5.12)

$$\bar{X} = \frac{X_1 + X_2}{2}$$

$$Y = \frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2}{2} = \frac{\left(\frac{X_1 - X_2}{2}\right)^2 + \left(\frac{X_2 - X_1}{2}\right)^2}{2} = \frac{(X_1 - X_2)^2}{4}$$

می‌دانیم اگر X_1 و X_2 نرمال و مستقل باشند $X_1 + X_2$ و $X_1 - X_2$ نیز مستقلند (در مثال‌ها ضمن درس دیدیم یا کتاب آمده P.5.11)

پس هر $g(X_1 + X_2)$ از هر $h(X_1 - X_2)$ نیز مستقل است ازجمله $\frac{X_1 + X_2}{2}$ از $\frac{(X_1 - X_2)^2}{4}$ مستقل است

یعنی $\bar{X}$ و Y مستقلند

5.15)

$$g(x,y) \simeq g(\eta_x, \eta_y) + (x - \eta_x)\frac{\partial g}{\partial x} + (y - \eta_y)\frac{\partial g}{\partial y}$$

$$\rightarrow E[g(x,y)] \simeq g(\eta_x, \eta_y) + 0 + 0 = g(\eta_x, \eta_y)$$

$$\rightarrow g(x,y) - E(g(x,y)) \simeq (x - \eta_x)\frac{\partial g}{\partial x} + (y - \eta_y)\frac{\partial g}{\partial y}$$

$$\sigma^2_{g(x,y)} = E\left(g(x,y) - E(g(x,y))\right)^2$$

$$\simeq E\left[(x - \eta_x)\frac{\partial g}{\partial x} + (y - \eta_y)\frac{\partial g}{\partial y}\right]^2$$

$$= \left(\frac{\partial g}{\partial x}\right)^2 \underbrace{E(x - \eta_x)^2}_{\sigma_x^2} + \left(\frac{\partial g}{\partial y}\right)^2 \underbrace{E(y - \eta_y)^2}_{\sigma_y^2} + 2\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\underbrace{E(x - \eta_x)(y - \eta_y)}_{\sigma_{xy}}$$

$$= \sigma_x^2\left(\frac{\partial g}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial g}{\partial y}\right)^2 + 2r\sigma_x\sigma_y\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)$$

5.16)

$$w = g(v, i) = vi \qquad \frac{\partial^{۲} g}{\partial v^{۲}} = ۰ \;;\; \frac{\partial^{۲} g}{\partial i^{۲}} = ۰ ,\; \frac{\partial g}{\partial v \partial i} = 1$$

$$(5.52) \rightarrow \eta_w \simeq \eta_v \eta_i + \frac{1}{۲}\left(\sigma_v^{۲} \times 0 + ۲\underbrace{\rho_{vi}}_{۰} \times 1 + \sigma_i^{۲} \times 0\right)$$

$$= ۱۱۰ \times ۲$$

$$= ۲۲۰ \quad W$$

$$\left.\frac{\partial g}{\partial i}\right|_{\eta_v,\eta_i} = \left. v \right|_{\eta_v} = \eta_v \qquad \left.\frac{\partial g}{\partial v}\right|_{\eta_v,\eta_i} = \left. i \right|_{\eta_i} = \eta_i$$

$$p.5.15 \rightarrow \sigma_w^{۲} \simeq \sigma_v^{۲} (\eta_i)^{۲} + \sigma_i^{۲} (\eta_v)^{۲} + ۲\underbrace{\rho_{vi}}_{۰} (\eta_i)(\eta_v)$$

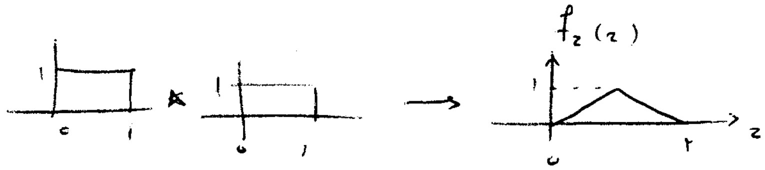
$$= ۳ (۲)^{۲} + (۰/۰۱)(۱۱۰)^{۲} = ۱۳۷$$

$$\rightarrow \sigma_w \simeq ۱۱/۷ \quad W$$

5.22)

