1 Bessel, Neumann, and Hankel Functions: $J_\nu(x)$, $N_\nu(x)$, $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$

Bessel functions are solutions of the following differential equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1.1}$$

Any two of the following functions are linearly independent solutions of (1.1)

$$J_\nu(x) \qquad N_\nu(x) \qquad H_\nu^{(1)}(x) \qquad H_\nu^{(2)}(x)$$

when ν is not an integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are also linearly independent principal solutions of (1.1). The Neumann function $N_\nu(x)$ is related to J_ν and $J_{-\nu}$:

$$N_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \tag{1.2}$$

$$N_n(x) = \lim_{\nu\to n} \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \tag{1.3}$$

in some books Neumann functions are denoted by $Y_\nu(x)$ instead of $N_\nu(x)$. Hankel functions of the first and second kind are related to Bessel and Neumann functions:

$$H_\nu^{(1)}(z) \triangleq J_\nu(z) + jN_\nu(z) = j\frac{e^{-j\nu\pi}J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi} \tag{1.4}$$

$$H_\nu^{(2)}(z) \triangleq J_\nu(z) - jN_\nu(z) = \frac{e^{j\nu\pi}J_\nu(z) - J_{-\nu}(z)}{j\sin\nu\pi} \tag{1.5}$$

With a variable transformation $x = \kappa\rho$ equation (1.1) can be transformed into:

$$\rho^2y'' + \rho y' + (\kappa^2\rho^2 - \nu^2)y = 0 \tag{1.6}$$

When $\nu = n$ is an integer J_n and J_{-n} are *not independent* anymore and we have:

$$J_{-n}(x) = (-1)^nJ_n(x) \qquad N_{-n}(x) = (-1)^nN_n(x) \tag{1.7}$$

Plots of the first three Bessel and Neumann functions are shown in Fig. 1.1 and Fig. 1.2, respectively.

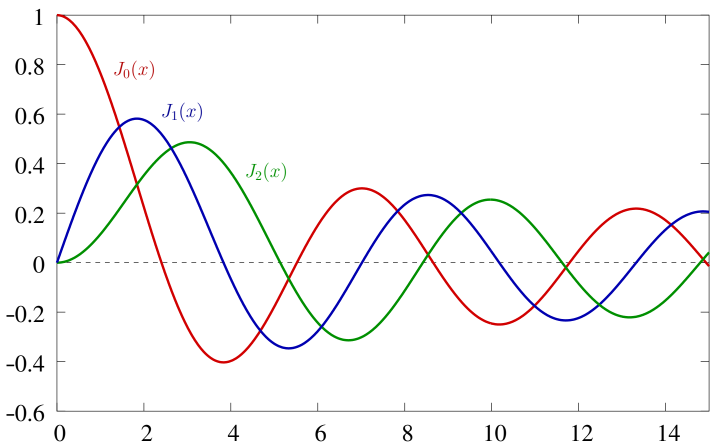

Figure 1.1: Bessel functions of the first kind

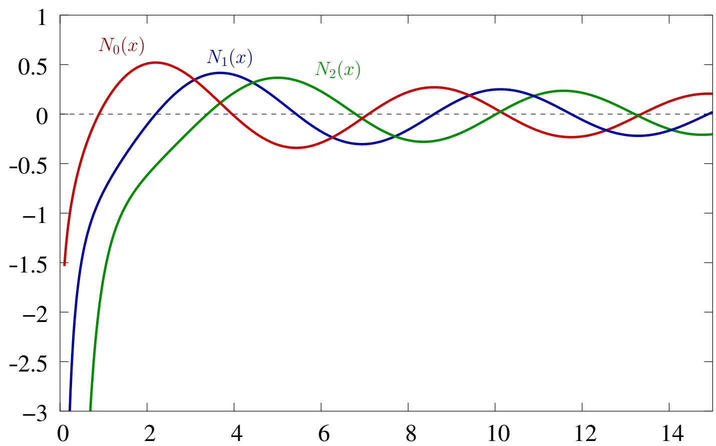

Figure 1.2: Bessel functions of the second kind

In general for arbitrary ν we have

$$J_\nu(e^{\pm j\pi}x) = e^{\pm j\nu\pi}J_\nu(x) \qquad N_\nu(e^{\pm j\pi}x) = e^{\mp j\nu\pi}N_\nu(x) \pm 2j\cos\nu\pi\, J_\nu(x) \tag{1.8}$$

