1

$\lambda = k^2$

$y = c_1 \sin kx + c_2 \cos kx$

$y' = k c_1 \cos kx - k c_2 \sin kx$

$$\begin{cases} 2c_2 + c_1 \sin k - c_2 \cos k = 0 \rightarrow c_1 = \left(\dfrac{2 - \cos k}{\sin k}\right) c_2 \\ 2\cancel{k} c_1 + \cancel{k} c_1 \cos k - \cancel{k} c_2 \sin k = 0 \end{cases}$$

$$\rightarrow \left(\frac{4 - 2\cos k}{\sin k}\right) c_2 + c_2 \left(\frac{2\cos k - \cos^2 k}{\sin k}\right)$$

$$- c_2 \left(\frac{\sin^2 k}{\sin k}\right) = \frac{3}{\sin k} c_2 = 0 \rightarrow c_2 = 0$$

$\rightarrow c_1 = 0 \rightarrow y = 0 \rightarrow$ بدیهی

$\lambda = 0 \rightarrow y = c_1 x + c_2$

$$\begin{cases} 2c_2 - c_1 - c_2 = 0 \\ 2c_1 + c_1 = 0 \end{cases} \longrightarrow c_1 = c_2 = 0$$

$\downarrow$

بدیهی

$\lambda = -k^2$

$y = 0$

۱-۲) $\underbrace{(py')' + (qy)}_{Ly} = -\lambda r y \quad a \leq x \leq b$

محقق است شرایط مرزی

$u, v \xrightarrow{\text{تعریف خودالحاق}} \int_a^b vLu - uLv \, dx = \int_a^b v(pu')' - u(pv')' \, dx = \int_a^b (pu'v - puv')' \, dx$

$= (pu'v - puv')\Big|_a^b = 0 \xrightarrow{\text{شرایط مرزی عادی}}$ اثبات می‌شود

۱-۳) $f_{(x)} = x^2 + x + 1 \leadsto f_{(x)} = C_0 P_{0(x)} + C_1 P_{1(x)} + C_2 P_{2(x)}$

$\rightarrow x^2 + x + 1 = C_0 + C_1 x + C_2 \left(\frac{3x^2 - 1}{2}\right) \longrightarrow \begin{cases} 1 = C_0 - \frac{C_2}{2} \rightarrow C_0 = \frac{4}{3} \\ C_1 = 1 \longrightarrow C_1 = 1 \\ \frac{3}{2} C_2 = 1 \longrightarrow C_2 = \frac{2}{3} \end{cases}$

۱-۴) $\langle f, g \rangle = \int_a^b w_{(x)} f_{(x)} g_{(x)} \, dx , \quad \|f\| = \langle f, f \rangle^{\frac{1}{2}}$

$\|f - \lambda g\|^2 = \|f\|^2 + \lambda^2 \|g\|^2 - 2\lambda \langle f, g \rangle \xrightarrow{\lambda = \frac{\langle f, g \rangle}{\|g\|^2}}$

$= \|f\|^2 + \frac{(\langle f, g \rangle)^2}{\|g\|^2} - \frac{2(\langle f, g \rangle)^2}{\|g\|^2} \geq 0 \leadsto \|f\|^2 \geq \frac{(\langle f, g \rangle)^2}{\|g\|^2} \leadsto \|f\| . \|g\| \geq |\langle f, g \rangle|$

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۱-۵) $\|f + g\| \leq \|f\| + \|g\|$

* در سوال ۴ اثبات شد ✓

$\longrightarrow \|f\|^2 + \|g\|^2 + 2\langle f, g \rangle \leq \|f\|^2 + \|g\|^2 + 2\|f\| . \|g\| \leadsto \cancel{2}\langle f, g \rangle \leq \cancel{2}\|f\| . \|g\|$

۱-۶) $y'' + \lambda y = 0 \begin{cases} y_{(0)} = 0 \\ y_{(1)} + h y'_{(1)} = 0 \end{cases} , h > 0$

$\lambda > 0 \leadsto \lambda = k^2 \rightarrow y = B \cos kx + A \sin kx \longrightarrow B = 0 \leadsto y = A \sin kx$ ✓

$\rightarrow A \sin k + hkA \cos k = 0 \xrightarrow{k \neq \frac{\pi}{2} + n\pi} \tan k + hk = 0 \leadsto \tan k_n = -hk_n \leadsto$ مقادیر ویژه $\leadsto \lambda_n = k_n^2$

$\rightarrow$ توابع ویژه $\leadsto y = A \sin k_n x$

Subject:
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۱-۷) $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0$, $1<x<e \xrightarrow{x=e^t} 0<t<1$, $\frac{dy}{dx}\times\frac{dx}{dt}=\frac{dy}{dt}$

$$\hookrightarrow \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\to x\frac{dy}{dx}=\cancel{e^t}\times\frac{\frac{dy}{dt}}{\frac{dx}{dt}\cancel{e^t}}=\frac{dy}{dt} \rightsquigarrow \frac{d}{dx}\left[\frac{dy}{dt}\right]=\frac{\frac{d}{dt}\left[\frac{dy}{dt}\right]}{e^t}=\frac{\frac{d^2y}{dt^2}}{e^t}$$

$$\to e^{-t}\frac{d^2y}{dt^2}+\lambda e^{-t}y=0 \rightsquigarrow \frac{d^2y}{dt^2}+\lambda y=0 \qquad s^2+\lambda=0$$

= طبق ۱۰

۱-۸) $(py')'+(q+\lambda r)y=0 \quad a<x<b$
$$\begin{cases} a_1y_{(a)}+a_2y'_{(a)}=0\\ b_1y_{(b)}+b_2y'_{(b)}=0\\ p_{(x)},\ r_{(x)}\geq 0\\ q_{(x)}\leq 0\\ a_1a_2\leq 0\\ b_1b_2\geq 0\end{cases}$$

$$\frac{d^2y}{dt^2}+\lambda y=0 \qquad s^2+\lambda=0$$

۱-۸) $(py')'+(q+\lambda r)y=0 \quad a<x<b$
$$\begin{cases} a_1y_{(a)}+a_2y'_{(a)}=0\\ b_1y_{(b)}+b_2y'_{(b)}=0\\ p_{(x)},\ r_{(x)}\geq 0\\ q_{(x)}\leq 0\\ a_1a_2\leq 0\\ b_1b_2\geq 0\end{cases}$$

$$\xrightarrow{*\text{hint}} \int_a^b y(py')'dx+\int_a^b qy^2dx+\lambda\int_a^b ry^2dx=0$$

$$\to \lambda\underbrace{\int_a^b ry^2dx}_{\text{مثبت}}=\underbrace{-\underbrace{\int_a^b qy^2dx}_{\text{منفی}}}_{\text{مثبت}}-\int_a^b y(py')'dx$$

$$\int_a^b y(py')'dx = -y(py')\Big|_a^b+\underbrace{\int_a^b py'^2dx}_{\text{مثبت}}$$

$$-y(py')\Big|_a^b = p_{(a)}y_{(a)}y'_{(a)}-p_{(b)}y_{(b)}y'_{(b)}$$

$$=p_{(a)}y_{(a)}\cdot-\frac{a_1}{a_2}y_{(a)}-p_{(b)}y_{(b)}\cdot-\frac{b_1}{b_2}y_{(b)}$$

$$=\underbrace{-p_{(a)}y^2_{(a)}\frac{a_1}{a_2}+p_{(b)}y^2_{(b)}\frac{b_1}{b_2}}_{\text{مثبت}}$$

$$\to \lambda \rightsquigarrow \text{مثبت}$$