty} x^n \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} dx = \frac{1}{2\alpha^2}\int_{-\infty}^{+\infty} |x|^{n+1} e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \sqrt{\frac{\pi}{2}}\frac{1}{\alpha}\underbrace{\left(\frac{1}{\sqrt{2\pi}\,\alpha}\int_{-\infty}^{+\infty}|x|^{n+1} e^{-\frac{x^2}{2\alpha^2}} dx\right)}_{\text{گشتاور } n+1 \text{ ام } |x| \text{ برای } x\sim N(0,\alpha)}$$

پس با توجه به مسئله 58 کتاب داریم

$$m_n = \begin{cases} \sqrt{\dfrac{\pi}{2}}\, 1\times 3\times \cdots \times n\; \alpha^n & n = 2k+1 \\ 2^k k!\, \alpha^n & n = 2k \end{cases}$$

(9

$$m_n = E(X^n) = \int_0^{\infty} x^n \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} dx = \frac{1}{2\alpha^2} \int_{-\infty}^{+\infty} |x|^{n+1} e^{\frac{-x^2}{2\alpha^2}} dx$$

$$\sqrt{\frac{\pi}{2}} \frac{1}{\alpha} \quad \underbrace{\left(\frac{1}{\sqrt{2\pi}\alpha} \int_{-\infty}^{+\infty} |x|^{n+1} e^{\frac{-x^2}{2\alpha^2}} dx\right)}_{moment\ of\ \deg ree\ n+1\ for\ |X|\ when\ X\sim N(0,1)}$$

So referring to the answer of $P5.8$:

$$m_n = \begin{cases} \sqrt{\frac{\pi}{2}} 1\times 3\times \cdots \times n\alpha^n & n = 2k+1 \\ 2^k k!\alpha^n & n = 2k \end{cases}$$

10) توزیع گاما

$$f_x(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)} u(x)$$

$$\int_0^{\infty} \frac{\lambda^r}{\Gamma(r)} e^{j\omega x} x^{r-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \underbrace{\int_0^{\infty} x^{r-1} e^{-(\lambda - j\omega)x} dx}_{\frac{\Gamma(r)}{(\lambda - j\omega)^r}}$$

$$\Rightarrow \Phi(j\omega) = \left(\frac{\lambda}{\lambda - j\omega}\right)^r$$

$$\frac{d\Phi(j\omega)}{dj\omega} = \frac{-r(\lambda - j\omega)^{r-1}(-1)\lambda^r}{(\lambda - j\omega)^{2r}}$$

$$= \frac{r\lambda^r}{(\lambda - j\omega)^{r+1}}$$

$$\eta = \left.\frac{d\Phi(j\omega)}{dj\omega}\right|_{\omega = 0} = \frac{r\lambda^r}{\lambda^{r+1}} = \frac{r}{\lambda}$$

$$\frac{d^2\Phi(j\omega)}{d(j\omega)^2} = \frac{-r\lambda^r (r+1)(\lambda - j\omega)^r (-1)}{(\lambda - j\omega)^{2r+2}} = \frac{r(r+1)\lambda^r}{(\lambda - j\omega)^{r+2}}$$

$$E(X^2) = \frac{d^2\Phi(j\omega)}{d(j\omega)^2}\bigg|_{\omega=0} = \frac{r(r+1)\lambda^r}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\sigma^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda^2}$$

توزیع هندسی :

$$f(k) = pq^k, \qquad k = 0,1,2,\cdots$$

$$\Phi(j\omega) = \sum_{k=0}^{\infty} e^{j\omega k} pq^k = p\sum_{k=0}^{\infty}(qe^{j\omega})^k = \frac{p}{1-qe^{j\omega}}$$

$$\frac{d\Phi(j\omega)}{dj\omega} = \frac{-p(-qe^{j\omega})}{(1-qe^{j\omega})^2} = \frac{pqe^{j\omega}}{(1-qe^{j\omega})^2}$$

$$\eta = \frac{d\Phi(j\omega)}{dj\omega}\bigg|_{\omega=0} = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