1.1 Asymptotic Approximations

1.1.1 Small Argument Limit $|x| \longrightarrow 0$

$$J_0(x) \approx 1 - \frac{x^2}{4} \approx 1 \tag{1.9}$$

$$J_\nu(x) \approx \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)} \longrightarrow \frac{1}{n!}\left(\frac{x}{2}\right)^n \tag{1.10}$$

$$N_\nu(x) \approx -\frac{1}{\pi}\Gamma(\nu)\left(\frac{2}{x}\right)^\nu \longrightarrow -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n \qquad n \neq 0 \tag{1.11}$$

$$N_0(x) \approx \frac{2}{\pi}\ln\frac{\gamma x}{2} \qquad \gamma = 1.78107241799\ldots \text{ Euler's constant} \tag{1.12}$$

1.1.2 Large Argument Limit $|x| \gg |\nu|$, $-\pi < \arg(x) < \pi$

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi\, x}}\cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1.13}$$

$$N_\nu(x) \approx \sqrt{\frac{2}{\pi\, x}}\sin\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1.14}$$

1.1.3 Wronskian relations

The wronskian between two functions is defined by

$$W\{f,g\} \triangleq f(x)g'(x) - f'(x)g(x) \tag{1.15}$$

$$W\{J_\nu, N_\nu\} = J_{\nu+1}N_\nu - J_\nu N_{\nu+1} = \frac{2}{\pi x} \tag{1.16}$$

which is independent of ν

1.2 Integral Representations

When n is an integer:

$$J_n(x) = \frac{e^{-jn\left(\alpha+\frac{\pi}{2}\right)}}{2\pi}\int_0^{2\pi} e^{jx\cos(\phi-\alpha)}e^{jn\phi}d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\sin\phi}e^{-jn\phi}d\phi \tag{1.17}$$

$$J_n(x) = \frac{e^{-jn\left(\alpha+\frac{\pi}{2}\right)}}{2\pi}\int_{-\pi}^{\pi} e^{jx\cos(\phi-\alpha)}e^{jn\phi}d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jx\sin\phi}e^{-jn\phi}d\phi \tag{1.18}$$

$$J_n(x) = \frac{e^{-j\frac{n\pi}{2}}}{\pi}\int_0^{\pi}\cos(n\phi)e^{jx\cos\phi}d\phi \tag{1.19}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos\left(x\sin(\phi) - n\phi\right)\,d\phi \tag{1.20}$$

$$J_0(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos\left(x\sin(\alpha)\right)\,d\alpha = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos\left(x\cos(\alpha)\right)\,d\alpha \tag{1.21}$$

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos\left(x\sin(\alpha)\right)\,d\alpha = \frac{1}{\pi}\int_0^{\pi}\cos\left(x\cos(\alpha)\right)\,d\alpha \tag{1.22}$$

In general for arbitrary ν

$$J_\nu(x) = \frac{1}{\pi}\int_0^{\pi}\cos\left(x\sin(\phi) - \nu\phi\right)\,d\phi - \frac{\sin\nu\pi}{\pi}\int_0^{\infty} e^{-x\sinh t - \nu t}dt \qquad \Re\{x\} > 0 \tag{1.23}$$

$$J_\nu(x) = \frac{2}{\pi}\int_0^{\infty}\sin\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh\nu t\,dt \tag{1.24}$$

$$N_\nu(x) = \frac{1}{\pi}\int_0^{\pi}\sin\left(x\sin(\phi) - \nu\phi\right)\,d\phi - \frac{1}{\pi}\int_0^{\infty}\left(e^{\nu t} + e^{-\nu t}\cos\nu\pi\right)e^{-x\sinh t}dt \qquad \Re\{x\} > 0 \tag{1.25}$$

1.3 Orthogonality Relationships and Fourier-Bessel Series

Bessel equation (1.6) can be written in the following form:

$$-(\rho y')' + \frac{\nu^2}{\rho} y - \kappa^2 \rho y = 0 \Longrightarrow L[y] = \kappa^2\, y \qquad L = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\nu^2}{\rho^2} \tag{1.26}$$

This is a Sturm-Liouville equation with $p(\rho) = \rho$, $q(\rho) = \frac{n^2}{\rho}$, $w(\rho) = \rho$, and $\lambda = \kappa^2$. With appropriate boundary conditions on a finite interval such as $\rho \in [a, b]$ we can have a Sturm-Liouville eigenvalue problem (regular if $a > 0$ and irregular if $a = 0$). **Here n can be any real and non-negative number**.

