

On Competitive Permutations for Set Cover by Intervals

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1. Introduction

In the $\max k$ cover problem, the input is a ground set P of n elements, and a family \mathcal{F} of m subsets of P , and an integer k . The task is to pick k sets of \mathcal{F} , that maximize the total number of elements of P covered. This problem is **NP-HARD** [GJ90], and the standard greedy algorithm, of repeatedly picking the set covering the largest number of elements not covered yet, has approximation ratio of $1 - 1/e$ [HP98]. If one wants to cover all the elements of P (but minimize the number of sets used), this is the **set cover** problem. In this case, the greedy algorithm provides a $(1 + \ln n)$ approximation. It is known that (essentially) no better approximation is possible [DS13, DS14] unless $\mathbf{P} = \mathbf{NP}$.

Diminishing returns. A natural property of the greedy solution is that the benefit of the i th set in the cover declines as i increases – that is, the i th greedy set covers no more elements than the previous sets. This phenomena is known as *diminishing returns*. Somewhat surprisingly, this phenomena does not hold for the optimal partial solutions. Specifically, the k th marginal value, is the increase in coverage as one moves from the optimal $(k - 1)$ -cover to the optimal k -cover. For the general set cover problem, the sequence of marginal values is not monotone, as we show in [Section 4](#).

Interval scheduling. A variant of the problem is where the ground set is on the real line, and the sets are intervals. This problem rises naturally in scheduling (i.e., time is the x -axis), known as **interval scheduling**. See [LT94, Spi99, CJST07, CFFN08] for related work.

Interval cover. The set cover problem becomes significantly easier if one is restricted to points and intervals on the real line. It is well known that the greedy algorithm adding intervals to the covers from left to right, always adding the one extending furthest to the right, computes the optimal solution. The k -cover variant can be solved using dynamic programming in $O(nk)$ time [GKK⁺09, EGK13, LLDL20].

Edwards *et al.* [EGK13] studied the **partial interval cover** problem, where needs to cover a specified number of points using minimum number of intervals. They also show that that the diminishing return property holds in this case for the optimal solution(s). Using this together with partitioning, they provide an $(1 + \epsilon)$ -approximation algorithm to the number of intervals. Their algorithm runs in $O(\epsilon^{-1}(n + m))$ time.

Competitive ratio for interval cover. A natural question is how to order the input sets (i.e., intervals in our case) so that they provide the best coverage for any prefix of this ordering. Conceptually, we are interested in ordering the intervals by the “usefulness” of the coverage they provide. The greedy algorithm naturally provides such an ordering, known as the *greedy permutation*. In particular, when

considering the first k intervals of the greedy permutation, and comparing it to the optimal k -cover, what is this competitive ratio? What is the competitive ratio when we consider all k ?

Since the optimal cover is unstable, and changes as one increases the number of intervals used, it is not a priori clear what is the best ordering if one wants to minimize the competitive ratio.

Our results.

We prove that the diminishing returns holds for intervals – we were unaware of the work by Edwards *et al.* [EGK13] who already proved it. Our proof is somewhat different, and we include it in the [Appendix A](#).

In [Section 3](#), we prove that the greedy permutation (for the case of intervals) provides a competitive ratio of $3/4$. We provide an example showing that this analysis is tight. This compares favorably with the general case, where the competitive ratio is $1 - 1/e \approx 0.6321 \ll 3/4$.

We provide an $(1 - \epsilon)$ -approximate algorithm with a complexity of $O(\frac{n}{\epsilon} + k \log(\epsilon k))$. The algorithm is also based on the diminishing returns property, and the partitioning method has been altered to adapt for a non-discrete model. In addition to the approximate algorithm, our research provided a tight approximate ratio of 0.75 for the greedy k interval cover, as a comparison to the approximate algorithm.

2. Preliminaries

2.1. Definitions and problem statement

Notations. In the following we deal with set systems. A set system (P, \mathcal{J}) has a ground set P (which is finite in our case), and a family \mathcal{J} of subsets of P . For a family of sets $\mathcal{X} \subseteq \mathcal{J}$, let $\cup \mathcal{X} = \cup_{X \in \mathcal{X}} X$. For a set $Y \subseteq P$, let

$$\mathcal{X} \sqcap Y = \{x \in Y \mid x \in \cup \mathcal{X}\} \subseteq P.$$

In the following, given a set X , and an element x , we use the notation $X - x = X \setminus \{x\}$, and similarly $X + x = X \cup \{x\}$.

For a set X , its *measure* is $\mu(X) = |X \cap P|$. For a set of intervals \mathcal{J} , its *measure* is $\mu(\mathcal{J}) = \mu(\cup \mathcal{J})$.

Definition 2.1. An instance of *interval cover* is a pair $\mathcal{J} = (P, \mathcal{J})$, where $P \subseteq \mathbb{R}$ is a set of points on the real line, and \mathcal{J} is a set of intervals.

The input. The input is an instance $\mathcal{J} = (P, \mathcal{J})$ of interval cover, where $n = |P|$ and $m = |\mathcal{J}|$. Specifically, let $\mathcal{J} = \{t_1, t_2, \dots, t_m\}$ be the set of intervals, sorted from left to right by their right endpoints (for simplicity of exposition, we assume all endpoints are distinct). Furthermore, we assume that no two intervals contains the same subset of points of P , and that no interval is contained inside another interval (as one would also use the bigger interval in a cover, and the smaller interval is as such redundant). Thus, the order of the intervals by their right endpoints, or by their left endpoints, is the same.

For two intervals t and j , let $t \prec j$ indicates that the left endpoint of t is to the left of the left endpoint of j – that is, t is to the *left* of j .

Definition 2.2. Let $\mathcal{J} = (P, \mathcal{J})$ the given instance of interval cover. A set $\mathcal{Y} \subseteq \mathcal{J}$ is *optimal k -cover*, if $|\mathcal{Y}| = k$, and the measure of \mathcal{Y} is maximum among all such sets of intervals of size k . Let $\mathcal{O}_k = \mathcal{O}_k(\mathcal{J}) = \mathcal{O}_k(P)$ denote such an optimal k -cover of the point set P . Let $\nu_k(\mathcal{J}) = \mu(\mathcal{O}_k(\mathcal{J}))$.

Problem definition. For a permutation π of the intervals of \mathcal{J} , its *k -competitive ratio*, for any $k > 0$, is

$$\rho_k(\pi) = \frac{\mu(\cup_{i=1}^k t_{\pi(i)})}{\nu_k}, \quad \text{where} \quad \nu_k = \mu(\mathcal{O}_k).$$

Its overall *competitive ratio* is $\rho(\pi) = \min_k \rho_k(\pi)$. The task at hand is to compute the permutation with competitive ratio as close to one as possible.

The greedy algorithm. This algorithm repeatedly picks the interval that covers the most points not yet covered, and add it to the current cover. Let g_1, g_2, \dots be the input intervals as ordered by the greedy algorithm. If there are several candidate intervals that cover the same number of points, the greedy algorithm always pick the leftmost such interval.

Diminishing returns and submodularity. An important property of the greedy algorithm is that the contribution of each added set decreases.

Definition 2.3. For a set of intervals \mathcal{X} , and an interval $t \in \mathcal{X}$, its *marginal value* is

$$\Delta(t, \mathcal{X}) = \mu(\mathcal{X}) - \mu(\mathcal{X} - t).$$

Definition 2.4. The *i th marginal profit* of an instance $J = (P, \mathcal{J})$ is the added value to the optimal solution by increasing the optimal solution to be of size i . Formally, it is the quantity

$$p_i(J) = \nu_i(J) - \nu_{i-1}(J)$$

see Definition 2.2. Observe that $\nu_k(J) = \sum_{i=1}^k p_i(J)$.

The following straightforward lemma shows that diminishing returns property holds for the greedy solution.