$Z = X + Y$

مستقل X, Y $\rightarrow f_Z = f_X * f_Y = U(۰,۱) * U(۰,۱)$

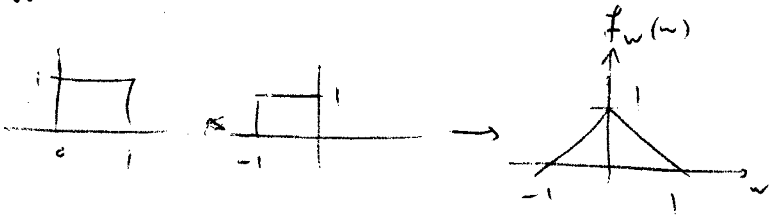


$W = X - Y = X + V \rightarrow f_W = f_X * f_V = U(۰,۱) * U(-۱,۰)$

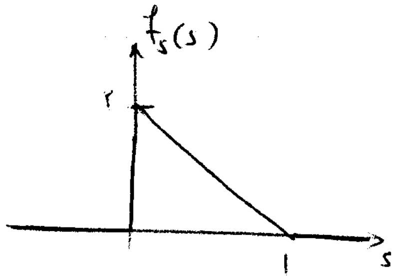
مستقل X, V

$X \sim U(۰,۱)$

$V = -Y \rightarrow f_V(v) = f_Y(-y) \rightarrow V \sim U(-۱,۰)$

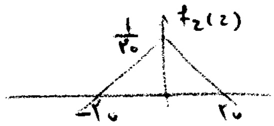


$S = |X-Y| = |W| \rightarrow f_S(s) = 2 f_W(s)\, u(s)$



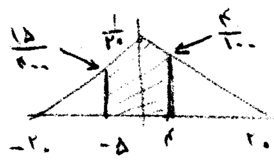
5.24) $Z = X - Y \rightarrow f_Z = f_X * f_V$ (a
مستقل $X, V \Leftarrow$ مستقل X, Y ، $V = -Y$

$= U(0,20) * U(-20,0)$



(b ملاقات تنها درصورتی خواهد بود $X \le Y + 4$ و $Y \le X + 5$ یعنی $-5 \le X - Y \le 4$

یعنی $-5 \le Z \le 4$



$$P(-5 \le Z \le 4) = \frac{\frac{1}{20} + \frac{4}{100}}{2} \times 4 + \frac{\frac{1}{20} + \frac{15}{400}}{2} \times 5 = 0{,}18 + 0{,}21875 = 0{,}39875$$

5.27)

$$\Phi = \arctan \frac{Y}{X} \qquad -\pi < \Phi < \pi$$
$$r = \sqrt{X^2 + Y^2} \qquad r \geq 0$$

دستگاه $\begin{cases} \Phi = \arctan \frac{y}{x} \\ r = \sqrt{x^2 + y^2} \end{cases}$ را حل میکنیم. ریشه دستگاه:

$$x_1 = r \cos\varphi$$
$$y_1 = r \sin\varphi$$

$$f_{r\varphi}(r,\varphi) = \frac{f_{XY}(x_1, y_1)}{|J(x_1, y_1)|} = f_{XY}(r\cos\varphi, r\sin\varphi) \left| \det \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \varphi} \\ \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \varphi} \end{bmatrix} \right|$$

$$= f_{XY}(r\cos\varphi, r\sin\varphi) \left| \det \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix} \right|$$

$$= r f_{XY}(r\cos\varphi, r\sin\varphi)$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

حالت خاص

$$\to f_{r\varphi}(r,\varphi) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad -\pi < \varphi < \pi \quad , r \geq 0$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\right) = f_\varphi(\varphi)\, f_r(r)$$

