باسمه تعالی

Eng. Math – HW4 سید محمدرضا موسوی گلان

① $$u_t(r,t) = c^2\left[u_{rr}(r,t) + \frac{1}{r}u_r(r,t)\right]$$

$$u(r,t) = R(r)T(t)$$

$$R(r)\,T'(t) = c^2\left[R''(r)T(t) + \frac{1}{r}R'(r)T(t)\right]$$

$$\Rightarrow \frac{T'(t)}{c^2T(t)} = \frac{R''(r) + r^{-1}R'(r)}{R(r)} = \begin{cases} \lambda^2 \\ -\lambda^2 \\ 0 \end{cases}$$

1) $\lambda^2 \leftarrow$ نمای T مثبت می‌شود و دما به بی‌نهایت میل می‌کند

2) $0 \leftarrow$ T ثابت است $\leftarrow f''(r) + \frac{1}{r}f'(r) = 0 \rightarrow u(r,t) = R(r)$

اگر این شرط برقرار باشد این نیز جواب مسأله خواهد بود. $R(r) = f(r)$

$$\rightarrow u(r,t) = f(r)$$

3)
$$rR'' + R' + \lambda^2 rR = 0$$
$$T' + \lambda c^2 T = 0$$

$$\Rightarrow R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$

در $r=0$، u متناهی است $\leftarrow c_2 = 0$

$$R(a) = 0 \rightarrow \boxed{J_0(a\lambda) = 0}$$

$\leftarrow a\lambda$ ریشه‌های J_0 است.

اگر n امین ریشه را با λ_n نمایش دهیم

$$\rightarrow u_n(r,t) = J_0(\lambda_n r) e^{-\lambda_n^2 c^2 t}$$

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 c^2 t}$$

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = f(r)$$

$$A_n = \frac{2}{c^2 [J_1(\lambda_n c)]^2} \int_0^c r J_0(\lambda_n r) f(r) dr .$$ ضرایب سری بسل

(2)

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0$$

$$u(r,\varphi) = R(r)\Phi(\varphi)$$

$$\rightarrow \Phi \frac{1}{r} \frac{d}{dr}\left(r \frac{dR}{dr}\right) + \frac{1}{r^2} R \frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{r}{R} \frac{d}{dr}\left(r \frac{dR}{dr}\right) = \frac{-\Phi''}{\Phi} = \begin{cases} k^2 \\ 0 \\ -k^2 \times \end{cases}$$

$1) -k^2 \rightarrow \Phi(\varphi) = Ae^{-k\varphi} + Be^{k\varphi}$ پریودیک نیست.

$$2) k^2 \rightarrow \Phi(\varphi) = A\sin k\varphi + B\cos k\varphi$$

Φ ، 2π متناوب است $\leftarrow$ k صحیح

$$\Phi(\varphi) = A\sin n\varphi + B\cos n\varphi$$
$$n = 1, 2, \ldots$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=k^2 \rightarrow r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-k^2R=0$$

باسخ توانی r^{α}

$$r^2\alpha(\alpha-1)r^{\alpha-2}+r\alpha r^{\alpha-1}-k^2r^{\alpha}=0$$

$$\alpha(\alpha-1)+\alpha-k^2=0 \rightarrow \alpha^2=k^2 \rightarrow \alpha=\pm k$$

$$\Rightarrow \boxed{R_n(r)=A_nr^n+B_nr^{-n}}$$

3) حالت $\underline{0}$: $\phi=A_0\varphi+B_0 \longrightarrow \phi=B_0$ بریودیک

$$\frac{d}{dr}\Big(\underbrace{r\frac{dR}{dr}}_{\text{ثابت}}\Big)=0 \rightarrow r\frac{dR}{dr}=A \rightarrow \frac{dR}{dr}=\frac{A}{r}$$

$$R(r)=A\ln r+B$$

$\Rightarrow$

n	R	ϕ
0	$A\ln r+B$	B_0
$n\neq 0$	$A_nr^n+B_nr^{-n}$	$A\sin n\varphi+B\cos n\varphi$

$$u(r,\varphi)=a_0+b_0\ln r+\sum_{n=1}^{\infty}\left(A_nr^n+B_nr^{-n}\right)\left(C_n\cos n\varphi+D_n\sin n\varphi\right)$$

$$=a_0+b_0\ln r+\sum_{n=1}^{\infty}\left[r^n(a_n\cos n\varphi+b_n\sin n\varphi)+r^{-n}(c_n\cos n\varphi+d_n\sin n\varphi)\right]$$

$b_0=0\ ,\ c_n=d_n=0 \leftarrow$ $r=0$ جز منطقه حل است

$$u(r,\varphi)=a_0+\sum_{n=1}^{\infty} r^n(a_n\cos n\varphi+b_n\sin n\varphi)$$

$$u(a,\varphi)=f(\varphi)\rightarrow a_0=\frac{1}{2\pi}\int_0^{2\pi} f(\varphi)d\varphi$$

$$a_n\times a^n=\frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi\, d\varphi$$

$$b_n\times a^n=\frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin n\varphi\, d\varphi$$

اگر مسئله درخارج از دایره بود

$\leftarrow \quad b_0=0,\ a_n=b_n=0$

اگر مسئله حلقوی بود هیچکدام صفر نمی‌شوند.