Lemma 2.5. *Consider any instance of set cover (not necessarily of points and intervals). Let $\mathcal{G}_i = \{g_1, \dots, g_i\}$ be the prefix of the first i intervals computed by the greedy algorithm. For all i , we have the *diminishing returns* property that*

$$\mu(\mathcal{G}_i) - \mu(\mathcal{G}_{i-1}) \geq \mu(\mathcal{G}_{i+1}) - \mu(\mathcal{G}_i).$$

Proof: If the diminishing property fails, then $\Delta(g_i, \mathcal{G}_i) < \Delta(g_{i+1}, \mathcal{G}_{i+1})$. This implies that $\mathcal{G}_i - g_i + g_{i+1}$ would cover more elements than \mathcal{G}_i , which is impossible as the greedy algorithm chooses g_i as the set that covers the largest number of elements that are yet uncovered by $\mathcal{G}_{i-1} = \mathcal{G}_i - g_i$. ■

Surprisingly, diminishing returns does not hold for the optimal solution – see Section 4. Our target function is submodular in the sense that an interval $t \in \mathcal{J}$ has lesser value as we add it into a bigger solution. Formally, $\mu(\cdot)$ is *submodular* if

$$\forall \mathcal{X}, \mathcal{Y}, t \quad \mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{J}, \text{ and } t \in \mathcal{J} \quad \mu(\mathcal{X} + t) - \mu(\mathcal{X}) \geq \mu(\mathcal{Y} + t) - \mu(\mathcal{Y}).$$

3. Competitiveness of the greedy algorithm

3.1. Extremality and allowable patterns

Observation 3.1. (A) For an optimal solution \mathcal{O}_t , and for any two distinct intervals $t, j \in \mathcal{O}_t$ that intersects, one can assume that they are extremal to each other. Specifically, if $t \prec j$, then one can assume that j is the right most interval that intersect t . This can be enforced by applying a greedy replacement of intervals on the optimal solution from left to right. Similarly, one can assume that t is the leftmost interval that intersects j . An optimal solution that has this property is **extremal**.

From this point on, we assume that all optimal solutions under discussion are extremal.

Lemma 3.2. Let \mathcal{O}_t be an (extremal) optimal solution, and let t be an interval not in \mathcal{O}_t . Then, t intersects at most two intervals of \mathcal{O}_t .

Proof: Any interval (that is not t) that intersects t must cover one of its endpoints, as no intervals contain each other. As such, if t intersects three intervals of \mathcal{O}_t , then two of them, say j and k , must cover one of the endpoints of t (say the left one). Furthermore, assume that $j \prec k$, see **Figure 1**.

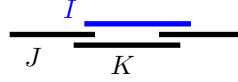


Figure 1

But then, as $t \notin \mathcal{O}_t$, it must be that $k \neq t$ and $k \prec t$. This contradicts the extremality property for $j \in \mathcal{O}_t$, as one can replace k by t in the optimal solution. ■

Observation 3.3. The extremality property implies that for any two intersecting intervals t, j in an optimal solution \mathcal{O}_t , with $t \prec j$, we have that no other interval $k \in \mathcal{O}_t$ intersects j , and $k \prec j$.

Lemma 3.4. Consider two optimal extremal solutions \mathcal{O} and \mathcal{P} , and let $t, j, k \in \mathcal{O} \cup \mathcal{P}$ be three consecutive intervals such that $t \prec j \prec k$ and $t \cap k \neq \emptyset$. Then, there are only two possibilities:

- (I) $t \in \mathcal{O} \setminus \mathcal{P}$, $j \in \mathcal{P} \setminus \mathcal{O}$, and $k \in \mathcal{O} \setminus \mathcal{P}$, or
- (II) $t \in \mathcal{P} \setminus \mathcal{O}$, $j \in \mathcal{O} \setminus \mathcal{P}$, and $k \in \mathcal{P} \setminus \mathcal{O}$.

Proof: The proof is by straightforward case analysis:

- (i) $t, j \in \mathcal{O}$. This is impossible as k can replace j in \mathcal{O} , contradicting the extremality of \mathcal{O} .

This implies that the following cases are impossible by symmetry:

- (ii) $t, j \in \mathcal{P}$.
- (iii) $j, k \in \mathcal{O}$.
- (iv) $j, k \in \mathcal{P}$.

This readily implies that it is impossible that $t \in \mathcal{O} \cap \mathcal{P}$, and the same holds for j and k . Thus, the only remaining possibilities are the ones stated in the lemma. ■

3.2. Competitive ratios

The following is well known and is included for the sake of completeness.

Lemma 3.5. *The competitive ratio of the greedy algorithm for set cover is $\geq 1 - 1/e$.*

Proof: Let \mathcal{O}_k denote an optimal solution of size k . In the beginning of the i th iteration, let $\Delta_i = \nu_k - \mu(\mathcal{G}_{i-1})$ be the deficit. There must be a set in \mathcal{O}_k that covers at least Δ_i/k elements that are not covered by the first $i - 1$ greedy sets. As such, the greedy algorithm picks a set that cover at least this number of elements (and potentially many more). As such, we have

$$\begin{aligned}\Delta_{i+1} &= \nu_k - \mu(\mathcal{G}_i) = \nu_k - \mu(\mathcal{G}_{i-1}) - \mu(\mathcal{G}_i \setminus \cup \mathcal{G}_{i-1}) \leq \Delta_i - \Delta_i/k = (1 - 1/k)\Delta_i \\ &\leq (1 - 1/k)^i \Delta_1 = (1 - 1/k)^i \nu_k.\end{aligned}$$

As such, we have

$$\rho(k) = \frac{\mu(\cup_{i=1}^k \mathcal{G}_i)}{\nu_k} = \frac{\nu_k - \Delta_{k+1}}{\nu_k} \geq \frac{\nu_k - (1 - 1/k)^k \nu_k}{\nu_k} \geq 1 - \frac{1}{e},$$

since $1 - x \leq \exp(-x)$, for $x \geq 0$. ■

Lemma 3.6. (A) *Consider an optimal k covering $\mathcal{O} = \mathcal{O}_k = \{\mathfrak{o}_1, \mathfrak{o}_2, \dots, \mathfrak{o}_k\} \subseteq \mathcal{I}$. For any interval $\mathfrak{o} \in \mathcal{O}$, we have that $\mathcal{O} - \mathfrak{o}$ is an optimal cover of $P \setminus \mathfrak{o}$ by $k - 1$ intervals.*

(B) *The set $\{\mathfrak{o}_{q+1}, \dots, \mathfrak{o}_k\}$ is an optimal cover by $k - q$ intervals of $P \setminus \bigcup_{i=1}^q \mathfrak{o}_i$.*

Proof: (A) Let \mathcal{O}' be an optimal cover of $P \setminus \mathfrak{o}$ by $k - 1$ intervals. If

$$|\mathcal{O}' \cap (P \setminus \mathfrak{o})| > |(\mathcal{O} - \mathfrak{o}) \cap (P - \mathfrak{o})|,$$

then $\mathcal{O}' + \mathfrak{o}$ covers more points of P than \mathcal{O} , which is a contradiction to the optimality of \mathcal{O} .

(B) Follows by repeated application of (A). ■

We remind the reader that the set of input intervals \mathcal{I} is made out intervals $\mathfrak{t}_1, \dots, \mathfrak{t}_m$, and they are sorted in increasing order by their left endpoint.