یونیفرم بین $-\pi$ و π رالی

5.28)

a) $Z = X + Y$

مستقل X, Y $\to \Phi_Z(j\omega) = \Phi_X(j\omega)\Phi_Y(j\omega)$

$$f_X(x) = c e^{-cx} u(x) \to \Phi_X(j\omega) = \frac{c}{c - j\omega}$$

$$f_Z(z) = c^2 z e^{-cz} u(z) \to \Phi_Z(j\omega) = \frac{c^2}{(c - j\omega)^2}$$

$$e^{-cx}u(x) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{c + j\omega}$$ یادآوری

$$x e^{-cx} u(x) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(c + j\omega)^2}$$

$$\Rightarrow \Phi_Y(j\omega) = \frac{\Phi_Z(j\omega)}{\Phi_X(j\omega)} = \frac{c}{c - j\omega} \longrightarrow f_Y(y) = c e^{-cy} u(y)$$

b) $Z = X + Y$

مستقل X, Y $\to f_Z = f_X * f_Y$ $f_X(x) = c e^{-cx}, \; x \geq 0$

$f_Y(y) = 1 \quad , \; 0 < x \leq 1$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y)\, dy$$

$$= \int_0^1 f_X(z-y)\, dy = -\int_z^{z-1} f_X(u)\, du = F_X(z) - F_X(z-1)$$

5.29)

$$Z = X - Y = X + V$$

$$V = -Y$$

$\Phi_Z = \Phi_X \Phi_V \leftarrow$ X و V مستقل $\leftarrow$ X و Y مستقل

$$\Phi_X(j\omega) = \frac{c}{c - j\omega}$$

می‌دانیم $Y = aX + b \leftarrow \Phi_Y(j\omega) = e^{bj\omega}\Phi_X(aj\omega)$ لذا

$$\Phi_Y(j\omega) = \frac{c}{c - j\omega} \rightarrow \Phi_V(j\omega) = \Phi_Y(-j\omega) = \frac{c}{c + j\omega}$$

$$\Phi_Z(j\omega) = \Phi_X(j\omega)\Phi_V(j\omega) = \frac{c^2}{c^2 + \omega^2} \xrightarrow[P.\ 5.18]{} f_Z(z) = \frac{c}{2} e^{-c|z|}$$

راه دیگر

$Z = X + V \rightarrow f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_V(z - x)\,dx$
X و V مستقل

$f_V(v) = f_Y(-v)$ لذا $V = -Y$

$$\rightarrow f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(x - z)\,dx$$

$$= \begin{cases} \int_0^{\infty} c\,e^{-cx}\, c\,e^{-c(x-z)}\,dx & z < 0 \\ \int_z^{\infty} c\,e^{-cx}\, c\,e^{-c(x-z)}\,dx & z > 0 \end{cases}$$

$$= \begin{cases} c^2 e^{cz} \frac{1}{(-2c)} \left(e^{-2cx}\right)\Big|_0^{\infty} = \frac{c}{2} e^{cz} & , z < 0 \\ c^2 e^{cz} \frac{1}{(-2c)} \left(e^{-2cx}\right)\Big|_z^{\infty} = \frac{c}{2} e^{-cz} & , z > 0 \end{cases}$$

$$= \frac{c}{2} e^{-c|z|}$$

تمرین نهم

$$z = \frac{X}{Y}$$

$$(5.101) \longrightarrow f_z(z) = \int_{-\infty}^{+\infty} |w|\, f_{xy}(zw, w)\, dw$$

در اینجا چون $f_{xy}(-x,-y) = f_{xy}(x,y)$

$$f_z(z) = 2\int_0^{\infty} w\, f_{xy}(zw, w)\, dw$$

$$= \frac{2}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \int_0^{\infty} w\, e^{-\frac{w^2}{2(1-r^2)}\left[\frac{z^2}{\sigma_x^2} - \frac{2rz}{\sigma_x\sigma_y} + \frac{1}{\sigma_y^2}\right]} dw$$

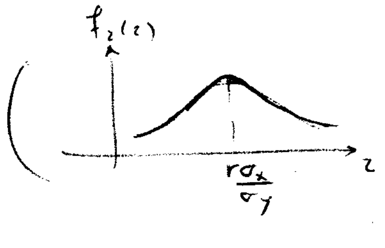
با

$$\int_0^{\infty} w\, e^{-\frac{w^2}{2a^2}}\, dw = a^2\int_0^{\infty} e^{-u}\, du = a^2$$