Case I: Consider (1.26) on the interval $0 \le \rho \le b$ with boundary condition $y(b) = 0$. At $\rho \to 0$ we require the solution to be bounded. Thus, we can not have $N_n(\kappa\rho)$ and eigenfunctions must be in the form of $J_n(\kappa\rho)$. Eigenvalues are:

$$J_n(\kappa b) = 0 \Longrightarrow \kappa_m = \frac{\nu_{nm}}{b} \Longrightarrow \lambda_m = (\kappa_m)^2 = \left(\frac{\nu_{nm}}{b}\right)^2 \tag{1.27}$$

in which ν_{nm} is the m^{th} root of the Bessel function $J_n(x) = 0$, *i.e.* $J_n(\nu_{nm}) = 0$. The following orthogonality property exists:

$$\int_0^b J_n\left(\frac{\nu_{nm}}{b}\rho\right) J_n\left(\frac{\nu_{nk}}{b}\rho\right) \rho\, d\rho = \begin{cases} 0, & m \neq k \\ \\ \dfrac{b^2}{2}\left[J_{n+1}(\nu_{nm})\right]^2, & m = k \end{cases} \tag{1.28}$$

For any piecewise continuous function $f(\rho)$ we have:

$$f(\rho) \sim \sum_{m=1}^{\infty} F_m J_n\left(\frac{\nu_{nm}}{b}\rho\right) \tag{1.29}$$

in which the coefficients F_m are obtained by using the orthogonality property (1.28)

$$F_m = \frac{2}{b^2\left[J_{n+1}(\nu_{nm})\right]^2} \int_0^b f(\rho) J_n\left(\frac{\nu_{nm}}{b}\rho\right) \rho\, d\rho \tag{1.30}$$

Expression (1.29) is called the **Fourier-Bessel Series** expansion of $f(\rho)$. Note that the series always converges to *zero* at $\rho = b$.

Case II: Consider (1.26) on the interval $0 \le \rho \le b$ with boundary condition $y'(b) = 0$. At $\rho \to 0$ we require the solution to be bounded. Again eigenfunctions must be in the form of $J_n(\kappa\rho)$ and since $y'(b) = 0$ we obtain the eigenvalues:

$$J_n'(\kappa b) = \left.\frac{dJ_n(\kappa\rho)}{d\rho}\right|_{\rho=b} = 0 \Longrightarrow \kappa_m = \frac{\nu'_{nm}}{b} \Longrightarrow \lambda_m = (\kappa_m)^2 = \left(\frac{\nu'_{nm}}{b}\right)^2 \tag{1.31}$$

in which ν'_{nm} is the m^{th} root of the derivative of Bessel function $J_n'(x) = 0$, *i.e.* $J_n'(\nu'_{nm}) = 0$. The following orthogonality property holds:

$$\int_0^b J_n\left(\frac{\nu'_{nm}}{b}\rho\right) J_n\left(\frac{\nu'_{nk}}{b}\rho\right) \rho\, d\rho = \begin{cases} 0, & m \neq k \\ \\ \dfrac{b^2}{2}\left(1 - \dfrac{n^2}{\nu'^2_{nm}}\right)\left[J_n(\nu'_{nm})\right]^2, & m = k \end{cases} \tag{1.32}$$

For any piecewise continuous function $f(\rho)$ we can write:

$$f(\rho) \sim \sum_{m=1}^{\infty} F_m J_n\left(\frac{\nu'_{nm}}{b}\rho\right) \tag{1.33}$$

in which the coefficients F_m are obtained by using the orthogonality property (1.32)

$$F_m = \frac{2}{b^2\left(1 - \dfrac{n^2}{\nu'^2_{nm}}\right)\left[J_n(\nu'_{nm})\right]^2} \int_0^b f(\rho) J_n\left(\frac{\nu'_{nm}}{b}\rho\right) \rho\, d\rho \tag{1.34}$$

Again this is called Fourier-Bessel expansion of $f(\rho)$. Note that the derivative of the series always converges to zero at $\rho = b$.