Lemma 3.7. *Let $\mathfrak{t} \in \mathcal{I}$ be an interval that covers the maximum number of points of P among all the intervals of \mathcal{I} . The interval \mathfrak{t} is either (i) in one of the optimal k covers, or (ii) alternatively, exactly two intervals $\mathfrak{j}, \mathfrak{j}' \in \mathcal{I}$ of an optimal cover overlap it, where $\mathfrak{j} \prec \mathfrak{t} \prec \mathfrak{j}'$.*

Proof: If \mathfrak{t} appears in an optimal k cover, then we are done. So assume \mathfrak{t} is not in any optimal cover, for all optimal k -covers. Fix such an optimal cover $\mathcal{O} = \mathcal{O}_k$. If the right endpoint of \mathfrak{t} is covered by two intervals of \mathcal{O} then one of them can be replaced by \mathfrak{t} , and yields an equivalent solution, a contradiction. Similarly, if the left endpoint of \mathfrak{t} is covered by two intervals in \mathcal{O} , the same argument applies. As \mathfrak{t} can not contain fully any interval of \mathcal{O} it follows that it intersects at most two such intervals.

If there was only one such interval, then one could just replace this interval by \mathfrak{t} , yielding an equivalent or better solution, which would imply (i).

The property that $\mathfrak{j} \prec \mathfrak{t} \prec \mathfrak{j}'$ readily follows, as one of the two intervals covers the left endpoint of \mathfrak{t} , and the other one covers the right endpoint of \mathfrak{t} . ■

Lemma 3.8. *We have $\mu(\mathcal{G}_k) \geq \mu(\mathcal{O}_{2k})/2$.*

Proof: If $k = 1$ the claim is immediate, as $\mu(\mathcal{O}_2) \leq 2\mu(\mathcal{O}_1) = 2\mu(\mathcal{G}_1)$. Consider the optimal $2k - 2$ cover \mathcal{O}'_{2k-2} of $P - \mathbf{g}_1$. There are several possibilities to consider:

If $\mathbf{g}_1 \in \mathcal{O}_{2k}$: Let \mathbf{o}_j be to any interval in $\mathcal{O}_{2k} \setminus \{\mathbf{g}_1\}$. Set \mathcal{O}''_{2k-2} to be the optimal $2k - 2$ cover of $P - \mathbf{g}_1 - \mathbf{o}_j$. By the optimality of \mathcal{O}'_{2k-2} , we have

$$\mu_{P-\mathbf{g}_1}(\mathcal{O}'_{2k-2}) \geq \mu_{P-\mathbf{g}_1-\mathbf{o}_j}(\mathcal{O}''_{2k-2}).$$

Recall that $2\mu(\mathbf{g}_1)$ is larger or equal to the coverage provided by any two intervals of \mathcal{O}_{2k} . By induction on the point set $P - \mathbf{g}_1$, we have by [Lemma 3.6](#) that

$$\begin{aligned} \mu(\mathcal{O}_{2k}) &= \mu(\mathbf{g}_1 \cup \mathbf{o}_j) + \mu_{P-\mathbf{g}_1-\mathbf{o}_j}(\mathcal{O}''_{2k-2}) \leq 2\mu(\mathbf{g}_1) + \mu_{P-\mathbf{g}_1}(\mathcal{O}'_{2k-2}) \\ &\leq 2\mu(\mathbf{g}_1) + 2\text{greedy}(P - \mathbf{g}_1, k - 1) = 2\mu(\mathcal{G}_k). \end{aligned}$$

Otherwise, $\mathbf{g}_1 \notin \mathcal{O}_{2k}$. By [Lemma 3.2](#), \mathbf{g}_1 intersects at most two intervals of \mathcal{O}_{2k} . Let \mathbf{o}_1 and \mathbf{o}'_1 be these two intervals. Set $\mathcal{O}'' = \mathcal{O}_{2k} - \mathbf{o}_1 - \mathbf{o}'_1$. By [Lemma 3.6](#), \mathcal{O}'' is an optimal $2k - 2$ cover of $P - \mathbf{o}_1 - \mathbf{o}'_1$, and by construction it does not cover any point of \mathbf{g}_1 . As such

$$\mu_{P-\mathbf{g}_1}(\mathcal{O}'_{2k-2}) \geq \mu_{P-\mathbf{g}_1}(\mathcal{O}'') = \mu_P(\mathcal{O}'') \geq \mu_{P-\mathbf{o}_1-\mathbf{o}'_1}(\mathcal{O}'').$$

As $2\mu(\mathbf{g}_1) \geq \mu(\mathbf{o}_1 \cup \mathbf{o}'_1)$, by induction we have

$$\begin{aligned} \mu(\mathcal{O}_{2k}) &= \mu(\mathbf{o}_1 \cup \mathbf{o}'_1) + \mu_{P-\mathbf{o}_1-\mathbf{o}'_1}(\mathcal{O}'') \leq 2\mu(\mathbf{g}_1) + \mu_{P-\mathbf{g}_1}(\mathcal{O}'_{2k-2}) \\ &\leq 2\mu(\mathbf{g}_1) + 2\text{greedy}(P - \mathbf{g}_1, k - 1) = 2\mu(\mathcal{G}_k). \end{aligned} \quad \blacksquare$$

3.2.1. The even case

Lemma 3.9. *If k is even, we have $\mu(\mathcal{G}_k) \geq 3\nu_k/4$, where ν_k is the optimal coverage by k intervals.*

Proof: Break the greedy k cover into two parts $\mathcal{G}' = \{\mathbf{g}_1, \dots, \mathbf{g}_{k/2}\}$ and $\mathcal{G}'' = \{\mathbf{g}_{k/2+1}, \dots, \mathbf{g}_k\}$. By [Lemma 3.8](#), $\mu(\mathcal{G}')/\nu_k \geq 1/2$.

Let $P' = P \setminus \bigcup \mathcal{G}'$. Observe that \mathcal{G}'' is the greedy $k/2$ cover of P' . Now, the optimal k cover of P' has value at least $\nu_k - \mu(\mathcal{G}')$. By [Lemma 3.8](#), we have $\mu_{P'}(\mathcal{G}'') \geq (\nu_k - \mu(\mathcal{G}'))/2$. As such, we have

$$\frac{\mu(\mathcal{G}_k)}{\nu_k} = \frac{\mu(\mathcal{G}') + \mu_{P'}(\mathcal{G}'')}{\nu_k} \geq \frac{\mu(\mathcal{G}') + (\nu_k - \mu(\mathcal{G}'))/2}{\nu_k} = \frac{\nu_k/2 + \mu(\mathcal{G}')/2}{\nu_k} \geq \frac{1}{2} + \frac{\mu(\mathcal{G}')}{2\nu_k} \geq \frac{3}{4},$$

since, by [Lemma 3.8](#), $\mu(\mathcal{G}')/\nu_k \geq 1/2$. \blacksquare

3.2.2. The odd case

Lemma 3.10. $\mu(\mathcal{G}_{k+1}) \geq \nu_{2k+1}/2 + \mu(\mathbf{g}_1)/4$.

Proof: Consider the interval \mathbf{g}_1 , by greedyness, it is the interval that covers the most points in the input \mathcal{I} .

