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۹۲۱۱۴۴۹   [illegible]

۱. f در دامنه D تحلیلی $\longrightarrow e^{f} = e^{u+iv} = e^{u}(\cos v + i\sin v)$

$= U + iV$

با توجه به اینکه e^{f} در دامنه D تحلیلی است پس با توجه به قضیه U و V در معادلات لاپلاس صدق کرده و همسازند.

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0 \\ \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0 \end{cases}$$

با توجه به تحلیلی بودن e^{f}، U و V در معادلات کوشی ریمان صدق کرده و بنا بر قضیه U و V همساز مزدوج هستند.

۲. $f(z) = u(z) + i v(z) = u(x+iy) + i v(x+iy)$

$= U_X(x)\,U_Y(y) + i\,V_X(x)\,V_Y(y)$

$\rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = U_Y \dfrac{dU_X}{dx} + i V_Y \dfrac{dV_X}{dx}$ *

$= \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y} = V_X \dfrac{dV_Y}{dy} - i U_X \dfrac{dU_Y}{dy}$ **

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$f(z) = f'(z) \xrightarrow{*} \frac{dU_x}{dx} = U_x \;,\; \frac{dV_x}{dx} = V_x$

$\longrightarrow U_x = e^x \quad V_x = e^x \longrightarrow f(x+iy) = U_y e^x + i V_y e^x$

$\xrightarrow[**]{*} f(z) = U_y e^x + i V_y e^x = e^x \frac{dV_y}{dy} - i e^x \frac{dU_y}{dy}$

$\longrightarrow \begin{cases} \frac{dV_y}{dy} = U_y \\ \frac{-dU_y}{dy} = V_y \end{cases} \longrightarrow -\frac{d^2U_y}{dy^2} = U_y \;,\; -\frac{d^2V_y}{dy^2} = V_y$

$U_y = a \cos y + b \sin y \quad V_y = \frac{dU_y}{dy} = -a \sin y + b \cos y$

$\longrightarrow f(x+iy) = e^x (a \cos y + b \sin y) + i e^x (b \cos y - a \sin y)$

$\xrightarrow{y=0} a e^x + i b e^x = e^x \longrightarrow a = 1 \;,\; b = 0$

$f(x+iy) = f(z) = e^x \cos y + i e^x \sin y$

۳. $|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| = \left| \frac{e^{iz} - e^{-iz}}{2} \right|$

$= \left| \frac{e^{(-y+ix)} - e^{(y-ix)}}{2} \right| = \left| \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2} \right|$

$\left| \frac{(e^{-y} - e^{y}) \cos x + i (e^{-y} + e^{y}) \sin x}{2} \right| =$

$$\frac{1}{2}\sqrt{(e^{-y}-e^{y})^{2}\cos^{2}x+(e^{-y}+e^{y})^{2}\sin^{2}x}=\frac{1}{2}\sqrt{e^{-2y}+e^{2y}-2\cos^{2}x+2\sin^{2}x}$$

$$=\frac{1}{2}\sqrt{e^{-2y}+e^{2y}-2-4\cos^{2}x}\geq|\sin x|$$

$$\iff \frac{1}{4}\left(e^{-2y}+e^{2y}+2-4\cos^{2}x\right)\geq 1-\cos^{2}x$$

$$\iff \frac{e^{-2y}+e^{2y}}{4}\geq\frac{1}{2}$$

$$\iff \frac{e^{-2y}+e^{2y}}{4}\geq\frac{2\sqrt{e^{-2y}\times e^{2y}}}{4}=\frac{1}{2}\quad ✓$$

$$|\cos z|=\left|\frac{e^{iz}+e^{-iz}}{2}\right|=\frac{1}{2}\left|e^{(-y+ix)}+e^{(y-ix)}\right|$$

$$=\frac{1}{2}\left|e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)\right|$$

$$=\frac{1}{2}\left|\cos x(e^{-y}+e^{y})+i\sin x(e^{-y}-e^{y})\right|$$

$$=\frac{1}{2}\sqrt{\cos^{2}x(e^{-y}+e^{y})^{2}+\sin^{2}x(e^{-y}-e^{y})^{2}}$$

$$=\frac{1}{2}\sqrt{e^{-2y}+e^{2y}+2\cos^{2}x-2\sin^{2}x}=\frac{1}{2}\sqrt{e^{-2y}+e^{2y}+4\cos^{2}x-2}$$

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$$\geq |\cos x| \iff \frac{1}{4}\left(e^{-2y}+e^{2y}+4\cos^2 x-2\right) \geq \cos^2 x$$

$$\iff \frac{e^{-2y}+e^{2y}}{4} \geq \frac{2\sqrt{e^{-2y}\times e^{2y}}}{4} = \frac{1}{2} \checkmark$$