$$\frac{d^2\Phi(j\omega)}{dj\omega} = \frac{pqe^{j\omega}(1-qe^{j\omega})^2 - pqe^{j\omega}2(1-qe^{j\omega})(-qe^{j\omega})}{(1-qe^{j\omega})^4} = \frac{pqe^{j\omega} + pq^2e^{2j\omega}}{(1-qe^{j\omega})^3}$$

$$E(X^2) = \frac{d^2\Phi(j\omega)}{dj\omega}\bigg|_{\omega=0} = \frac{pq+pq^2}{(1-q)^3} = \frac{q+q^2}{p^2}$$

$$\sigma^2 = \frac{q+q^2}{p^2} - \frac{q^2}{p^2} = \frac{q}{p^2}$$

(11

الف) برای توزیع متقارن $\mu_{2k+1} = 0$ از جمله $\mu_3 = 0$ در نتیجه S=0 .

ب) برای توزیع نرمال $\mu_4 = 3\sigma^4 \rightarrow K = \frac{3\sigma^4}{\sigma^4} - 3 = 0$

ج) با توجه به اینکه گشتاورهای توزیع ریلی را قبلا بدست آوردیم

$$m_1 = \eta = \sqrt{\frac{\pi}{2}}\alpha$$

$$m_2 = 2!\alpha^2 = 2\alpha^2 \qquad \rightarrow \sigma^2 = 2\alpha^2 - \frac{\pi}{2}\alpha^2 = (2-\frac{\pi}{2})\alpha^2 \rightarrow \sigma = \sqrt{2-\frac{\pi}{2}}\alpha$$

$$m_3 = \sqrt{\frac{\pi}{2}}3\alpha^3$$

$$m_4 = 2^2 2!\alpha^4 = 8\alpha^4$$

(14

اگر چه برای هر نقطه داخل دایره واحد f_{xy} جداپذیر است ولی با توجه به صفر بودن f_{xy} در خارج دایره واحد

جداپذ یر بودن به طور کلی وجود ندارد و X,Y مستقل نیستند. چون با توجه به اینکه (x,y) نمی توانند خارج دایره واحد باشند برای Y=y داده شده مقادیر ممکنه برای X به y بستگی دارد.(بین $-\sqrt{1-y^2}$ و $\sqrt{1-y^2}$

مثلا اگر y=0 آنگاه $-1 \le X \le 1$ و اگر $y=\frac{1}{2}$ آنگاه $\frac{-\sqrt{3}}{2} \le X \le \frac{\sqrt{3}}{2}$

15)
الف)

$$f_y(y) = \frac{1}{\sigma\sqrt{2\pi}y} e^{-\frac{\ln y - \eta}{2\sigma^2}} u(y) \qquad Y = e^x, \quad X \sim N(\eta, \sigma)$$

$$m_n(y) = E(Y^n) = E(e^{nX}) = \int_{-\infty}^{+\infty} e^{nx} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\eta)^2}{2\sigma^2}} dx$$

$$nx - \frac{(x-\eta)^2}{2\sigma^2} = -\frac{(x-(\eta+n\sigma^2))^2 + \eta^2 - (\eta + n\sigma^2)^2}{2\sigma^2}$$

$$= -\frac{(x-(\eta+n\sigma^2))^2 + -(n^2\sigma^4 + 2n\eta\sigma^2)}{2\sigma^2}$$

$$m_n(y) = e^{\frac{n^2\sigma^4 + 2n\eta\sigma^2}{2\sigma^2}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-(\eta+n\sigma^2))^2}{2\sigma^2}} dx}_{1}$$

$$\underline{\underline{m_n(y) = e^{(\frac{n^2\alpha^2}{2} + n\eta)}}}$$

ب)