First consider the case that $\mathbf{g}_1 \in \mathcal{O}_{2k+1}$. After removing \mathbf{g}_1 from P , the intervals $\mathbf{g}_2, \dots, \mathbf{g}_{k+1}$ are the greedy k cover of $P - \mathbf{g}_1$. For $\nu_{2k+1} = \nu_{2k+1}(P) = \mu(\mathcal{O}_{2k+1}(P))$, observe that $\nu_{2k+1} - \mu(\mathbf{g}_1) \leq \nu_{2k}(P - \mathbf{g}_1)$. By [Lemma 3.8](#) we have:

$$\mu(\mathcal{G}_{k+1}) = \mu(\text{greedy}(P - \mathbf{g}_1, k)) + \mu(\mathbf{g}_1) \geq \frac{\nu_{2k+1} - \mu(\mathbf{g}_1)}{2} + \mu(\mathbf{g}_1) \geq \frac{\nu_{2k+1} + \mu(\mathbf{g}_1)}{2}.$$

The other possibility is that $\mathfrak{g}_1 \notin \mathcal{O}_{2k+1}$. Then by [Lemma 3.7](#), \mathfrak{g}_1 intersects at most two intervals \mathfrak{o}_1 and \mathfrak{o}'_1 , where $\mathfrak{o}_1, \mathfrak{o}'_1 \in \mathcal{O}_{2k+1}$. Let $\mathfrak{m} = (\mathfrak{g}_1 \cap \mathfrak{o}_1) \setminus \mathfrak{o}'_1$ and $\mathfrak{m}' = (\mathfrak{g}_1 \cap \mathfrak{o}'_1) \setminus \mathfrak{o}_1$, and observe that they both contained in \mathfrak{g}_1 , and are disjoint. Assume, with loss of generality, that $\mu(\mathfrak{m}') \leq \mu(\mathfrak{m})$. This implies that $\mu(\mathfrak{m}') \leq \mu(\mathfrak{g}_1)/2$. Thus, we have

$$\begin{aligned} \mu(\cup \mathcal{O}_{2k+1} \setminus (\mathfrak{g}_1 \cup \mathfrak{o}_1)) &= \mu(\cup \mathcal{O}_{2k+1} \setminus ((\mathfrak{g}_1 \cap \mathfrak{o}_1) \cup (\mathfrak{g}_1 \cap \mathfrak{o}'_1) \cup \mathfrak{o}_1)) \\ &= \nu_{2k+1} - \mu(\mathfrak{o}_1) - \mu(\mathfrak{m}') \geq \nu_{2k+1} - \mu(\mathfrak{g}_1) - \frac{\mu(\mathfrak{g}_1)}{2} = \nu_{2k+1} - \frac{3}{2}\mu(\mathfrak{g}_1), \end{aligned}$$

as $\mu(\mathfrak{g}_1) \geq \mu(\mathfrak{o}_1)$.

Thus when the points covered by \mathfrak{g}_1 are removed from P , the union size of the remaining optimal $2k$ cover is at least $\mu(\cup \mathcal{O}_{2k+1} - (\mathfrak{g}_1 \cup \mathfrak{o}_1))/2$. By [Lemma 3.8](#), we have

$$\mu(\text{greedy}(P - \mathfrak{g}_1, 2k)) \geq \frac{\mu(\mathcal{O}_{2k}(P - \mathfrak{g}_1))}{2} \geq \frac{\mu(\cup \mathcal{O}_{2k+1} - (\mathfrak{g}_1 \cup \mathfrak{o}_1))}{2} \geq \frac{\nu_{2k+1}}{2} - \frac{3\mu(\mathfrak{g}_1)}{4}.$$

As such, we have

$$\mu(\mathcal{G}_{k+1}) = \mu(\mathfrak{g}_1) + \mu(\text{greedy}(P - \mathfrak{g}_1, k)) \geq \mu(\mathfrak{g}_1) + \frac{\nu_{2k+1}}{2} - \frac{3\mu(\mathfrak{g}_1)}{4} \geq \frac{\nu_{2k+1}}{2} + \frac{\mu(\mathfrak{g}_1)}{4}. \quad \blacksquare$$

Definition 3.11. For a set of intervals \mathcal{X} , and an interval $t \in \mathcal{X}$, its *marginal interval* is

$$\nabla t = \nabla_{\mathcal{X}} t = t - \bigcup (\mathcal{X} \setminus \{t\}).$$

Lemma 3.12. For a set of intervals $\mathcal{X} \subset \mathcal{I}$, such that no interval of \mathcal{I} contains another interval of \mathcal{I} , any interval $t \in \mathcal{I}$ intersects at most two marginal intervals of \mathcal{X} .

Proof: The marginal intervals of \mathcal{X} are disjoint. As such, an interval t intersecting three marginal intervals $\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \mathfrak{m}_3$, would have to contain the original interval of \mathcal{X} inducing \mathfrak{m}_2 , which is impossible. \blacksquare

Lemma 3.13. For any non-negative integer k , we have $\mu(\mathcal{G}_{2k+1}) \geq 3\nu_{2k+1}/4$.

Proof: If $k = 0$, the claim is immediate because $\mu(\mathcal{G}_1) = \nu_1$.

If $k \geq 1$, we separate the greedy $2k + 1$ cover into two parts: $\mathcal{G}_k = \{\mathfrak{g}_1, \dots, \mathfrak{g}_k\}$ and $\mathcal{G}'' = \{\mathfrak{g}_{k+1}, \dots, \mathfrak{g}_{2k+1}\}$. By [Lemma 3.12](#), we have that $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ intersect at most $2k$ marginal intervals of \mathcal{O}_{2k+1} . Let t be the interval of \mathcal{O}_{2k+1} , such that $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ do no overlap its marginal interval $\mathfrak{m} = \nabla_{\mathcal{O}_{2k+1}} t_x$.

By [Lemma 3.8](#), we have that

$$\mu(\mathcal{G}_k) \geq \frac{\nu_{2k}}{2} \geq \frac{\nu_{2k+1} - \Delta(t, \mathcal{O}_{2k+1})}{2} = \frac{\nu_{2k+1} - \mu(\mathfrak{m})}{2}. \quad (1)$$

Let $P' = P - \bigcup \mathcal{G}_k$. Since \mathfrak{m} does not intersect any intervals in \mathcal{G}_k , we have that the largest of the remaining intervals over P' is at least of size $\Delta(t_x, \mathcal{O}_{2k+1})$. Thus by [Lemma 3.10](#),

$$\beta = \mu(\text{greedy}(P', k+1)) \geq \frac{\nu_{2k+1}(P')}{2} + \frac{\mu(\mathfrak{g}_{k+1})}{4} \geq \frac{\mu_{P'}(\mathcal{O}_{2k+1})}{2} + \frac{\mu(\mathfrak{m})}{4} \geq \frac{\nu_{2k+1} - \mu(\mathcal{G}_k)}{2} + \frac{\mu(\mathfrak{m})}{4}.$$

Thus, by [Eq. \(1\)](#), we have

$$\mu(\mathcal{G}_{2k+1}) = \mu(\mathcal{G}_k) + \beta \geq \frac{\nu_{2k+1}}{2} + \frac{\mu(\mathcal{G}_k)}{2} + \frac{\mu(\mathfrak{m})}{4} \geq \frac{\nu_{2k+1}}{2} + \frac{\nu_{2k+1} - \mu(\mathfrak{m})}{4} + \frac{\mu(\mathfrak{m})}{4} \geq \frac{3}{4}\nu_{2k+1}. \quad \blacksquare$$

3.3. The result

Theorem 3.14. *The greedy algorithm for cover by intervals has competitive ratio at least $3/4$, for any prefix of the greedy permutation computed by the algorithm.*

Proof: Combining Lemma 3.13 and Lemma 3.9, implies that for any k , we have $\mu(\mathcal{G}_k) \geq (3/4)\nu_k$. ■

Lemma 3.15. *The $3/4$ competitive ratio of the greedy algorithm is tight.*

Proof: Consider three intervals t_1, t_2, t_3 , where $\mu(t_1) = \mu(t_2) = s$, and $\mu(t_3) = s + \varepsilon$. In this example, t_1 and t_2 are connected end-to-end. Let the connecting point of t_1 and t_2 be the median point of t_3 . As such, the greedy two cover would first include t_3 , then one of t_1 and t_2 . The union size of greedy two cover is $\frac{3s+\varepsilon}{2}$. The optimal two cover has a union size of $2s$. The competitive ratio is $\frac{3}{4} + \frac{\varepsilon}{2s}$. ■

4. Diminishing returns do not hold for optimal set cover

4.1. The construction

Let $\tau > 1$ be some arbitrary integer. In the following, pick some arbitrary rational numbers $\alpha, \beta, \gamma \in (0, 1)$, such that $0 < \alpha < \beta < \gamma$,

$$\beta - \alpha < \gamma - \beta, \quad \text{and} \quad \frac{\gamma}{\tau + 2} < \frac{\beta}{\tau + 1} < \frac{\alpha}{\tau}.$$

We have a ground set U – this set is going to be a sufficiently large finite set (more on that below). For a set $X \subseteq U$, its *measure* is

$$\mu(X) = \frac{|X|}{|U|}.$$

In the following we pick some sets from the ground set – how exactly we do that so that we have the desired properties listed below is described in Section 4.3.