If the interval is $[a, b]$ and $a > 0$, then the SLP is regular and the general form of eigenfunctions would be $A_m J_n(\kappa_m \rho) + B_m N_n(\kappa_m \rho)$. The boundary conditions at $\rho = a$ and $\rho = b$ will determine the eigenvalues κ_m and we have similar orthogonality property between the eigenfunctions as well.

1.4 Recursion Relationships

Consider $Z_\nu(x)$ to be $J_\nu(x)$ or $N_\nu(x)$ or $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$ or any linear combination of these functions. Then, the following recursive formulas are applicable (ν can be any number):

$$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x) \tag{1.35}$$

$$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z'_\nu(x) \tag{1.36}$$

$$Z'_\nu(x) + \frac{\nu}{x} Z_\nu(x) = Z_{\nu-1}(x) \tag{1.37}$$

$$Z'_\nu(x) - \frac{\nu}{x} Z_\nu(x) = -Z_{\nu+1}(x) \tag{1.38}$$

$$[x^\nu Z_\nu(x)]' = x^\nu Z_{\nu-1}(x) \tag{1.39}$$

$$\left[x^{-\nu} Z_\nu(x)\right]' = -x^{-\nu} Z_{\nu+1}(x) \tag{1.40}$$

in particular $Z'_0(x) = -Z_1(x)$. Equation (1.39) and (1.40) are very useful when integrating over Bessel functions.

1.5 Series and Integral Relationships

$$e^{-jk\rho\cos\phi} = \sum_{n=-\infty}^{+\infty} (-j)^n J_n(k\rho) e^{jn\phi} \qquad e^{jk\rho\cos\phi} = \sum_{n=-\infty}^{+\infty} j^n J_n(k\rho) e^{jn\phi} \tag{1.41}$$

$$e^{-jk\rho\sin\phi} = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(k\rho) e^{jn\phi} \qquad e^{jk\rho\sin\phi} = \sum_{n=-\infty}^{+\infty} J_n(k\rho) e^{jn\phi} \tag{1.42}$$

In the following expressions $Z_n(x)$ and $B_n(x)$ can be any of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$, $H_n^{(2)}(x)$ or linear combinations of them. m, n, α, β are arbitrary real numbers.

$$\int Z_n(\alpha x) B_n(\beta x) x \, dx = x \, \frac{\beta Z_n(\alpha x) B_{n-1}(\beta x) - \alpha Z_{n-1}(\alpha x) B_n(\beta x)}{\alpha^2 - \beta^2} \tag{1.43}$$

$$= x \, \frac{\alpha Z_{n+1}(\alpha x) B_n(\beta x) - \beta Z_n(\alpha x) B_{n+1}(\beta x)}{\alpha^2 - \beta^2} \tag{1.44}$$

$$\int Z_n^2(\alpha x) x \, dx = \frac{x^2}{2} \left[Z_n^2(\alpha x) - Z_{n-1}(\alpha x) Z_{n+1}(\alpha x)\right] \tag{1.45}$$

$$\int x^{n+1} Z_n(x) \, dx = x^{n+1} Z_{n+1}(x) \tag{1.46}$$

$$\int x^{-n+1} Z_n(x) \, dx = -x^{-n+1} Z_{n+1}(x) \tag{1.47}$$

$$\int \frac{1}{x} Z_n(\alpha x) B_m(\alpha x) dx = \alpha x \frac{Z_n(\alpha x) B_{m+1}(\alpha x) - Z_{n+1}(\alpha x) B_m(\alpha x)}{n^2 - m^2} + \frac{Z_n(\alpha x) B_m(\alpha x)}{n + m} \tag{1.48}$$

$$\int Z_1(x) dx = -Z_0(x) \tag{1.49}$$

$$\int x Z_0(x) dx = x Z_1(x) \tag{1.50}$$

2 Modified Bessel Functions $I_\nu(x)$ and $K_\nu(x)$

Modified Bessel functions are solutions of the following differential equation:

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0 \tag{2.1}$$

which is called the modified Bessel's differential equation. The general solution of (2.1) can be written as a linear combination of the modified Bessel functions of the first and second kind:

$$AI_\nu(x) + BK_\nu(x)$$

When ν is not an integer ($\nu \neq n$) I_ν and $I_{-\nu}$ are linearly independent (principal) solutions of (2.1), however, we usually use $I_\nu(x)$ and $K_\nu(x)$ (also called Kelvin function) which is related to I_ν and $I_{-\nu}$:

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left[I_{-\nu}(x) - I_\nu(x)\right] \tag{2.2}$$

With a variable transformation $x = \kappa\rho$ equation (2.1) can be transformed into:

$$\rho^2 y'' + \rho y' - (\kappa^2\rho^2 + \nu^2)y = 0 \tag{2.3}$$

whose independent solutions are $I_\nu(\kappa\rho)$ and $K_\nu(\kappa\rho)$. When ν is an integer I_n and I_{-n} are *not independent* anymore and we have $I_{-n}(x) = I_n(x)$. Furthermore, for arbitrary ν we always have:

$$I_\nu(e^{\pm j\pi}x) = e^{\pm j\nu\pi} I_\nu(x) \qquad K_\nu(e^{\pm j\pi}x) = \mp j\pi\, I_\nu(x) + e^{\mp j\nu\pi} K_\nu(x) \tag{2.4}$$

Plots of the first three modified Bessel functions of the first and second kind are shown in Fig. 2.1 and Fig. 2.2, respectively.

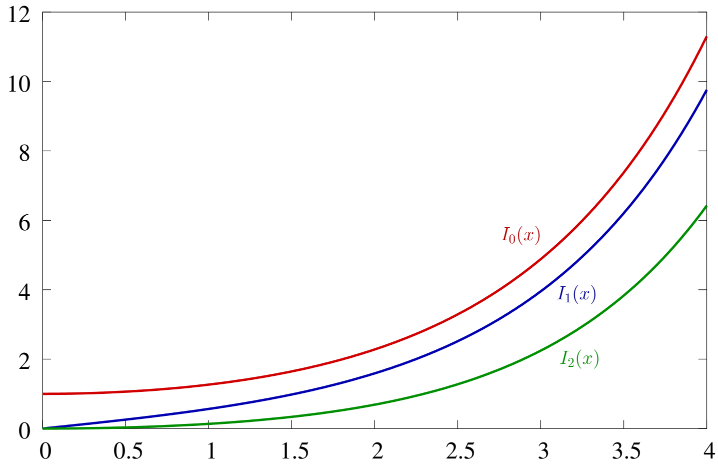

Figure 2.1: Modified Bessel functions of the first kind

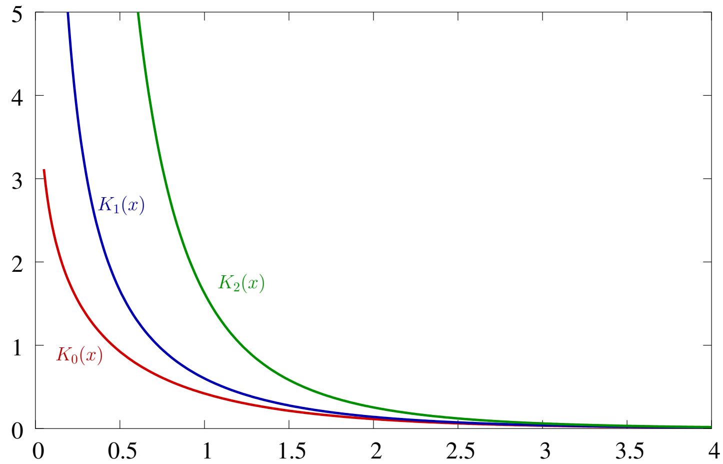

Figure 2.2: Modified Bessel functions of the second kind

2.1 Small and Large Argument Approximations

2.1.1 Small Argument Limit $|x| \longrightarrow 0$

$$I_0 \approx 1 + \frac{x^2}{4} \approx 1 \tag{2.5}$$

$$I_\nu(x) \approx \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)} \longrightarrow \frac{1}{n!}\left(\frac{x}{2}\right)^n \tag{2.6}$$

$$K_\nu(x) \approx \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \longrightarrow \frac{(n-1)!}{2}\left(\frac{2}{x}\right)^n \qquad \nu \neq 0 \tag{2.7}$$

$$K_0(x) \approx -\ln\frac{\gamma x}{2} \qquad \gamma = 1.78107241799\ldots \text{ Euler's constant} \tag{2.8}$$

2.1.2 Large Argument Limit $|x| \longrightarrow \infty$

$$I_\nu(x) \approx \frac{1}{\sqrt{2\pi x}}\, e^x \tag{2.9}$$

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{2.10}$$