$$\eta_y = m_1 = e^{\frac{\sigma^2}{2} + \eta}$$

$$m_2 = e^{2\sigma^2 + 2\eta}$$

$$\to \sigma_y^2 = e^{2\eta}(e^{2\sigma^2} - e^{\sigma^2})$$

$$m_3 = e^{\frac{9}{2}\sigma^2 + 3\eta}$$

$$m_4 = e^{8\sigma^2 + 4\eta}$$

$$\mu_3 = m_3 - 3m_2m_1 + 2m_1^{\;3} = e^{\frac{8}{2}\sigma^2 + 3\eta} - 3e^{\frac{5}{2}\sigma^2 + 3\eta} + 2e^{\frac{3}{2}\sigma^2 + 3\eta} = e^{3\eta}(e^{\frac{9}{2}\sigma^2} - 3e^{\frac{5}{2}\sigma^2} + 2e^{\frac{3}{2}\sigma^2})$$

$$\mu_4 = m_4 - 4m_3m_1 + 6m_2{m_1}^2 - 3{m_1}^4 = e^{8\sigma^2+4\eta} - 4e^{5\sigma^2+4\eta} + 6e^{3\sigma^2+4\eta} - 3e^{2\sigma^2+4\eta}$$
$$= e^{4\eta}(e^{8\sigma^2} - 4e^{5\sigma^2} + 6e^{3\sigma^2} - 3e^{2\sigma^2})$$
$$S = \frac{\mu_3(y)}{\sigma_y^3} = \frac{e^{3\eta}(e^{\frac{9}{2}\sigma^2} - 3e^{\frac{5}{2}\sigma^2} + 2e^{\frac{3}{2}\sigma^2})}{e^{3\eta}(e^{2\sigma^2} - e^{\sigma^2})^{\frac{3}{2}}}$$
$$K = \frac{\mu_4(y)}{\sigma_y^4} - 3 = \frac{e^{4\eta}(e^{8\sigma^2} - 4e^{5\sigma^2} + 6e^{3\sigma^2} - 3e^{2\sigma^2})}{e^{4\eta}(e^{2\sigma^2} - e^{\sigma^2})^2} - 3 = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 6$$

(16

$$\mu_k(x) = \sigma^k m_k(z) \qquad Z = \frac{X-\eta}{\sigma}$$
$$we\ know:$$
$$m_k(z) = j^k H_k(0) \qquad H_k(x) = xH_{k-1}(x) - (k-1)H_{k-2}(x)$$
$$H_k(0) = -(k-1)H_{k-2}(0)$$
$$m_k(z) = -j^k(k-1)H_{k-2}(0) = j^k(k-1)H_{k-2}(0) = (k-1)m_{k-2}(z)$$
$$\mu_k(x) = \sigma^k m_k(z) = \sigma^2(k-1)\sigma^{k-2}m_{k-2}(z) = (k-1)\sigma^2\mu_{k-2}(x)$$

(17

الف)

$$|\Phi(j\omega)| = \left|\int_{-\infty}^{+\infty} e^{j\omega x} f(x)dx\right| \le \int_{-\infty}^{+\infty} |e^{j\omega x}| f(x)dx = \int_{-\infty}^{+\infty} f(x)dx = 1 = \Phi(0)$$

ب) به ازای $\forall a_1, a_2, \cdots, a_n$ مختلط و $\forall \omega_1, \omega_2, \cdots \omega_n$ حقیقی

$\left|\sum_{m=1}^{n} a_m e^{j\omega_m X}\right|^2$ همواره مثبت است. پس

$$\left|\sum_{m=1}^{n} a_m e^{j\omega_m X}\right|^2 \ge 0$$
$$E(\sum_{m=1}^{n} a_m e^{j\omega_m X})(\sum_{k=1}^{n} a_k e^{-j\omega_m X}) \ge 0$$
$$E\sum_{m=1}^{n}\sum_{k=1}^{n} a_m a_k e^{(j\omega_m - j\omega_k)X} \ge 0$$
$$\sum_{m=1}^{n}\sum_{k=1}^{n} a_m a_k E e^{(j\omega_m - j\omega_k)X} \ge 0 \rightarrow \sum_{m=1}^{n}\sum_{k=1}^{n} a_m a_k \Phi_X(j\omega_m - j\omega_k) \ge 0$$