We pick τ disjoint sets B_1, \dots, B_τ from U , each one of measure α/τ .

Next, we pick $\tau + 1$ disjoint sets $C_1, \dots, C_{\tau+1}$ each one of measure $\beta/(\tau + 1)$, such that for all i, j , we have

$$\mu(B_i \cap C_j) = \frac{\alpha}{\tau(\tau + 1)}.$$

Finally, we pick disjoint sets $D_1, \dots, D_{\tau+2}$ each one of measure $\gamma/(\tau + 2)$, such that for all i and j we have

$$\mu(B_i \cap D_j) = \frac{\alpha}{\tau(\tau + 2)}.$$

Similarly, for all i and j , we require that

$$\mu(C_i \cap D_j) = \frac{\beta}{(\tau + 1)(\tau + 2)}.$$

4.1.1. Realization in three dimensions

The above construction can be realized in three dimensions using axis-parallel cubes if one uses volume for measure. So, assume $\alpha = 1$. Consider the unit cube in three dimensions $[0, 1]^3$. We set

$$B_i = [(i-1)/\tau, i/\tau] \times [0, 1]^2, \quad \text{for } i = 1, \dots, \tau.$$

That is, B_1, \dots, B_τ slices the unit cube into equal boxes along the x -axis.

Next, consider the enlarged cube $[0, \beta] \times [0, 1] \times [0, 1]$. We set $y = 1/(\tau + 1)$ and

$$C_i = [0, \beta] \times [(i-1)y, iy] \times [0, 1], \quad \text{for } i = 1, \dots, \tau + 1.$$

This is illustrated in **Figure 2**.

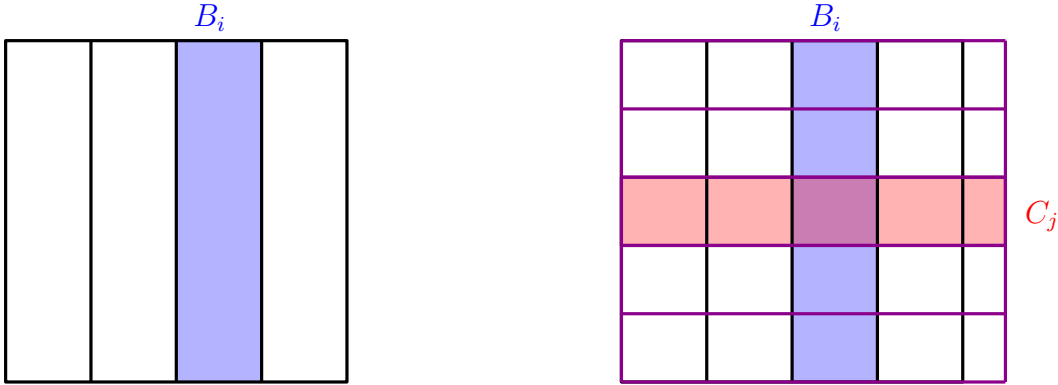


Figure 2

Finally, consider the cube $[0, \beta] \times [0, \gamma/\beta] \times [0, 1]$. We set $z = 1/(\tau + 2)$ and

$$D_i = [0, \beta] \times [0, \gamma/\beta] \times [(i-1)z, i/z], \quad \text{for } i = 1, \dots, \tau + 2.$$

It is easy to verify that this construction has the required properties from above.

4.2. Some properties

Here is a list of some easy properties that the construction has:

(I) For any i , we have that $\cup_j (B_i \cap C_j) = B_i$. Indeed, since the C_j s are disjoint, we have

$$\frac{\alpha}{k} = \mu(B_i) \geq \sum_j \mu(B_i \cap C_j) = (\tau + 1) \frac{\alpha}{\tau(\tau + 1)} = \frac{\alpha}{k}.$$

(II) $\cup_i B_i \subseteq \cup_j C_j$.

(III) Similarly, $\cup_j C_j \subseteq \cup_k D_k$.

Lemma 4.1. *The optimal cover by τ sets is B_1, \dots, B_{uu} .*

Proof: Indeed, the sets $C_1, \dots, C_{\tau+1}, D_1, \dots, D_{\tau+2}$ are smaller than B_i , for all i . Furthermore, the sets B_1, \dots, B_τ are disjoint, which implies that it is indeed the largest possible cover by τ sets. ■

Lemma 4.2. *The optimal cover by $\tau + 1$ sets is C_1, \dots, C_{uu+1} .*

Proof: Indeed, the set D_k is smaller than C_j , for all j and k . Furthermore, the sets $C_1, \dots, C_{\tau+1}$ are disjoint, which implies that any optimal cover by $\tau + 1$ sets can involve only B s and C s.

Consider a set C_j , and observe that

$$\mu(C_j \setminus \cup_i B_i) = \frac{\beta}{\tau + 1} - \sum_{i=1}^{\tau} \mu(B_i \cap C_j) = \frac{\beta}{\tau + 1} - \tau \frac{\alpha}{\tau(\tau + 1)} = \frac{\beta - \alpha}{\tau + 1} > 0,$$

since $\beta > \alpha$. This implies that each C_j contains elements that are not in any of the B_i s.

As such, any cover by $\tau + 1$ sets (made out of B_i s and C_j s) that does not include all sets of $C_1, \dots, C_{\tau+1}$, must fail to cover some element in $\cup_j C_j$. This implies that $\cup_j C_j$ is an optimal cover. ■

Lemma 4.3. *The optimal cover by $\tau + 2$ sets is D_1, \dots, D_{uu+2} .*

Proof: Consider a set D_k , and observe that

$$\mu(D_k \setminus \cup_j C_j) = \frac{\gamma}{\tau + 2} - \sum_{j=1}^{\tau+1} \mu(C_j \cap D_k) = \frac{\gamma}{\tau + 2} - (\tau + 1) \frac{\beta}{(\tau + 1)(\tau + 2)} = \frac{\gamma - \beta}{\tau + 2} > 0,$$

since $\gamma > \beta$. This implies that each D_k contains elements that are not in any of the C_j s (and thus also elements not covered by any of the B_i s). As such, any other cover by $\tau + 2$ sets fails to cover some element of $\cup_k D_k$, which being contained in this union, thus implying the claim. ■

4.3. How to pick the sets exactly

One can explicitly describe how to pick the sets, but we instead are going to use an existential argument that is easier to see. Pick n to be a sufficiently large, such that $\alpha n, \beta n, \gamma n$ are all integer numbers divisible by $\tau, \tau + 1$ and $\tau + 2$. Let $U = \{1, \dots, n\}$. Let $b = \alpha n / \tau$, and we set $B_i = \{(i - 1)b + 1, \dots, ib - 1\}$, for $i = 1, \dots, \tau$.