لذا

$$f_z(z) = \frac{1}{\pi\sigma_x\sigma_y\sqrt{1-r^2}} \times \frac{1-r^2}{\frac{z^2}{\sigma_x^2} - \frac{2rz}{\sigma_x\sigma_y} + \frac{1}{\sigma_y^2}}$$

$$= \frac{1}{\pi}\, \frac{\sqrt{1-r^2}\,\sigma_x / \sigma_y}{z^2 - 2r\frac{\sigma_x}{\sigma_y} z + \frac{\sigma_x^2}{\sigma_y^2}} = \frac{1}{\pi}\, \frac{\sqrt{1-r^2}\,\frac{\sigma_x}{\sigma_y}}{\left(z - \frac{r\sigma_x}{\sigma_y}\right)^2 + \frac{\sigma_x^2}{\sigma_y^2}(1-r^2)}$$



(توزیع کوشی که مرکز آن به $z = \frac{r\sigma_x}{\sigma_y}$ منتقل شده است)

$$f_{xy}(-x,-y) = f_{xy}(x,y) \longrightarrow \begin{array}{l} m_1 = m_3 \\ m_2 = m_4 \end{array}$$

$$m_1 = P\{x > 0,\ y > 0\}$$
$$m_3 = P\{x < 0,\ y < 0\}$$
$$P\left\{\frac{X}{Y} > 0\right\} = P\{x > 0, y > 0\} + P\{x < 0, y < 0\}$$

$$P\left\{\frac{X}{Y}<0\right\}=P\{X<0,\,Y>0\}+P\{X>0,\,Y<0\}=m_2+m_4$$ همچنین

پس

$$F_Z(0)=m_2+m_4 \longrightarrow m_2=m_4=\frac{1}{2}F_Z(0)$$
$$1-F_Z(0)=m_1+m_3 \longrightarrow m_1=m_3=\frac{1}{2}-\frac{1}{2}F_Z(0)$$

$$F_Z(0)=\int_{-\infty}^{0} f_Z(z)\,dz=\int_{-\infty}^{0}\frac{1}{\pi}\,\frac{\frac{\sigma_x}{\sigma_y}\sqrt{1-r^2}}{\left(z-\frac{r\sigma_x}{\sigma_y}\right)^2+\frac{\sigma_x^2}{\sigma_y^2}(1-r^2)}\,dz$$

$$=\frac{1}{\pi}\,\operatorname{arctg}\left(\frac{z-\frac{r\sigma_x}{\sigma_y}}{\frac{\sigma_x}{\sigma_y}\sqrt{1-r^2}}\right)\Bigg|_{-\infty}^{0}=\frac{1}{2}-\frac{1}{\pi}\operatorname{arctg}\frac{r}{\sqrt{1-r^2}}$$
$$=\frac{1}{2}-\frac{1}{\pi}\arcsin r$$

$$\Rightarrow\quad m_1=m_3=\frac{1}{4}+\frac{1}{2\pi}\arcsin r$$
$$m_2=m_4=\frac{1}{4}-\frac{1}{2\pi}\arcsin r$$

(یعنی برای $r\simeq 0$ حجم احتمال در چهار ربع یکسان است برای $r>0$ در ربع اول و سوم بیشتر است برای $r<0$ در ربع دوم و چهارم
بیشتر خواهد بود)

5.26) $\begin{aligned} z &= x^{۲} \\ w &= y^{۳} \end{aligned}$

دستگاه $\begin{cases} z = x^{۲} \\ w = y^{۳} \end{cases}$ دو ریشه دارد

برای $z>0$

$$\begin{cases} x_{۱} = \sqrt{z} \\ y_{۱} = \sqrt[۳]{w} \end{cases} , \quad \begin{cases} x_{۲} = -\sqrt{z} \\ y_{۲} = \sqrt[۳]{w} \end{cases}$$

$$J(x,y) = \det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \det \begin{bmatrix} ۲x & ۰ \\ ۰ & ۳y^{۲} \end{bmatrix} = ۶xy^{۲}$$

$$J(x_{۱},y_{۱}) = ۶\sqrt{z}\,\sqrt[۳]{w^{۲}}$$

$$J(x_{۲},y_{۲}) = -۶\sqrt{z}\,\sqrt[۳]{w^{۲}}$$

$$f_{zw}(z,w) = \frac{f_{xy}(x_{۱},y_{۱})}{|J(x_{۱},y_{۱})|} + \frac{f_{xy}(x_{۲},y_{۲})}{|J(x_{۲},y_{۲})|}$$

$$= \frac{1}{۶\sqrt{z}\,\sqrt[۳]{w^{۲}}}\left[f_{xy}(\sqrt{z},\sqrt[۳]{w}) + f_{xy}(-\sqrt{z},\sqrt[۳]{w})\right] u(z)$$