4. $$\frac{e^{iz}+e^{-iz}}{2} = 2 \longrightarrow e^{iz}-4+e^{-iz} = 0$$

$$\longrightarrow \left(e^{iz}\right)^2 - 4\left(e^{iz}\right) + 1 = 0 \longrightarrow e^{iz} = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 \pm \sqrt{3} \longrightarrow \ln\left(2 \pm \sqrt{3}\right) + 2n\pi i = iz$$

$$\longrightarrow 2n\pi - i\ln\left(2 \pm \sqrt{3}\right) = z$$

5. $$\sin \bar{z} = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \frac{e^{i(x-iy)} - e^{-i(x-iy)}}{2i}$$

$$= \frac{e^{y+ix} - e^{-y-ix}}{2i} = \frac{1}{2i}\Big(e^{y}(\cos x + i\sin x) -$$

$$e^{-y}(\cos x - i\sin x)\Big) = \frac{1}{2}\Big(\underbrace{(e^{y}+e^{-y})}\sin x + i\underbrace{(e^{-y}-e^{y})}\cos x\Big)$$

$$u_x = (e^{y}+e^{-y})\cos x = v_y = -(e^{-y}+e^{y})\cos x \rightarrow \begin{cases} \cos x = 0 \checkmark \\ e^{-y}+e^{y} = 0 \times \end{cases}$$

$$u_y = (e^{y}-e^{-y})\sin x = -v_x = -(e^{y}-e^{-y})\sin x \rightarrow \begin{cases} \sin x = 0 \\ e^{y} = e^{-y} \end{cases}$$

$\cos x = 0$ ✗

$$\longrightarrow \boxed{x = \frac{\pi}{2} + n\pi \,,\; y = 0}$$

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$$\cos\bar{z} = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} = \frac{e^{i(x-iy)} + e^{-i(x-iy)}}{2}$$

$$= \frac{e^{y+ix} + e^{-y-ix}}{2} = \frac{e^{y}(\cos x + i\sin x) + e^{-y}(\cos x - i\sin x)}{2}$$

$$= \underbrace{\frac{1}{2}(e^{y}+e^{-y})\cos x}_{u} + \frac{1}{2}i\,\underbrace{\sin x\,(e^{y}-e^{-y})}_{v}$$

$$\frac{\partial u}{\partial x} = -(e^{y}+e^{-y})\sin x = \frac{\partial v}{\partial y} = (e^{y}+e^{-y})\sin x$$

$$\longrightarrow \sin x = 0 \ \checkmark \quad ! \quad e^{y}+e^{-y} = 0 \longrightarrow \times$$

$$\frac{\partial u}{\partial y} = (e^{y}-e^{-y})\cos x = -\frac{\partial v}{\partial x} = -\cos x\,(e^{y}-e^{-y})$$

$$\longrightarrow \cos x = 0 \xrightarrow{\sin x = 0} \times \qquad e^{y} = e^{-y} \longrightarrow y = 0 \longrightarrow \checkmark$$

$$\longrightarrow \boxed{x = n\pi \qquad y = 0}$$

4.

$$\mathrm{Re}\{z_1\} > 0 \longrightarrow -\frac{\pi}{2} < \mathrm{Arg}(z_1) < \frac{\pi}{2}$$

$$\mathrm{Re}\{z_2\} > 0 \longrightarrow -\frac{\pi}{2} < \mathrm{Arg}(z_2) < \frac{\pi}{2}$$

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$$\operatorname{Log}(z_1 z_2) = \ln|z_1 z_2| + \operatorname{Arg}(z_1 z_2)$$

$$= \ln|z_1| + \ln|z_2| + \operatorname{Arg}(z_1 z_2)$$

$\arg z_1 z_2$ برابر است با مجموعه‌ی حاصل از ضرب z_1, z_2 که برابر با جمع دو زاویه است حال آن که جمع $\operatorname{Arg} z_1 + \operatorname{Arg} z_2$ به بازه‌ی $(-\pi, \pi)$ تعلق نمی‌گیرد پس جمع $\operatorname{Arg} z_1$ و $\operatorname{Arg} z_2$ بجای $\operatorname{Arg} z_1 z_2$ می‌باشد.

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الف) $e^z = -3 \longrightarrow z = \log(-3) = \ln 3 + i(2n+1)\pi$

ب) $\log z = \frac{\pi i}{2} \longrightarrow z = e^{\frac{\pi i}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

۸. $(i+1)^i = e^{i\log(i+1)} = e^{i\left(\ln\sqrt{2} + i\frac{\pi}{4} + 2n\pi i\right)}$

$= e^{-\frac{\pi}{4} + i\ln\sqrt{2} + 2k\pi}$ $\quad k \in z$

مقدار اصلی $= e^{-\frac{\pi}{4} + i\ln\sqrt{2}}$

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.۹ $|i^c| = |e^{c\log i}| = |e^{c(\ln|1| + i\frac{\pi}{2} + n\pi i)}|$

$c = x+iy$

$= e^{(x+iy)(i\frac{\pi}{2} + in\pi)} = |e^{-\frac{\pi}{2}y - n\pi y}(\cos k + i\sin k)|$

$= e^{-\frac{\pi}{2}y - n\pi y} = Cte \longrightarrow y = 0 \longrightarrow$ c باید

بخش موهومی نداشته باشد.