Next, we random assign each element of $\{1, \dots, \beta n\}$ to $C_1, \dots, C_{\tau+1}$ with probability $1/(\tau + 1)$. Standard application of Chernoff's inequality implies that

$$\Pr \left[\left| \mu(C_j) - \frac{\beta}{\tau + 1} \right| > \sqrt{\frac{c \log n}{n}} \right] < \frac{1}{n^{O(1)}}.$$

Namely, by picking n to be sufficiently large enough, we can assume the measure of the C_j s are arbitrarily close to the desired measure. Furthermore, using the same argumentation we have

$$\Pr \left[\left| \mu(B_i \cap C_j) - \frac{1}{\tau(\tau + 1)} \right| > \sqrt{\frac{c \log n}{n}} \right] < \frac{1}{n^{O(1)}}.$$

Finally, we chose the sets $D_1, \dots, D_{\tau+2}$, by assigning each element of $\{1, \dots, \gamma n\}$ to one of these sets with equal probability. Again, a Chernoff type argument implies that all the desired measures hold as desired within additive error that is arbitrarily small. Picking all these additive errors to be smaller than $(\beta - \alpha)/(\tau + 2)^{10}$ (say), implies that the above example implies the desired properties.

5. Algorithms

The input is a set of n points $P \subseteq \mathbb{R}$, and a set of m intervals \mathcal{J} . For simplicity of exposition, we assume that $m = O(n)$. Here, the measure of the coverage is $\mu_P(\mathfrak{t}) = |P \cap \mathfrak{t}|$.

5.1. Dynamic programming algorithm

The task at hand is to compute (exactly) the cover by k intervals of \mathcal{I} that maximizes the coverage

The algorithm starts by sorting, in $O(n \log n)$ time, all the intervals in \mathcal{I} in non-decreasing order of their left endpoint. For an interval t .

Definition 5.1. For an interval t , let t_{\leftarrow} be the first interval intersecting t in this sorted order. Let $\text{pred}(t)$ be the predecessor of t in this order – it is the interval immediately to the left of t .

Let $\mathcal{I}_{\leq t}$ be all the intervals of \mathcal{I} that are before t in this order (including t). Let $\text{opt}(t, t)$ to be the optimal cover by t intervals of $\mathcal{I}_{\leq t}$. Similarly, let $\text{opt}_{\in}(t, t)$ to be the optimal cover by t intervals of $\mathcal{I}_{\leq t}$ that must contain t in the cover.

Observation 5.2. Consider an interval $t \in \mathcal{I}$, and the cover $\mathcal{O} = \text{opt}_{\in}(t, t)$. Then, if there is an interval $t^+ \in \mathcal{O}$ to the left of t that intersects t , then $\mathcal{O} - t^+ + t_{\leftarrow}$ is an equivalent solution providing the same coverage, as $t^+ \subseteq t_{\leftarrow} \cup t$.

Note that t_{\leftarrow} can be precomputed, in linear time, for each interval $t \in \mathcal{I}$ by linear scanning (after the sorting).

When our algorithm is scanning from first interval to the last, it would monotonically discard all the intervals to the left of current t_{\leftarrow} .

After sorting and pre-processing comes to the dynamic programming algorithm. Let $d_{\in}(t, t) = \mu(\text{opt}_{\in}(t, t))$ and $d(t, t) = \mu(\text{opt}(t, t))$. We get the following recursive definitions of these two quantities

$$d_{\in}(t, t) = \max \begin{cases} \mu(t) & t = 1 \\ d_{\in}(t_{\leftarrow}, t-1) + \mu(t \setminus t_{\leftarrow}) & t > 1 \text{ and } t_{\leftarrow} \text{ is defined} \\ d(\text{pred}(t_{\leftarrow}), t-1) + \mu(t) & t > 1 \text{ and } \text{pred}(t_{\leftarrow}) \text{ is defined,} \end{cases}$$

and

$$d(t, t) = \max \begin{cases} d_{\in}(t, t) \\ d(\text{pred}(t), t). \end{cases}$$

Theorem 5.3. Given a set P of n points on the line, a set \mathcal{I} of m intervals, and a parameter k , one can compute the optimal k -cover of P by k intervals of \mathcal{I} , in $O(mk)$ time, assuming the intervals and points are presorted. Otherwise, the running time is $O(mk + (m+n) \log(m+n))$.

Furthermore, the execution of this algorithm can be resumed, to compute the optimal $k+1$ -cover in additional $O(m)$ time, and this can be done repeatedly, to compute an optimal t -cover, for any $t > k$, in $O((t-k)m)$ additional time.

Proof: Observe that, by sweeping from left to right, one can compute for all intervals $t \in \mathcal{I}$, the quantities $\mu(t) = |P \cap t|$ and $\mu(t \setminus t_{\leftarrow}) = |P \cap (t \setminus t_{\leftarrow})|$. This takes $O(n+m)$ time. The algorithm then follows by using dynamic programming using the recursive formalns provided above together with memoization. ■

5.2. An approximate algorithm for maximum k -coverage

5.2.1. Merging optimal solutions from disjoint instances.

Given two instances of interval cover $J = (P, \mathcal{J})$ and $J' = (P', \mathcal{J}')$ that are disjoint (that is $(\cup J) \cap (\cup J') = \emptyset$), consider computing the optimal marginal profits for the two instances. That is, for all i , let

$$q_i = p_i(J) \quad \text{and} \quad q'_i = p_i(J').$$

see [Definition 2.4](#). By [Theorem A.10](#) the sequences $\Pi \equiv q_1, q_2, \dots$ and $\Pi' \equiv q'_1, q'_2, \dots$ are non-increasing. As a reminder, the value of the optimal k -cover of J , is the sum of the first k elements in Π .

Next, consider the merged instance $K = (P \cup P', \mathcal{J} \cup \mathcal{J}')$, and the sorted non-decreasing sequence $\alpha_1 \geq \alpha_2 \geq \dots$ formed by merging Π and Π' .

Claim 5.4. *For any i , we have that $\alpha_i = p_i$, where $p_i = p_i(K)$.*

Proof: Consider the optimal i -cover of K . It uses α intervals of J and β intervals of J' , where $\alpha + \beta = i$. As such, the value of this optimal solution is

$$\sum_{i=1}^{\alpha} q_i + \sum_{i=1}^{\beta} q'_i.$$

If $q_{\alpha+1} > q_{\beta}$, then a better solution is formed by taking $\alpha + 1$ intervals of J and $\beta - 1$ intervals of J' . Similarly, if $q_{\alpha} < q_{\beta+1}$, then a better solution is formed by taking $\alpha - 1$ intervals of J and $\beta + 1$ intervals of J' . It follows that the optimal strategy is to always partition the optimal solution into the two subproblems according to the merged (sorted) sequence, which implies the claim. \blacksquare

5.2.2. Settings

The input is an instance $J = (P, \mathcal{J})$ of interval cover, and parameter ε and k . Here $n = |P|$ and $m = |\mathcal{J}|$, with $\mathcal{J} = \{t_1, \dots, t_m\}$.

The task at is to approximate the maximum coverage provided by k intervals of \mathcal{J} – specifically, the approximation algorithm would output a cover with $(1 + \varepsilon)k$ intervals, that has value better than optimal k -cover. In the following, assume the intervals of \mathcal{J} are sorted from left to right by their left endpoint, and the points of P are sorted similarly.

5.2.3. The algorithm

Let $\Delta = \lceil m/(\varepsilon k) \rceil$. For $K = \lfloor \varepsilon k \rfloor$, the algorithm takes the set of intervals

$$\mathcal{X} = \{t_{j\Delta} \mid j = 1, \dots, K\}$$

into the computed cover. Since no input interval contains another interval, this breaks the given instance into $K + 1$ instances, where the set of intervals for the j th instance is

$$\mathcal{J}_j = \{t_{j\Delta+1}, \dots, t_{(j+1)\Delta-1}\},$$

for $j = 0, \dots, K$. The algorithm computes the set $P' = P \setminus \cup \mathcal{X}$. For each set of intervals \mathcal{J}_j , the corresponding point set is

$$P_j = \mathcal{J}_j \cap P'.$$

Now, we have $K + 1$ “parallel” disjoint instances of interval cover. Formally, the i th instance is $J_i = (J_i, P_i)$, for $i = 0, \dots, K$. Each instance induces a sequence of marginal profits. Specifically, the sequence Π_i for the i th instance is $p_1(J_i) \geq p_2(J_i) \geq \dots$. By [Claim 5.4](#), we need to compute the first k elements in the merged (in non-increasing order) sequence formed by these $K + 1$ sequences.