اگر x و y مستقل باشند

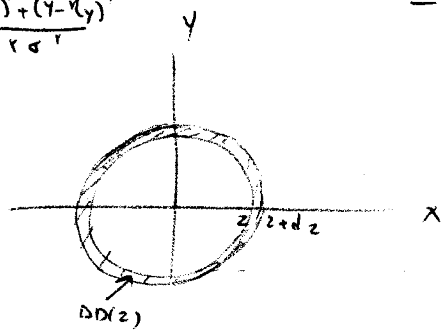
$$f_{zw}(z,w) = \frac{1}{۶\sqrt{z}\,\sqrt[۳]{w^{۲}}}\left[f_x(\sqrt{z})f_y(\sqrt[۳]{w}) + f_x(-\sqrt{z})f_y(\sqrt[۳]{w})\right] u(z)$$

$$= \underbrace{\frac{f_y(\sqrt[۳]{w})}{۳\sqrt[۳]{w^{۲}}}}_{f_w(w)} \; \underbrace{\frac{1}{۲\sqrt{z}}\left(f_x(\sqrt{z}) + f_x(-\sqrt{z})\right) u(z)}_{f_z(z)}$$

پس w و z مستقلند

— توزیع رایس

$$Z = \sqrt{X^2 + Y^2} \qquad f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\eta_x)^2+(y-\eta_y)^2}{2\sigma^2}}$$



$$f_Z(z)dz = \iint_{\Delta D(z)} f_{XY}(x,y)\, dx\, dy \qquad \text{برای } z \geq 0$$

$$= \int_z^{z+dz} \int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{(r\cos\theta-\eta_x)^2+(r\sin\theta-\eta_y)^2}{2\sigma^2}} r\, d\theta\, dr \quad , z \geq 0$$

اما

$$(r\cos\theta - \eta_x)^2 + (r\sin\theta - \eta_y)^2$$

$$= r^2 - 2r(\eta_x \cos\theta + \eta_y \sin\theta) + \eta_x^2 + \eta_y^2$$

$$\eta_x = \eta\cos\alpha \qquad \text{یعنی} \quad \alpha = tg^{-1}\frac{\eta_y}{\eta_x} \quad , \quad \eta = \sqrt{\eta_x^2+\eta_y^2} \qquad \text{اگر تعریف کنیم}$$

$$\eta_y = \eta\sin\alpha$$

عبارت فوق برابر است با

$$= r^2 - 2r(\eta\cos\alpha\cos\theta + \eta\sin\alpha\sin\theta) + \eta^2 = r^2 - 2r\eta\cos(\theta-\alpha) + \eta^2$$

در نتیجه

$$f_Z(z)dz = dz \int_0^{2\pi} \frac{z}{2\pi\sigma^2} e^{-\frac{z^2-2z\eta\cos(\theta-\alpha)+\eta^2}{2\sigma^2}} d\theta \quad , z \geq 0$$

$$f_Z(z) = \frac{z}{\sigma^2} e^{-\frac{z^2+\eta^2}{2\sigma^2}} \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{z\eta\cos(\theta-\alpha)}{\sigma^2}} d\theta \quad , z \geq 0$$

$$\Rightarrow f_Z(z) = \frac{z}{\sigma^2} e^{-\frac{z^2+\eta^2}{2\sigma^2}} I_0\left(\frac{\eta z}{\sigma^2}\right) u(z)$$

$$Z = \min(X,Y) \qquad W = \max(X,Y)$$

الف - روشن است که max همواره بزرگتر از min آنهاست پس

$$f_{ZW}(z,w) = 0 \quad , \quad w<z$$

برای $w>z$ دستگاه $\begin{cases} z=\min(x,y) \\ w=\max(x,y) \end{cases}$ دو جواب دارد $\begin{cases} x_2=w \\ y_2=z \end{cases}$ و $\begin{cases} x_1=z \\ y_1=w \end{cases}$

$$\det\begin{bmatrix} \frac{\partial x_2}{\partial z} & \frac{\partial x_2}{\partial w} \\ \frac{\partial y_2}{\partial z} & \frac{\partial y_2}{\partial w} \end{bmatrix} = \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

$$\det\begin{bmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$\Rightarrow f_{ZW}(z,w) = f_{XY}(z,w) + f_{XY}(w,z) \quad , \quad w>z$$