3

$$\nabla^2 u=0\rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

$$u=X(x)Y(y)Z(z)$$

$$\rightarrow X''YZ+XY''Z+XYZ''=0$$

$$\rightarrow \frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\frac{X''}{X}=-\frac{Y''}{Y}-\frac{Z''}{Z}=-k_x^2$$

$$\rightarrow X=A\cos k_x x+B\sin k_x x\rightarrow \begin{matrix}X(0)=0\\X(1)=0\end{matrix}$$

$$X_m(x) = B_m \sin(m\pi x)$$

$$k_x = m\pi \qquad m = 1, 2, \dots$$

$$\frac{Y''}{Y} = -k_y^2 \rightarrow Y(y) = D_n \sin(n\pi y)$$

$$k_y = n\pi \qquad n = 1, 2, \dots$$

$$\frac{Z''}{Z} = k_y^2 + k_x^2 \rightarrow Z'' = Z(\underbrace{k_y^2 + k_x^2}_{\lambda_{mn}^2})$$

$$\lambda_{mn} = \left[(m\pi)^2 + (n\pi)^2 \right]^{1/2}$$

$$Z = E_{mn} e^{z\lambda_{mn}} + F_{mn} e^{-\lambda_{mn} z} \qquad Z(0) = 0 \rightarrow E = -F$$

$$\rightarrow u_{mn}(x,y,z) = E_{mn} e^{\lambda_{mn} z} - E_{mn} e^{-\lambda_{mn} z}) \sin(m\pi x) \sin(n\pi y)$$

$$\rightarrow u(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} e^{\lambda_{mn} z} - A_{mn} e^{-\lambda_{mn} z}) \sin(m\pi x) \sin(n\pi y)$$

$$u(x,y,1) = f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \underbrace{A_{mn} (e^{\lambda_{mn}} - e^{-\lambda_{mn}})}_{G_{mn}} \sin(m\pi x) \sin(n\pi y)$$

سری فوریه دوگانه

$$\rightarrow f(x,y) = \sum_{m=1}^{\infty} \sin(m\pi x) \underbrace{\sum_{n=1}^{\infty} G_{mn} \sin(n\pi y)}_{I(m)}$$

$$= \sum_{m=1}^{\infty} I(m) \sin(m\pi x)$$

$$I(m) = \frac{2}{1}\int_0^1 f(x,y)\sin(m\pi x)\,dx$$

$$I(m) = \sum_{n=1}^{\infty} G_{mn}\sin(n\pi y)$$

$$\to G_{mn} = \frac{2}{1}\int_0^1 I(m)\sin(n\pi y)\,dy$$

$$= 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$A_{mn} = \frac{G_{mn}}{e^{\lambda_{mn}} - e^{-\lambda_{mn}}}$$ ✓

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$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = xt \qquad u(x,0)=0 \quad x \geqslant 0$$
$$u(0,t)=0 \quad t \geqslant 0$$

$$\xrightarrow{\mathcal{L}\ \text{طرفین}} x\frac{d}{dx}u(x,s) + \Big[sU(x,s) - \underbrace{u(x,0)}_{0}\Big] = \frac{x}{s^2}$$

$$\to xu' + sU = \frac{x}{s^2} \qquad \Big/ \qquad \frac{du}{dx} = \frac{du}{dt}\,\frac{dt}{dx}$$

$$\xrightarrow{x=e^t} e^t\left(\frac{du}{dt}\right)e^{-t} + sU = \frac{e^t}{s^2}$$

$$\to u' + su = \frac{e^t}{s^2}$$ ضرایب ثابت

همگن: $u' + su = 0 \to u = Ae^{-st}$

جواب خصوصی : $\frac{1}{s^2(s+1)}e^t \to u = Ae^{-st} + \frac{e^t}{s^2(s+1)}$

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$$u = A x^{-s} + \frac{x}{s^2(s+1)} = A e^{-\ln x\, s} + \frac{x}{s^2(s+1)}$$

$$\xrightarrow{\mathcal{L}^{-1}} u(x,t) = A\,\delta(t - \ln x) + x(e^{-t} + t - 1)$$

$$u(x,0) = 0 \xrightarrow{x=1} A\,\delta(t) = 0 \quad t \geqslant 0 \longrightarrow A = 0$$

$$\longrightarrow u(x,t) = x(e^{-t} + t - 1)$$

$u(x,0) = 0$
$u(0,t) = 0$ شرایط مرزی