Interpreting these sequences as streams, where we compute a number only if we need it, and using heap to perform these $K + 1$ -way merge results in the desired optimal solution. Specifically, for each sequence Π_i if the algorithm uses t_i elements from it in the current merged sequence, then the algorithm computes $t_i + 1$ elements of Π_i using [Theorem 5.3](#) on J_i . Whenever, a number in a sequence is being consumed by the algorithm, we compute the next number in the sequence using the “resume” procedure provided by [Theorem 5.3](#).

5.2.4. Analysis

The algorithm outputs a cover by $(1 + \varepsilon)k$ intervals.

Let ν be the value of the optimal cover by k intervals of $P' = P \setminus \cup \mathcal{X}$. Clearly, the algorithm outputs a cover of P' of the same value. It follows that the value of the solution computed by the algorithm is

$$\nu + \mu(\mathcal{X}) \geq \nu_k(J).$$

As for the running time, let e_i be the number of elements computed in the i th sequence by the end of the execution of the algorithm. We have that $u = \sum_i e_i = k + K = (1 + \varepsilon)k$. Computing an element in such a sequence takes $T = O(m/K) = O(m/(\varepsilon k))$. It follows that the running time, ignoring presorting, is $O(Tu) = O(m/\varepsilon)$.

5.2.5. The result

Theorem 5.5. *Given an instance (P, \mathcal{I}) of n points and m intervals, and parameters k and ε , one can a (partial) cover of P by $(1 + \varepsilon)k$ intervals, in $O(n + \frac{m}{\varepsilon} + k \log k)$ time, and $O(n + m/\varepsilon)$ space. If the points and intervals are not presorted, the running time becomes $O((n + m) \log(n + m) + \frac{m}{\varepsilon})$.*

The computed solution covers at least as many points of P as the optimal k -cover.

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A. Diminishing returns for optimal covers by intervals

In the following, we use \mathcal{O}_t to denote an optimal cover with t intervals.

A.1. Definitions and basic properties

A.1.1. Diminishing returns, and marginal value

For two sets \mathcal{X} and \mathcal{Y} , let $\mathcal{X} \oplus \mathcal{Y} = (\mathcal{X} \setminus \mathcal{Y}) \cup (\mathcal{Y} \setminus \mathcal{X})$ denote their symmetric difference.

Observation A.1. *For any sets $X, Y \subseteq \mathbb{R}$, we have that $\mu(X \cup Y) \leq \mu(X) + \mu(Y)$. As such, we have $\mu(X \cup Y) - \mu(X) \leq \mu(Y)$.*

Definition A.2. For an instance $J = (P, \mathcal{J})$ of interval cover, the **diminishing return** property at step k , for the optimal solution, states that $\mathbf{p}_k \geq \mathbf{p}_{k+1}$, where $\mathbf{p}_k = \mathbf{p}_k(J) = \mu(\mathcal{O}_k) - \mu(\mathcal{O}_{k-1})$.

Lemma A.3. *For any i and any $\mathbf{t} \in \mathcal{O}_t$, we have that $\Delta(\mathbf{t}, \mathcal{O}_i) \geq \mathbf{p}_i$, see [Definition 2.3](#).*

Proof: Observe that $\mu(\mathcal{O}_t - \mathbf{t}) \leq \mu(\mathcal{O}_{t-1})$ and $\mu(\mathcal{O}_t - \mathbf{t}) = \mu(\mathcal{O}_t) - \Delta(\mathbf{t}, \mathcal{O}_t)$. Combining we have $\mu(\mathcal{O}_{t-1}) \geq \mu(\mathcal{O}_t) - \Delta(\mathbf{t}, \mathcal{O}_t)$. ■

A.1.2. Decomposing the optimal solutions into runs

Let

$$\mathcal{X} = \mathcal{O}_{k+2} \cap \mathcal{O}_k$$

be the set of intervals that appear in both \mathcal{O}_k and \mathcal{O}_{k+2} . Specifically, let $k' = k - |\mathcal{X}|$, and let

$$\Xi = (\mathcal{O}_{k+2} \cup \mathcal{O}_k) \setminus \mathcal{X} = \{j_1, \dots, j_{2k'+2}\},$$

where the intervals are sorted in increasing order of their left endpoints. A **run** is a consecutive sequence of intervals $t_u, \dots, t_v \in \Xi$, such that:

- (i) any two consecutive intervals in the run intersect, and
- (ii) the odd intervals belong to \mathcal{O}_k and the even intervals belong to \mathcal{O}_{k+2} (or vice versa).

We partition Ξ into the unique maximal runs from left to right, and let $\Gamma_1, \dots, \Gamma_t$ be the resulting partition of Ξ . The uniqueness follows as the breakpoints between runs are pre-determined.

Definition A.4. The *balance* of a run Γ is $\chi(\Gamma) = |\Gamma \cap \mathcal{O}_{k+2}| - |\Gamma \cap \mathcal{O}_k|$ and it is either $-1, 0, +1$. For a single interval $t \in \mathcal{O}_k \cup \mathcal{O}_{k+2}$, we denote $\chi(t) = \chi(\{t\})$.

Observation A.5. Since $|\mathcal{O}_{k+2}| = |\mathcal{O}_k| + 2$, we have that $\sum_i \chi(\Gamma_i) = 2$. As such, there must be at least two runs in $\Gamma_1, \dots, \Gamma_t$ with balance one. That is, at least two runs that starts and ends with an interval of \mathcal{O}_{k+2} .

A.2. Proving the diminishing returns property

Lemma A.6. If there is an interval $t \in \mathcal{O}_{k+2}$ that does not intersect any interval of \mathcal{O}_k , then *Definition A.2* holds.

Proof: If $\Delta = \Delta(t, \mathcal{O}_{k+2}) < \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1})$ then

$$\mu(\mathcal{O}_{k+2} - t) = \mu(\mathcal{O}_{k+2}) - \Delta > \mu(\mathcal{O}_{k+2}) - (\mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1})) = \mu(\mathcal{O}_{k+1}),$$

which is a contradiction to the optimality of \mathcal{O}_{k+1} . Thus, $\Delta \geq \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1})$ and we have

$$\mu(\mathcal{O}_{k+1}) \geq \mu(\mathcal{O}_k + t) = \nu_k + \mu(t) \geq \nu_k + \Delta \geq \nu_k + \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1}),$$

which implies the claim. ■

Lemma A.7. If there is a run Γ of length one with $\chi(\Gamma) = 1$, then *Definition A.2* holds.

Proof: Let $\Gamma = \{t\}$, with $t \in \mathcal{O}_{k+2} \setminus \mathcal{X}$. The case not handled by *Lemma A.6*, is if t intersects some interval $j \in \mathcal{O}_k$. Assume that $j \prec t$ (the other case is handled symmetrically), and j is the rightmost interval with this property. If $j \in \mathcal{O}_k \setminus \mathcal{O}_{k+2}$, then there must be an interval $k \in \mathcal{O}_{k+2} \setminus \mathcal{O}_k$ such that $j \prec k \prec t$, as otherwise j and t would be in the same run, which contradicts Γ being of size one. However, this case is impossible by *Lemma 3.4*.