پس

$$f_{ZW}(z,w) = \begin{cases} f_{XY}(z,w) + f_{XY}(w,z) & , \quad w>z \\ (f_{XY}(z,z) & , \quad w=z) \\ 0 & , \quad w<z \end{cases}$$

اگر X و Y مستقل باشند

$$f_{ZW}(z,w) = \begin{cases} f_X(z) f_Y(w) + f_X(w) f_Y(z) & w>z \\ 0 & , \quad w<z \\ (f_X(z) f_Y(z) & , \quad w=z) \end{cases} \qquad (1)$$

داریم

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w)\,dw = \int_z^{\infty} [f_X(z) f_Y(w) + f_X(w) f_Y(z)]\,dw$$

$$= f_X(z)[1-F_Y(z)] + f_Y(z)[1-F_X(z)] \qquad (2)$$

شـ ۳

$$f_W(w)=\int_{-\infty}^{+\infty} f_{ZW}(z,w)\,dz=\int_{-\infty}^{w}\left[f_X(z)f_Y(w)+f_X(w)f_Y(z)\right]dz$$

$$=f_Y(w)F_X(w)+f_X(w)F_Y(w) \qquad (3)$$

الف - برای توزیع نمایی

برای $f_X(x)=\alpha e^{-\alpha x}u(x)$, $f_Y(y)=\beta e^{-\beta x}u(x)$ از ① نتیجه می‌شود

$$f_{ZW}(z,w)=\alpha e^{-\alpha z}\beta e^{-\beta w}+\alpha e^{-\alpha w}\beta e^{-\beta z} \quad , \begin{matrix} z\geq 0\\ w\geq 0\\ w>z\end{matrix}$$

$$\Rightarrow f_{ZW}(z,w)=\begin{cases}\alpha\beta\, e^{-(\alpha+\beta)(z+w)} & , \begin{matrix} z\geq 0\\ w\geq 0\\ w>z\end{matrix}\\ \left(\alpha\beta\, e^{-(\alpha+\beta)z}\right) & (z\geq 0, w\geq 0, w=z)\\ 0 & \end{cases}$$

از ② نتیجه می‌شود

$$f_Z(z)=\alpha e^{-\alpha z}e^{-\beta z}+\beta e^{-\beta z}e^{-\alpha z} \quad , z\geq 0$$

$$\Rightarrow f_Z(z)=(\alpha+\beta)\,e^{-(\alpha+\beta)z}u(z)$$

از ③ نتیجه می‌شود

$$f_W(w)=\beta e^{-\beta w}(1-e^{-\alpha w})+\alpha e^{-\alpha w}(1-e^{-\beta w}) \quad , w\geq 0$$

$$\Rightarrow f_W(w)=\left[\alpha e^{-\alpha w}+\beta e^{-\beta w}-(\alpha+\beta)e^{-(\alpha+\beta)w}\right]u(w)$$

ب - برای توزیع پواسن

مشابه آنچه برای توابع چگالی بدست آوردیم (رابطه ①) برای تابع احتمال نیز بدست می‌آوریم یعنی

تابع احتمال $f_{ZW}(z,w)=P\{Z=z,W=w\}=\begin{cases} f_X(z)f_Y(w)+f_X(w)f_Y(z) & , w>z\\ 0 & , w<z\\ f_X(z)f_Y(z) & , w=z\end{cases}$