Thus, $j \in \mathcal{X}$. There are two possibilities:

- (I) There is another interval $\ell \in \mathcal{O}_k$ that intersects t . By the same argument $\ell \in \mathcal{X}$. But then, the interval t has the same two neighbors in \mathcal{O}_k and \mathcal{O}_{k+2} . By [Lemma 3.2](#) it has no other neighbors in \mathcal{O}_k or \mathcal{O}_{k+2} . Thus, removing t from \mathcal{O}_{k+2} , and adding it to \mathcal{O}_k results in two covers of size $k+1$. By [Lemma A.3](#), we have

$$\begin{aligned} \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1}) &\leq \Delta(t, \mathcal{O}_{k+2}) = \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+2} - t) = \mu(\mathcal{O}_k + t) - \nu_k \\ &\leq \mu(\mathcal{O}_{k+1}) - \nu_k, \end{aligned} \tag{2}$$

which establish the claim in this case.

- (II) There is no other interval $\ell \in \mathcal{O}_k$ that intersects t . Observe that

$$\Delta(t, \mathcal{O}_{k+2}) = \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+2} - t) \leq \mu(\mathcal{O}_k + t) - \nu_k \leq \mu(\mathcal{O}_{k+1}) - \nu_k$$

as \mathcal{O}_{k+2} potentially has one more interval that covers portions of t that is not present in \mathcal{O}_k . The claim now follows by [Lemma A.3](#) and arguing as in [Eq. \(2\)](#). \blacksquare

Lemma A.8. *Consider a run Γ , with $|\Gamma| > 1$, and $\chi(\Gamma) = 1$. Then any interval of $\mathcal{O}_k \setminus \Gamma$ that intersects Γ is in \mathcal{X} .*

Proof: Assume that Γ intersects some interval $j \in \mathcal{O}_k \setminus \Gamma$. And assume for the sake of contradiction that $j \notin \mathcal{X}$.

If j was to the left of all the intervals of Γ , then one assume that j is the rightmost such interval. Let $\ell \in \mathcal{O}_{k+2}$ be the first interval of Γ . There must be another interval $k \in \mathcal{O}_{k+2} \setminus \mathcal{X}$ such that $j \prec k \prec \ell$, as otherwise j would be part of the run. However, this situation is impossible by [Lemma 3.4](#), as $j \in \mathcal{O}_k$, $k, \ell \in \mathcal{O}_{k+2}$, and $j \cap \ell \neq \emptyset$.

Symmetric argument implies that j can not be to the right of all the intervals of Γ .

If j is in the middle between two consecutive intervals of the run, then we would have broken the run at j , which is again impossible. \blacksquare

Lemma A.9. *Let Γ be a run of balance one, with $|\Gamma| > 1$, then Γ can intersect at most two intervals of $t_{\leftarrow}, t_{\rightarrow} \in \mathcal{X}$, where $t_{\leftarrow} \prec t_{\rightarrow}$. Furthermore, t_{\leftarrow} (resp., t_{\rightarrow}) intersects only the first (resp. last) interval of Γ .*

Proof: Assume for contradiction that t_{\leftarrow} intersects some (internal) interval $j \in \Gamma \cap \mathcal{O}_k$, and observe that this would imply that j intersects three intervals of \mathcal{O}_{k+2} , which is impossible by [Lemma 3.2](#). The same analysis applies if $j \in \Gamma \cap \mathcal{O}_{k+2}$ and j is not the first or last interval of Γ .

Thus, it must be that j is the last or first interval of Γ , and both these intervals belong to \mathcal{O}_{k+2} . Same analysis applies to t_{\rightarrow} , implying that only the first and last intervals in Γ can intersect intervals of Γ .

As for the second part, assume for contradiction that the (say) the first interval j in Γ intersects two intervals $t, t' \in \mathcal{X}$, and observe that j then intersects three intervals that belongs to \mathcal{O}_k , which is impossible by [Lemma 3.2](#). The same argument applies to the last interval in the run Γ . \blacksquare

A.3. The result

Theorem A.10. *Let $J = (P, \mathcal{J})$ be an instance of interval cover. We have that $p_1 \geq p_2 \geq \dots$, where p_i is the i th marginal profit of J (see [Definition 2.4](#)).*

Proof: Consider the optimal solutions, $\mathcal{O}_k, \mathcal{O}_{k+1}, \mathcal{O}_{k+2}$, and their decomposition into runs. If there is a run of length one in this decomposition, with balance one, then the claim holds by [Lemma A.7](#). As there are at least two runs in the decomposition with balance one, by [Observation A.5](#), we consider such a run Γ , with $\chi(\Gamma) = 1$, and $|\Gamma| > 1$.

By [Lemma A.9](#) such a run can intersect at most two intervals of \mathcal{X} , and if so they intersect only the first and last intervals in the run. Let $t_{\leftarrow}, t_{\rightarrow} \in \mathcal{X}$ be these two intervals, and assume that $t_{\leftarrow} \prec t_{\rightarrow}$, with t_{\leftarrow} (resp., t_{\rightarrow}) intersect the first (resp., last) interval of Γ . If there is no interval t_{\leftarrow} (resp., t_{\rightarrow}) with this property, we take t_{\leftarrow} (resp., t_{\rightarrow}) to be the empty set.

The argument used in the proof of [Lemma A.8](#) implies that Γ does not intersect any interval of $(\mathcal{O}_k - t_{\leftarrow} - t_{\rightarrow}) \setminus \Gamma$.

If there is an interval $j \in \mathcal{O}_{k+2} \setminus \Gamma$ that intersects an interval $k \in \Gamma \cap \mathcal{O}_k$, then there are three intervals of \mathcal{O}_{k+2} that intersect k (two in the run, and j), which is impossible by [Lemma 3.2](#). So consider the sets

$$L_k = \cup((\mathcal{O}_k - t_{\leftarrow} - t_{\rightarrow}) \setminus \Gamma), \quad L_{k+2} = \cup(\mathcal{O}_{k+2} \setminus \Gamma), \quad R_k = \cup(\mathcal{O}_k \cap \Gamma), \quad \text{and} \quad R_{k+2} = \cup(\mathcal{O}_{k+2} \cap \Gamma).$$

The above argument implies that L_k is disjoint from both R_k and R_{k+2} , and similarly, L_{k+2} and R_k are disjoint. As such $\mu(L_k \cup R_{k+2}) = \mu(L_k) + \mu(R_{k+2})$, $\mu(L_k \cup R_k) = \mu(L_k) + \mu(R_k)$, and $\mu(L_{k+2} \cup R_k) = \mu(L_{k+2}) + \mu(R_k)$.

Consider the two covers $\mathcal{O}_k^+ = \mathcal{O}_k \oplus \Gamma$ and $\mathcal{O}_{k+2}^- = \mathcal{O}_{k+2} \oplus \Gamma$. We have that $|\mathcal{O}_k^+| = |\mathcal{O}_{k+2}^-| = k + 1$. Observe that $t_{\leftarrow}, t_{\rightarrow} \subseteq L_{k+2}$, and let $t = t_{\leftarrow} \cup t_{\rightarrow}$. We have that

$$\begin{aligned} \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1}) &\leq \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+2}^-) = \mu(L_{k+2} \cup R_{k+2}) - \mu(L_{k+2} \cup R_k) \\ &= \mu(L_{k+2} \cup (R_{k+2} \setminus t)) - \mu(L_{k+2}) - \mu(R_k) \\ &\leq \mu(R_{k+2} \setminus t) - \mu(R_k). \end{aligned}$$

Now, as $L_k \cup t$ is disjoint from $R_{k+2} \setminus t$ and R_k , we have

$$\begin{aligned} \mu(\mathcal{O}_{k+2}) - \mu(\mathcal{O}_{k+1}) &\leq \mu(L_k \cup t \cup (R_{k+2} \setminus t)) - \mu(L_k \cup t \cup R_k) \\ &= \mu(L_k \cup t \cup R_{k+2}) - \mu(L_k \cup t \cup R_k) = \mu(\mathcal{O}_k^+) - \nu_k \\ &\leq \mu(\mathcal{O}_{k+1}) - \nu_k, \end{aligned}$$

■