که در آن $f(\cdot)$ بیانگر تابع احتمالات (نه چگالی)

$$f_X(x) = P\{X = x\} = \begin{cases} e^{-\alpha} \dfrac{\alpha^x}{x!} & , \quad x = 0,1,2,\dots \\ 0 & \text{و.ا،} \end{cases}$$

$$f_Y(y) = \begin{cases} e^{-\beta} \dfrac{\beta^y}{y!} & , \quad y = 0,1,2,\dots \\ 0 & \text{و.ا،} \end{cases}$$

$$f_{ZW}(z,w) = e^{-\alpha} \frac{\alpha^z}{z!} e^{-\beta} \frac{\beta^w}{w!} + e^{-\alpha} \frac{\alpha^w}{w!} e^{-\beta} \frac{\beta^z}{z!} \quad , \quad \begin{matrix} z, w \in \mathbb{N} \\ w > z \end{matrix}$$

$$\Rightarrow f_{ZW}(z,w) = \begin{cases} e^{-(\alpha+\beta)} \dfrac{\alpha^z \beta^w + \alpha^w \beta^z}{z!\, w!} & , \quad \begin{matrix} z, w = 0,1,2,\dots \\ w > z \end{matrix} \\ e^{-(\alpha+\beta)} \dfrac{(\alpha \beta)^z}{(z!)^2} & , \quad w = z \in \mathbb{N} \\ 0 & \text{و.ا،} \end{cases}$$

$$Y_1 = a_{11} X_1 + a_{12} X_2$$
$$Y_2 = a_{21} X_1 + a_{22} X_2$$

یعنی $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ —

$$\Phi_{Y_1 Y_2}(j\omega_1, j\omega_2) = E\, e^{j(\omega_1 y_1 + \omega_2 y_2)} = E\, e^{j[\omega_1(a_{11}x_1 + a_{12}x_2) + \omega_2(a_{21}x_1 + a_{22}x_2)]}$$

$$= E\, e^{j[(a_{11}\omega_1 + a_{21}\omega_2)x_1 + (a_{12}\omega_1 + a_{22}\omega_2)x_2]} = \Phi_{X_1 X_2}\big(j(a_{11}\omega_1 + a_{21}\omega_2), j(a_{12}\omega_1 + a_{22}\omega_2)\big)$$

$\varphi \sim U(0, 2\pi)$, $Z = X\cos(\omega t + \varphi)$ —

اگر تعریف کنیم $Y = \cos(\omega t + \varphi)$ طبق تمرین پنجم سری چهارم داریم

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & \text{اگر } |y|<1 \\ 0 & \text{والا} \end{cases}$$

X و Φ مستقل ← X و Y مستقل ←

$$f_{XY}(x,y) = \begin{cases} \dfrac{f_X(x)}{\pi\sqrt{1-y^2}} , & |y|<1 \\ 0 & \text{والا} \end{cases}$$

$Z = XY$ لذا طبق یکی از مثال‌های سر کلاس داریم:

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{|u|} f_{XY}\left(u, \frac{z}{u}\right) du$$

ولی

$$f_{XY}\left(u, \frac{z}{u}\right) = \begin{cases} \dfrac{f_X(u)}{\pi\sqrt{1-\left(\frac{z}{u}\right)^2}} & \left|\frac{z}{u}\right|<1 \\ 0 & \text{والا} \end{cases}$$

$|u| > |z|$ پس $\frac{|z|}{|u|} < 1$ پس $\left|\frac{z}{u}\right| < 1$

$$\frac{1}{|u|} f_{XY}\left(u, \frac{z}{u}\right) = \begin{cases} \dfrac{f_X(u)}{\pi\sqrt{u^2 - z^2}} & |u| > |z| \\ 0 & \text{و الا} \end{cases}$$

[illegible]

$$f_Z(z) = \int_{-\infty}^{-|z|} \frac{f_X(u)}{\pi\sqrt{u^2 - z^2}}\, du + \int_{|z|}^{\infty} \frac{f_X(u)}{\pi\sqrt{u^2 - z^2}}\, du$$

$$f(x,y) = x + y \qquad 0 < x \le 1,\quad 0 < y \le 1$$

$$f(x) = \int_0^1 (x+y)\, dy = \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{1} = x + \frac{1}{2}, \qquad 0 < x \le 1$$

$$f(y) = \int_0^1 (x+y)\, dx = y + \frac{1}{2}, \qquad 0 \le y \le 1$$

$$g(x,y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \qquad 0 < x \le 1,\quad 0 < y \le 1$$

$$g(x) = \int_0^1 \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) dy = \left(x + \frac{1}{2}\right)\left(\frac{y^2}{2} + \frac{y}{2}\right)\Big|_{y=0}^{1} = x + \frac{1}{2}, \qquad 0 < x \le 1$$

$$g(y) = \int_0^1 \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) dx = y + \frac{1}{2}, \qquad 0 \le y \le 1$$

از این مثال نتیجه می‌گیریم که صرف داشتن توابع چگالی حاشیه‌ای یکسان از متغیرهای تصادفی برای توصیف رفتار مشترک آنها کفایت

نمی‌کند (دو f_{xy} کاملاً متفاوت می‌توانند توابع حاشیه‌ای یکسان داشته باشند)

جزوه

— قضیهٔ پرایس

الف   حالتم

$$f_{xy}(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_{xy}(\omega_1,\omega_2)\,e^{-j(\omega_1x+\omega_2y)}\,d\omega_1\,d\omega_2$$

برای نرمال دو متغیره :

$$\Phi_{xy}(\omega_1,\omega_2)=e^{j(\omega_1\eta_x+\omega_2\eta_y)}\,e^{-\frac{1}{2}(\sigma_x^2\omega_1^2+2\mu_{xy}\omega_1\omega_2+\sigma_y^2\omega_2^2)}$$

لذا

$$\frac{\partial^n}{\partial\mu_{xy}^n}\Phi_{xy}(\omega_1,\omega_2)=(-1)^n\,\omega_1^n\,\omega_2^n\,\Phi_{xy}(\omega_1,\omega_2)$$

با توجه به این رابطه می‌توانیم بنویسیم :

$$\frac{\partial^n E(g(x,y))}{\partial\mu_{xy}^n}=\frac{\partial^n}{\partial\mu_{xy}^n}\iint_{-\infty\,-\infty}^{+\infty\,+\infty} g(x,y)\,f_{xy}(x,y)\,dx\,dy$$

$$=\iint_{-\infty\,-\infty}^{+\infty\,+\infty} g(x,y)\,\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial^n\Phi(\omega_1,\omega_2)}{\partial\mu_{xy}^n}\,e^{-j(\omega_1x+\omega_2y)}\,d\omega_1\,d\omega_2\,dx\,dy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\,\underbrace{\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-1)^n\,\omega_1^n\,\omega_2^n\,e^{-j(\omega_1x+\omega_2y)}\,d\omega_1\,d\omega_2}_{\frac{\partial^{2n}f_{xy}(x,y)}{\partial x^n\,\partial y^n}}\,dx\,dy$$

با یادآوری خاصیت مشتق فوریه

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\,\frac{\partial^{2n}f_{xy}(x,y)}{\partial x^n\,\partial y^n}\,dx\,dy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial^{2n}g(x,y)}{\partial x^n\,\partial y^n}\,f_{xy}(x,y)\,dx\,dy$$

$$=E\left[\frac{\partial^{2n}g(x,y)}{\partial x^n\,\partial y^n}\right]$$

صفحه سه

ب ـ اگر در مثال الف بگیریم $g(x,y) = x^k y^r$

از مثال الف نتیجه می‌شود

$$\frac{\partial E(x^k y^r)}{\partial \mu_{xy}} = kr\,E\left(x^{k-1} y^{r-1}\right)$$

از طرفی

$$E(x^k y^r)\Big|_{\mu_{xy}=0} = E(x^k)\,E(y^r)$$

لذا

$$E(x^k y^r) = kr \int_0^{\mu_{xy}} E\left(x^{k-1} y^{r-1}\right) d\mu + E(x^k)\,E(y^r)$$

ج ـ برای $k=r=2$

$$E(x^2 y^2) = 4 \int_0^{\mu_{xy}} E(xy)\, d\mu + \eta_x \eta_y$$

پس

$$Exy = \mu_{xy} + \eta_x \eta_y$$

$$\rightarrow E(x^2 y^2) = 4 \int_0^{\mu_{xy}} (\mu + \eta_x \eta_y)\, d\mu + \eta_x \eta_y$$

$$= 4 \left(\frac{\mu^2}{2} + \eta_x \eta_y \mu \right)_{\mu=0}^{\mu_{xy}} + \eta_x \eta_y$$

$$= 2\mu_{xy}^2 + 4\mu_{xy}\eta_x \eta_y + \eta_x^2 \eta_y^2$$